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Pathway Fractional Integral Operator Associated with Certain Special Functions

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7. Mathai, A.M., A pathway to matrix variate gamma and normal densities, Linear Algebra and its Applications,396 (2005), 317-328.

Pathway Fractional Integral Operator Associated with Certain Special Functions

V.B.L.Chaurasia [α] & Jaswant Singh [σ]

Abstract - The aim; of the present paper is to study a pathway fractional integral operator concerning with the pathway model and pathway probability density of some product of special functions. The results derived here are quite general in nature, and hence encompass several cases of interest.

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I. Introduction

Nair [11] introduced the Pathway fractional integral operator which is defined in the following manner

$$\left(P_{0+}^{(\eta,\alpha)}f\right)(x) = x^{\eta}\int_0^{\left[\frac{x}{a(1-\alpha)}\right]}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)\,dt, \tag{1.1}$$

where $f(x) \in L(a,b)$, $\eta \in C$, $Re(\eta) > 0$, $a > 0$ and pathway parameter $\alpha < 1$.

Mathai [7] introduced the pathway model and further studied by Mathai and Haubold ([8], [9]). For real scalar α, the pathway model for scalar random variables is denoted by following probability density function (p.d.f.)W.

$$f(x) = c\,|x|^{\gamma-1}[1-a(1-\alpha)|x|^{\delta}]^{\frac{\beta}{1-\alpha}}, \tag{1.2}$$

where $\gamma > 0, \delta>0, \beta\geq 0, \{1-a(1-\alpha)|x|^{\delta}\}>0, \gamma>0, -\infty < x < \infty$, c is the normalizing constant and α is known as pathway parameter. The normalizing constant, for real α, is as follows:

$$c = \frac{1}{2}\frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\gamma}{\delta}+\frac{\beta}{1-\alpha}+1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{1-\alpha}+1\right)},\ \alpha<1 \tag{1.3}$$

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$$=\frac{1}{2}\frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\beta}{1-\alpha}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{1-\alpha}-\frac{\gamma}{\delta}\right)}, \text{ for } \frac{1}{\alpha-1}-\frac{\gamma}{\delta}>0,\ \alpha>1, \tag{1.4}$$

$$=\frac{1}{2}\frac{\delta(a\beta)^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)} \text{ for } \alpha\to 1. \tag{1.5}$$

It is a finite range density with $\{1-a(1-\alpha)\,|\,x\,|^{\delta} > 0$, for $\alpha < 1$, and (1.2) remains in the extended generalized type – 1 beta family. For $\alpha < 1$, the pathway density in (1.2) includes the extended type – 1 beta density, the triangular density, the uniform density and many other p.d.f.

We have, for $\alpha > 1$,

$$f(x)=c\,|\,x\,|^{\gamma-1}\,[1+a\,(\alpha-1)\,|\,x\,|^{\delta}\,]^{\frac{\beta}{\alpha-1}}, \tag{1.6}$$

where $\alpha > 1$, $\delta > 0$, $\beta \geq 0$, $-\infty < x < \infty$, which is extended generalized type – 2 beta model for real x. It includes the type – 2 beta density, the F-density, the student –t density, the Cauchy density and many more. The pathway parameter $\alpha < 1$ has only been considered here. For $\alpha \to 1$, (1.2) and (1.6) take the exponential form, since

$$\lim_{\alpha\to 1} c|x|^{\gamma-1}\,[1-a(1-\alpha)|\,x\,|^{\delta}\,]^{\frac{\eta}{1-\alpha}}$$

$$=\lim_{\alpha\to 1} c\,|\,x\,|^{\gamma-1}\,[1+a(\alpha-1)|\,x\,|^{\delta}\,]^{\frac{\eta}{\alpha-1}}$$

$$=c\,|\,x|^{\gamma-1}\,e^{-a\eta|x|^{\delta}}. \tag{1.7}$$

This includes generalized Gamma-, the Weibull-, the Chi-square, the Laplace-, the Maxwell-Boltzmann and other related density.

For $\alpha\to 1_{-}$, $\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}} \to e^{-\frac{a\,\eta}{x}t}$ U, the operator (1.1) reduces to the well known Laplace integral transform of f with parameter $\frac{a\eta}{x}$

$$(P_{0+}^{(\eta,1)}f)x = x^{\eta}\int_0^{\infty} e^{-\frac{a\eta}{x}t}\,f(t)\,dt$$

Notes

$$= x^{\eta} L_r\left(\frac{a^{\eta}}{x}\right) \tag{1.8}$$

For $\alpha = 0$, $a = 1$, then replacing η by $\eta-1$ in (1.1) the integral operator reduces to Riemann-Liouville fraction integral operator.
Sharma [13] introduced the generalized M-series as follows

$${}_{\rho}\overset{\alpha',\beta'}{M}{}_{\sigma}(z) = \sum_{k=0}^{\infty} \frac{(a_1')_k \ldots (a_\rho')_k}{(b_1')_k \ldots (b_\sigma')_k} \frac{z^k}{\Gamma(\alpha' k + \beta')},$$

$$= \psi_1(k) \tag{1.9}$$

where $z, \alpha', \beta' \in C$, $\mathrm{Re}(\alpha') > 0$, $\forall z$ if $\rho \le \sigma$, $|z| < \alpha'^{\alpha'}$, for other details see [13].

