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Solving Third Order Three-Point Boundary Value Problem on Time Scales by Solution Matching Using Differential Inequalities

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Strictly as per the compliance and regulations of :

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K. R. Prasad[α], N. V. V. S. Suryanarayana[σ], & P. Murali[ρ]

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I. Introduction

In this paper we consider, the existence and uniqueness of solutions of the three point boundary value problems associated with the differential equation on time scales

$$y^{\Delta^3}(t) = f(t, y(t), y^{\Delta}(t), y^{\Delta^2}(t)) \tag{1.1}$$

With

$$y(t_1) = y_1,\ y(t_2) = y_2,\ y(\sigma^3(t_3)) = y_3 \tag{1.2}$$

where $f \in C_{rd}\,[[t_1, \sigma^3(t_3)] \times \mathbb{R}^3, \mathbb{R}]$ and we assume through out that solutions of initial value problems associated with (1.1) exist, are unique and extend through out a fixed interval of $\mathbb{R}$. A monotonicity restriction on f assumes that the two point boundary value problem for (1.1) satisfying any one of

$$y(t_1) = y_1,\ y(t_2) = y_2,\ y^{\Delta}(t_2) = m \tag{1.3}$$

$$y(t_1) = y_1,\ y(t_2) = y_2,\ y^{\Delta^2}(t_2) = m \tag{1.4}$$

$$y(t_2) = y_2,\ y^{\Delta}(t_2) = m,\ y(\sigma^3(t_3)) = y_3 \tag{1.5}$$

or

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$$y(t_2) = y_2,\ y^{\Delta^2}(t_2) = m,\ y(\sigma^3(t_3)) = y_3 \qquad (1.6)$$

have at most one solution and with added hypothesis, a unique solution of the three point boundary value problem (1.1), (1.2) is constructed by using differential inequalities. This is acheived by matching solutions of the boundary value problem (1.1), (1.3) with solutions of (1.1), (1.5) or solutions of the boundary value problem (1.1), (1.4) with solutions of (1.1), (1.6).

The technique of matching solutions was discussed by Bailey, Shamphine and Waltman [2] to obtain solutions of two-point boundary value problems for the second order equation by matching solutions of initial value problems. Later, many authors like Barr and Sherman [4], Barr and Miletta [3], Das and Lalli [8], Henderson [10, 11], Henderson and Taunton [13], Lakshmikantham and Murty [16], Moorti and Garner [17], Rao, Murty and Rao [18] have used this technique and obtained solutions three point bound-ary value problems by matching solutions of two two-point boundary value problems for ordinary differential equations. Henderson and Prasad [12] and Eggensperger, Kaufmann and Kasmatov [9] obtained solutions of three point boundary value problems using matching methods for boundary value problems on time scales.

In this paper, we are concerned with the existence and uniqueness of solutions of three point boundary value problems for a differential equation on time scales using differential inequalities. We state some basic definitions of time scales for ready reference.

Definition 1.1. A nonempty closed subset of $\mathbb{R}$ is called a time scale. It is denoted by $\mathbb{T}$. By an interval we mean the intersection of the given interval with a time scale. For $t < \sup\mathbb{T}$ and $r > \inf \mathbb{T}$, define the forward jump operator, σ and backward jump operator, ρ, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T},$$

$$\rho(r) = \sup\{s \in \mathbb{T} : s < r\} \in \mathbb{T},$$

for all $t, r \in \mathbb{T}$. If $\sigma(t) = t$, t is said to be right dense,(otherwise t is said to be right scattered) and if $\rho(r) = r$, r is said to be left dense, (otherwise r is said to be left scattered).

Definition 1.2. For $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$ (if $t = \sup\mathbb{T}$, assume t is not left scattered), define the delta derivative of $x(t)$, denoted by $x^{\Delta}(t)$, to be the number(when it exists), with the property that, for any $\epsilon > 0$, there is a neighborhood U of t such that

$$|\, [x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]\, | \leq \epsilon\, |\, \sigma(t) - s\, |,$$

for all $s \in U$.

If x is delta differentiable for every $t \in \mathbb{T}$; we say that $x : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$.

Definition 1.3. If the time scale $\mathbb{T}$ has a maximal element which is also left scattered, that point is called a degenerate point. Any subset of non- degenerate points of $\mathbb{T}$ is denoted by $\mathbb{T}^k$.

Definition 1.4. A function $x : \mathbb{T} \to \mathbb{R}$ is right dense continuous (rd- continuous) if it is continuous at every right dense point $t \in \mathbb{T}$ and its left hand limit exists at each left dense point $t \in \mathbb{T}$.

Ref.

