Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 14 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Intuitionistic L-Fuzzy Rings

By K. Meena & K. V. Thomas
Bharata Mata College, Thrikkakara

Abstract - In this paper we study some generalized properties of Intuitionistic L-fuzzy subrings. In this direction the concept of image and inverse image of an Intuitionistic L-fuzzy set under ring homomorphism are discussed. Further the concept of Intuitionistic L-fuzzy quotient subring and Intuitionistic L-fuzzy ideal of an Intuitionistic L-fuzzy subring are studied. Finally, weak homomorphism, weak isomorphism, homomorphism and isomorphism of an Intuitionistic L-fuzzy subring are introduced and some results are established in this direction.

Keywords : intuitionistic l-fuzzy quotient subring, intuitionistic l-fuzzy ideal of an intuitionistic l-fuzzy subring.

GJSFR-F Classification : MSC 2010: 16P70

INTUITIONISTIC L-FUZZY RINGS

Strictly as per the compliance and regulations of :

Intuitionistic L-Fuzzy Rings

K. Meena [α] & K. V. Thomas [σ]

Abstract - In this paper we study some generalized properties of Intuitionistic L-fuzzy subrings. In this direction the concept of image and inverse image of an Intuitionistic L-fuzzy set under ring homomorphism are discussed. Further the concept of Intuitionistic L-fuzzy quotient subring and Intuitionistic L-fuzzy ideal of an Intuitionistic L-fuzzy subring are studied. Finally, weak homomorphism, weak isomorphism, homomorphism and isomorphism of an Intuitionistic L-fuzzy subring are introduced and some results are established in this direction.

Keywords : intuitionistic l-fuzzy quotient subring, intuitionistic l-fuzzy ideal of an intuitionistic l-fuzzy subring.

I. Introduction

The theory of Intuitionistic fuzzy sets plays an important role in modern mathematics. The idea of Intuitionistic L-fuzzy set (ILFS) was introduced by Atanassov (1986) [3–5] as a generalisation of Zadeh's (1965) [15] fuzzy sets. Rosenfeld (1971) [14] has applied the concept of fuzzy sets to the theory of groups. Many researchers [1, 2, 6–10] applied the notion of Intuitionistic fuzzy concepts to set theory, relation, group theory, topological space, knowledge engineering, natural language, neural network etc. This paper is a continuation of our earlier paper [13]. Along with some basic results, we introduce and study Intuitionistic fuzzy quotient subrings and Intuitionistic fuzzy ideal of an Intuitionistic fuzzy subring of a ring. Further we define homomorphism and isomorphism of Intuitionistic fuzzy subrings of any two rings. Using this we establish the fundamental theorem of ring homomorphism and the third isomorphism theorem of rings for Intuitionistic fuzzy subrings. Infact, we emphasize the truth of the results relating to the non-membership function of an Intuitionistic fuzzy subring on a lattice $(L, \leq, \wedge, \vee)$. The proof of the results on the membership function of Intuitionistic fuzzy subrings, Intuitionistic fuzzy ideals of an Intuitionistic fuzzy subring and Intuitionistic fuzzy Quotient ring are omitted to avoid repetitions which are already done by researchers Malik D.S and Mordeson J.N. [11, 12].

II. Preliminaries

In this section we list some basic concepts and well known results of Intuitionistic L-fuzzy sets, Intuitionistic L-fuzzy subrings and Intuitionistic L-fuzzy ideals [13].

Author α σ : P.G. Research Centre in Mathematics, Bharata Mata College, Thrikkakara.

[1] Adam Niewiadomski and Eulalia Szmidt, Handling Uncertainty in natural sentences via IFS, *Notes on Intuitionistic Fuzzy Sets*, **7** (2001), No.1, 22–29.

Throughout this paper $(L, \leq, \wedge, \vee)$ denotes a complete distributive lattice with maximal element 1 and minimal element 0, respectively. Let R and S be commutative rings with binary operations + and $\cdot$.

