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Global Existence of Classical Solutions for a Class Nonlinear Parabolic Equations

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GLOBAL EXISTENCE OF CLASSICAL SOLUTIONS FOR A CLASS NONLINEAR PARABOLIC EQUATIONS

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I. Introduction

In this paper we investigate the Cauchy problem

$$u_t - (\Delta + m)u = \lambda |u|^{p-1}u, \quad t \in [0,\infty), x \in \mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $n \geq 2$ is fixed, $\Delta = \partial^2_{x_1} + \cdots + \partial^2_{x_n}$, $\lambda, m \in \mathbb{R}$, $p \geq 1$, $u_0(x) \in \mathcal{C}^2(\mathbb{R}^n)$, $|u_0(x)| \leq P$, $|(u_0)_{x_i}(x)| \leq P$ for every $i = 1, 2, \ldots, n$ and for every $x \in \mathbb{R}^n$, $P > 0$ is given constant. Here we propose new approach for invetsigating of this problem which gives new results.

The Cauchy problem for the equation (1.1) is investigated by many authors. For instance see [1] and references therein. Obviously the problem for existence of solutions to the Cauchy problem (1.1), (1.2) is connected with the Fujita's exponent, which depends of the dimension n and the values of the parameter p. Here we propose new conception about this problem. Our thesis is that this problem depends only of the integral representation for the solutions which is used. In this work we propose new integral representation. Our conception tell us that there are cases in which there is a global existence of solutions under specific set and local existence and blow up under another specific set. We will illustrate our new conception with the following example.

Example. Let $\lambda = -1$, $p = 3$, $m = 0$. Then

$$u(t,x) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{t+1}}$$

is a solution to the problem

$$\begin{aligned} u_t - \Delta u = -|u|^2 u, \quad t \in [0,\infty), \quad x \in \mathbb{R}^n, \\ u(0,x) = \tfrac{1}{\sqrt{2}}, \quad x \in \mathbb{R}^n. \end{aligned} \tag{1.3}$$

Really, for $u(t,x) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{t+1}}$ we have

$$u_t = -\frac{1}{2\sqrt{2}(t+1)^{\frac{3}{2}}},$$

$$u_t - \Delta u = -|u|^2 u \Longleftrightarrow$$

$$-\frac{1}{2\sqrt{2}(t+1)^{\frac{3}{2}}} = -\frac{1}{2(t+1)} \frac{1}{\sqrt{2}\sqrt{t+1}}.$$

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Ref.

1. Fujita, H. On the blowing up of solutions of the Cauchy problem for ut = u + u1+, J. Fac. Sci. Univ. Tokyo Sect. I 13 (1966), 109124.

This example and our main result we can consider as counter - example of the well known theory.

Our new nice result is due to our new integral representation.

This approach is used for hyperbolic equations in [2].

Our main result is

Theorem 1.1. *Let $n \geq 2$ be fixed, $p \geq 1$ be fixed, $P > 0$ be fixed, $\lambda, m \in \mathbb{R}$ be fixed, $u_0(x) \in \mathcal{C}^2(\mathbb{R}^n)$, $|u_0(x)| \leq P$, $|(u_0)_{x_i}(x)| \leq P$ for every $i = 1, 2, \ldots, n$ and for every $x \in \mathbb{R}^n$. Then the Cauchy problem* (1.1), (1.2) *has a solution $u \in \mathcal{C}^1([0, \infty), \mathcal{C}^2(\mathbb{R}^n))$.*

II. Proof of Theorem 1.1

Let ϵ is fixed so that $1 > \epsilon > 0$. For fixed $P > 0$ we choose the constants A_i, $i = 1, 2, \ldots, n$, so that

$$
\begin{aligned}
&|\lambda|(A_1A_2\ldots A_n)^2P^{p-1} + (3+|m|)(A_1A_2\ldots A_n)^2\\
&+(A_1A_2\ldots A_{n-1})^2 + (A_1A_2\ldots A_{n-2}A_n)^2 + \cdots + (A_2A_3\ldots A_n)^2 \leq (1-\epsilon),\\
&\Big(|\lambda|(A_1A_2\cdots A_{j-1}A_{j+1}\cdots A_n)^2P^{p-1} + (4+|m|)(A_1A_2\cdots A_{j-1}A_{j+1}\cdots A_n)^2\\
&+(A_1A_2\cdots A_{j-1}A_{j+1}\cdots A_{n-1})^2 + \cdots + (A_2\cdots A_{j-1}A_{j+1}\cdots A_n)^2\Big)A_j \leq (1-\epsilon)\\
&\forall j = 1, 2, \ldots, n,
\end{aligned} \tag{2.1}
$$

where $A_0 = A_{n+1} = 1$. There exist such constants A_1, A_2, ..., A_n, for which the inequalities (2.1) are possible. For instance when $1 > A_i > 0$ are enough small, $i = 1, 2, \ldots, n$.

With B_1 we will denote the set

$$B_1 = \Big\{x \in \mathbb{R}^n : 0 \leq x_i \leq A_i, \quad i = 1, 2, \ldots, n\Big\}.$$

Firstly we will prove that the Cauchy problem

$$u_t - (\Delta + m)u = \lambda |u|^{p-1}u, \quad t \in [0, 1], x \in B_1, \tag{2.2}$$

$$u(0, x) = u_0(x), \quad x \in B_1, \tag{2.3}$$

has a solution u for which $u \in \mathcal{C}^1([0, 1], \mathcal{C}^2(B_1))$. For this purpose we will use fixed point arguments. Therefore we have a need to define an operator whose fixed points satisfy the above Cauchy problem.

Our observation is

Lemma 2.1. *If $u \in \mathcal{C}^1([0, 1], \mathcal{C}^2(B_1))$ satisfies the integral equation*

$$
\begin{aligned}
&\lambda \int_0^t \int_x^A \int_z^A |u|^{p-1}udsdzdy + m\int_0^t\int_x^A\int_z^A u(y, s)dsdzdy + \sum_{i=1}^n \int_0^t \int_{\overline{x}_i}^A \int_{\overline{z}_i}^A u(y, \hat{s}_i)d\overline{s}_i d\overline{z}_i dy\\
&-\int_x^A\int_z^A u(t, s)dsdz + \int_x^A\int_z^A u_0(s)dsdz = 0, t \in [0, 1], x \in B_1,
\end{aligned} \tag{2.4}
$$

then u is a solution to the Cauchy problem (2.2), (2.3).

