Solution of Fractional Kinetic Equation with Laplace and Fourier Transform

By Satendra Kumar Tripathi & Renu Jain

Jiwaji university, India

Abstract - In earlier paper Saxena et al.(2002,2003)[18],[19] derived the solutions of a number of fractional kinetic equations in terms of generalized Mittage-Leffler functions which extended the work of Haubold and Mathai (2000)[5]. The objects of present paper is to investigate the solution of fractional diffusion equation involving Mittag-Leffler functions. The method involves simultaneous application of Laplace and Fourier transforms with time and space variable respectively. The results obtained are in a form of H-function.

Keywords : Mittage-Leffler function, Fractional Kinetic Equation, Laplace Transform, Fourier Transform and H-functions.

GJSFR-F Classification : MSC 2010: 65T50, 44A10
Solution of Fractional Kinetic Equation with Laplace and Fourier Transform

Satendra Kumar Tripathi & Renu Jain

Abstract - In earlier paper Saxena et al. (2002, 2003) [18, 19] derived the solutions of a number of fractional kinetic equations in terms of generalized Mittag-Leffler functions which extended the work of Haubold and Mathai (2000) [5]. The objects of present paper is to investigate the solution of fractional diffusion equation involving Mittag-Leffler functions. The method involves simultaneous application of Laplace and Fourier transforms with time and space variable respectively. The results obtained are in a form of H-function.

Keywords: Mittage-Leffler function, Fractional Kinetic Equation, Laplace Transform, Fourier Transform and H-functions.

I. Introduction

Fundamental law of physics are written as equations for the time evolution of a quantity X(t), dX(t)/dt=-AX(t), where this could be Maxwell’s equation or Schrödinger’s equation (If A is limited to linear operators), or it could be Newton’s law of motion or Einstein’s equations for geodesics (If A may also be a non linear operator). The mathematical solution (for linear operators) is X(t) = X(0) Exp{-At}. The initial value of the quantity at t=0 is given by X(0).

The same exponential behavior referred to above arises if X(t) represents the scalar number density of species at time t that do not interact with each other. If one denote $A_p$ the production rate and $A_d$ the destruction rate, respectively, the number density obey an exponential equation where the coefficient A is equal to the differen t of $A_p - A_d$. Subsequently, $A_p^{-1}$ is the average time between production and $A_d^{-1}$ is the average time between destruction. This type of behavior arises frequently in biology, chemistry and physics (Hilfer, 2000; Metzler and Klafter, 2000) [6, 12]. This paper in Section 2 summarizes mathematical result concerning solution of the diffusion equations in section 3 and section 4 respectively, widely distributed in the literature or of very recent origin. These involve the Mittage-Leffler function, H-function and the application of fractional calculus, Fourier transform and Laplace transform to them.

The section 3 and section 4 presented in a closed form solution of a fractional diffusion equation in terms of H-function.
II. Mathematical Prerequisites

A generalization of the Mittage-Leffler function (Mittage-Leffler, 1903, 1905)[9],[10]

\[ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, (\alpha \in C, Re(\alpha) > 0) \]  \hspace{1cm} (1)

was introduced by Wiman (1905)[20] in the general form

\[ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, (\alpha, \beta \in C, Re(\alpha) > 0) \]  \hspace{1cm} (2)

The main result of these functions are available in the handbook of Erdelyi Magnus, Oberhettinger and Tricomi (1955, Section 18.1)[4] and the monographs written by Dzherbashyas (1966, 1993)[1][2], Prabhakar (1971)[14] introduced a generalization of (2) in the form

\[ E_{\alpha,\beta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(n\alpha + \beta)n!}, (\alpha, \beta, \gamma \in C, Re(\alpha) > 0) \]  \hspace{1cm} (3)

Where

\[(\gamma)_0 = 1, (\gamma)_k = \gamma(\gamma + 1)(\gamma + 2) \ldots \ldots \ldots (\gamma + k - 1) \quad (k = 1, 2 \ldots\ldots) \quad \gamma \neq 0 \]  \hspace{1cm} (4)

For \( \gamma = 1 \)

\[ E_{1,\beta}(z) = E_{\alpha,\beta}(z), \]  \hspace{1cm} (5)

For \( \gamma = 1, \beta = 1 \)

\[ E_{1,1}(z) = E_{\alpha}(z) \]  \hspace{1cm} (5)