3. D. Barr and P. Miletta, An existence and uniqueness criterion for solutions of boundary value problems *J. Diff. Eqns.*, 16(1974), 460-471.
4. D. Barr and T. Sherman, Existence and uniqueness of solutions of three point boundary value problems, *J. Diff. Eqns.*, 13(1973), 197-212.

The forward jump operator $\sigma : \mathbb{T} \to \mathbb{R}$ is right dense continuous and more generally if $x : \mathbb{T} \to \mathbb{R}$ is continuous, then $x(\sigma) : \mathbb{T} \to \mathbb{R}$ is right dense continuous. moreover, we say that f is delta differentiable on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$: The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is then called the delta derivative of f on $\mathbb{T}^k$.

Definition 1.5. A function $\mathbb{F} : \mathbb{T}^k \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T}^k \to \mathbb{R}$ provided $\mathbb{F}^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$: We then define the integral of f by

Notes

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

Definition 1.6. The point t_0 is a generalized zero of the function $y(t)$ if either $y(t_0) = 0$ or $y(t_0)y(\sigma(t_0)) < 0$.

Theorem 1.1. Mean value theorem: if $y : \mathbb{T} \to \mathbb{R}$ is continuous and $y(t)$ has generalized zeros at a and b, then there exists $p \in [a, b]$ such that y^{Δ} has a generalized zero at p.

Proof. We refer to Bohner and Eloe [5].

II. Differential Inequalities

In this section, we develop the theory of differential inequalities on time scales associated with the second order differential equation

$$y^{\Delta^2}(t) = f(t, y(t), y^{\Delta}(t)) \tag{2.1}$$

For this, we need the following set.

Definition 2.1. Let $y \in C^2_{rd}[[t_1, \sigma^2(t_2)], \mathbb{R}]$. We say that a point $t_0 \in (t_1, \sigma^2(t_2))$ is in the set Ω if $y(t_0) \leq 0$ and y^{Δ} has a generalized zero at t_0.

Lemma 2.1. Assume that $y \in C^2_{rd}[[t_1, \sigma 2(t_2)], \mathbb{R}]$ and y has a generalized zero at t_1 and suppose that $y^{\Delta 2}(t_0) < 0$ whenever $t_0 \in \Omega$. If $y(t_0) \neq 0$ on $[t_1, \sigma^2(t_2))$, then y^{Δ} has a generalized zero at t_2 if and only if y has a generalized zero at t_2.

Proof. Suppose that y has a generalized zero at t_2 and $y(t) \neq 0$ on $[t_1, \sigma 2(t_2))$. For the sake of contradiction, we assume that y^{Δ} has no generalized zero at t_2. Since y has generalized zeros at t_1 and t_2, y^{Δ} will have a generalized zero at some $r \in (t_1; \sigma 2(t_2))$ such that y^{Δ} has no generalized zero in $(r, \sigma^2(t_2)) \cap \mathbb{T}$. From the definition of generalized zero, we have either $y^{\Delta}(r) = 0$ or $y^{\Delta}(r)y^{\Delta}(\sigma(r)) < 0$. If r is right dense, then $y^{\Delta}(r) = 0$ and if r is right scattered, then $y^{\Delta}(r)y^{\Delta}(\sigma(r)) < 0$. Let $y^{\Delta}(t) = 0$ on $(r, \sigma^2(t_2)] \cap \mathbb{T}$. (otherwise use $-y^{\Delta}(t)$.) Then $0 < \int_r^{t2} y^{\Delta}(t)\Delta(t) = y(t_2) - y(r) \leq -y(r)$ which implies $y(r) < 0$. Since $y(r) < 0$, y^{Δ} has a generalized zero at r and hence $r \in \Omega$ which implies by hypothesis that$y^{\Delta 2}(r) < 0$. However, if r is right dense (i.e. $\sigma(r) = r$),then

$$y^{\Delta^2}(r) = \lim_{t \to \sigma(r)} \frac{y^{\Delta}(t)}{t - r} > 0$$

and if r is right scattered (i.e. $\sigma(r) > r$) then

$$y^{\Delta^2}(r) = \frac{y^{\Delta}(\sigma(r)) - y^{\Delta}(r)}{\sigma(r) - r} > 0$$

Thus, in either case,we have obtained $y^{\Delta 2}(r) > 0$, which is a contradiction. Thus, y^{Δ} has a generalized zero at t_2. A similar argument holds if y^{Δ} has a generalized zero at t_2.

Lemma 2.2. Assume that $y \in C^2_{rd}[[t_1, \sigma^2(t_2)],R]$ and y has a generalized zero at t_2 and further suppose that $y^{\Delta 2}(r) < 0$ whenever $r \in \Omega$. If $y(t) \neq 0$ on $[t_1, \sigma^2(t_2))$ then y^{Δ} has a generalized zero at t_1 if and only if y has a generalized zero t_1.

