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New Representations in Terms of q-Product Identities for Ramanujan's Results

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Strictly as per the compliance and regulations of :

Ref.

2. B.C. Berndt; *What is a q-series?*, preprint.
3. B.C. Berndt; *Ramanujan's notebook Part III*, Springer-Verlag, New York, 1991.

New Representations in Terms of q-Product Identities for Ramanujan's Results

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Abstract - In this paper author has established six q-product identities, which are presumably new, and not available in the literature.

Keywords : Euler's pentagonal number theorem, triple product identities.

I. Introduction

For $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) \tag{1.1}$$

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{(n-1)}) \tag{1.2}$$

$$(a_1,a_2,a_3,...,a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty...(a_k;q)_\infty \tag{1.3}$$

Ramanujan [2, p.1(1.2)]has defined general theta function, as

$$f(a,b) = \sum_{-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \quad ; \quad |ab| < 1, \tag{1.4}$$

Jacobi's triple product identity [3,p.35] is given, as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{1.5}$$

Special cases of Jacobi's triple products identity are given, as

$$\phi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty \tag{1.6}$$

$$(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.7}$$

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.8}$$

Equation (1.8) is known as Euler's pentagonal number theorem. Euler's another well known identity is as

$$(q;q^2)_\infty^{-1} = (-q;q)_\infty \tag{1.9}$$

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Throughout this paper we use the following representations

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty=(q^a,q^b,q^c\cdots q^t;q^n)_\infty \tag{1.10}$$

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty=(q^a,q^b,q^c\cdots q^t;q^n)_\infty \tag{1.11}$$

$$(-q^a;q^n)_\infty(-q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty=(-q^a,-q^b,q^c\cdots q^t;q^n)_\infty \tag{1.12}$$

Now we can have following q-products identities, as

$$(q^2;q^2)_\infty=\prod_{n=0}^{\infty}(1-q^{2n+2})$$

$$=\prod_{n=0}^{\infty}(1-q^{2(4n)+2})\times\prod_{n=0}^{\infty}(1-q^{2(4n+1)+2})\times\prod_{n=0}^{\infty}(1-q^{2(4n+2)+2})\times\prod_{n=0}^{\infty}(1-q^{2(4n+3)+2})$$

$$=\prod_{n=0}^{\infty}(1-q^{8n+2})\times\prod_{n=0}^{\infty}(1-q^{8n+4})\times\prod_{n=0}^{\infty}(1-q^{8n+6})\times\prod_{n=0}^{\infty}(1-q^{8n+8})$$

or,

$$(q^2;q^2)_\infty=(q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty=(q^2,q^4,q^6,q^8;q^8)_\infty \tag{1.13}$$

$$(q^4;q^4)_\infty=\prod_{n=0}^{\infty}(1-q^{4n+4})$$

$$=\prod_{n=0}^{\infty}(1-q^{4(3n)+4})\times\prod_{n=0}^{\infty}(1-q^{4(3n+1)+4})\times\prod_{n=0}^{\infty}(1-q^{4(3n+2)+4})$$

$$=\prod_{n=0}^{\infty}(1-q^{12n+4})\times\prod_{n=0}^{\infty}(1-q^{12n+8})\times\prod_{n=0}^{\infty}(1-q^{12n+12})$$

or,

$$(q^4;q^4)_\infty=(q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty=(q^4,q^8,q^{12};q^{12})_\infty \tag{1.14}$$

$$(q^4;q^{12})_\infty=\prod_{n=0}^{\infty}(1-q^{12n+4})=\prod_{n=0}^{\infty}(1-q^{12(5n)+4})\times\prod_{n=0}^{\infty}(1-q^{12(5n+1)+4})\times$$

$$\times\prod_{n=0}^{\infty}(1-q^{12(5n+2)+4})\times\prod_{n=0}^{\infty}(1-q^{12(5n+3)+4})\times\prod_{n=0}^{\infty}(1-q^{12(5n+4)+4})$$

$$=\prod_{n=0}^{\infty}(1-q^{60n+4})\times\prod_{n=0}^{\infty}(1-q^{60n+16})\times\prod_{n=0}^{\infty}(1-q^{60n+28})\times\prod_{n=0}^{\infty}(1-q^{60n+40})\times\prod_{n=0}^{\infty}(1-q^{60n+52})$$