The Mellin-Barnes integral representation for this function follows from the integral

\[ E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} \frac{1}{2\pi i} \int_{\Omega} \frac{\Gamma(-\xi)\Gamma(\gamma + \xi)(-z)^{\xi}}{\Gamma(\beta + \xi\alpha)} d\xi \]  \hspace{1cm} (6)

where \( \omega = (-1)^{1/2} \) The contour \( \Omega \) is straight line parallel to the imaginary axis at a distance ‘c’ from the origin and separating the poles of \( \Gamma(-\xi) \) at the point \( \xi = \nu(\nu = 0, 1, 2 \ldots) \) from those of \( \Gamma(\gamma + \xi) \) at the points \( \xi = -\gamma - \nu(\nu = 0, 1, 2 \ldots) \). If we calculate the residues at the poles of \( \Gamma(\gamma + \xi) \) at the points \( \xi = -\gamma - \nu(\nu = 0, 1, 2 \ldots) \) then it gives the analytic continuation formula of this function in the form[2]

\[ E_{\alpha,\beta}(z) = \frac{(-z)^{-\gamma}}{\Gamma(\gamma)} \sum_{\nu=0}^{\infty} \frac{\Gamma(\gamma + \nu)}{\Gamma(\beta - \alpha\gamma - \alpha\nu)} \frac{(-z)^{-\nu}}{\nu!}, |z| > 1 \]  \hspace{1cm} (7)
From (7) it follows that for large \( z \) its behavior is given by

\[ E^\gamma_{\alpha,\beta}(z) \sim O(|z|^{-\gamma}), |z| > 1 \]  

(8)

The H-function is defined by means of Mellin-Barnes type integral in the following manner (Mathai and Saxena, 1978 p-2)[8]

\[ H_{p,q}^{m,n}(z) = H_{p,q}^{m,n} \left[ z \left( a_p, A_p \right) \right] = H_{p,q}^{m,n} \left[ z \left( a_1, A_1 \ldots, a_p, A_p \right) \right] \]

\[ \frac{\Gamma(b_l + B_j \xi)}{\Gamma(1 - b_l - B_j \xi)} \prod_{j=m+1}^{n} \Gamma(a_j + A_j \xi) \]

\[ \theta(\xi) = \prod_{j=1}^{m+1} \Gamma(b_j + B_j \xi) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j \xi) \]

(9)

where \( \theta(\xi) = \frac{\prod_{j=m+1}^{n} \Gamma(b_j + B_j \xi) \prod_{j=m+1}^{n} \Gamma(1 - a_j - A_j \xi)}{\prod_{j=1}^{m+1} \Gamma(1 - b_j - B_j \xi) \prod_{j=m+1}^{n} \Gamma(a_j + A_j \xi)} \) 

(10)

\[ m, n, p, q \in N_0 \text{ with } 1 \leq n \leq p, 1 \leq m \leq q, A_j, B_j \in R_+ a_j, b_j \in R \]

\[ (i = 1,2 \ldots, p, j = 1,2 \ldots, q) \]

\[ A_i(b_j + k) \neq B_j(a_i - l - 1) \quad (k, l \in N_0 ; i = 1,2 \ldots, n, j = 1,2 \ldots, m) \]  

(11)

Where we employ the usual notations \( N_0 = (0,1,2 \ldots) \) \( R = (-\infty, \infty) \) \( R_+ = (0, \infty) \) and \( C \) defines the complex number field. \( \Omega \) is a suitable contour separating the poles of \( \Gamma(b_j + B_j \xi) \) from those of \( \Gamma(1 - a_j - A_j \xi) \).

A detailed and comprehensive account of the H-function along with convergence condition is available from Mathai and Saxena (1978)[8]

It follows from (7) that the generalized Mittag-Leffler function

\[ E^\gamma_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} H_{1,2}^{1,1} \left[ -z \left( 1 - \gamma, 1 \right) \left( 0, 1 \right) \left( 1 - \beta, \alpha \right) \right] (\alpha, \beta, \gamma \in C, Re(\alpha) > 0) \]  

(12)

Putting \( \gamma = 1 \) in (12)

\[ E^1_{\alpha,\beta}(z) = H_{1,2}^{1,1} \left[ -z \left( 0, 1 \right) \left( 0, 1 \right) \left( 1 - \beta, \alpha \right) \right] \]

(13)

If we further take \( \beta = 1 \) in (13) we get

\[ E^1_{\alpha}(z) = H_{1,2}^{1,1} \left[ -z \left( 0, 1 \right) \left( 0, \alpha \right) \right] \]

(14)

From Prudnikov, A.P., Brychkov, Yu.A. and Marichev, O.I (1989,p.355.eq2.25.3.2) [15] and Mathai and Saxena(1978,p.49)[8] it follows that the cosine transform of the H-function is given