Proof is analogous to the proof of the Lemma 2.1.

Lemma 2.3. If $y(t)$ is any solution of (2.1) such that y has generalized zeros at t_1 and t_2 and if $y^{\Delta 2}(t_0) < 0$ whenever $t_0 \in \Omega$, then $y(t) = 0$ on $[t_1, \sigma^2(t_2)]$.

Proof. For the sake of contradiction, we assume that $y(t) \neq 0$ on $[t_1, \sigma^2(t_2)]$. Since $y(t) \neq 0$ at any point in $(t_1, \sigma^2(t_2))$, y must have a non zero extremum in $(t_1, \sigma^2(t_2))$ =) y^{Δ} has a generalized zero at some $t_0 \in (t_1, \sigma(t_2))$. i.e. either $y^{\Delta}(t_0) = 0$ or $y^{\Delta}(t_0)y^{\Delta}(\sigma(t_0)) < 0$. If t_0 is right dense, then $y^{\Delta}(t\ ) = 0$

and

if t_0 is right scattered, then $y^{\Delta}(t_0)y^{\Delta}(\sigma(t_0)) < 0$. Assume with out loss generality that $y^{\Delta}(t_0) > 0$ on $(t_0, œ(t_2)]$. Then $0 < \int_{t0}^{t2} y^{\Delta}(t)\Delta(t) = y(t_2) - y(t_0) \leq -y(t_0)$ which implies $y(t_0) < 0$. Now $y(t_0) < 0$ and $y^{\Delta}(t_0) \geq 0$ which implies by hypothesis that $y^{\Delta 2}(t_0) < 0$. How ever, if t_0 is right dense ,then

$$y^{\Delta^2}(t_0) = \lim_{t \to \sigma(t_0)} \frac{y^{\Delta}(t)}{t - t_0} > 0$$

and if t_0 is right scattered,then

$$y^{\Delta^2}(t_0) = \frac{y^{\Delta}(\sigma(t_0)) - y^{\Delta}(t_0)}{\sigma(t_0) - t_0} > 0.$$

Hence, a contradiction. Thus, $y(t) = 0$ on $[t_1, \sigma^2(t_2)]$.

Consider the boundary value problem

$$\begin{aligned} y^{\Delta^2}(t) &= f(t, y(t), y^{\Delta}(t)) \\ y(t_1) &= y_1, \ y(\sigma^2(t_2)) = y_2 \end{aligned} \tag{2.3}$$

Suppose $\Phi(t)$ and $\Psi(t)$ are two solutions of the above boundary value problem. Write $\chi(t) = \Phi(t) - \Psi(t)$. Then

$$\begin{aligned} \chi^{\Delta^2}(t) &= \Phi^{\Delta^2}(t) - \Psi^{\Delta^2}(t) \\ &= f(t, \Phi(t), \Phi^{\Delta}(t)) - f(t, \Psi(t), \Psi^{\Delta}(t)) \\ &= f(t, \chi(t) + \Psi(t), \chi^{\Delta}(t) + \psi^{\Delta}(t)) - f(t, \Psi(t), \Psi^{\Delta}(t)) \\ &= F(t, \chi(t), \chi^{\Delta}(t)) \end{aligned}$$

Clearly $F(t, 0, 0) = 0$, $\chi(t_1) = 0$, $\chi(t_2) = 0$. Thus, we have the following theorem.

Notes

Theorem 2.1. The boundary value problem

$$y^{\Delta^2}(t) = F(t, y(t), y^{\Delta}(t))$$
$$y(t_1) = 0,\ \ y(\sigma^2(t_2)) = 0 \tag{2.2}$$

where F(t, 0, 0) = 0 has only the trivial solution if and only if the following boundary value problem

Notes

$$y^{\Delta^2}(t) = f(t, y(t), y^{\Delta}(t))$$
$$y(t_1) = y_1,\ \ y(\sigma^2(t_2)) = y_2 \tag{2.3}$$

has a unique solution.

Proof. Suppose (2.2) has only the trivial solution $\chi(t)$. Then $\chi(t) = 0\ \forall\ t \in [t_1, \sigma^2(t_2)]$ and hence $\Phi(t) = \Psi(t)$. Conversely, suppose that (2.3) has a unique solution. Then, $\chi(t) = \Phi(t) - \Psi(t)$. Obviously $\chi(t_1) = \chi(t_2) = 0$ and $\chi^{\Delta 2}(t) = F(t, \chi(t), \chi^{\Delta}(t))$ and $F(t, 0, 0) = 0$. Hence $\chi(t)$ is the only solution of (2.2). Thus, the proof of the theorem is complete.

We now develop the theory of differential inequalities associated with the third order differential equation. For this, we need the following sets and classes of functions.

