On The Response of a Non-Uniform Beam Transvered by Mobile Distributed Loads

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Abstract – The problem being investigated in this paper is that of the response of non-uniform beam under tensile stress and resting on an elastic foundation. The fourth order partial differential equation governing the problem is solved when the beam is transverse by mobile distributed loads. The elastic properties of the beam, the flexible rigidity, and the mass per unit length are expressed as functions of the spatial variable using Struble's method. It is observed that the deflection of non-uniform beam under the action of moving masses is higher than the deflection of moving force when only the force effects of the moving load are considered. From the analysis, the response amplitudes of both moving force and moving mass problems decrease with increasing foundation constant.

Keywords : Distributed Load, Non-uniform, Elastic Foundation, moving Mass.

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ON THE RESPONSE OF A NON-UNIFORM BEAM TRANSVERED BY MOBILE DISTRIBUTED LOADS

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I. INTRODUCTION

Structural engineers usually encountered problem that arises especially when a beam is being transverse by a moving load. The theory of vibration of structures has treated some of these problem i.e vibrations of turbines, hulls of shills and bridge girders of variable dept etc. Beam on elastic foundation subjected to moving masses have received extensive attention in the literature.

Kolousek et al [3] used normal mode analysis to address the problem of flexible vibration of non-uniform beam. This was followed by Sadiku and Leipholz [6] who only studied the dynamics of a uniform beam by considering the inertia effect of a moving mass and later developed the Green's function of the associated differential problem thereby obtained a closed form solution.

In a later development, Oni [10] presented the problem of dynamic analysis of a non uniform beam to several moving masses under concentrated load. The beam considered is under tensile stress and by the method of Galarkin, the result is obtained for the first mode response of the beam. Chau and Seng [8] worked on the static response of beams on non-linear elastic foundation where the deformed shape of the structure was represented by a Fourier series, and thereafter, the giving equation is reduced to a set of second order simultaneous equations using Galarkin's method. In all the aforementioned works, the practical cases where the elastic systems are of variable cross section and of distributed moving loads use not considered.

The paper therefore presents the problem of dynamic response of a non-uniform beam to moving masses on elastic foundation traversed by mobile distributed load.

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II. DERIVATION AND ASSEMBLY OF THE GOVERNING EQUATION

Consider a moving load $\Delta(x,t)$ of mass M acting on a Bernoulli-Euler beam (Non-uniform) uniformly loaded and move at a constant velocity c as shown below:
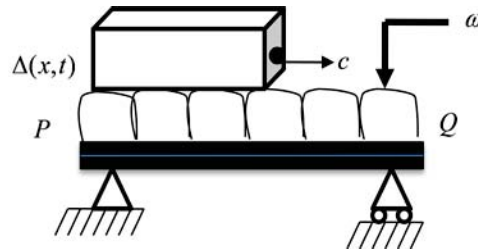
Figure 1: Uniformly distributed load on simply supported beam.

In the structure above, the displacement is governed by the equation

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 \overline{U}(x,t)}{\partial x^2}\right]+\alpha^m(x)\frac{\partial^2 \overline{U}(x,t)}{\partial t^2}-N\frac{\partial^2 \overline{U}(x,t)}{\partial x^2}+k(x)\overline{U}(x,t)=\Delta_f(x,t)\overline{U}(x,t)\left[1-\frac{\Delta^*}{g}\left(\overline{U}(x,t)\right)\right]$$

where $U(x,t)$ is transverse displacement, E is the Young modulus, $I(x)$ is variable moment of inertia, $EI(x)$ is flexible rigidity, α^m, Δ_f is the substantive acceleration operator, g is the acceleration due to gravity.

For the non-uniform beam such as above, its properties such as moment of inertia I and the mass per unit length of the beam α_m vary along the span of L of the beam.

The structure under consideration is simply supported and carrying an arbitrary number of masses M moving with constant velocities.

The Operator Δ^* is defined as

$$\Delta^* = \frac{\partial^2}{\partial t^2}+2c\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x^2} \qquad (2.2)$$

and the load $\Delta(x,t)$ is given as

$$\Delta_f(x,t)=MH(x-ct)\left[g-\frac{\partial^2}{\partial t^2}+2c\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x^2}\right] \qquad (2.3)$$

where $H(x-ct)$ is the Heaviside function.