© 2012 Global Journals Inc. (US)
\[
\int_0^\infty t^{\rho - 1} \cos kt H_{p,q}^{m,n} \left[ a^\alpha \right] \left( a_p, A_p \right) \left( b_q, B_q \right) \right] dt \\
= \frac{\pi}{k^\rho} H_{q+1,p+2}^{n+1,m} \left[ \frac{k^\mu}{\alpha} \right] \left( 1 - b_q, B_q \right) \left( \frac{1}{2} + \frac{\rho}{2}, \frac{\mu}{2} \right) \left( \rho, \mu \right) \left( 1 - a_p, A_p \right) \left( \frac{1}{2} + \frac{\rho}{2}, \frac{\mu}{2} \right)
\]

(15)

The Riemann-Liouville fractional integral of order \( v \in \mathbb{C} \) is defined by Miller and Ross (1993, p.45; [11]) see also Srivastva and saxena (2001)[17]

\[
_0 D_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t - u)^{v-1} f(u) du
\]

(16)

where \( Re(v) > 0 \) following Samko, S.G., Kilbas, A. A. and Marichev, O.I. (1993, p.37)[16] we define the fractional derivative for \( \alpha > 0 \) in the form

\[
_0 D_t^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(u)}{(t - u)^{\alpha-n+1}} du, (n = [Re(\alpha)] + 1)
\]

(17)

where \([Re(\alpha)]\) means the integral part of \( Re(\alpha) \).

In particular, if \( 0 < \alpha < 1 \)

\[
_0 D_t^{\alpha} f(t) = \frac{d}{dt} \int_0^t \frac{f(u)}{(t - u)^{\alpha}} du
\]

(18)

And in \( \alpha = n \in \mathbb{N} \) then

\[
_0 D_t^{\alpha} f(t) = D^n f(t)
\]

(19)

is the usual derivative of \( n \).

From Erdelyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F.G (1954, p.182) [3] we have

\[
L \{ _0 D_t^{-v} f(t) \} = s^{-v} F(s)
\]

(20)

\[
F(s) = L \{ f(t); s \} = \int_0^\infty e^{-st} f(t) dt
\]

(21)

where \( Re(s) > 0 \)

The Laplace transform of the fractional derivative is given by Oldham and spanier (1974, p.134, eq 8.1.3; [13]) see also (srivastva and saxena 2001)[17]

\[
L \{ _0 D_t^{-v} f(t) \} = s^\alpha F(s) - \sum_{k=1}^{n} s^{k-1} _0 D_t^{\alpha-k} f(t) \mid_{t=0}
\]

(22)
In this we present solution of the fractional diffusion equation given by (Metzler and Klafter 2000; Jorgenson and Lang, 2001)[12][7]

Theorem 1. Consider the fractional diffusion equation

\[ N(x, t) - N_0 t^{\mu - 1} = -c^v \int_0^t \int_0^x D_x^{\nu} N(x, t) \] \hspace{1cm} (23)

with initial condition

\[ 0D_t^{-k} N(x, t)|_{t=0} = 0 \text{ and } 0D_t^{-\nu-k} N(x, t)|_{x=0} = 0, k = 1, 2 \ldots n \] \hspace{1cm} (24)

Where \( n = [\text{Re}(\nu)] + 1; c^v \) is diffusion constant then for the solution of (23) is given by

\[ N(x, t) = \frac{N_0 \Gamma(\mu)}{c^t} H_{1,1}^{1,0} \left( \frac{|x|^\nu}{(ct)^\nu} \right) (1 + \nu, \nu) \] \hspace{1cm} (25)

Proof-

\[ N(x, t) - N_0 t^{\mu - 1} = -c^v \int_0^t \int_0^x D_x^{\nu} N(x, t) \]

Apply Laplace and fourier transform with time variable and space variable respectively to (23) we get

\[ N^*(k, s) - N_0 \frac{\Gamma(\mu)}{s^\mu} = -c^v k^v s^{-\nu} N^*(k, s) \]

\[ N^*(k, s) \{1 + (s/c)^{-\nu} k^v\} = N_0 s^{-\mu} \Gamma(\mu) \]

\[ N^*(k, s) = N_0 s^{-\mu} \Gamma(\mu) \left\{1 + \left(\frac{s}{kc}\right)^{-\nu}\right\}^{-1} \]

\[ = N_0 s^{-\mu} \Gamma(\mu) \sum_{r=0}^{\infty} \frac{(1)_r \left(-\left(\frac{s}{kc}\right)^{-\nu}\right)^r}{r!} \]

\[ = N_0 \Gamma(\mu) \sum_{r=0}^{\infty} \frac{(1)_r (kc)^{rv} (-1)^r r!}{r!} s^{-\nu r - \mu} \]

where \( N^*(k, s) \) Laplace and Fourier transform of \( N(x, t) \)