Definition 2.2. Let $y \in C^3_{rd}[[t_1, \sigma^3(t_3)],\mathbb{R}]$. We say that a point $t_0 \in \Omega_1$ if $y(t_0) \leq 0$, $y^{\Delta}(t_0) > 0$ and $y^{\Delta 2}$ has a generalized zero at t_0 for some $t_0 \in [t_1, t_2]$ and $t_0 \in \Omega_2$ if $y(t_0) \geq 0$, $y^{\Delta}(t_0) > 0$ and $y^{\Delta 2}$ has a generalized zero at t_0 for some $t_0 \in [t_2, \sigma^3(t_3)]$.

Definition 2.3. We say that a function

$f(t, y(t), y^{\Delta}(t), y^{\Delta 2}(t)) \in \mathbb{C}rd[[t_1, \sigma^3(t_3)] \times \mathbb{R}^3,\mathbb{R}]$ is in the set G_1 if $f(t, y(t), y^{\Delta}(t), y^{\Delta 2}(t)) \geq 0\ \forall\ t \in [t_1, t_2]$, $f(t, y(t), y^{\Delta}(t), y^{\Delta 2}(t))$ is non decreasing in y and strictly increasing in y^{Δ} and belongs to the set G_2 if $f(t, y(t), y^{\Delta}(t), y^{\Delta 2}(t)) \geq 0\ \forall\ t \in [t_2, \sigma^3(t_3)]$, $f(t, y(t), y^{\Delta}(t), y^{\Delta 2}(t))$ is non decreasing in y and strictly increasing in y^{Δ}.

Lemma 2.4. Let $y(t)$ be a solution of (1.1) such that y has generalized zeros at t_1 and t_2 with $f(t, 0, 0, 0) = 0$. Further suppose that $y^{\Delta 3}(t_0) > 0$ whenever $t_0 \in \Omega_1$. If either y^{Δ} or $y^{\Delta 2}$ has a generalized zero at t_2, then $y(t) = 0$ for all $t \in [t_1, t_2]$.

Proof. We first suppose that y^{Δ} has a generalized zero at t_2. Then we claim that $y^{\Delta 2}$ also has a generalized zero at t_2. To the contrary, assume that $y^{\Delta 2}$ has no generalized zero at t_2. With out loss of generality we can assume that $y^{\Delta 2}(t2) < 0$. So, $\exists$ a $q \in [t_1, t_2)$ such that $y^{\Delta 2}$ has a generalized zero at q and $y^{\Delta 2}(t) < 0\ \forall\ t \in (q, t_2]$. Then

$$0 > \int_t^{t_2} y^{\Delta^2}(t)\Delta t = y^{\Delta}(t_2) - y^{\Delta}(t) \geq -y^{\Delta}(t)$$

which implies $y^{\Delta}(t) > 0\ \forall\ t \in (q, t_2]$. Since $y^{\Delta 2}(t) < 0\ \forall\ t \in (q, t_2]$, it follows that $y^{\Delta}(t)$ is decreasing for $t > q$ and $y^{\Delta}(q)$ is positive. Again $0 < \int_q^{t2} y^{\Delta}(t)\Delta t = y(t2) - y(q) \leq -y(q)$ which implies $y(q) < 0$. Thus, $y(q) < 0$, $y^{\Delta}(q) > 0$ and $y^{\Delta 2}$ has a generalized zero at q, which implies $q \in \Omega_1$ which implies by hypothesis that $y^{\Delta 3}(q) > 0$.

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However, if q is right dense,

$$y^{\Delta^3}(q) = \lim_{t\to\sigma(q)} \frac{y^{\Delta^2}(t) - y^{\Delta^2}(q)}{t-q} = \lim_{t\to\sigma(q)} \frac{y^{\Delta^2}(t)}{t-q} \le 0$$

and if q is right scattered,

$$y^{\Delta^3}(q) = \frac{y^{\Delta^2}(\sigma(q)) - y^{\Delta^2}(q)}{\sigma(q)-q} \le 0.$$

Notes

Hence, a contradiction.

Thus, $y^{\Delta 2}$ has a generalized zero at t_2. Since y has a generalized zero at t_2, y^{Δ} has a generalized zero at t_2, $y^{\Delta 2}$ has a generalized zero at t_2 and $f(t, 0, 0, 0) = 0$, it follows that $y(t) = 0 \ \forall \ t \in [t_1, t_2]$.