Furthermore, the boundary condition for the dynamical system is taken to be arbitrary and the initial condition of the motion is

$$\overline{U}(x,t)=0=\frac{\partial}{\partial t}\overline{U}(x,t) \qquad (2.4)$$

Substituting equations (2.2), (2.3), into (2.1), the governing of motion takes the form

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\overline{U}(x,t)\right]+\alpha^m(x)\frac{\partial^2}{\partial t^2}\overline{U}(x,t)-N\frac{\partial^2}{\partial x^2}\overline{U}(x,t)+K(x)\overline{U}(x,t)$$

$$+MH(x-ct)\left[\frac{\partial^2}{\partial t^2}+2c\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x^2}\right]\overline{U}(x,t)=MgH(x-ct) \qquad (2.5)$$

Equation 2.5 can be further be simplified to give further simplification yields;

$$N_1\left(1+Sin\frac{\pi x}{L}\right)^3\frac{\partial^4}{\partial x^4}\overline{U}(x,t) + N_2\left(1+Sin\frac{\pi x}{L}\right)^2 Cos^2\frac{\pi x}{L}\frac{\partial^3}{\partial x^3}\overline{U}(x,t)$$

$$+\left[N_3\left(1+Sin\frac{\pi x}{L}\right)Cos^2\frac{\pi x}{L}-N_4(1+sin\frac{\pi x}{L})^2 Sin\frac{\pi x}{L}-N_5\right]\frac{\partial^2\overline{U}(x,t)}{\partial x^2}$$

$$+\left(1+Sin\frac{\pi x}{L}\right)\frac{\partial^2\overline{U}(x,t)}{\partial t^2}+N_6\overline{U}(x,t)+\frac{M}{\alpha^m_o}H(x-ct)\left[\frac{\partial^2}{\partial t^2}\overline{U}(x-ct)+2c\frac{\partial^2\overline{U}(x,t)}{\partial x\partial t}+c^2\frac{\partial^2\overline{U}(x,t)}{\partial x^2}\right]$$

$$=\frac{mg}{\alpha^m_o}H(x-ct) \tag{2.6}$$

where,

$$N_1=\frac{EI_o\pi}{\alpha^m_o}, N_2=\frac{6\pi EI_o}{\alpha^m_o L}, N_3=\frac{6\pi^2 EI_o}{\alpha^m_o L^2}, N_4=\frac{3\pi^2 EI_o}{\alpha^m_o L} N_5=\frac{N}{\alpha^m_o}, N_6=\frac{K_o}{\alpha^m_o} \tag{2.7}$$

Equation (2.6) is a non-homogenous partial differential equation with variable coefficients. Clearly, it is seen that the closed from solution does not exists.

III. SOLUTION PROCEDURE

To solve equation (2.6), an approximate solution is sought. One of the approximate methods best suited to solve diverse problems in dynamics of structures is the Galarkin's method [7]. This method requires that the solution of equation (2.6) be of the form

$$\overline{U}_n=\sum_{m=1}^{n}Y_m(t)X_m(x) \tag{3.1}$$

where $X_m(x)$ is chosen such that all the boundary conditions are satisfied. Equation (3.1) when substituted into equation (2.6) yields;

$$\sum_{m=1}^{n}\left[N_1\left(1+Sin\frac{\pi x}{L}\right)^3 Y_m(t)X_m^{IV}(x)+N_2\left(1+Sin\frac{\pi x}{L}\right)^2 Cos^2\frac{\pi x}{L}Y_m(t)X_m^{III}(x)\right.$$

$$+\left[N_3\left(1+Sin\frac{\pi x}{L}\right)Cos^2\frac{\pi x}{L}-N_4(1+sin\frac{\pi x}{L})^2 Sin\frac{\pi x}{L}-N_5\right]Y_m(t)X_m^{II}(x)$$

$$\left.+\left(1+Sin\frac{\pi x}{L}\right)\ddot{Y}_m(t)X_m(x)+N_6 Y_m(t)X_m(x)+\frac{M}{\alpha^m_o}H(x-ct)\left[\ddot{Y}_m(t)X_m(x)+2c\dot{Y}_m(t)X_m^{I}(x)+c^2 Y_m(t)X_m^{II}(x)\right]\right]$$

$$-\frac{Mg}{\alpha^m_o}H(x-ct)=0 \tag{3.2}$$

7. Wh, J.S et al (1987):The Dynamic analysis of a flat plate under a moving load by a definite element method. International Journal of Numerical methods in Engineering 24,743-762.