Definition 2.1. *Let X be a non-empty set. A L-fuzzy set μ of X is a function $\mu : X \to L$.*

Definition 2.2. *Let $(L, \leq)$ be the lattice with an involutive order reversing operation $N : L \to L$. Let X be a non-empty set. An Intuitionistic L-fuzzy set (ILFS) A in X is defined as an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$$

where $\mu_A : X \to L$ and $\nu_A : X \to L$ define the degree of membership and the degree of non membership for every $x \in X$ satisfying $\mu_A(x) \leq N(\nu_A(x))$.

Definition 2.3. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in X\}$ be two Intuitionistic L-fuzzy sets of X. Then we define*

(i) $A \subseteq B$ iff for all $x \in X$, $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$
(ii) $A = B$ iff for all $x \in X$, $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$
(iii) $A \cup B = \{\langle x, (\mu_A \cup \mu_B)(x), (\nu_A \cap \nu_B)(x)\rangle / x \in X\}$ where $\mu_A \cup \mu_B = \mu_A \vee \mu_B$, $\nu_A \cap \nu_B = \nu_A \wedge \nu_B$
(iv) $A \cap B = \{\langle x, (\mu_A \cap \mu_B)(x), (\nu_A \cup \nu_B)(x)\rangle / x \in X\}$ where $\mu_A \cap \mu_B = \mu_A \wedge \mu_B$, $\nu_A \cup \nu_B = \nu_A \vee \nu_B$.

Definition 2.4. *An Intuitionistic L-fuzzy subset $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ of R is said to be an Intuitionistic L-fuzzy subring of R (ILFSR) if for all $x, y \in R$,*
(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$
(ii) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$
(iii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$
(iv) $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$.

Proposition 2.5. *If $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ is an ILFSR. Then*
(i) $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in R$
(ii) if R is a ring with identity 1 then $\mu_A(1) \leq \mu_A(x)$ and $\nu_A(1) \geq \nu_A(x)$, for all $x \in R$.

Theorem 2.6. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ be two ILFSR. Then $A \cap B$ is an ILFSR of R.*

Definition 2.7. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ be an ILFSR of R. Then A is called an Intuitionistic L-fuzzy ideal of R (ILFI) if,*

Notes

Notes

(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$
(ii) $\mu_A(xy) \geq \mu_A(x)$
(iii) $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$
(iv) $\nu_A(xy) \leq \nu_A(x)$, *for all* $x, y \in R$.

Definition 2.8. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be an ILFI of* R. *Then we define*

$$(\mu_A)_* = \{x \in R/\mu_A(x) = \mu_A(0)\}$$

$$(\nu_A)_* = \{x \in R/\nu_A(x) > \nu_A(0)\}.$$

Proposition 2.9. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be an ILFI. If* $\mu_A(x-y) = \mu_A(0)$ *then* $\mu_A(x) = \mu_A(y)$ *and if* $\nu_A(x-y) = \nu_A(0)$ *then* $\nu_A(x) = \nu_A(y)$, *for all* $x, y \in R$.

Proposition 2.10. *Every ILFI is an ILFSR.*

Definition 2.11. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be an ILFSR. Let* $x \in R$, *then*

$$C = \{\langle x, (\mu_A(0)_{\{x\}} + \mu_A)(x), (\nu_A(0)_{\{x\}} + \nu_A)(x)\rangle / x \in R\}$$

is called an Intuitionistic L*-fuzzy coset (ILFC) of* A *and is denoted as*

$$C = \{\langle x, (x + \mu_A)(x), (x + \nu_A)(x)\rangle / x \in R\}.$$

Definition 2.12. Let R/A={(x+μ_A),(x+ν_A)/x belongs to R} *be an ILFI.*
Let $R/A = \{(x + \mu_A, x + \nu_A)/x \in R\}$. *Define* $+$ *and* $\cdot$ *on* R/A *by*

(i) $(x + \mu_A) + (y + \mu_A) = x + y + \mu_A$,
(ii) $(x + \nu_A) + (y + \nu_A) = x + y + \nu_A$, *for all* $x, y \in R$, *and*
(iii) $(x + \mu_A) \cdot (y + \mu_A) = xy + \mu_A$,
(iv) $(x + \nu_A) \cdot (y + \nu_A) = xy + \nu_A$, *for all* $x, y \in R$.