Next, we suppose that $y^{\Delta 2}$ has a generalized zero at t_2. Then it is claimed that y^{Δ} has a generalized zero at t_2. To the contrary, suppose that y^{Δ} has no generalized zero at t_2. With out loss of generality we can assume that $y^{\Delta}(t_2) > 0$. Since y has generalized zeros at t_1 and t_2, it follows from mean value theorem that there exists an $r \in (t_1, t_2)$ such that y^{Δ} has a generalized zero at r and y^{Δ} has no generalized zero in (r, t_2). Assume without loss of generality that $y^{\Delta}(t) > 0$ forall $t \in (r, t_2)$. We claim that there exists a $p \in [r, t_2)$ such that $y^{\Delta 2}(p) > 0$. To the contrary, suppose that $y^{\Delta 2}(p) \le 0$. Then $0 \ge \int_r^t y^{\Delta 2}(t)\Delta t = y^{\Delta}(t) - y^{\Delta}(r) \ge -y^{\Delta}(r)$ which implies $y^{\Delta}(t) \le 0 \ \forall \ t \in (r, t_2)$, which is a contradiction. Hence, the claim. Now, there exists a $q \in (p, t_2)$ such that $y^{\Delta 2}$ has a generalized zero at q and $y^{\Delta 2}(t) > 0$ forall $t \in (p, q)$. Again $0 < \int_q^{t2} y^{\Delta}(t)\Delta t = y(t_2) - y(q) \ge -y(q)$.

Thus $y(q) < 0$, $y^{\Delta}(q) > 0$ and $y^{\Delta 2}$ has a generalized zero at q and hence $q \in \Omega_1$ which implies by hypothesis, $y^{\Delta 3}(q) > 0$. However, if q is right dense,

$$y^{\Delta^3}(q) = \lim_{t\to\rho(q)} \frac{y^{\Delta^2}(t) - y^{\Delta^2}(q)}{t-q} = \lim_{t\to\rho(q)} \frac{y^{\Delta^2}(t)}{t-q} \le 0$$

and if q is right scattered,

$$y^{\Delta^3}(q) = \frac{y^{\Delta^2}(\rho(q)) - y^{\Delta^2}(q)}{\rho(q)-q} \le 0.$$

Hence, a contradiction. Thus y^{Δ}has a generalized zero at t_2. Since y, y^{Δ}, $y^{\Delta 2}$ has generalized zeros at t_2 and $f(t, 0, 0, 0) = 0$, it follows that $y(t) = 0 \ \forall \ t \in [t_1, t_2]$.

Lemma 2.5. Let $y(t)$ be a solution of (1.1) such that y has a generalized zero at t_1 and t_2 with $f(t, 0, 0, 0) = 0$. Further suppose that $y^{\Delta 3}(t_0) > 0$ whenever $t_0 \in \Omega_2$. If either y^{Δ} or $y^{\Delta 2}$ has a generalized zero at t_1, then $y(t) = 0$ for all $t \in [t_1, t_2]$.

Proof. We first suppose that y^{Δ} has a generalized zero at t_1. Then we claim that $y^{\Delta 2}$ also has a generalized zero at t_1. To the contrary, assume that $y^{\Delta 2}$ has no generalized zero at t_1. With out loss of generality we can assume that $y^{\Delta 2}(t_1) > 0$. So, $\exists$ a $q \in [t_1, t_2)$ such that $y^{\Delta 2}$ has a generalized zero at q and $y^{\Delta 2}(t) > 0 \ \forall \ t \in [t_1, q)$. Then

$$0 < \int_{t_1}^{t} y^{\Delta^2}(t)\Delta t = y^{\Delta}(t) - y^{\Delta}(t_1) \le y^{\Delta}(t)$$

which implies $y^{\Delta}(\mathrm{t}) > 0\ \forall\ \mathrm{t} \in [\mathrm{t}_1, q)$. Since $y^{\Delta\Delta}(\mathrm{t}) > 0\ \forall\ \mathrm{t} \in [\mathrm{t}_1, q)$, it follows that $y^{\Delta}(\mathrm{t})$ is decreasing for $\mathrm{t} < q$ and $y^{\Delta}(\mathrm{t})$ is positive. Again $0 < \int_{t1}^{q} y^{\Delta}(\mathrm{t})\Delta\mathrm{t} = y(q) - y(\mathrm{t}_1) \leq y(q)$ which implies $y(q) > 0$. Thus, $y(q) > 0$, $y^{\Delta}(q) > 0$ and $y^{\Delta 2}$ has a generalized zero at q, which implies $q \in \Omega_2$ which implies by hypothesis that $y^{\Delta 3}(q) > 0$.

