Note on Elzaki Transform of Distributions and Certain Space of Boehmians

By S.K.Q.Al-Omari

Al-Balqa Applied University, Amman, Jordan

Abstract - The Elzaki transform transform was discussed in [19] as a motivation of the classical Sumudu transform. In this article, we extend the Elzaki transform to a space of tempered distributions (distributions of slow growth) by known kernel method. Further, we establish two spaces of Boehmians so that the Elzaki transform is well defined. Certain theorems are established in some details.

Keywords and phrases : Generalized function; Elzaki Transform; Sumudu Transform; Tempered Distribution; Boehmian Space.

1991 Mathematics Subject Classification. Primary 54C40, 14E20; Secondary 46E25, 20C20.
Note on Elzaki Transform of Distributions and Certain Space of Boehmians

S.K.Q. Al-Omari

Abstract - The Elzaki transform transform was discussed in [19] as a motivation of the classical Sumudu transform. In this article, we extend the Elzaki transform to a space of tempered distributions (distributions of slow growth) by known kernel method. Further, we establish two spaces of Boehmians so that the Elzaki transform is well defined. Certain theorems are established in some details.

Keywords and phrases: Generalized function; Elzaki Transform; Sumudu Transform; Tempered Distribution; Boehmian Space.

I. INTRODUCTION

In order to solve differential equations, several integral transforms were extensively used and applied in theory and application such as the Laplace, Fourier, Mellin, Hankel and Sumudu transforms, to name but a few. In the sequence of these transforms, recently, Elzaki, T. and Elzaki, S. [17, 18, 19] introduced a motivation of the Sumudu transform [14-16] and applied it to the solution of ordinary and partial differential equations as well.

The Elzaki transform over the set functions is defined by

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{t/\tau_2}, t \in (-1)^i \times (0, \infty) \right\}$$  (1)

by the formula

$$\hat{f}(z) = Ef(z) = \int_{-\tau_1}^{\tau_2} f(t) e^{zt} dt, z \in (-\tau_1, \tau_2).$$  (2)

The general properties of Elzaki transforms are found in above citations. In fact there is a relationship between Elzaki transform and some other transforms. In particular, the strong relationship between the Elzaki transform and Laplace transform was already proved in [19] which can be described as follows. Let $f$ be a function of exponential order and $Lf$ and $Ef$ be the Laplace and Elzaki transforms of $f$, respectively, then

$$Ef(z) = zLf\left(\frac{1}{z}\right).$$

and hence

$$Lf\left(\frac{1}{z}\right) = zE\left(\frac{1}{z}\right).$$

The following are needful in the sequel.

(1) If $a$ and $b$ are non-negative real numbers then

$$E(af(t) + bg(t)) (z) = aEf(z) + bEg(z).$$

(2) $\lim_{t \to 0} f(t) = \lim_{z \to 0} Ef(z) = f(0)$.

II. ELZAKI TRANSFORM OF BOEHMIANS

The minimal structure necessary for the construction of Boehmians consists of the following: (1) A nonempty set $A$; (2) A commutative semigroup $(B, *)$; (3) An operation $\ast : A \times B \to A$ such that for each $x \in A$ and $s_1, s_2, \in B, x \ast (s_1 \ast s_2) = (x \ast s_1) \ast s_2$; (3) A collection $\Delta \subset B^N$ such that: (a) $x, y \in A, (s_n) \in \Delta, x \ast s_n = y \ast s_n$ for all then $x = y$; (b) if $(s_n), (t_n) \in \Delta$, then $(s_n \ast t_n) \in \Delta$.

Elements of $\Delta$ are called delta sequences. Consider

$$Q = \{(x_n, s_n) : x_n \in A, (s_n) \in \Delta, x_n \ast s_m = x_m \ast s_n, \forall m, n \in N\}.$$
If \((x_n, s_n), (y_n, t_n) \in Q, x_n \ast t_m = y_m \ast s_n, \forall m, n \in \mathbb{N}\), then we say \((x_n, s_n) \sim (y_n, t_n)\). The relation \(\sim\) is an equivalence relation in \(Q\). The space of equivalence classes in \(Q\) is denoted by \(\beta\). Elements of \(\beta\) are called Boehmians. Between \(A\) and \(\beta\) there is a canonical embedding expressed as \(x \rightarrow \frac{x \ast s_n}{s_n}\). The operation \(\ast\) can be extended to \(\beta \times A\) by \(\frac{x \ast s_n}{s_n} \ast t = \frac{x \ast s_n t}{s_n}\). The relationship between the notion of convergence and the product \(\ast\) is given by:

1. If \(f_n \rightarrow f\) as \(n \rightarrow \infty\) in \(A\) and, \(\phi \in B\) is any fixed element, then \(f_n \ast \phi \rightarrow f \ast \phi\) in \(A\) (as \(n \rightarrow \infty\));
2. If \(f_n \rightarrow f\) as \(n \rightarrow \infty\) in \(A\) and \((\delta_n) \in \Delta\), then \(f_n \ast \delta_n \rightarrow f\) in \(A\) (as \(n \rightarrow \infty\)).

The operation \(\ast\) is extended to \(\beta \times B\) as follows: If \(\frac{L_n}{s_n} \in \beta\) and \(\phi \in B\), then \(\frac{L_n \ast \phi}{s_n}\).

Convergence in \(\beta\) is defined as

1. A sequence \((h_n)\) in \(\beta\) is said to be \(\delta\) convergent to \(h\) in \(\beta\), \(h_n \xrightarrow{\delta} h\), if there exists \((s_n) \in \Delta\) such that \((h_n \ast s_n), (h \ast s_n) \in A, \forall k, n \in \mathbb{N}\), and \((h_n \ast s_k) \rightarrow (h \ast s_k)\) as \(n \rightarrow \infty, \in A\), for every \(k \in \mathbb{N}\).
2. A sequence \((h_n)\) in \(\beta\) is said to be \(\Delta\) convergent to \(h\) in \(\beta\), \(h_n \xrightarrow{\Delta} h\), if there exists \((s_n) \in \Delta\) such that \((h_n - h) \ast s_n \in A, \forall n \in \mathbb{N}\), and \((h_n - h) \ast s_n \rightarrow 0\) as \(n \rightarrow \infty, \in A\). For further details, we refer to \([1 - 8, 10, 11, 13]\).

The convolution product between two functions \(u\) and \(v\) is given by the integral

\[
(u \ast v)(y) = \int_{0}^{\infty} u(y - x) v(x) \, dx
\]

or, equivalently,

\[
(u \ast v)(y) = \int u(t) \tau_y \tilde{v}(t) \, dt,
\]

where

\[
\tilde{v}(t) = v(-t) \quad \text{and} \quad \tau_y v(t) = v(t - y).
\]

**Lemma 2.1.** \(E(u \ast v)(z) = \frac{1}{2}(Eu)(z)(Ev)(z)\).

**Proof** See \([19, \text{Thm.2-6}]\):

Denote by \(S\) the space of all complex valued functions \(s(t)\) that are infinitely smooth and are such that, as \(|t| \rightarrow \infty\), they and their partial derivatives decrease to zero faster than every power of \(\frac{1}{|t|}\). This required behaviour as \(|t| \rightarrow \infty\) can also be stated in the following alternative way. For \(t\) one-dimensional, every function \(s(t) \in S\) satisfies the infinite set of inequalities

\[
|t^m s^{(k)}(t)| \leq C_{mk}, t \in (0, \infty),
\]

where \(m\) and \(k\) run through all non negative integers. The elements of \(S\) are called testing functions of rapid descents. \(S\) is a linear space. The dual space of \(S\) is denoted by \(\hat{S}\). A distribution \(u \in \hat{S}\) is said to be tempered distribution or distribution of slow growth.

Let \(\mathbb{R}_+\) be the field of positive real numbers and \(z\) be arbitrary but fixed in \(\mathbb{R}_+\), then

\[
D_t^k \left(ze^{\frac{-t}{z}}\right) = (-1)^k z^{1-k} e^{\frac{-t}{z}}, k = 1, 2, ...
\]

Hence for arbitrary but fixed \(z \in \mathbb{R}_+\), we get

\[
|t^m D_t^k \left(ze^{\frac{-t}{z}}\right)| = |t^m z^{1-k} e^{\frac{-t}{z}}| < \infty, 0 < t < \infty.
\]