Then R/A *is a ring with respect to* $+$ *and* $\cdot$ *and is called Quotient ring of* R *by* μ_A *and* ν_A.

III. Correspondence Theorem for Intuitionistic L-Fuzzy Ideal

Here we define the image and inverse image of ILFS under ring homomorphism and study their elementary properties. Using this, Correspondence Theorem for ILFI is established. This section also provides a definition for an Intuitionistic L-fuzzy quotient subring of an ILFSR relative to an ordinary ideal of a ring.

Definition 3.1. *Let $f : R \to S$ be a ring homomorphism.*
Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in S\}$ be ILFS. Then $C = \{\langle y, f(\mu_A)(y), f(\nu_A)(y)\rangle / y \in S\}$ is called Intuitionistic Image of A, where

$$f(\mu_A)(y) = \begin{cases} \vee\{\mu_A(x)/x \in R, f(x) = y\} & \textit{if } f^{-1}(y) \neq \emptyset \\ 0 & \textit{otherwise} \end{cases}$$

$$f(\nu_A)(y) = \begin{cases} \wedge\{\nu_A(x)/x \in R, f(x) = y\} & \textit{if } f^{-1}(y) \neq \emptyset \\ 0 & \textit{otherwise} \end{cases}$$

for all $y \in S$;
and $D = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\nu_B)(x)\rangle / x \in R\}$ is called Intuitionistic Inverse Image of B, where

$$f^{-1}(\mu_B)(x) = \mu_B(f(x))$$

$$f^{-1}(\nu_B)(x) = \nu_B(f(x))$$

for all $x \in R$.
Here $f(\mu_A)$ and $f(\nu_A)$ are called the image of μ_A and ν_A under f. Also $f^{-1}(\mu_B)$ and $f^{-1}(\nu_B)$ are called the inverse image of μ_B and ν_B under f.

The proof of the following result is direct.

Lemma 3.2. *Let $f : R \to S$ be a ring homomorphism.*
Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in S\}$ be ILFI of R and S respectively. Then

(i) $f(\nu_A)(0') = \nu_A(0)$ where $0'$ is the zero element of S and 0 is the zero element of R.
(ii) $f(\nu_A)_* \subseteq (f(\nu_A))_*$;
(iii) If ν_A has the infimum property, then $f(\nu_A)_* = (f(\nu_A))_*$;
(iv) If ν_A is constant on Ker f, then $(f(\nu_A))(f(x)) = \nu_A(x)$ for all $x \in R$.

Theorem 3.3. *Let $f : R \to S$ be a ring homomorphism.*
Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in S\}$ be ILFI of R and S. Then

(i) $D = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\nu_B)(x)\rangle / x \in R\}$ is an ILFI of R which is a constant on Ker f;
(ii) $f^{-1}(\nu_B)_* = (f^{-1}(\nu_B))_*$;
(iii) If f is onto then $(f \circ f^{-1})(\nu_B) = \nu_B$;
(iv) If ν_A is constant on Ker f, then $(f^{-1} \circ f)(\nu_A) = \nu_A$.

Notes

Notes

Proof.