The following series representation of H-function [12] will be required

$$H_{P,Q}^{M,N}\left[z\left|\begin{matrix}(e_P, E_P)\\(f_Q, F_Q)\end{matrix}\right.\right] = \sum_{h=1}^{N} \sum_{v=0}^{\infty} \frac{(-1)^v \chi(\xi)}{v!\, E_h}\left(\frac{1}{z}\right)^{\xi}, \tag{1.10}$$

where $\xi = \dfrac{e_h - 1 - v}{E_h}$ and $(h = 1, \ldots, N)$

and

$$\chi(\xi) = \frac{\prod_{j=1}^{M} \Gamma(f_j + F_j \xi) \prod_{\substack{j=1\\ j \ne h}}^{N} \Gamma(1 - e_j - E_j \xi)}{\prod_{j=M+1}^{Q} \Gamma(1 - f_j - F_j \xi) \prod_{j=N+1}^{P} \Gamma(e_j + \xi E_j)},$$

$$= \psi_2(\xi) \tag{1.11}$$

for convergence condition and other details see ([4] and [13]).
For the sake of brevity

$$T_1 = \sum_{i=1}^{N} E_i - \sum_{i=N+1}^{P} E_i + \sum_{i=1}^{M} F_i - \sum_{i=M+1}^{Q} F_i \tag{1.12}$$

$$T_2 = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{q} \alpha_i + \sum_{i=1}^{m} \beta_i - \sum_{i=m+1}^{q} \beta_i \tag{1.13}$$

Ref.

13. Sharma, Manoj and Jain, Renu, A note on a generalized M-series as a special function of fractional calculus, Fract. Cal. Appl. Anal., 12(4) (2009), 449-452.

II. Main Results

Theorem 1. Let $\eta,\omega \in C, Re(\beta) > 0, Re(\delta) > 0, Re\left(1+\frac{h}{1-\alpha}\right)>0, Re(\rho)>0, \alpha<1, b\in R,$

$$c\in R, Re\left(\omega+\delta\frac{f_j}{F_j}\right)>0, Re\left(\omega+\beta\frac{b'_j}{\beta'_j}\right)>0, |\arg c|<\frac{1}{2}T_1\pi, |\arg b|<\frac{1}{2}T_2\pi, \beta^*>0,$$

$$T_1, T_2>0, \rho\le\sigma, |d|<\alpha'^{\alpha'}, \beta^*>0, j=1,\ldots,Q;\ j'=1,\ldots,q.$$

Then

$$P_{0+}^{(\eta,\alpha)}\left\{t^{\omega-1}\ {}_{\rho}\overset{\alpha',\beta'}{M}_{\sigma}[dt^{-\beta^*}]H_{P,Q}^{M,N}\left[c\,t^{\delta}\middle|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right]H_{p,q}^{m,n}\left[bt^{\beta}\middle|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right]\right\}$$

$$=\psi_1(k)\frac{d^k x^{\eta+\omega-\beta^*k}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\omega-\beta^*k}}H_{P,Q}^{M,N}\left[\frac{c\,x^{\delta}}{[a(1-\alpha)]^{\delta}}\middle|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right]$$

$$.\ H_{p+1,q+1}^{m,n+1}\left[\frac{b\,x^{\beta}}{a(1-\alpha)^{\beta}}\middle|\begin{matrix}(1-\omega+\delta\xi+\beta^*k,\beta),(a_p,\alpha_p)\\(b_q,\beta_q),\left(-\omega-\delta\xi+\beta^*k-\frac{\eta}{1-\alpha},\beta\right)\end{matrix}\right], \qquad (2.1)$$

Proof. Making use of (1.9), (1.10) and (1.1) and appealing to a known result [11], we arrive at the desired result (2.1).

Theorem 2. Let $\eta,\gamma,\delta,\beta,T_1,T_2>0, Re(\eta)>0, Re(\gamma)>0, Re(\omega)>0, Re\left(1+\frac{\eta}{1-\alpha}\right)>\max.$

$$[0,-Re(\omega)], b, c\in R, \alpha<1, Re\left(\omega+\delta\frac{f_j}{F_j}\right)>0, |\arg c|<\frac{1}{2}T_1\pi, \rho\le\sigma \text{ and } |d|<, \beta^*>0,$$

$j = 1,\ldots,Q$.