Here

$$\overline{s}_i = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n), \quad \hat{s}_i = (s_1, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_n),$$

$$\int_x^A = \int_{x_1}^{A_1}\cdots\int_{x_n}^{A_n}, \quad \int_{\overline{x}_i}^A = \int_{x_1}^{A_1}\cdots\int_{x_{i-1}}^{A_{i-1}}\int_{x_{i+1}}^{A_{i+1}}\cdots\int_{x_n}^{A_n}.$$

Proof. We differentiate in t the equality (2.4) and obtain

$$
\begin{aligned}
&\lambda \int_x^A\int_z^A |u|^{p-1}udsdz + m\int_x^A\int_z^A u(t, s)dsdz + \sum_{i=1}^n \int_{\overline{x}_i}^A\int_{\overline{z}_i}^A u(t, \hat{s}_i)d\overline{s}_i d\overline{z}_i\\
&-\int_x^A\int_z^A u_t(t, s)dsdz = 0.
\end{aligned}
$$

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Ref.
2. Georgiev, S. Global existence for nonlinear wave equation. Dyn. PDE, Vol. 7, No. 3, pp. 207-216.

Now we differentiate twice in x_1, after which twice in x_2 and etc. twice in x_n and we obtain

$$u_t - (\Delta + m)u = \lambda |u|^{p-1}u.$$

We put $t = 0$ in (2.4) and we obtain

$$-\int_x^A \int_z^A u(0,s)dsdz + \int_x^A \int_z^A u_0(s)dsdz = 0.$$

After we differentiate the last equality twice in x_1, after which twice in x_2 and etc. twice in x_n we obtain

$$u_0(x) = u(0,x).$$

Consequently $u(t,x)$ is a solution to the Cauchy problem (2.2), (2.3).

The above lemma motivate us to define the integral operator

$$L_{11}(u) = u(t,x) + m\int_0^t \int_x^A \int_z^A u(y,s)dsdzdy + \lambda \int_0^t \int_x^A \int_z^A |u|^{p-1}udsdzdy$$

$$+\sum_{i=1}^n \int_0^t \int_{\overline{x}_i}^A \int_{\overline{z}_i}^A u(y,\hat{s}_i)d\overline{s}_i d\overline{z}_i dy - \int_x^A \int_z^A \Big(u(t,s) - u_0(s)\Big)dsdz, \quad t \in [0,1], \quad x \in B_1.$$

Our aim is to prove that the operator L_{11} has a fixed point. We will use the following fixed point theorem.

Theorem 2.2. *(see [3], Corrolary 2.4, pp. 3231) Let X be a nonempty closed convex subset of a Banach space Y. Suppose that T and S map X into Y such that*

(i) *S is continuous, $S(X)$ resides in a compact subset of Y;*

(ii) *$T : X \longrightarrow Y$ is expansive and onto.*

Then there exists a point $x^\star \in X$ with $Sx^\star + Tx^\star = x^\star$.

Here we will use the following definition for expansive operator.

Definition. (see [3], pp. 3230) Let (X,d) be a metric space and M be a subset of X. The mapping $T : M \longrightarrow X$ is said to be expansive, if there exists a constant $h > 1$ such that

$$d(Tx,Ty) \geq hd(x,y) \quad \forall x,y \in M.$$

For this purpose we will use the representation of the operator L_{11} as follows

$$L_{11}(u) = T_{11}(u) + S_{11}(u),$$

where

$$T_{11}(u) = (1+\epsilon)u(t,x),$$

$$S_{11}(u) = -\epsilon u(t,x) + m\int_0^t \int_x^A \int_z^A u(y,s)dsdzdy + \lambda \int_0^t \int_x^A \int_z^A |u|^{p-1}udsdzdy$$

$$+\sum_{i=1}^n \int_0^t \int_{\overline{x}_i}^A \int_{\overline{z}_i}^A u(y,\hat{s}_i)d\overline{s}_i d\overline{z}_i dy - \int_x^A \int_z^A \Big(u(t,s) - u_0(s)\Big)dsdz.$$

Also, we define the sets

$$M_{11} = \Big\{u(t,x) \in \mathcal{C}^1([0,1],\mathcal{C}^2(B_1)), \quad \max_{t\in[0,1]} \max_{x\in B_1} \Big|u(t,x)\Big| \leq P,$$

$$\max_{t\in[0,1]} \max_{x\in B_1} \Big|u_{x_i}(t,x)\Big| \leq P, \quad i = 0,1,2,\dots,n\Big\},$$

$$N_{11} = \Big\{u(t,x) \in \mathcal{C}^1([0,1],\mathcal{C}^2(B_1)), \quad \max_{t\in[0,1]} \max_{x\in B_1} \Big|u(t,x)\Big| \leq (1+\epsilon)P,$$

$$\max_{t\in[0,1]} \max_{x\in B_1} \Big|u_{x_i}(t,x)\Big| \leq (1+\epsilon)P, \quad i = 0,1,2,\dots,n\Big\},$$

Ref.

3. Xiang, T., Rong Yuan. A class of expansive - type Krasnosel'skii ˙xed point theorems. Nonlinear Analysis, 71(2009), 3229-3239.

where $u_{x_0} = u_t$. In these sets we define a norm as follows

$$||u||_2 = \sup\Big\{\Big|u(t,x)\Big| : (t,x) \in [0,1] \times B_1\Big\}.$$

Lemma 2.3. *The sets M_{11} and N_{11} are closed, compact and convex spaces in $\mathcal{C}([0,1] \times B_1)$ in the sense of norm $|| \cdot ||_2$.*

Proof. We will prove our assertion for M_{11}.

Let $\{u_n\}$ is a sequence of elements of M_{11} and $u_n \longrightarrow_{n\longrightarrow\infty} u$ in the sense of the norm $|| \cdot ||_2$. Evidently $u \in \mathcal{C}([0,1] \times B_1)$ and $|u(t,x)| \le P$ for every $(t,x) \in [0,1] \times B_1$.