(i) Let $x, y \in R$. Then

$$\begin{aligned} f^{-1}(\nu_B)(x-y) &= \nu_B(f(x-y)) \\ &= \nu_B(f(x)-f(y)) \\ &\le \nu_B(f(x)) \vee \nu_B(f(y)) \\ &= f^{-1}(\nu_B)(x) \vee f^{-1}(\nu_B)(y) \end{aligned}$$

and $$\begin{aligned} f^{-1}(\nu_B)(xy) &= \nu_B(f(xy)) \\ &= \nu_B(f(x)f(y)) \\ &\le \nu_B(f(x)) \wedge \nu_B(f(y)) \\ &= f^{-1}(\nu_B)(x) \wedge f^{-1}(\nu_B)(y). \end{aligned}$$

Hence D is an ILFI of R.

Let $x \in \text{Ker } f$. Then

$$\begin{aligned} f^{-1}(\nu_B)(x) &= \nu_B(f(x)) \\ &= \nu_B(f(0)) \\ &= \nu_B(0'). \end{aligned}$$

Hence $f^{-1}(\nu_B)$ is constant on Ker f.

(ii) Let $x \in R$. Then

$$\begin{aligned} x \in f^{-1}(\nu_B)_* &\Leftrightarrow \nu_B(f(x)) > \nu_B(0') = \nu_B(f(0)) \\ &\Leftrightarrow f^{-1}(\nu_B)(x) > f^{-1}(\nu_B)(0) \\ &\Leftrightarrow x \in (f^{-1}(\nu_B))_*. \end{aligned}$$

Hence $f^{-1}(\nu_B)_* = (f^{-1}(\nu_B))_*$.

(iii) Let $y \in S$. Then $y = f(x)$ for some $x \in R$, so that

$$\begin{aligned} (f \circ f^{-1})(\nu_B)(y) &= f(f^{-1}(\nu_B))(y) \\ &= f(f^{-1}(\nu_B))(f(x)) \\ &= f^{-1}(\nu_B)(x) \\ &= \nu_B(f(x)) \\ &= \nu_B(y). \end{aligned}$$

Hence $(f \circ f^{-1})(\nu_B) = \nu_B$.

(iv) Let $x \in R$. Then

$$(f^{-1} \circ f)(\nu_A)(x) = f^{-1}(f(\nu_A))(x)$$
$$= f(\nu_A)(f(x))$$
$$= \nu_A(x).$$

Hence $(f^{-1} \circ f)(\nu_A) = \nu_A$.

Theorem 3.4. *Let $f : R \to S$ be an onto ring homomorphism.*
Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ be an ILFI of R.
Then $C = \{\langle y, f(\mu_A)(y), f(\nu_A)(y)\rangle / y \in S\}$ is an ILFI of S. If ν_A is a constant on Ker f, then $f(\nu_A)_* = (f(\nu_A))_*$.

Proof. Let $s_1, s_2 \in S$. Then $s_1 = f(r_1)$, $s_2 = f(r_2)$ for some $r_1, r_2 \in R$. Now

$$\begin{aligned} f(\nu_A)(s_1 - s_2) &= \wedge\{\nu_A(x)/x \in R, f(x) = s_1 - s_2\} \\ &\leq \wedge\{\nu_A(r_1 - r_2)/r_1, r_2 \in R, f(r_1) = s_1, f(r_2) = s_2\} \\ &\leq \wedge\{\nu_A(r_1) \vee \nu_A(r_2)/r_1, r_2 \in R, f(r_1) = s_1, f(r_2) = s_2\} \\ &= (\wedge\{\nu_A(r_1)/r_1 \in R, f(r_1) = s_1\}) \vee (\wedge\{\nu_A(r_2)/r_2 \in R, f(r_2) = s_2\}) \\ &= f(\nu_A)(s_1) \vee f(\nu_A)(s_2). \end{aligned}$$

Also

$$\begin{aligned} f(\nu_A)(s_1 s_2) &= \wedge\{\nu_A(x)/x \in R, f(x) = s_1 s_2\} \\ &\leq \wedge\{\nu_A(r_1 r_2)/r_1, r_2 \in R, f(r_1) = s_1, f(r_2) = s_2\} \\ &\leq \wedge\{\nu_A(r_1) \wedge \nu_A(r_2)/r_1, r_2 \in R, f(r_1) = s_1, f(r_2) = s_2\} \\ &= (\wedge\{\nu_A(r_1)/r_1 \in R, f(r_1) = s_1\}) \wedge (\wedge\{\nu_A(r_2)/r_2 \in R, f(r_2) = s_2\}) \\ &= f(\nu_A)(s_1) \wedge f(\nu_A)(s_2). \end{aligned}$$

Hence $C = \{\langle y, f(\mu_A)(y), f(\nu_A)(y)\rangle / y \in S\}$ is an ILFI(S).