Then

$$P_{0+}^{(\eta,\alpha)}\left\{t^{\omega-1}\ {}_{\rho}\overset{\alpha',\beta'}{M}_{\sigma}[dt^{-\beta^*}]H_{P,Q}^{M,N}\left[c\,t^{\delta}\middle|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right]E_{\beta,\rho}^{\gamma}(b\,t^{\beta})\right\}$$

Ref.

11. Nair, Seema S., Pathway fractional integration operator, Fract. Cal. Appl. Anal., 12(3) (2009), 237-259.

$$=\psi_1(k)\,\frac{d^k\, x^{\eta+\omega-\beta*k}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma(\gamma)\,\Gamma[a(1-\alpha)]^{\omega-\beta*k}}\;{}_2\psi_2\left[\frac{b\,x^{\beta}}{a(1-\alpha)^{\beta}}\left|\begin{matrix}(\omega-\delta\xi+\beta*k,\beta),(\gamma,1)\\(\omega,\beta),\left(1+\omega+\frac{\eta}{1-\alpha}-\delta\xi-\beta*k,,\beta\right)\end{matrix}\right.\right],$$

$$.\;H^{M,N}_{P,Q}\left[\frac{c\,x^{\delta}}{[a(1-\alpha)]^{\delta}}\left|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right.\right], \tag{2.2}$$

where $E^{\gamma}_{\beta,\omega}(b)$ is the generalized Mittag-Leffler function (see [14],[15]).

Proof. The result in (2.2) can be derived from Theorem 1 by taking m =1 = n, p = 1, q = 2, $b_1=0,\beta_1=0, b_2=1-\omega,\beta_2=\beta,\alpha_1=1-\gamma$ and $\alpha_1=1$. We have the required result.

Theorem 3. Let $\eta,\gamma,v\in C,\delta>0,\alpha<1,\rho\le\sigma,|d|<\alpha'^{\alpha'},\mathrm{Re}(\eta)>0,c\in R,\mathrm{Re}\,(\gamma+v)>0,$

$$\mathrm{Re}\left(1+\frac{\eta}{1-\alpha}\right)>0,\mathrm{Re}\left(\gamma+\delta\frac{f_j}{F_j}\right)>0,|\arg c|<\frac{1}{2}T_1\pi,T_1>0,\beta^*>0,\ j=1,\ldots,Q.$$

Then

$$P^{(\eta,\alpha)}_{0+}\left\{\left(\frac{t}{2}\right)^{\gamma-1}{}_{\rho}M^{\alpha',\beta'}_{\sigma}\left[d\left(\frac{t}{2}\right)^{-\beta*}\right]J_v(t)\,H^{M,N}_{P,Q}\left[c\left(\frac{t}{2}\right)^{\delta}\left|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right.\right]\right\}$$

$$=\psi_1(k)\,\frac{d^k\,x^{\eta+v+\gamma-\beta*k}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\gamma+v-\beta*k}\,2^{\gamma+v+\eta-\beta*k}}\;H^{M,N}_{P,Q}\left[\frac{c\,x^{\delta}}{[2a(1-\alpha)]^{\delta}}\left|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right.\right]$$

$$.\;{}_1\psi_2\left[-\frac{x^2}{4a^2(1-\alpha)^2}\left|\begin{matrix}(\gamma+v-\delta\xi-\beta*k,2)\\(v+1,1),\left(1+v+\gamma-\delta\xi-\beta*k+\frac{\eta}{1-\alpha},2\right)\end{matrix}\right.\right]. \tag{2.3}$$

Here ${}_p\psi_q$ denotes the generalized Wright hypergeometric function ([14],[15]).

Proof. The result in (2.3) can be established by taking m =1, n = 0, p = 0, q = -2, b₁=0, $\beta_1=0, b_2=-v,\beta_2=1,\omega=\gamma+v, b'=1,\beta=2$ and replacing t by $\frac{t}{2}$ after a little simplification, we have the desired result.

Ref.
14. Wright, E.M., The asymptotic expansion of the generalized Bessel function, Proc. London Math. Soc., 38 (1934), 257-270.

III. Special Cases

1. Letting $\beta^* \to 0$ in the result (2.1), we get the result recently obtained by Chaurasia and Ghiya [1] for ρ, ρ_1 and $\rho_2 \to 0$.
2. Making $\beta^*, \delta \to 0$ in the results (2.1) through (2.3), we have the results recently derived by Chaurasia and Gill in [2].
3. Giving suitable values to the parameters in the results (2.1) through (2.3), we get the results recently obtained by Nair in [11].

A large number of simpler corresponding results pertaining to simpler functions can be obtained easily merely by specializing the parameters in them.

Notes

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