However, if q is right dense,

Notes

$$y^{\Delta^3}(q) = \lim_{t\to\rho(q)} \frac{y^{\Delta^2}(t) - y^{\Delta^2}(q)}{t-q} = \lim_{t\to\rho(q)} \frac{y^{\Delta^2}(t)}{t-q} \leq 0$$

and if q is right scattered,

$$y^{\Delta^3}(q) = \frac{y^{\Delta^2}(\rho(q)) - y^{\Delta^2}(q)}{\rho(q)-q} \leq 0.$$

Hence, a contradiction. Thus, $y^{\Delta 2}$ has a generalized zero at t_1. Since y has a generalized zero at t_1, y^{Δ} has a generalized zero at t_1, $y^{\Delta 2}$ has a generalized zero at t_1 and $f(\mathrm{t}, 0, 0, 0) = 0$, it follows that $y(\mathrm{t}) = 0\ \forall\ \mathrm{t} \in [\mathrm{t}_1, \mathrm{t}_2]$. Next, we suppose that $y^{\Delta 2}$ has a generalized zero at t_1. Then it is claimed that y^{Δ} has a generalized zero at t_1. To the contrary, suppose that y^{Δ} has a generalized zero at t_1. With out loss of generality we can assume that $y^{\Delta}(\mathrm{t}_1) > 0$. Since y has generalized zeros at t_1 and t_2, it follows from mean value theorem that there exists an $r \in (\mathrm{t}_1, \mathrm{t}_2)$ such that y^{Δ} has a generalized zero at r and y^{Δ} has no generalized zero in $[\mathrm{t}_1, r)$. Assume with out loss of generality that $y^{\Delta}(\mathrm{t}) > 0$ forall t 2 $[\mathrm{t}_1, r)$. We claim that $\exists$ a $p \in [\mathrm{t}_1, r)$ such that $y^{\Delta 2}(\mathrm{p}) < 0$. To the contrary, suppose that $y^{\Delta 2}(p) \geq 0$. Then

$$0 \leq \int_t^r y^{\Delta^2}(t)\Delta t = y^{\Delta}(r) - y^{\Delta}(t) \leq -y^{\Delta}(t)$$

which implies $y^{\Delta}(\mathrm{t}) \leq 0\ \forall\ \mathrm{t} \in [\mathrm{t}_1, r)$, which is a contradiction. Hence, the claim. Now, $\exists$ a $q \in [\mathrm{t}_1, p)$ such that $y^{\Delta 2}$ has a generalized zero at q and $y^{\Delta 2}(\mathrm{t}) < 0\ \forall\ \mathrm{t} \in (q, p)$. Again

$$0 < \int_{t_1}^q y^{\Delta}(t)\Delta t = y(q) - y(t_1) \leq y(q).$$

Thus $y(q) > 0$, $y^{\Delta}(q) > 0$ and $y^{\Delta 2}$ has a generalized zero at q and hence $q \in \Omega_2$, which implies by hypothesis that $y^{\Delta 3}(q) > 0$. However, if q is right dense,

$$y^{\Delta^3}(q) = \lim_{t\to\sigma(q)} \frac{y^{\Delta^2}(t) - y^{\Delta^2}(q)}{t-q} = \lim_{t\to\sigma(q)} \frac{y^{\Delta^2}(t)}{t-q} \leq 0$$

and if q is right scattered,

$$y^{\Delta^3}(q) = \frac{y^{\Delta^2}(\sigma(q)) - y^{\Delta^2}(q)}{\sigma(q)-q} \leq 0.$$

Hence, a contradiction. Thus y^{Δ} has a generalized zero at t_1. Since y, y^{Δ}, $y^{\Delta 2}$ has generalized zeros at t_1 and $f(\mathrm{t}, 0, 0, 0) = 0$, it follows that $y(\mathrm{t}) = 0$ forall $\mathrm{t} \in [\mathrm{t}_1, \mathrm{t}_2]$.

Lemma 2.6. Let $y(\mathrm{t})$ be a solution of (1.1) such that y has a generalized zero at t_2 and t_3 with $f(\mathrm{t}, 0, 0, 0) = 0$. Further suppose that $y^{\Delta 3}(\mathrm{t}_0) > 0$ whenever $\mathrm{t}_0 \in \Omega_2$. If either y^{Δ} or $y^{\Delta 2}$ has a generalized zero at t_2, then $y(\mathrm{t}) = 0$ for all $\mathrm{t} \in [\mathrm{t}_2, \sigma^3(\mathrm{t}_3)]$.

Lemma 2.7. Let $y(t) \in C^3_{rd}[[t_1, t_2],\mathbb{R}] \ni y$ has generalized zeros at t_1 and t_2, $y^{\Delta 3}(t_0) > 0$ for some $t_0 \in \Omega_1$ and either $y^{\Delta}(t_2) < 0$ or $y^{\Delta 2}(t_2) < 0$. Then, $\exists$ a $p \in [t_1, t_2)$ such that y^{Δ} has a generalized zero at p and $y^{\Delta}(t) < 0$ on $(p, t_2]$ and $y(t) > 0$ on $[p, t_2)$.