By aid of (6) we define the Elzaki transform of \(u \in \hat{S}\) by kernel method as

\[
Ef(z) = \left<u(t), ze^{\frac{-t}{z}}\right>.
\]
Denote by $D$ the space of test functions of compact supports on $\mathbb{R}_+$ then

**Definition 2.2** Let $u \in \mathcal{S}$ and $s \in D$ then we define the convolution $u \ast s$ to be $C^\infty$ function such that

$$
(u \ast v)(y) = \langle u, \tau_y \delta \rangle,
$$

where $\delta(t) = v(-t)$ and $\tau_y v(t) = v(t-y), t \in \mathbb{R}_+$. Equ(8) can also be written as

$$
(u \ast v)(y) = \langle u(t), v(t-y) \rangle
$$

**Definition 2.3.** The convolution of two tempered distributions $u, v \in \mathcal{S}$ is defined as an element in $\mathcal{S}$ through

$$
\langle (u \ast v), s \rangle = \langle u(y), \langle v(t), \phi(t+y) \rangle \rangle, s \in D.
$$

It can be noted that if $u \in \mathcal{S}, v \in S$ then $u \ast v \in O_m$ where $O_m$ is the space of multipliers for $\mathcal{S}$. In fact $O_m \subset \mathcal{S}$. This, establishes the following lemma.

**Lemma 2.4.** If $u \in \mathcal{S}, s \in D$ then $u \ast s \in \mathcal{S}$.

**Lemma 2.5.** If $u \in \mathcal{S}, s_1, s_2 \in D$ then

$$
(u \ast s_1) \ast s_2 = u \ast (s_1 \ast s_2).
$$

**Proof.** Since $D \subset S, u \ast s_1 \in C^\infty$ and hence $(u \ast s_1) \ast s_2 \in C^\infty, u \ast (s_1 \ast s_2) \in C^\infty$. Also, $u \in \mathcal{S}, s_1 \in D \subset S \subset \mathcal{S}$, implies $u \ast s_1 \in \mathcal{S}$.

We write

$$
\langle (u \ast s_1) \ast s_2, \rangle(y) = \langle u \ast s_1, \tau_y \delta \rangle
$$

$$
= \langle u(t), \delta(\tau_y \delta) \rangle
$$

$$
= \langle u(t), \delta(x) \rangle
$$

$$
= \langle u \ast s_1, s_2(y-t) \rangle
$$

$$
= \langle u \ast s_1 \ast s_2, s \rangle(y).
$$

Hence

$$
(u \ast s_1) \ast s_2 = u \ast (s_1 \ast s_2).
$$

This completes the proof.

**Lemma 2.6.** If $u_1, u_2 \in \mathcal{S}, s \in D$ and $\alpha \in R$ then we have (1) $(u_1 + u_2) \ast s = u_1 \ast s + u_2 \ast s$; (2) $\alpha (u_1 \ast s) = (\alpha u_1) \ast s = u_1 \ast (\alpha s)$. Let $\Delta$ be the collection of all sequences $(r_n)$ from $D$ such that Equ. (11 - 13) satisfies.

$$
\int_{\mathbb{R}_+} r_n(t) \, dt = 1
$$

(11)

$$
\int_{\mathbb{R}_+} |r_n(t)| \, dt < M, M \in \mathbb{R}_+
$$

(12)

$$
\text{supp} r_n(t) \rightarrow 0 \text{ as } n \rightarrow \infty
$$

(13)

Sequences from $\Delta$ are called delta sequences.

**Lemma 2.7.** If $u_n \rightarrow u$ is $S$ as $n \rightarrow \infty$ then

$$
\text{if } u_n \ast s \rightarrow u \ast s \text{ as } n \rightarrow \infty \text{ in } \mathcal{S}, s \in D.
$$

**Lemma 2.8.** If $u_n \rightarrow u$ in $\mathcal{S}$ as $n \rightarrow \infty$ then $u_n \ast r_n \rightarrow u$ as $n \rightarrow \infty$ for each $(r_n) \in \Delta$.

The described Boehmian space is denoted by $O(\mathcal{S}, D, \Delta)^\ast$. Next, we describe another Boehmian space as follows.