We suppose that $u \notin \mathcal{C}^1([0,1] \times B_1)$. Then there exists $j \in \{0,1,2,\ldots,n\}$ and $\epsilon > 0$ so that for every $\delta_1 = \delta_1(\epsilon) > 0$ and $|h| < \delta_1$, $h \neq 0$, $(x_0, x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n) \in [0,1] \times B_1$, we have

$$\Big|\frac{u(x_0, x_1, \cdots, x_{j-1}, x_j + h, x_{j+1}, \cdots, x_n) - u(x_0, x_1, \cdots, x_n)}{h}\Big| > \epsilon. \tag{2.5}$$

On the other hand since $u_n \in \mathcal{C}^1([0,1], \mathcal{C}^2(B_1))$ we have that there exists $\delta_2 = \delta_2(\epsilon) > 0$ so that from $|h| < \delta_2$, $h \neq 0$, $(x_0, x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n) \in [0,1] \times B_1$, we have

$$\Big|\frac{u_n(x_1, \cdots, x_{j-1}, x_j + h, x_{j+1}, \cdots, x_n) - u_n(x_1, \cdots, x_n)}{h}\Big| < \frac{\epsilon}{3}. \tag{2.6}$$

Also, from $u_n \longrightarrow_{n\longrightarrow\infty} u$ in the sense of the norm $|| \cdot ||_2$ we have for enough large n and $|h| < \min\{\delta_1, \delta_2\}$, $h \neq 0$, $(x_0, x_1, x_2, \cdots, x_{j-1}, x_j + h, x_{j+1}, \cdots, x_n) \in [0,1] \times B_1$ that

$$\Big|\frac{u_n(x_0,x_1,\cdots,x_{j-1},x_j+h,x_{j+1},\cdots,x_n)-u_n(x_0,x_1,\cdots,x_n)}{h} - \frac{u(x_0,x_1,\cdots,x_{j-1},x_j+h,x_{j+1},\cdots,x_n)-u(x_0,x_1,\cdots,x_n)}{h}\Big| < \frac{\epsilon}{3}. \tag{2.7}$$

Then from (2.7), (2.6), (2.5) we obtain for $|h| < \min\{\delta_1, \delta_2\}$, $h \neq 0$, $(x_0, x_1, x_2, \cdots, x_{j-1}, x_j + h, x_{j+1}, \cdots, x_n) \in [0,1] \times B_1$, for enough large n,

$$\epsilon < \Big|\frac{u(x_0,x_1,\cdots,x_{j-1},x_j+h,x_{j+1},\cdots,x_n)-u(x_0,x_1,\cdots,x_n)}{h}\Big|$$

$$\le \Big|\frac{u_n(x_0,x_1,\cdots,x_{j-1},x_j+h,x_{j+1},\cdots,x_n)-u_n(x_0,x_1,\cdots,x_n)}{h}\Big|$$

$$+\Big|\frac{u_n(x_0,x_1,\cdots,x_{j-1},x_j+h,x_{j+1},\cdots,x_n)-u_n(x_0,x_1,\cdots,x_n)}{h} - \frac{u(x_0,x_1,\cdots,x_{j-1},x_j+h,x_{j+1},\cdots,x_n)-u(x_0,x_1,\cdots,x_n)}{h}\Big| < 2\frac{\epsilon}{3},$$

which is a contradiction with our assumption that $u \notin \mathcal{C}^1([0,1] \times B_1)$.

Therefore $u \in \mathcal{C}^1([0,1] \times B_1)$

We suppose that $u \notin \mathcal{C}^1([0,1], \mathcal{C}^2(B_1))$. Then there exists $j \in \{1,2,\ldots,n\}$ and $\epsilon_1 > 0$ so that for every $\delta_3 = \delta_3(\epsilon_1) > 0$ and $|h| < \delta_3$, $h \neq 0$, $(x_0, x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n) \in [0,1] \times B_1$ we have

$$\Big|\frac{u_{x_j}(x_0, x_1, \cdots, x_{j-1}, x_j + h, x_{j+1}, \cdots, x_n) - u_{x_j}(x_0, x_1, \cdots, x_n)}{h}\Big| > \epsilon_1. \tag{2.8}$$

On the other hand since $u_n \in \mathcal{C}^1([0,1], \mathcal{C}^2(B_1))$ we have that there exists $\delta_4 = \delta_4(\epsilon_1) > 0$ so that from $|h| < \delta_2$, $h \neq 0$, $(x_0, x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n) \in [0,1] \times B_1$ we have

$$\Big|\frac{(u_n)_{x_j}(x_0, x_1, \cdots, x_{j-1}, x_j + h, x_{j+1}, \cdots, x_n) - (u_n)_{x_j}(x_0, x_1, \cdots, x_n)}{h}\Big| < \frac{\epsilon}{3}. \tag{2.9}$$

Notes

Also, from $u_n \longrightarrow_{n\longrightarrow\infty} u$ in the sense of the norm $||\cdot||_2$ we have for enough large n and $|h| < \min\{\delta_3, \delta_4\}$, $h \neq 0$, $(x_1, x_2, \cdots, x_{j-1}, x_j + h, x_{j+1}, \cdots, x_n) \in B_1$ that

$$\left|\frac{(u_n)_{x_j}(x_0,x_1,\cdots,x_{j-1},x_j+h,x_{j+1},\cdots,x_n)-(u_n)_{x_j}(x_0,x_1,\cdots,x_n)}{h} - \frac{u_{x_j}(x_0,x_1,\cdots,x_{j-1},x_j+h,x_{j+1},\cdots,x_n)-u_{x_j}(x_0,x_1,\cdots,x_n)}{h}\right| < \frac{\epsilon_1}{3}. \tag{2.10}$$

Then from (2.10), (2.9), (2.8) we obtain for $|h| < \min\{\delta_3, \delta_4\}$, $h \neq 0$, $(x_1, x_2, \cdots, x_{j-1}, x_j + h, x_{j+1}, \cdots, x_n) \in B_1$, for enough large n,

Notes

$$\epsilon_1 < \left|\frac{u_{x_j}(x_0,x_1,\cdots,x_{j-1},x_j+h,x_{j+1},\cdots,x_n)-u_{x_j}(x_0,x_1,\cdots,x_n)}{h}\right|$$

$$\leq \left|\frac{(u_n)_{x_j}(x_0,x_1,\cdots,x_{j-1},x_j+h,x_{j+1},\cdots,x_n)-(u_n)_{x_j}(x_0,x_1,\cdots,x_n)}{h}\right|$$

$$+\left|\frac{(u_n)_{x_j}(x_0,x_1,\cdots,x_{j-1},x_j+h,x_{j+1},\cdots,x_n)-(u_n)_{x_j}(x_0,x_1,\cdots,x_n)}{h} - \frac{u_{x_j}(x_0,x_1,\cdots,x_{j-1},x_j+h,x_{j+1},\cdots,x_n)-u_{x_j}(x_0,x_1,\cdots,x_n)}{h}\right| < 2\frac{\epsilon_1}{3},$$

which is a contradiction with our assumption that $u \notin \mathcal{C}^1([0,1], \mathcal{C}^2(B_1))$.