or,

$$(q^4;q^{12})_\infty=(q^4;q^{60})_\infty(q^{16};q^{60})_\infty(q^{28};q^{60})_\infty(q^{40};q^{60})_\infty(q^{52};q^{60})_\infty$$

$$=(q^4,q^{16},q^{28},q^{40},q^{52};q^{60})_\infty \tag{1.15}$$

Notes

Similarly we can compute following as

$$(q^5;q^5)_\infty = (q^5;q^{15})_\infty (q^{10};q^{15})_\infty (q^{15};q^{15})_\infty \tag{1.16}$$

$$(q^6;q^6)_\infty = (q^6;q^{24})_\infty (q^{12};q^{24})_\infty (q^{18};q^{24})_\infty (q^{24};q^{24})_\infty = (q^6,q^{12},q^{18},q^{24};q^{24})_\infty \tag{1.17}$$

$$\begin{aligned}(q^6;q^{12})_\infty &= (q^6;q^{60})_\infty (q^{18};q^{60})_\infty (q^{30};q^{60})_\infty (q^{42};q^{60})_\infty (q^{54};q^{60})_\infty \\ &= (q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty \end{aligned} \tag{1.18}$$

The outline of this paper is as follows. In sections 2, some recent results obtained by the author in [1], and also some well known results are recorded, which are useful to the rest of the paper. In section 3, we state and prove six new q-product identities, which are not recorded in the literature.

II. Preliminaries

In [1], following identities are being established

$$(q^2,q^4,q^6;q^8)_\infty [(-q;q^2)^2_\infty + (q;q^2)^2_\infty] = 2(-q^4;q^8)^2_\infty \tag{2.1}$$

$$(q^2,q^4,q^6,q^8;q^8)_\infty [(-q;q^2)^2_\infty - (q;q^2)^2_\infty] = 4q\frac{(q^{16},q^{32},q^{48};q^{48})_\infty}{(q^8,q^{24},q^{40};q^{48})_\infty} \tag{2.2}$$

$$\frac{(-q;q^2)^2_\infty + (q;q^2)^2_\infty}{(-q;q^2)^2_\infty - (q;q^2)^2_\infty} = \frac{(-q^4;q^8)^2_\infty (q^8,q^8,q^{24},q^{24},q^{40},q^{40};q^{48})_\infty}{2q} \tag{2.3}$$

$$(-q;q^2)^2_\infty (q;q^2)^2_\infty (q^2;q^2)^2_\infty = (q^2,q^2,q^4;q^4)_\infty \tag{2.4}$$

$$\frac{(-q;q^2)_\infty(-q^3;q^6)_\infty - (q;q^2)_\infty(q^3;q^6)_\infty}{(-q;q^2)_\infty \times (-q^3;q^6)_\infty \times (q;q^2)_\infty \times (q^3;q^6)_\infty} = \frac{2q(-q^2;q^4)^2_\infty (q^4,q^8,q^{16},q^{20},q^{24};q^{24})_\infty}{(q^2,q^4,q^6,q^8;q^8)_\infty (q^6,q^{12},q^{18};q^{24})_\infty} \tag{2.5}$$

$$\begin{aligned}\frac{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty - (q^3;q^6)_\infty(q^5;q^{10})_\infty}{(-q^3;q^6)_\infty \times (-q^5;q^{10})_\infty \times (q^3;q^6)_\infty \times (q^5;q^{10})_\infty} &= \frac{(q^4,q^8,q^{12};q^{12})_\infty}{(q^6,q^{12},q^{18},q^{24};q^{24})_\infty} \times \\ &\times \frac{2q^3}{(q^2,q^6,q^{10};q^{12})_\infty (q^{10},q^{20},q^{30},q^{30},q^{40},q^{50};q^{60})_\infty}\end{aligned} \tag{2.6}$$

$$\begin{aligned}\frac{[(q;q^2)_\infty(q^{15};q^{30})_\infty] + [(-q;q^2)_\infty(-q^{15};q^{30})_\infty]}{[(q;q^2)_\infty(q^{15};q^{30})_\infty][(-q;q^2)_\infty(-q^{15};q^{30})_\infty]} &= \frac{(q^{12},q^{20},q^{24},q^{36},q^{40},q^{48},q^{60},q^{60};q^{60})_\infty}{(q^{10},q^{30},q^{30},q^{50},q^{60};q^{60})_\infty} \times \\ &\times \frac{2}{(q^2,q^4,q^6,q^8,q^8;q^8)_\infty (q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty}\end{aligned} \tag{2.7}$$