Ref.

In order to determine $Y_m(t)$, it is required that the expression on the left hand side of equation (3.2) be orthogonal to function $X_m(x)$. Hence,

$$\int_0^L \left\{ \sum_{m=1}^n \left[N_1\left(1+Sin\frac{\pi x}{L}\right)^3 Y_m(t)X_m^{IV}(x) + N_2\left(1+Sin\frac{\pi x}{L}\right)^2 Cos^2\frac{\pi x}{L}Y_m(t)X_m^{III}(x) \right. \right.$$

$$+ \left[N_3\left(1+Sin\frac{\pi x}{L}\right)Cos^2\frac{\pi x}{L} - N_4(1+sin\frac{\pi x}{L})^2 Sin\frac{\pi x}{L} - N_5 \right]Y_m(t)X_m^{II}(x)$$

$$\left. + \left(1+Sin\frac{\pi x}{L}\right)\ddot{Y}_m(t)X_m(x) + N_6 Y_m(t)X_m(x) + \frac{M}{\alpha^m_o}H(x-ct)\left[\ddot{Y}_m(t)X_m(x) + 2c\dot{Y}_m(t)X_m^{I}(x) + c^2 Y_m(t)X_m^{II}(x)\right]\right]$$

$$\left. - \frac{Mg}{\alpha^m_o}H(x-ct)\right\}X_k(x)dx = 0$$

Since our dynamical system has simple supports at the edges $x = 0$ and $x = L$, we choose;

$$X_m(x) = Sin\frac{m\pi x}{L} \tag{3.3}$$

Consequently, using (3.4) in (3.3) gives

$$\sum_{m=1}^n \left\{ Ha\ddot{Y}_m(t) + H_b Y_m(t) + \frac{M}{\alpha^m_o}\left[H_c(t)\ddot{Y}_m(t) + 2cH_d(t)\dot{Y}_m(t) + c^2 H_e(t)Y_m(t)\right]\right\} = \frac{Mg}{\alpha^m_o}H_f(t) \tag{3.5}$$

where

$$H_a = \int_0^L \left(1+Sin\frac{\pi x}{L}\right)Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx, \qquad H_b = Q_1 + Q_2 + Q_3 - Q_4 - Q_5 + Q_6$$

$$H_c(t) = \int_0^L H(x-ct)Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx, \qquad H_d(t) = \frac{m\pi}{L}\int_0^L H(x-ct)Cos\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx$$

$$H_e(t) = \frac{m^2\pi^2}{L^2}\int_0^L H(x-ct)Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx, \quad H_f(t) = \int_0^L H(x-ct)Sin\frac{k\pi x}{L}dx \tag{3.6}$$

and

$$Q_1 = \frac{m^4\pi^4}{L^4}N_1\int_0^L\left(1+sin\frac{\pi x}{L}\right)^3 Sin\frac{\pi x}{L}Sin\frac{k\pi x}{L}dx, \qquad Q_2 = \frac{m^3\pi^3}{L^3}N_2\int_0^L\left(1+sin\frac{\pi x}{L}\right)^2 Cos\frac{\pi x}{L}Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx$$

$$Q_3 = \frac{m^2\pi^2}{L^2}N_3\int_0^L\left(1+sin\frac{\pi x}{L}\right)Cos^2\frac{\pi x}{L}Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx, \quad Q_4 = \frac{m^2\pi^2}{L^2}N_4\int_0^L\left(1+sin\frac{\pi x}{L}\right)^2 Sin\frac{\pi x}{L}Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx$$

$$Q_5 = N_5\int_0^L Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx, \qquad Q_6 = N_6\int_0^L sin\frac{m\pi x}{L}sin k\frac{\pi x}{L}dx \tag{3.7}$$

When the integrals (3.6) and (3.7) are evaluated, the result is a series of coupled differential equations called Galarkin's equations for n-degree of freedom system governing

6. Sadiku S Leiophtz H.H.E (1987): On the dynamics of elastic systems with moving concentrated masses. Ing. Archiv 57, 223-242.
9. Smith, J.W (1988): ibration of structures. Application in Civil Engineering Design. Chapman and hall Ltd London.

the coefficients of all lower and higher modes of the beam. Thus, restricting ourselves to the analysis of the first mode response, we set $m = 1$ and $n = 1$ in equation (3.5) for analytical approximation.