Therefore $u \in \mathcal{C}^1([0,1], \mathcal{C}^2(B_1))$.

Now we suppose that there exists $j \in \{0, 1, \ldots, n\}$ and $(\tilde{t}, \tilde{x}) \in [0,1] \times B_1$ so that

$$|u_{x_j}(\tilde{t}, \tilde{x})| > P.$$

Then there exists $\epsilon_2 > 0$ so that

$$|u_{x_j}(\tilde{t}, \tilde{x})| \geq P + \epsilon_2.$$

From here there exists $\delta_5 = \delta_5(\epsilon_2) > 0$ such that from $|h| < \delta_5$, $h \neq 0$, $(\tilde{t}, \tilde{x}_1, \ldots, \tilde{x}_{j-1}, \tilde{x}_j + h, \tilde{x}_{j+1}, \ldots \tilde{x}_n) \in [0,1] \times B_1$ we have

$$\left|\frac{u(\tilde{t}, \tilde{x}_1, \ldots, \tilde{x}_{j-1}, \tilde{x}_j + h, \tilde{x}_{j+1}, \ldots \tilde{x}_n) - u(\tilde{t}, \tilde{x})}{h}\right| \geq P + \epsilon_2.$$

On the other hand, since $u_n(\tilde{t}, \tilde{x}) \longrightarrow u(\tilde{t}, \tilde{x})$ in sense of $||\cdot||_2$, as $n \longrightarrow \infty$, follows that there exists $\delta_6 = \delta_6(\epsilon_2) > 0$ so that we have from $|h| < \delta_6$, $h \neq 0$, $(\tilde{t}, \tilde{x}_1, \ldots, \tilde{x}_{j-1}, \tilde{x}_j + h, \tilde{x}_{j+1}, \ldots \tilde{x}_n) \in [0,1] \times B_1$

$$\left|\frac{u_n(\tilde{t},\tilde{x}_1,\ldots,\tilde{x}_{j-1},\tilde{x}_j+h,\tilde{x}_{j+1},\ldots\tilde{x}_n)-u_n(\tilde{t},\tilde{x})}{h} - \frac{u(\tilde{t},\tilde{x}_1,\ldots,\tilde{x}_{j-1},\tilde{x}_j+h,\tilde{x}_{j+1},\ldots\tilde{x}_n)-u(\tilde{t},\tilde{x})}{h}\right| < \epsilon_2$$

and since $|(u_n)_{x_j}| \leq P$ in $[0,1] \times B_1$

$$\left|\frac{u_n(\tilde{t}, \tilde{x}_1, \ldots, \tilde{x}_{j-1}, \tilde{x}_j + h, \tilde{x}_{j+1}, \ldots \tilde{x}_n) - u_n(\tilde{t}, \tilde{x})}{h}\right| \leq P$$

for enough large n. From here, for enough large n and for $|h| < \min\{\delta_5, \delta_6\}$, $h \neq 0$, $(\tilde{t}, \tilde{x}_1, \ldots, \tilde{x}_{j-1}, \tilde{x}_j + h, \tilde{x}_{j+1}, \ldots \tilde{x}_n) \in [0,1] \times B_1$ we have

$$\epsilon_2 = P + \epsilon_2 - P$$

$$\leq \left|\frac{u(\tilde{t},\tilde{x}_1,\ldots,\tilde{x}_{j-1},\tilde{x}_j+h,\tilde{x}_{j+1},\ldots\tilde{x}_n)-u(\tilde{t},\tilde{x})}{h}\right| - \left|\frac{u_n(\tilde{t},\tilde{x}_1,\ldots,\tilde{x}_{j-1},\tilde{x}_j+h,\tilde{x}_{j+1},\ldots\tilde{x}_n)-u_n(\tilde{t},\tilde{x})}{h}\right|$$

$$\leq \left|\frac{u(\tilde{t},\tilde{x}_1,\ldots,\tilde{x}_{j-1},\tilde{x}_j+h,\tilde{x}_{j+1},\ldots\tilde{x}_n)-u(\tilde{t},\tilde{x})}{h} - \frac{u_n(\tilde{t},\tilde{x}_1,\ldots,\tilde{x}_{j-1},\tilde{x}_j+h,\tilde{x}_{j+1},\ldots\tilde{x}_n)-u_n(\tilde{t},\tilde{x})}{h}\right| < \epsilon_2,$$

which is a contradiction. Therefore $|u_{x_j}| \leq P$ in $[0,1] \times B_1$ for every $j = 0, 1, \ldots, n$. Consequently $u \in M_{11}$ and M_{11} is closed in $\mathcal{C}([0,1] \times B_1)$ in sense of $||\cdot||_2$. Using Arzela - Ascoli Theorem the set M_{11} is a compact set in $\mathcal{C}([0,1] \times B_1)$ in sense of $||\cdot||_2$.

Let now $\lambda \in [0,1]$ is arbitrary chosen and fixed and $u_1, u_2 \in M_{11}$. Then for $(t,x) \in [0,1] \times B_1$ we have $\lambda u_1(t,x) + (1-\lambda)u_2(t,x) \in \mathcal{C}^1([0,1], \mathcal{C}^2(B_1))$ and

$$|u_i(t,x)| \le P, |u_{ix_j}(t,x)| \le P \quad \text{for} \quad j = 0,1,\dots,n, i = 1,2,$$

$$|\lambda u_1(t,x) + (1-\lambda)u_2(t,x)| \le \lambda|u_1(t,x)| + (1-\lambda)|u_2(t,x)| \le \lambda P + (1-\lambda)P = P,$$

$$|\lambda u_{1x_j}(t,x) + (1-\lambda)u_{2x_j}(t,x)| \le \lambda|u_{1x_j}(t,x)| + (1-\lambda)|u_{2x_j}(t,x)| \le \lambda P + (1-\lambda)P = P, \quad j = 0,1,\dots,n.$$

Therefore M_{11} is convex.

As in above we can prove that N_{11} is closed, compact and convex in $\mathcal{C}([0,1] \times B_1)$ in sense of $||\cdot||_2$.