Let $H$ be the set of all Elzaki transforms of tempered distributions from $\mathcal{S}$. That is, for each $h \in H$, there is $u \in \mathcal{S}$ such that $h = Eu$. Moreover, $h_n \rightarrow h$ in $H$ if there are $u_n, u \in \mathcal{S}$ such that $u_n \rightarrow u$ in $\mathcal{S}$.

Define a mapping $\cdot$ between $h \in H$ and $s \in D$ by

$$
(h \cdot s)(z) = h(z) \int e^{-z \tau} s(t) \, dt
$$

(14)

**Lemma 2.9.** Let $h \in H$ such that $h = Eu, u \in \mathcal{S}$ and $s \in D$ then

$$
E(u \ast s)(z) = (h \cdot s)(z).
$$
Hence the Lemma.

Following lemmas are straightforward. We avoid some details.

**Lemma 2.10.** If \( h \in H, s \in D \) then \( h \ast s \in H \).

Note that if \( h \in H \) then \( h = Eu \), for some \( u \in \hat{S} \). Therefore \( h \ast s = Eu \ast s = E(u \ast s) \), by Lemma 2.9. Since \( u \ast s \in \hat{S} \), the lemma follows.

**Lemma 2.11.** If \( h \in H, s \in D \) then \( E^{-1}(h \ast s) = E^{-1}h \ast \phi \) where \( E^{-1} \) is the inverse Elzaki transform.

Proof. Let \( u \in \hat{S} \) such that \( Eu = h \) then

\[
E(u \ast s) = h \ast s.
\]

Hence, employing \( E^{-1} \) on both sides yields \( E^{-1}(h \ast s) = u \ast s = E^{-1}h \ast s \).

**Lemma 2.12.** If \( h_1, h_2 \in H, s_1, s_2 \in D \) then

\[
(h_1 + h_2) \ast s = h_1 \ast s + h_2 \ast s; \quad (2) h \ast (s_1 \ast s_2) = (h \ast s_1) \ast s_2.
\]

**Lemma 2.13.** If \( h_n \to h \) and \( s \in D \) then \( h_n \ast s \to h \ast s \).

**Lemma 2.14.** If \( h_n \to h \) in \( H \) and \( (r_n) \in \Delta \) then

\[
h_n \ast r_n \to h \text{ as } n \to \infty.
\]

The space \( O(H, D, \Delta) \) can therefore be regarded as a Boehmian space.

### III. Elzaki Transform of Boehmians

Let \( \beta_1 = \left[ \frac{u_n}{s_n} \right] \in O(S, D, \Delta) \) then we define the extended Elzaki transform of \( \beta_1 \) as

\[
\hat{E} \left[ \frac{u_n}{r_n} \right] = \left[ \frac{Eu_n}{r_n} \right] \in O(H, D, \Delta), \quad (15)
\]

where \( (r_n) \in \Delta \).

**Theorem 3.1.** \( \hat{E} : O(S, D, \Delta) \to O(H, D, \Delta) \) is well defined.

**Proof:** Let \( \left[ \frac{u_n}{r_n} \right] = \left[ \frac{v_n}{\psi_n} \right] \) in \( O(S, D, \Delta) \) then

\[
\nu_n \ast \psi_m = v_n \ast r_n = v_n \ast r_m.
\]

Employing \( E \) on both sides,

\[
(Eu_n)(z) \int \psi_m(t) e^{\frac{-rt}{\psi}} dt = (Ev_n)(z) \int r_m(t) e^{\frac{-rt}{\psi}} dt.
\]

Hence,

\[
Eu_n \ast \psi_m = Ev_n \ast r_m.
\]

That is,

\[
\frac{Eu_n}{r_n} \sim \frac{Ev_n}{\psi_n}.
\]

Therefore,

\[
\left[ \frac{Eu_n}{r_n} \right] = \left[ \frac{Ev_n}{\psi_n} \right].
\]
This completes the proof of the theorem.