Taking inverse Laplace transform

\[ N(k, t) = N_0 \Gamma(\mu) \sum_{r=0}^{\infty} (kc)^{rv} (-1)^r \frac{L^{-1}\{s^{-\nu r - \mu}\}}{r!} \]

\[ N(k, t) = N_0 \Gamma(\mu) \sum_{r=0}^{\infty} (kc)^r (-1)^r \frac{t^{\mu+r-1}}{\Gamma(r\nu + \mu)} \]

\[ = N_0 \Gamma(\mu) t^{\mu-1} E_{\nu,\mu}(-c^\nu k^\nu t^\nu) \]

which can we expressed in terms of H-function

\[ = N_0 \Gamma(\mu) t^{\mu-1} H_{1,2}^{1,1}

[ c^\nu k^\nu t^\nu ]

\( (0,1) \)

\( (0,1)(1 - \mu, \nu) \)

Now take inverse fourier transformation

\[ N(x, t) = \frac{1}{\pi} \int_0^\infty \cos kx \ t^{\mu-1} N_0 \Gamma(\mu) H_{1,2}^{1,1}

[ c^\nu k^\nu t^\nu ]

\( (0,1) \)

\( (0,1)(1 - \mu, \nu) \)

\( dk \)

\[ = \frac{t^{\mu-1} N_0 \Gamma(\mu) \pi}{\pi} \frac{\pi}{|x|} H_{3,3}^{2,1}

[ |x|^\nu ]

\( (ct)^\nu \)

\( (1,1)(\mu, \nu)(1, \nu/2) \)

\( (1,1)(1, \nu)(1, \nu/2) \)

Applying a result of Mathai and Saxena (1978, p.4.eq1.2.1) the above expression becomes

\[ N(x, t) = \frac{N_0 \Gamma(\mu)}{|x|} H_{2,2}^{2,0}

[ |x|^\nu ]

\( (ct)^\nu \)

\( (\mu, \nu)(1, \nu/2) \)

\( (1, \nu)(1, \nu/2) \)

If we employ the formula Mathai and Saxena (1978,p.4.eq1.2.4)

\[ x^\sigma H_{p,q}^{m,n}

[ z \]

\( (a_p, A_p) \)

\( (b_q, B_q) \)

\[ = H_{p,q}^{m,n}

[ z \]

\( (a_p + \sigma A_p, A_p) \)

\( (b_q + \sigma B_q, B_q) \)

\[ N(x, t) = \frac{N_0 \Gamma(\mu)}{ct} H_{2,2}^{2,0}

[ |x|^\nu ]

\( (ct)^\nu \)

\( (\mu + \nu, \nu)(1, \nu/2) \)

\( (1 + \nu, \nu)(1, \nu/2) \)

\[ N(x, t) = \frac{N_0 \Gamma(\mu)}{ct} H_{1,1}^{1,0}

[ |x|^\nu ]

\( (ct)^\nu \)

\( (\mu + \nu, \nu) \)

\( (1 + \nu, \nu) \)

Theorem 2- Consider the fractional diffusion equation (Metzler and Klafter 2000;Jorgenson and Long,2001)[12][7]

\[ 0D^\nu_t N(x, t) - E_\nu (-d^\nu t^\nu) = -c^\nu \frac{\partial^2}{\partial x^2} N(x, t) \quad (26) \]

with initial condition
\[ 0 D_t^{\nu-k} N(x, t) \big|_{t=0} = 0 \quad k = 1,2 \ldots n \]  

(27)

Where \( n = \text{Re}(\nu) + 1 \); \( c^\nu \) is diffusion constant.