Following the method of [9] where Heaviside function is expresses as Fourier cosine series. Thus, equation (3.5) leads to

$$\sum_{m=1}^{n} H_a \ddot{Y}_m(t) + H_b Y_m(t) + \Gamma_1 \left[\left(\frac{1}{L} I_1 + \frac{2}{n\pi L} \sum_{n=1}^{\infty} Cos \frac{n\pi ct}{L} I_2 + C^{\square} I_3 \right) \ddot{Y}_m(t) \right.$$

$$\left(\frac{2Cm\pi}{L^2} I_4 + \frac{4Cm\pi}{nL^2} \sum_{n=1}^{\infty} Cos \frac{n\pi ct}{L} I_5 + \frac{2CC^{\square}m\pi}{L^2} I_6 \right) \dot{Y}_m(t)$$

$$+ \left(\frac{C^2 m^2 \pi^2}{L^3} I_7 + \frac{2C^2 m^2 \pi}{nL^3} \sum_{n=1}^{\infty} Cos \frac{n\pi ct}{L} I_8 + \frac{C^2 m^2 \pi^2 C^{\square}}{L^2} I_9 \right) Y_m(t) = P_m \left[Cos \frac{\lambda_m ct}{L} - Cos \lambda_m \right] \qquad (3.8)$$

where $\Gamma_1 = \dfrac{M}{\alpha^m_0}$ and $P_m = \dfrac{MgL}{\alpha^m_0 \lambda_m}$ $\qquad (3.9)$

which is the transformed equation of the dynamical system.

IV.   Analytical Approximate Solution

a) Simply Supported Traversed By Moving Force

An approximate model of the system, when the inertia effect of the moving mass is neglected, is the moving force problem associated with the system. Setting $\Gamma_1 = 0$, we have

$$\ddot{Y}_m(t) + \beta_{mf}^2 Y_m(t) = P_m \left[Cos \frac{\lambda_m ct}{L} - Cos \lambda_m \right] \qquad (4.1)$$

where $\beta_{mf} = \dfrac{H_b}{H_a}$ $\qquad (4.2)$

Subjecting equation (4.2) to Laplace transform defined by

$$(\tilde{}) = \int_0^{\infty} e^{-st} dt \qquad (4.3)$$

where S is a Laplace transform. It yields,

$$\overline{U}_n(x,t) = \sum_{m=1}^{n} P_m \left[\frac{CosZ_k t - Cos\beta_{mf}t}{\beta_{mf}^2 - Z_k^2} - E(m) \frac{(1 - Cos\beta_{mf}t)}{\beta_{mf}} \right] \times Sin \frac{n\pi x}{L} \qquad (4.4)$$

where $Z_k = \dfrac{\lambda_m ct}{L}$ and $E(m) = Cos\lambda_m$

which is the response to moving force solution of the elastic system at constant velocity.

b) Simply Supported Traversed By Moving Mass

For the moving mass solution, we set $\Gamma_1 \neq 0$, in this case, the entire solution to the problem is sought. To this end, a modification of the asymptotic method of Struble[6] often used for treating weakly homogeneous and non-homogenous non-linear system is employed. Further arrangement of equation (3.8) yields

$$\left[H_a + \Gamma_1\left(\frac{1}{L}I_1 + \frac{2}{n\pi L}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}I_2 + C^\square I_3 \right) \right]\ddot{Y}_m(t)$$

$$+\left[\Gamma_1\left(\frac{2Cm\pi}{L^2}I_4 + \frac{4Cm\pi}{nL^2}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}I_5 + \frac{2CC^\square m\pi}{L^2}I_6 \right) \right]\dot{Y}_m(t)$$

$$+\left[H_b + \Gamma_1\left(\frac{C^2m^2\pi^2}{L^3}I_7 + \frac{2C^2m^2\pi}{nL^3}\sum_{n=1}^{\infty}Cos\frac{n\pi ct}{L}I_8 + \frac{C^2m^2\pi^2C^\square}{L^2}I_9 \right) \right]Y_m(t) = P_m\left[Cos\frac{\lambda_m ct}{L} - Cos\lambda_m \right]$$
(4.5)