Notes

Lemma 2.4. *The operator $T_{11} : M_{11} \longrightarrow N_{11}$ is an expansive operator and onto.*

Proof. Let $u \in M_{11}$. Then $u \in \mathcal{C}^1([0,1], \mathcal{C}^2(B_1))$, from here $(1+\epsilon)u \in \mathcal{C}^1([0,1], \mathcal{C}^2(B_1))$, i.e. $T_{11}(u) \in \mathcal{C}^1([0,1], \mathcal{C}^2(B_1))$ and $|T_{11}(u)| \le (1+\epsilon)P$, $|(T_{11}(u))_{x_i}| \le (1+\epsilon)P$, $i = 0,1,2,\dots,n$. Let now $u, v \in M_{11}$. Then we have

$$||T_{11}(u) - T_{11}(v)||_2 = (1+\epsilon)||u-v||_2.$$

From here and above follows that $T_{11} : M_{11} \longrightarrow N_{11}$ is an expansive operator.

Now we will see that $T_{11} : M_{11} \longrightarrow N_{11}$ is onto. Really, let $v \in N_{11}$, $v \neq 0$. Let also $u = \frac{v}{1+\epsilon}$. From here $\max_{t\in[0,1]} \max_{x\in B_1} |u| \le P$, $\max_{t\in[0,1]} \max_{x\in B_1} |u_{x_i}| \le P$ for $i = 0,1,2,\dots,n$. Therefore $u \in M_{11}$ and the operator $T_{11} : M_{11} \longrightarrow N_{11}$is onto.

Lemma 2.5. *We have*

$$S_{11} : M_{11} \longrightarrow M_{11}$$

is continuous.

Proof. Let $u \in M_{11}$ is arbitrary chosen element. Then, using the definition of the operator S_{11}, we have

$$|S_{11}(u)| \le \epsilon|u(t,x)| + |m| \int_0^t \int_x^A \int_z^A |u(y,s)|dsdzdy + |\lambda| \int_0^t \int_x^A \int_z^A |u|^p dsdzdy$$

$$+ \sum_{i=1}^n \int_0^t \int_{\overline{x}_i}^A \int_{\overline{z}_i}^A |u(y,\hat{s}_i)|d\overline{s}_i d\overline{z}_i dy + \int_x^A \int_z^A \Big(|u(t,s)| + |u_0(s)|\Big) dsdz$$

$$\le \epsilon P + |m|(A_1A_2\dots A_n)^2 P + |\lambda|(A_1A_2\dots A_n)^2 P^p + 3(A_1A_2\dots A_n)^2 P$$

$$+ P\Big((A_1A_2\dots A_{n-1})^2 + (A_1A_2\dots A_{n-2}A_n)^2 + \dots + (A_2A_3\dots A_n)^2\Big) \le \epsilon P + (1-\epsilon)P = P.$$

Also,

$$(S_{11}(u))_t = -\epsilon u_t(t,x) + m \int_x^A \int_z^A u(t,s)dsdz + \lambda \int_x^A \int_z^A |u|^{p-1} u dsdz + \sum_{i=1}^n \int_{\overline{x}_i}^A \int_{\overline{z}_i}^A u(t,\hat{s}_i) d\overline{s}_i d\overline{z}_i$$

$$- \int_x^A \int_z^A u_t(t,s)dsdz.$$

Then

$$|(S_{11}(u))_t| \le \epsilon|u_t(t,x)| + |m| \int_x^A \int_z^A |u(t,s)|dsdz + |\lambda| \int_x^A \int_z^A |u|^p dsdz$$

$$+ \sum_{i=1}^n \int_{\overline{x}_i}^A \int_{\overline{z}_i}^A |u(t,\hat{s}_i)|d\overline{s}_i d\overline{z}_i + \int_x^A \int_z^A |u_t(t,s)|dsdz$$

$$\le \epsilon P + |\lambda|(A_1A_2\dots A_n)^2 P^p + (2+|m|)(A_1A_2\dots A_n)^2 P$$

$$+ P\Big((A_1A_2\dots A_{n-1})^2 + (A_1A_2\dots A_{n-2}A_n)^2 + \dots + (A_2A_3\dots A_n)^2\Big)$$

$$\le \epsilon P + (1-\epsilon)P = P.$$

Notes

Let now $j=1,2,\cdots,n$, is arbitrary chosen and fixed. Then

$$\Big(S_{11}(u)\Big)_{x_j}=-\epsilon u_{x_j}(t,x)-m\int_0^t\int_{\overline{x}_j}^A\int_{\hat{z}_j}^A u(y,s)dsd\overline{z}_jdy-\lambda\int_0^t\int_{\overline{x}_j}^A\int_{\hat{z}_j}^A|u|^{p-1}udsd\overline{z}_jdy$$

$$-\sum_{i=1,i\neq j}^n\int_0^t\int_{\overline{\overline{x}}_{ij}}^A\int_{\hat{\overline{z}}_{ij}}^A u(y,\hat{s}_i)d\overline{s}_id\overline{z}_idy+\int_0^t\int_{\overline{x}_j}^A\int_{\overline{z}_j}^A u_{x_j}(y,\hat{s}_j)d\overline{s}_jd\overline{z}_jdy$$

$$+\int_{\overline{x}_j}^A\int_{\hat{z}_j}^A\Big(u(t,\hat{s}_j)-u_0(\hat{s}_j)\Big)d\hat{s}_jd\overline{z}_j,$$

where

$$\int_{\overline{\overline{x}}_{ij}}^A=\int_{x_1}^{A_1}\cdots\int_{x_{i-1}}^{A_{i-1}}\int_{x_{i+1}}^{A_{i+1}}\cdots\int_{x_{j-1}}^{A_{j-1}}\int_{x_{j+1}}^{A_{j+1}}\cdots\int_{x_n}^{A_n},$$

$$\int_{\hat{z}_j}^A=\int_{z_1}^{A_1}\cdots\int_{z_{j-1}}^{A_{j-1}}\int_{x_j}^{A_j}\int_{z_{j+1}}^{A_{j+1}}\cdots\int_{z_n}^{A_n},$$

and from here

$$\Big|\Big(S_{11}(u)\Big)_{x_j}\Big|\leq\epsilon|u_{x_j}(t,x)|+|m|\int_0^t\int_{\overline{x}_j}^A\int_{\hat{z}_j}^A|u(y,s)|dsd\overline{z}_jdy+|\lambda|\int_0^t\int_{\overline{x}_j}^A\int_{\hat{z}_j}^A|u|^pdsd\overline{z}_jdy$$

$$+\sum_{i=1,i\neq j}^n\int_0^t\int_{\overline{\overline{x}}_{ij}}^A\int_{\hat{\overline{z}}_{ij}}^A|u(y,\hat{s}_i)|d\overline{s}_id\overline{z}_idy+\int_0^t\int_{\overline{x}_j}^A\int_{\overline{z}_j}^A|u_{x_j}(y,\hat{s}_j)|d\overline{s}_jd\overline{z}_jdy$$

$$+\int_{\overline{x}_j}^A\int_{\hat{z}_j}^A\Big(|u(t,\hat{s}_j)|+|u_0(\hat{s}_j)|\Big)d\hat{s}_jd\overline{z}_j$$

$$\leq\epsilon P+|\lambda|(A_1A_2\cdots A_{j-1}A_{j+1}\cdots A_n)^2A_jP^p+(4+|m|)(A_1A_2\cdots A_{j-1}A_{j+1}\cdots A_n)^2A_jP$$

$$+\Big((A_1A_2\cdots A_{j-1}A_{j+1}\cdots A_{n-1})^2+\cdots+(A_2\cdots A_{j-1}A_{j+1}\cdots A_n)^2\Big)A_jP\leq\epsilon P+(1-\epsilon)P=P.$$