In Ramanujan's notebooks [6, p.240], the following entries are recorded as

$$f(q^3,q^6) = \quad (q) - q\psi(q^9) \tag{2.8}$$

$$f(-q^2,-q^3)f(-q,-q^4) = f(-q,-q^2)f(-q^5,-q^{10}) \tag{2.9}$$

In Ramanujan's notebooks [6, p.243], the following entries are recorded as

$$f(q,q^7)f(q^3,q^5) = \quad (q) \quad (q^4) \tag{2.10}$$

Ref.

1. M.P. Chaudhary; *On q-product identities*, preprint.
6. S. Ramanujan; *Notebooks (Volume I)*, Tata Institute of Fundamental Research, Bombay, 1957.

$$2f(q^3,q^5) = \quad (q^{\frac{1}{2}}) + \quad (-q^{\frac{1}{2}}) \tag{2.11}$$

$$2q^{\frac{1}{2}}f(q,q^7) = \quad (q^{\frac{1}{2}}) - \quad (-q^{\frac{1}{2}}) \tag{2.12}$$

III. Main Results

Notes

In this paper, we established following new results, which are not recorded in the literature of special functions

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = (-q^3,-q^6,q^9;q^9)_\infty + q\frac{(q^{18};q^{18})_\infty}{(q^9;q^{18})_\infty} \tag{3.1}$$

$$(q;q)_\infty = (q,q^2,q^3,q^4,q^5;q^5)_\infty \tag{3.2}$$

$$(q;q^2)_\infty = \frac{(q^2,q^6;q^8)_\infty}{(-q,-q^3,-q^5,-q^7;q^8)_\infty} \tag{3.3}$$

$$(-q^{\frac{1}{2}};q)_\infty + (q^{\frac{1}{2}};q)_\infty = \frac{2(-q^3,-q^5,q^8;q^8)_\infty(q^{\frac{1}{2}},-q^{\frac{1}{2}};q)_\infty}{(q,q^2,q^3,q^4,q^5;q^5)_\infty} \tag{3.4}$$

$$(-q^{\frac{1}{2}};q)_\infty - (q^{\frac{1}{2}};q)_\infty = \frac{2q^{\frac{1}{2}}(-q,-q^7,q^8;q^8)_\infty(q^{\frac{1}{2}},-q^{\frac{1}{2}};q)_\infty}{(q,q^2,q^3,q^4,q^5;q^5)_\infty} \tag{3.5}$$

$$\frac{(-q^{\frac{1}{2}};q)_\infty + (q^{\frac{1}{2}};q)_\infty}{(-q^{\frac{1}{2}};q)_\infty - (q^{\frac{1}{2}};q)_\infty} = \frac{(-q^3,-q^5;q^8)_\infty}{q^{\frac{1}{2}}(-q,-q^7;q^8)_\infty} \tag{3.6}$$

Proof of (3.1): By substituting $a = q^3$ and $b = q^6$ in (1.5), we have

$$f(q^3,q^6) = (-q^3;q^9)_\infty(-q^6;q^9)_\infty(q^9;q^9)_\infty$$

also, by substituting $q = q^9$ in (1.7), we get

$$(q^9) = \frac{(q^{18};q^{18})_\infty}{(q^9;q^{18})_\infty}$$

employing (1.7) and putting the values of $f(q^3,q^6)$ and $\;(q^9)$ in (2.8), we get

$$(-q^3;q^9)_\infty(-q^6;q^9)_\infty(q^9;q^9)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} - q\frac{(q^{18};q^{18})_\infty}{(q^9;q^{18})_\infty}$$

after simplification, we get

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = (-q^3,-q^6,q^9;q^9)_\infty + q\frac{(q^{18};q^{18})_\infty}{(q^9;q^{18})_\infty}$$

which established (3.1).