Proof. We first suppose that $y^{\Delta 2}(t_2) < 0$. Then, there exists a $q \in [t_1, t_2)$ such that $y^{\Delta 2}$ has a generalized zero at q and $y^{\Delta 2}(t) < 0\ \forall\ t \in (q, t_2]$.We first claim that $y^{\Delta}(t) < 0\ \forall\ t \in [q, t_2]$. To the contrary, suppose that $y^{\Delta}(t) > 0\ \forall\ t \in [q, t_2]$. Then $0 < \int_q^{t2} y^{\Delta}(t)\Delta t = y(t_2) - y(q) \leq - y(q)$ which implies $y(q) < 0$. Thus $y(q) < 0$, $y^{\Delta}(q) > 0$ and $y^{\Delta 2}$ has a generalized zero at q and hence $q \in \Omega_1$, implies by hypothesis that $y^{\Delta 3}(q) > 0$.
However, if q is right dense, then

$$y^{\Delta^3}(q) = \lim_{t\to\sigma(q)} \frac{y^{\Delta^2}(t) - y^{\Delta^2}(q)}{t-q} \leq 0$$

and if q is right scattered, then

$$y^{\Delta^3}(q) = \frac{y^{\Delta^2}(\sigma(q)) - y^{\Delta^2}(q)}{\sigma(q)-q} \leq 0,$$

which is a contradiction.Hence, $y^{\Delta}(t) \leq 0$ forall $t \in [q, t_2)$. therefore there exists *a* $p \in [q, t_2)$ such that y^{Δ} has a generalized zero at p and $y^{\Delta}(t) < 0$ forall $t \in (p, t_2]$ which implies $0 > \int_t^{t2} y^{\Delta}(t)\Delta t = y(t_2) - y(t) \geq -y(t)$ which implies $y(t) < 0$ on $[p, t_2)$. A similar argument holds if $y^{\Delta}(t_2) < 0$.

Lemma 2.8. Let $y(t) \in C^{3}_{rd}[[t_2, \sigma^3(t_3)],\mathbb{R}] \ni y$ has generalized zeros at t_2 and t_3, $y^{\Delta 3}(t_0) > 0$ for some $t_0 \in \Omega_2$ and either $y^{\Delta}(t_2) > 0$ or $y^{\Delta 2}(t_2) > 0$.Then, $\exists$ *a* $p \in (t_2, \sigma^3(t_3)]$ such that y^{Δ} has a generalized zero at p and $y^{\Delta}(t) < 0$ on $[t_2, p)$ and $y(t) < 0$ on $(t_2, p]$.

Proof. We first suppose that $y^{\Delta 2}(t_2) > 0$. Then, there exists a $q \in (t_2, \sigma^3(t_3)]$ such that $y^{\Delta 2}$ has a generalized zero at q and $y^{\Delta 2}(t) > 0$ forall $t \in [t_2, q)$. Then it is claimed that $y^{\Delta}(t) \leq 0$ forall $t \in (t_2, q]$. For the sake of contradiction, we assume that $y^{\Delta}(t) > 0\ \forall\ t \in (t_2, q]$. Then $0 < \int_{t2}^{q} y^{\Delta}(t)\Delta t = y(q) - y(t_2) \leq y(q)$ which implies $y(q) > 0$. Thus $y(q) > 0$, $y^{\Delta}(q) > 0$ and $y^{\Delta 2}$ has a generalized zero at q and hence $q \in \Omega_2$, implies by hypothesis that $y^{\Delta 3}(q) > 0$.
However, if q is right dense, then

$$y^{\Delta^3}(q) = \lim_{t\to\rho(q)} \frac{y^{\Delta^2}(t) - y^{\Delta^2}(q)}{t-q} \leq 0$$

and if q is right scattered, then

$$y^{\Delta^3}(q) = \frac{y^{\Delta^2}(\rho(q)) - y^{\Delta^2}(q)}{\rho(q)-q} \leq 0,$$

which is a contradiction. Hence, $y^{\Delta}(t) \leq 0$ forall $t \in (t_2, q]$. Therefore there exists a $p \in (t_2, q]$ such that y^{Δ} has a generalized zero at p and $y^{\Delta}(t) < 0$ forall $t \in [t_2, p)$ which implies $0 > \int_{t2}^{t} y^{\Delta}(t)\Delta t = y(t) - y(t_2) \geq y(t)$ which implies $y(t) < 0$ on $(t_2, p]$. A similar argument holds if $y^{\Delta}(t_2) > 0$.

Notes

III. Main Result

In this section we establish existence and uniqueness of solutions (1.1),(1.2). We first show that there exists at most one solution to (1.1) satisfying one of (1.3) ,(1.4) ,(1.5) or (1.6).

Lemma 3.1. Assume that $f \in C_{rd}[[t_1, \sigma^3(t_3)] \times \mathbb{R}_3, \mathbb{R}]$ and let $f \in G_1$, f $_2$ G_2.