Consequently

$$S_{11}:M_{11}\longrightarrow M_{11}.$$

From the above estimates for $|S_{11}(u)|$, $|(S_{11}(u))_{x_j}|$, $j=0,1,2,\dots,n$, follows that if $v_n\longrightarrow_{n\longrightarrow\infty}v$ in the sense of the topology of the set M_{11}, $v_n,v\in M_{11}$, we have that $S_{11}(v_n)\longrightarrow_{n\longrightarrow\infty}S_{11}(v)$ in the sense of topology of the set M_{11}. Therefore the operator $S_{11}:M_{11}\longrightarrow M_{11}$ is a continuous operator.

Using Lemma 2.3, Lemma 2.4, Lemma 2.5 we apply Theorem 2.2 as the operator S in Theorem 2.2 corresponds of S_{11}, the operator T in Theorem 2.2 corresponds of T_{11}, the set X in Theorem 2.2 corresponds of M_{11} and Y in Theorem 2.2 corresponds of N_{11} and therefore follows that there exists $u^{11}\in M_{11}$ so that $u^{11}=S(u^{11})+T(u^{11})$, i.e. u^{11} is a fixed point of the operator L_{11}. From here and Lemma 2.1 follows that u^{11} is a solution to the Cauchy problem (2.2), (2.3), for which $u^{11}\in\mathcal{C}^1([0,1],\mathcal{C}^2(B_1))$.

Now we define the set

$$B_2=\Big\{x\in\mathbb{R}^n:A_1\leq x_1\leq 2A_1,0\leq x_i\leq A_i,\quad i=2,\dots,n\Big\},$$

the operators

$$L_{12}(u)=u(t,x)+m\int_0^t\int_x^A\int_z^A u(y,s)dsdzdy+\lambda\int_0^t\int_x^A\int_z^A|u|^{p-1}udsdzdy$$

$$+\sum_{i=2}^n\int_0^t\int_{\overline{x}_i}^A\int_{\overline{z}_i}^A u(y,\hat{s}_i)d\overline{s}_id\overline{z}_idy$$

$$-\int_x^A\int_z^A\Big(u(t,s)-u^{11}(0,s)\Big)dsdz$$

$$+\int_0^t\int_{\overline{x}_1}^A\int_{\overline{z}_1}^A(u(y,x_1,s_2,\dots,s_n)-u^{11}(y,A_1,s_2,\dots,s_n)+(A_1-x_1)u^{11}_{x_1}(y,A_1,s_2,\dots,s_n))d\overline{s}_1d\overline{z}_1dy,$$

$t\in[0,1]$, $\quad x\in B_2$,

$$L_{12}(u)=T_{12}(u)+S_{12}(u),$$

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where

$T_{12}(u)=(1+\epsilon)u(t,x), \quad t\in[0,1], x\in B_2,$

$S_{12}(u)=-\epsilon u(t,x)+m\int_0^t\int_x^A\int_z^A u(y,s)dsdzdy+\lambda\int_0^t\int_x^A\int_z^A|u|^{p-1}udsdzdy$

$+\sum_{i=2}^n\int_0^t\int_{\overline{x}_i}^A\int_{\overline{z}_i}^A u(y,\hat{s}_i)d\overline{s}_id\overline{z}_idy$

$-\int_x^A\int_z^A\Big(u(t,s)-u^{11}(0,s)\Big)dsdz$

$+\int_0^t\int_{\overline{x}_1}^A\int_{\overline{z}_1}^A(u(y,x_1,s_2,\dots,s_n)-u^{11}(y,A_1,s_2,\dots,s_n)+(A_1-x_1)u^{11}_{x_1}(y,A_1,s_2,\dots,s_n))d\overline{s}_1d\overline{z}_1dy,$

$t\in[0,1], x\in B_2,$

the sets

$$M_{12}=\Big\{u(t,x)\in\mathcal{C}^1([0,1],\mathcal{C}^2(B_2)),\quad \max_{t\in[0,1]}\max_{x\in B_2}\Big|u(t,x)\Big|\le P,$$

$$\max_{t\in[0,1]}\max_{x\in B_2}\Big|u_{x_i}(t,x)\Big|\le P,\quad i=0,1,2,\dots,n\Big\},$$

$$N_{12}=\Big\{u(t,x)\in\mathcal{C}^1([0,1],\mathcal{C}^2(B_2)),\quad \max_{t\in[0,1]}\max_{x\in B_2}\Big|u(t,x)\Big|\le(1+\epsilon)P,$$

$$\max_{t\in[0,1]}\max_{x\in B_2}\Big|u_{x_i}(t,x)\Big|\le(1+\epsilon)P,\quad i=0,1,2,\dots,n\Big\},$$

in these sets we define a norm as follows

$$||u||_2=\sup\Big\{\Big|u(t,x)\Big|:(t,x)\in[0,1]\times B_2\Big\}.$$

The sets M_{12} and N_{12} are closed, convex and compact in $\mathcal{C}([0,1]\times B_2)$ in sense of $||\cdot||_2$.