Then for the solution of (26) is given by

\[
\frac{1}{2d^\nu/2} \sin(d^\nu/2x) * \frac{1}{(ct)^\nu} H^{1,0}_{1,1} \left[ x^2 \left| (1 - \nu/2, \nu) \right. \right] (0,2) \\
- \frac{1}{2d^\nu/2} \sin(d^\nu/2x) H^{1,1}_{1,2} \left[ d^\nu t^\nu \left| (0,1) \right. \right] (0,1)(0,\nu) \]  

(28)

Proof-

\[ 0 D_t^\nu N(x, t) - E^\nu (-d^\nu t^\nu) = -c^\nu \frac{\partial^2}{\partial x^2} N(x, t) \]

Applying the fourier transform with respect to the space variable \( x \) and the Laplace transform with respect to the time variable \( t \). we get

\[ s^\nu N^*(k, s) - \frac{s^{\nu-1}}{s^\nu + d^\nu} = -c^\nu k^2 N^*(k, s) \]

\[ \{s^\nu + c^\nu k^2\} N^*(k, s) = \frac{s^{\nu-1}}{s^\nu + d^\nu} \]

\[ N^*(k, s) = \frac{s^{\nu-1}}{s^\nu + d^\nu \{s^\nu + c^\nu k^2\}} \]

\[ = \frac{s^{\nu-1}}{c^\nu k^2 - d^\nu} \left[ \frac{1}{s^\nu + d^\nu} - \frac{1}{s^\nu + c^\nu k^2} \right] \ldots \]  

(29)

To invert equation(29).It is convenient to first invert the Laplace transformation and fourier transform.Apply inverse Laplace transform we obtain

\[ N(k, t) = \frac{1}{c^\nu k^2 t^\nu} \left[ E^\nu (-d^\nu t^\nu) - E^\nu (-c^\nu k^2 t^\nu) \right] \ldots \]  

(30)

Which can expressed in terms of H-function

\[ N(k, t) = \frac{1}{c^\nu k^2 - d^\nu} \left\{ H^{1,1}_{1,2} \left[ d^\nu t^\nu \left| (0,1) (0,\nu) \right. \right] - H^{1,1}_{1,2} \left[ c^\nu k^\nu t^\nu \left| (0,1) (0,\nu) \right. \right] \right\} \]  

(31)

Invert the fourier transform
\[ N(x, t) = \frac{1}{\pi} \int_{0}^{\infty} \cos k x \frac{1}{c^{\nu} k^{2} - d^{\nu}} \left\{ H^{1,1}_{1,2} \left[ d^{\nu} t^{\nu} \left| \frac{(0,1)}{(0,1)(0,\nu)} \right] \right\} dk \]

\[ - \frac{1}{\pi} \int_{0}^{\infty} \cos k x \frac{1}{c^{\nu} k^{2} - d^{\nu}} H^{1,1}_{1,2} \left[ c^{\nu} k^{\nu} t^{\nu} \left| \frac{(0,1)}{(0,1)(0,\nu)} \right] \right\} dk \]

\[ = - \frac{1}{2d^{\nu/2}} \sin(d^{\nu/2}x) H^{1,1}_{1,2} \left[ d^{\nu} t^{\nu} \left| \frac{(0,1)}{(0,1)(0,\nu)} \right] \right\] + \frac{1}{2d^{\nu/2}} \sin(d^{\nu/2}x) \]

\[ * \frac{1}{|x|} H^{2,1}_{2,3} \left[ \frac{|x|^2}{(ct)^{\nu}} \left| (1,1)(1,\nu)(1,1) \right| \right] \]

\[ = - \frac{1}{2d^{\nu/2}} \sin(d^{\nu/2}x) H^{1,1}_{1,2} \left[ d^{\nu} t^{\nu} \left| \frac{(0,1)}{(0,1)(0,\nu)} \right] \right\] + \frac{1}{2d^{\nu/2}} \sin(d^{\nu/2}x) \]

\[ \times \frac{1}{(c^{\nu} t^{\nu})^{1/2}} H^{2,0}_{2,2} \left[ \frac{|x|^2}{(ct)^{\nu}} \left| \frac{(1 - v/2, \nu)}{(0,2)} \left( \frac{1}{2}, 1 \right) \right| \right] \]

\[ = \frac{1}{2d^{\nu/2}} \sin(d^{\nu/2}x) * \frac{1}{(ct)^{\nu}} H^{1,0}_{1,1} \left[ \frac{|x|^2}{(ct)^{\nu}} \left| \frac{(1 - v/2, \nu)}{(0,2)} \right| \right] \]

\[ - \frac{1}{2d^{\nu/2}} \sin(d^{\nu/2}x) H^{1,1}_{1,2} \left[ d^{\nu} t^{\nu} \left| \frac{(0,1)}{(0,1)(0,\nu)} \right] \right\] \]

### III. Conclusion

The fractional kinetic equation has been extended to generalized fractional equation (23) and (26). Their respective solutions are given in terms of Mittag-Leffler function and their generalization, which can also be represented as Fox’s H-function.

### References Références Referencias