**Theorem 3.2.** \( \hat{E} : O \left( \hat{S}, D, \Delta \right) \to O \left( H, D, \Delta \right) \) is linear

**Proof.** Let \( \left[ \frac{u_n}{r_n} \right], \left[ \frac{v_n}{\psi_n} \right] \). From definitions and Equ.\( (15) \) we get

\[
\hat{E} \left( \left[ \frac{u_n}{r_n} \right] + \left[ \frac{v_n}{\psi_n} \right] \right) = \hat{E} \left( \left[ \frac{u_n \ast \psi_n + v_n \ast r_n}{r_n \ast \psi_n} \right] \right) = \hat{E} \left( \frac{E \left( u_n \ast \psi_n + v_n \ast r_n \right)}{r_n \ast \psi_n} \right) = \hat{E} \left( \frac{E u_n \bullet \psi_n + E v_n \bullet r_n}{r_n \ast \psi_n} \right) = \frac{E u_n}{r_n} + \frac{E v_n}{\psi_n}.
\]

Hence

\[
\hat{E} \left( \left[ \frac{u_n}{r_n} \right] + \left[ \frac{v_n}{\psi_n} \right] \right) = \hat{E} \left[ \frac{u_n}{r_n} \right] + \hat{E} \left[ \frac{v_n}{\psi_n} \right].
\]

Also, if \( \alpha \in \mathbb{R}_+ \) then

\[
\alpha \hat{E} \left[ \frac{u_n}{r_n} \right] = \alpha \left[ \frac{E u_n}{r_n} \right] = \frac{E \left( \alpha u_n \right)}{r_n}.
\]

Hence

\[
\alpha \hat{E} \left[ \frac{u_n}{r_n} \right] = \hat{E} \left( \alpha \left[ \frac{u_n}{r_n} \right] \right).
\]

This completes the proof.

**Theorem 3.3.** \( \hat{E} \) is one-one.

**Proof.** Let \( \beta_1, \beta_2 \in O \left( \hat{S}, D, \Delta \right) \) such \( \beta_1 = \left[ \frac{u_n}{r_n} \right] \) and \( \beta_2 = \left[ \frac{v_n}{\psi_n} \right] \).

Assume \( E \beta_1 = E \beta_2 \) then \( \left[ \frac{E u_n}{r_n} \right] = \left[ \frac{E v_n}{\psi_n} \right] \). That is,

\[
E u_n \bullet \psi_n = E v_m \bullet r_n.
\]

Using Lemma 2.9,

\[
E \left( u_n \ast \psi_m \right) = E \left( v_m \ast r_n \right).
\]

Therefore

\[
u_n \ast \psi_m = v_m \ast r_n.
\]

Hence

\[
\frac{u_n}{r_n} \sim \frac{v_n}{\psi_n}
\]

and

\[
\left[ \frac{u_n}{r_n} \right] \sim \left[ \frac{v_n}{\psi_n} \right].
\]

This completes the proof of the lemma.

**Theorem 3.4.** \( \hat{E} : O \left( \hat{S}, D, \Delta \right) \to O \left( H, D, \Delta \right) \) is onto.

**Proof.** Let \( \left[ \frac{h_n}{r_n} \right] \in O \left( H, D, \Delta \right) \) then

\[
h_n = E u_n.
\]
for all \( n, \left[ \frac{u_n}{r_n} \right] \) is in \( O \left( \hat{S}, D, \Delta \right) \) such that

\[
\hat{E} \left[ \frac{u_n}{r_n} \right] = \left[ \frac{E u_n}{r_n} \right] = \left[ \frac{h_n}{r_n} \right].
\]

Hence the theorem. Now, we define the inverse \( \hat{E}^{-1} \) by the relation

\[
\hat{E}^{-1} \left[ \frac{h_n}{r_n} \right] = \left[ \frac{E^{-1} h_n}{r_n} \right],
\]

for every \( h_n \in O \left( H, D, \Delta \right) \).

**Theorem 3.5.** \( \hat{E}^{-1} : O \left( H, D, \Delta \right) \to O \left( \hat{S}, D, \Delta \right) \) is well defined.

**Theorem 3.6.** \( \hat{E}^{-1} : O \left( H, D, \Delta \right) \to O \left( \hat{S}, D, \Delta \right) \) is linear.

**Theorem 3.7.** \( \hat{E}^{-1} : O \left( H, D, \Delta \right) \to O \left( \hat{S}, D, \Delta \right) \) is an isomorphism.

Proof of Theorem 3.5, 3.6, 3.7, are analogous to that of Theorem 3.1, 3.2, 3.3, and 3.4. Detailed proofs are avoided.

REFERENCES