Assume that when $u_1 \le u_2$, $v_1 > v_2$ and $w_1 = w_2$, then $f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2) \ge 0$ $\forall$ $t \in [t_1, t_2)$.

Notes

Also assume that when $u_1 \ge u_2$, $v_1 > v_2$ and $w_1 = w_2$, then $f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2) \ge 0$ $\forall$ $t \in (t_2, \sigma^3(t_3)]$.

Then for each $m \in \mathbb{R}$, $\exists$ at most one solution to (1.1) satisfying one of (1.3) ,(1.4) ,(1.5) or (1.6).

Proof. The proof of (1.1),(1.4) will be given. Similar argument holds for other boundary problems. Suppose that $\Phi(t)$ and $\Psi(t)$ are each solutions of the boundary value problem (1.1),(1.4).

Write $\chi(t) = \Phi(t) - \Psi(t)$.

Clearly $\chi(t_1) = 0$, $\chi(t_2) = 0$ and $\chi^{\Delta 2}(t_2) = 0$.

Hence

$$\chi^{\Delta^3}(t) = \Phi^{\Delta^3}(t) - \Psi^{\Delta^3}(t)$$
$$= f(t, \Phi(t), \Phi^{\Delta}(t), \Phi^{\Delta^2}(t)) - f(t, \Psi(t), \Psi^{\Delta}(t), \Psi^{\Delta^2}(t)) > 0.$$

Now $\chi(t)$ satisfies the hypothesis of Lemma 2.4.

So, $\chi(t) = 0$ or $\Phi(t) = \Psi(t)$.

Theorem 3.1. Assume that

(i) for each $m \in \mathbb{R}$, $\exists$ solutions for each of the boundary value problem (1.1) satisfying one of (1.3) ,(1.4), (1.5) or (1.6).

(ii) $f \in G_1$ and if $u_1 \le u_2$, $v_1 > v_2$ and $w_1 = w_2$, then $f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2) \ge 0$ $\forall$ $t \in [t_1, t_2)$

(iii) $f \in G_2$ and if $u_1 \ge u_2$, $v_1 > v_2$ and $w_1 = w_2$, then $f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2) \ge 0$ $\forall$ $t \in (t_2, \sigma 3(t_3)]$.

Then the boundary value problem (1.1),(1.2) has a unique solution.

Proof. By Lemma 3.1, the solutions of (1.1) satisfying one of (1.3) ,(1.4) ,(1.5) or (1.6), whenever they exists, are unique. Let $\Phi(t, m)$ denotes the solution of the boundary value problem (1.1), (1.4).

Set $\chi(t) = \Phi(t, m_1) - \Phi(t, m_2)$.

Clearly if $m_2 > m_1$, $\chi(t_1) = 0$, $\chi(t_2) = 0$, and $\chi^{\Delta 2}(t_2) = 0$. If $t \in \Omega_1$, then $\chi(t) \le 0$, $\chi^{\Delta}(t) > 0$ and $\chi^{\Delta 2}(t)$ has a generalized zero at t and hence using (ii),

$$\chi^{\Delta^3}(t) = f(t, \Phi(t, m_1), \Phi^{\Delta}(t, m_1), \Phi^{\Delta^2}(t, m_1))$$
$$- f(t, \Phi(t, m_2), \Phi^{\Delta}(t, m_2), \Phi^{\Delta^2}(t, m_2)) \ge 0$$

Thus Lemma 2.7 yields $\chi^{\Delta}(t) < 0$, $t \in (p, t_2]$. In particular,

$$\chi^{\Delta}(t_2) = \Phi^{\Delta}(t_2, m_1) - \Phi^{\Delta}(t_2, m_2) < 0.$$

Hence, it follows that $\Phi^{\Delta}(t_2, m)$ is a strictly increasing function of m. A similar reasoning given above demonstrates that $\Psi^{\Delta}(t_2, m)$ is a strictly decreasing function of m, where $\Psi(t, m)$ is the solution of the boundary value problem (1.1),(1.6).

It now follows from the fact that solutions of (1.1),(1.4) and (1.1),(1.6) are unique and the ranges of $\Phi^{\Delta}(t_2, m)$ and $\Psi^{\Delta}(t_2, m)$ are the set of all reals, that there exists a unique $m_0 \in \mathbb{R}$ such that $\Phi^{\Delta}(t_2, m_0) = \Psi^{\Delta}(t_2, m_0)$. Thus $y(t)$ defined by

$$y(t) = \begin{cases} \Phi(t, m), & t_1 \le t \le t_2, \\ \Psi(t, m), & t_2 \le t \le \sigma^3(t_3), \end{cases}$$

is a solution of (1.1),(1.2).

Notes

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