As in above we conclude that there exists $u^{12}\in M_{12}$ so that $L_{12}u^{12}=u^{12}$, i.e. u^{12} is a solution to the Cauchy problem

$$u_t-(\Delta+m)u=\lambda|u|^{p-1}u,\quad t\in[0,1], x\in B_2,$$
$$u(0,x)=u^{11}(0,x), x\in B_2.$$

For u^{12} we have

$m\int_0^t\int_x^A\int_z^A u^{12}(y,s)dsdzdy+\lambda\int_0^t\int_x^A\int_z^A|u^{12}|^{p-1}u^{12}dsdzdy$

$+\sum_{i=2}^n\int_0^t\int_{\overline{x}_i}^A\int_{\overline{z}_i}^A u^{12}(y,\hat{s}_i)d\overline{s}_id\overline{z}_idy$

$-\int_x^A\int_z^A\Big(u^{12}(t,s)-u^{11}(0,s)\Big)dsdz$

$+\int_0^t\int_{\overline{x}_1}^A\int_{\overline{z}_1}^A(u^{12}(y,x_1,s_2,\dots,s_n)-u^{11}(y,A_1,s_2,\dots,s_n)+(A_1-x_1)u^{11}_{x_1}(y,A_1,s_2,\dots,s_n))d\overline{s}_1d\overline{z}_1dy=0.$ (2.11)

Now we put $x_1=A_1$ in the last equality and we obtain

$$\int_0^t\int_{\overline{x}_1}^A\int_{\overline{z}_1}^A(u^{12}(y,A_1,s_2,\dots,s_n)-u^{11}(y,A_1,s_2,\dots,s_n))d\overline{s}_1d\overline{z}_1dy=0$$

and we differentiate it in t, twice in x_2 and etc. twice in x_n and we obtain

$$u^{12}_{|x_1=A_1}=u^{11}_{|x_1=A_1}.$$

Now we differentiate in x_1 the equality (2.11), after which we put $x_1=A_1$ and we obtain

$$\int_0^t\int_{\overline{x}_1}^A\int_{\overline{z}_1}^A(u^{12}_{x_1}(y,A_1,s_2,\dots,s_n)-u^{11}_{x_1}(y,A_1,s_2,\dots,s_n))d\overline{s}_1d\overline{z}_1dy=0$$

Notes

and we differentiate it in t, twice in x_2 and etc. twice in x_n and we obtain

$$(u^{12}_{x_1})_{|x_1=A_1} = (u^{11}_{x_1})_{|x_1=A_1}.$$

Using the equalities

$$u^{11}_t - (\Delta+m)u^{11} = \lambda|u^{11}|^{p-1}u^{11}, \quad u^{12}_t - (\Delta+m)u^{12} = \lambda|u^{12}|^{p-1}u^{12},$$

$$u^{11}_{|x_1=A_1} = u^{12}_{|x_1=A_1}, (u^{11}_{x_1})_{|x_1=A_1} = (u^{12}_{x_1})_{|x_1=A_1}$$

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we conclude that

$$(u^{11}_{x_1x_1})_{|x_1=A_1} = (u^{12}_{x_1x_1})_{|x_1=A_1}.$$

In this way we obtain that the function

$$u = \begin{cases} u^{11} & t\in[0,1], 0\le x_1\le A_1, 0\le x_2\le A_2,\dots,0\le x_n\le A_n,\\ u^{12} & t\in[0,1], A_1\le x_1\le 2A_1, 0\le x_2\le A_2,\dots,0\le x_n\le A_n\end{cases}$$

is a solution to the Cauchy problem

$$u_t - (\Delta+m)u = \lambda|u|^{p-1}u, \quad t\in[0,1], 0\le x_1\le 2A_1, 0\le x_2\le A_2,\dots,0\le x_n\le A_n,$$

$$u(0,x) = u_0(x), 0\le x_1\le 2A_1, 0\le x_2\le A_2,\dots,0\le x_n\le A_n.$$

Repeat the above steps in x_1, x_2 and etc. x_n we obtain a solution u_1 to the Cauchy problem (1.1), (1.2) which belongs to the space $\mathcal{C}^1([0,1]),\mathcal{C}^2(\mathbb{R}^n))$.

Now we consider the Cauchy problem

$$\begin{aligned} &u_t - (\Delta+m)u = \lambda|u|^{p-1}u, \quad t\in[1,2], \quad x\in B_1,\\ &u(1,x) = u_1(1,x), \quad x\in B_1.\end{aligned} \tag{2.12}$$

For this purpose we consider the operator

$$L^1_{11}(u) = T^1_{11}(u) + S^1_{11}(u),$$

where

$$T^1_{11}(u) = (1+\epsilon)u(t,x),$$

$$S^1_{11}(u) = -\epsilon u(t,x) + m\int_1^t\int_x^A\int_z^A u(y,s)dsdzdy + \lambda\int_1^t\int_x^A\int_z^A |u|^{p-1}udsdzdy$$

$$+\sum_{i=1}^n \int_1^t\int_{\overline{x}_i}^A\int_{\overline{z}_i}^A u(y,\hat{s}_i)d\overline{s}_id\overline{z}_idy - \int_x^A\int_z^A\Big(u(t,s)-u_1(1,s)\Big)dsdz.$$

Also, we define the sets

$$M^1_{11} = \Big\{u(t,x)\in\mathcal{C}^1([1,2],\mathcal{C}^2(B_1)), \quad \max_{t\in[1,2]}\max_{x\in B_1}\Big|u(t,x)\Big|\le P,$$

$$\max_{t\in[1,2]}\max_{x\in B_1}\Big|u_{x_i}(t,x)\Big|\le P, \quad i=0,1,2,\dots,n\Big\},$$

$$N^1_{11} = \Big\{u(t,x)\in\mathcal{C}^1([1,2],\mathcal{C}^2(B_1)), \quad \max_{t\in[1,2]}\max_{x\in B_1}\Big|u(t,x)\Big|\le(1+\epsilon)P,$$

$$\max_{t\in[1,2]}\max_{x\in B_1}\Big|u_{x_i}(t,x)\Big|\le(1+\epsilon)P, \quad i=0,1,2,\dots,n\Big\}.$$

In these sets we define a norm as follows

$$||u||_2 = \sup\Big\{\Big|u(t,x)\Big| : (t,x)\in[1,2]\times B_1\Big\},$$

these sets are closed, convex and compact in $\mathcal{C}([1,2]\times B_1)$ in sense of $||\cdot||$.

As in above we conclude that the problem (2.12) has a solution $u_1^{11}\in\mathcal{C}^1([1,2],\mathcal{C}^2(B_1))$.