At this juncture, we seek the modified frequency corresponding to the frequency of the free system due to the presence of moving mass [8].To this end, the solution to equation (4.5) can be written as

$$Y_m(t) = N(m,t)\left[\beta_{mf}t - \varphi(m,t) \right]$$
(4.6)

where $\beta_{mf}t$ and $\varphi(m,t)$ are constants.

Therefore when the mass of the particle is considered, the first approximation to the homogeneous system is given as

$$Y_m(t) = D^\square(m,t)\left[\beta_{jj}t - \varphi(m,t) \right]$$
(4.7)

where

$$\beta_{jj} = \left\{ 1 - \frac{\Gamma_j}{H_a}\left[\beta_{mf}\left(\frac{1}{L}I_1 + C^\square I_3 \right) - \frac{C^2m^2\pi^2}{L^2}(2I_7 - I_9) \right] \right\}$$
(4.8)

Equation (4.8) is called the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass. Thus, the entire equation (4.5) takes the form

$$\frac{d^2}{dt^2}Y_m(t) + \beta_{mf}^2 Y_m(t) = \frac{P_m\Gamma_j}{H_a}\left[Cos\frac{\lambda_m ct}{L} - Cos\lambda_m \right]$$
(4.9)

which is a prototype of equation (4.1)and when inverted we have

$$\overline{U}_n(x,t) = \sum_{m=1}^{n}\frac{P_m\Gamma_j}{H_a}\left[\frac{CosZ_k t - Cos\beta_{jj}t}{\beta_{jj}^2 - Z_k^2} - E(m)\frac{(1 - Cos\beta_{jj}t)}{\beta_{jj}} \right] \times Sin\frac{n\pi x}{L}$$
(4.10)

Equation (4.10) is the transverse displacement response to moving mass solution for simply supported beam on elastic foundation.

V. Discussion of Results

Resonance condition

It is desirable to inspect closely the response amplitude of the dynamical system. Following [12], the moving force in equation (4.8) attains a resonance whenever

$$\beta_{mf} = \frac{m\pi c}{L}$$
(5.1)

while when

$$\beta_{jj} = \frac{m\pi c}{L}$$
(5.2)

Ref.

8. Chau, F.W and Seng, O.L (198): Beams on non-linear elastic foundation. Journal of Applied Mechanics. Vol. 2, Pp 587-600.

12. Oni S.T and Ogunyebi S.N (2008): Dynamical analysis of a prestressed elastic beam with general boundry conditions under the action of uniform distributed masses.

gives for the moving mass problem. Re-written equation (4.8) in the form

$$\beta_{jj} = \beta_{mf} \left\{ \frac{1}{\beta_{mf}} - \frac{\Gamma_j}{H_a} \left[\left(\frac{1}{L} I_1 + C^{\Box} I_3 \right) - \frac{C^2 m^2 \pi^2}{\beta_{mf} L^2} \left(2I_7 - I_9 \right) \right] \right\} \tag{5.3}$$

which implies

$$\beta_{mf} = \frac{L/m\pi c}{\beta_{mf} \left\{ \frac{1}{\beta_{mf}} - \frac{\Gamma_j}{H_a} \left[\left(\frac{1}{L} I_1 + C^{\Box} I_3 \right) - \frac{C^2 m^2 \pi^2}{\beta_{mf} L^2} \left(2I_7 - I_9 \right) \right] \right\}} \tag{5.4}$$

VI. Conclusion

In view of the condition for resonance established above, it is deduced that for the same natural frequency, the critical speed for the moving force simply supported beam is greater than that of the moving mass problem. Thus for the same natural frequency, resonance is reached earlier in the moving mass system than in the moving force system.

For practical purposes, a one dimensional structures (Beam) are used as mathematical models in the buildings and bridges construction. Hence appropriate precaution may now be taken by the structural engineers to forestall the occurrence of resonance in the structure by integrating the necessary vibration absorber into the model.

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