Proof of (3.2): By substituting $a = -q^2$ and $b = -q^3$ in (1.5), we have

$$f(-q^2,-q^3) = (q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty$$

Notes

by substituting $a=-q$ and $b=-q^4$ in (1.5), we have

$$f(-q,-q^4)=(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty$$

employing (1.8) and putting the values of $f(-q^2,-q^3)$ and $f(-q,-q^4)$ in (2.9), we get

$$(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty=(q;q)_\infty(q^5;q^5)_\infty$$

after simplification, we get

$$(q;q)_\infty=(q,q^2,q^3,q^4,q^5;q^5)_\infty$$

which established (3.2).

Proof of (3.3): By substituting $a=q$ and $b=q^7$ in (1.5), we have

$$f(q,q^7)=(-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty$$

by substituting $a=q^3$ and $b=q^5$ in (1.5), we have

$$f(q^3,q^5)=(-q^3;q^8)_\infty(-q^5;q^8)_\infty(q^8;q^8)_\infty$$

also by substituting $q=q^4$ in (1.7), we get

$$(q^4)=\frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty}$$

employing (1.7) and putting the values of $f(q,q^7)$, $f(q^3,q^5)$ and (q^4) in (2.10), we get

$$(-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty(-q^3;q^8)_\infty(-q^5;q^8)_\infty(q^8;q^8)_\infty=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\times\frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty}$$

further using (1.13), and after simplification, we get

$$(q;q^2)_\infty=\frac{(q^2,q^6;q^8)_\infty}{(-q,-q^3,-q^5,-q^7;q^8)_\infty}$$

which established (3.3).

Proof of (3.4): By putting $q=q^{\frac{1}{2}}$ and $q=-q^{\frac{1}{2}}$ in (1.7), we have

$$(q^{\frac{1}{2}})=\frac{(q;q)_\infty}{(q^{\frac{1}{2}};q)_\infty}\quad\text{and}\quad(-q^{\frac{1}{2}})=\frac{(q;q)_\infty}{(-q^{\frac{1}{2}};q)_\infty}$$

by substituting the values of $f(q^3,q^5)$ from proof of (3.3), $(q^{\frac{1}{2}})$ and $(-q^{\frac{1}{2}})$ in (2.11), we get

$$2(-q^3;q^8)_\infty(-q^5;q^8)_\infty(q^8;q^8)_\infty=\frac{(q;q)_\infty}{(q^{\frac{1}{2}};q)_\infty}+\frac{(q;q)_\infty}{(-q^{\frac{1}{2}};q)_\infty}$$

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after simplification, and employing (3.2), we have

$$(-q^{\frac{1}{2}};q)_\infty + (q^{\frac{1}{2}};q)_\infty = \frac{2(-q^3,-q^5,q^8;q^8)_\infty(q^{\frac{1}{2}},-q^{\frac{1}{2}};q)_\infty}{(q,q^2,q^3,q^4,q^5;q^5)_\infty}$$

which established (3.4).

Proof of (3.5): Substituting the values of $f(q,q^7)$, $(q^{\frac{1}{2}})$ and $(-q^{\frac{1}{2}})$ in (2.12), we get

$$2q^{\frac{1}{2}}(-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty = \frac{(q;q)_\infty}{(q^{\frac{1}{2}};q)_\infty} - \frac{(q;q)_\infty}{(-q^{\frac{1}{2}};q)_\infty}$$

after simplification, and employing (3.2), we have

$$(-q^{\frac{1}{2}};q)_\infty - (q^{\frac{1}{2}};q)_\infty = \frac{2q^{\frac{1}{2}}(-q,-q^7,q^8;q^8)_\infty(q^{\frac{1}{2}},-q^{\frac{1}{2}};q)_\infty}{(q,q^2,q^3,q^4,q^5;q^5)_\infty}$$

which established (3.5).

Proof of (3.6): Dividing (3.4) by (3.5), we get desired result.

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Notes