Now we consider the Cauchy problem

$$u_t-(\Delta+m)u=\lambda|u|^{p-1}u,\quad t\in[1,2],\quad x\in B_2, \qquad (2.13)$$
$$u(1,x)=u_1(1,x)$$

$$L_{12}^1(u)=u(t,x)+m\int_1^t\int_x^A\int_z^A u(y,s)dsdzdy+\lambda\int_1^t\int_x^A\int_z^A|u|^{p-1}udsdzdy$$

$$+\sum_{i=2}^n\int_1^t\int_{\overline{x}_i}^A\int_{\overline{z}_i}^A u(y,\hat{s}_i)d\overline{s}_id\overline{z}_idy$$

$$-\int_x^A\int_z^A\Big(u(t,s)-u_1(1,s)\Big)dsdz$$

$$+\int_1^t\int_{\overline{x}_1}^A\int_{\overline{z}_1}^A(u(y,x_1,s_2,\ldots,s_n)-u_1^{11}(y,A_1,s_2,\ldots,s_n)+(A_1-x_1)u_{x_1}^{11}(y,A_1,s_2,\ldots,s_n))d\overline{s}_1d\overline{z}_1dy,$$

$$t\in[1,2],\quad x\in B_2,$$

$$L_{12}^1(u)=T_{12}^1(u)+S_{12}^1(u),$$

where

$$T_{12}^1(u)=(1+\epsilon)u(t,x),\quad t\in[1,2],x\in B_2,$$

$$S_{12}^1(u)=-\epsilon u(t,x)+m\int_1^t\int_x^A\int_z^A u(y,s)dsdzdy+\lambda\int_1^t\int_x^A\int_z^A|u|^{p-1}udsdzdy$$

$$+\sum_{i=2}^n\int_1^t\int_{\overline{x}_i}^A\int_{\overline{z}_i}^A u(y,\hat{s}_i)d\overline{s}_id\overline{z}_idy$$

$$-\int_x^A\int_z^A\Big(u(t,s)-u_1(1,s)\Big)dsdz$$

$$+\int_1^t\int_{\overline{x}_1}^A\int_{\overline{z}_1}^A(u(y,x_1,s_2,\ldots,s_n)-u_1^{11}(y,A_1,s_2,\ldots,s_n)+(A_1-x_1)u_{1_{x_1}}^{11}(y,A_1,s_2,\ldots,s_n))d\overline{s}_1d\overline{z}_1dy,$$

$$t\in[1,2],x\in B_2,$$

the sets

$$M_{12}^1=\Big\{u(t,x)\in\mathcal{C}^1([1,2],\mathcal{C}^2(B_2)),\quad \max_{t\in[1,2]}\max_{x\in B_2}\Big|u(t,x)\Big|\le P,$$
$$\max_{t\in[1,2]}\max_{x\in B_2}\Big|u_{x_i}(t,x)\Big|\le P,\quad i=0,1,2,\ldots,n\Big\},$$

$$N_{12}^1=\Big\{u(t,x)\in\mathcal{C}^1([1,2],\mathcal{C}^2(B_2)),\quad \max_{t\in[1,2]}\max_{x\in B_2}\Big|u(t,x)\Big|\le(1+\epsilon)P,$$
$$\max_{t\in[1,2]}\max_{x\in B_2}\Big|u_{x_i}(t,x)\Big|\le(1+\epsilon)P,\quad i=0,1,2,\ldots,n\Big\},$$

in these sets we define a norm as follows

$$||u||_2=\sup\Big\{\Big|u(t,x)\Big|:(t,x)\in[1,2]\times B_2\Big\}.$$

The sets M_{12}^1 and N_{12}^1 are closed, convex and compact in $\mathcal{C}([1,2]\times B_2)$ in sense of $||\cdot||_2$. As in the step 1 we conclude that the problem (2.13) has a solution $u_1^{12}\in\mathcal{C}^1([1,2],\mathcal{C}^2(B_2))$.

Since $u_1^{12}(1,x)=u_1(1,x)$ for $x\in B_2$ and $u_1^{11}(1,x)=u_1(1,x)$ for $x\in B_1$ we have that

$$u_1^{12}{}_{|_{t=1,x_1=A_1}}=u_1{}_{|_{t=1,x_1=A_1}},$$

$$(u_1^{12})_{x_1}{}_{|_{t=1,x_1=A_1}}=(u_1)_{x_1}{}_{|_{t=1,x_1=A_1}},$$

Notes

$$\begin{aligned}
&(u_1^{12})_{x_1x_1|_{t=1,x_1=A_1}} = (u_1)_{x_1x_1|_{t=1,x_1=A_1}}, \\
&u_1^{11}{}_{|_{t=1,x_1=A_1}} = u_{1_{|_{t=1,x_1=A_1}}}, \\
&(u_1^{11})_{x_1|_{t=1,x_1=A_1}} = (u_1)_{x_1|_{t=1,x_1=A_1}}, \\
&(u_1^{11})_{x_1x_1|_{t=1,x_1=A_1}} = (u_1)_{x_1x_1|_{t=1,x_1=A_1}},
\end{aligned} \tag{2.14}$$

Notes

as in the case $t \in [0,1]$ we have

$$\begin{aligned}
&u_1^{12}{}_{|_{x_1=A_1}} = u^{11}_{1_{|_{x_1=A_1}}}, \\
&(u_1^{12})_{x_1|_{x_1=A_1}} = (u_1^{11})_{x_1|_{x_1=A_1}}, \\
&(u_1^{12})_{x_1x_1|_{x_1=A_1}} = (u_1^{11})_{x_1x_1|_{x_1=A_1}}
\end{aligned} \tag{2.15}$$

for $t \in [1,2]$ and etc. In this way we obtain a solution $u_2 \in \mathcal{C}^1([1,2],\mathcal{C}^2(\mathbb{R}^n))$ to the Cauchy problem

$$u_t - (\Delta + m)u = \lambda |u|^{p-1}u, \quad t \in [1,2], x \in \mathbb{R}^n,$$

$$u(1,x) = u_1(1,x), \quad x \in \mathbb{R}^n.$$

Using reasonings as (2.14), (2.15) we have that

$$\begin{cases} u_1 & t \in [0,1], x \in \mathbb{R}^n, \\ u_2 & t \in [1,2], x \in \mathbb{R}^n, \end{cases}$$

is a solution to the Cauchy problem

$$u_t - (\Delta + m)u = \lambda |u|^{p-1}u \quad t \in [0,2], x \in \mathbb{R}^n,$$

$$u(0,x) = u_0(x) \quad x \in \mathbb{R}^n$$

which belongs to the space $\mathcal{C}^1([0,2],\mathcal{C}^2(\mathbb{R}^n))$ and etc. we obtain a solution to the problem (1.1), (1.2) which belongs to the space $\mathcal{C}^1([0,\infty),\mathcal{C}^2(\mathbb{R}^n))$.

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Notes

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