Next, if ν_A is a constant on Ker f, then for $y \in (f(\nu_A))_*$, we have

$$f(\nu_A)(y) > f(\nu_A)(0') = \nu_A(0).$$

Since f is onto, $y = f(x)$ for some $x \in R$. Hence $f(\nu_A)(f(x)) = \nu_A(x) > \nu_A(0)$. Thus $\nu_A(x) > \nu_A(0)$ or $x \in (\nu_A)_*$. Hence $y = f(x) \in f(\nu_A)_*$. The remaining part of the proof follows from Lemma 3.2 (ii).

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue XIV Version I Year 2012

Notes

Let R be a ring. Let $R/A = \{(x + \mu_A), (x + \nu_A)/x \in R\}$ be a quotient ring by μ_A and ν_A where $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in R\}$ is an ILFI.

Define $A^{(*)} = \{\langle x, \mu_A^{(*)}(x), \nu_A^{(*)}(x)\rangle/x \in R/A\}$ as follows:

$$\mu_A^{(*)}(x + \mu_A) = \mu_A(x)$$

and

$$\nu_A^{(*)}(x + \nu_A) = \nu_A(x), \text{ for all } x \in R.$$

Obviously, $\nu_A^{(*)}$ and $\mu_A^{(*)}$ are well-defined. Also $A^{(*)}$ is an ILFS(R/A).

Theorem 3.5. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in R\}$ be an ILFI. Then $A^{(*)}$ is an ILFI of R/A, where $A^{(*)} = \{\langle x, \mu_A^{(*)}(x), \nu_A^{(*)}(x)\rangle/x \in R/A\}$ is defined by*

$$\mu_A^{(*)}(x + \mu_A) = \mu_A(x) \text{ and}$$

$$\nu_A^{(*)}(x + \nu_A) = \nu_A(x), \text{ for all } x \in R.$$

Proof. Let $x, y \in R$. Then

$$\begin{aligned}\nu_A^{(*)}((x + \nu_A) + (y + \nu_A)) &= \nu_A^{(*)}(x + y + \nu_A)\\ &= \nu_A(x + y)\\ &\le \nu_A(x) \vee \nu_A(y)\\ &= \nu_A^{(*)}(x + \nu_A) \vee \nu_A^{(*)}(y + \nu_A)\end{aligned}$$

and

$$\begin{aligned}\nu_A^{(*)}((x + \nu_A)(y + \nu_A)) &= \nu_A^{(*)}(xy + \nu_A)\\ &= \nu_A(xy)\\ &\le \nu_A(x) \wedge \nu_A(y)\\ &= \nu_A^{(*)}(x + \nu_A) \wedge \nu_A^{(*)}(y + \nu_A).\end{aligned}$$

Hence $A^{(*)}$ is an ILFI(R/A).

Theorem 3.6. (Correspondence Theorem for ILFI) *Let $f : R \to S$ be a ring epimorphism. Then there is a one-to-one order preserving correspondence between ILFI of S and the ILFI of R, which are constant on Ker f.*

Proof. Let $F(R)$ denote the set of ILFI of R which are constant on Ker f and $F(S)$ denote the set of ILFI of S. Define $\Phi : F(R) \to F(S)$ by $\Phi(A) = f(A)$, for all $A \in F(R)$ and $\Psi : F(S) \to F(R)$ by $\Psi(B) = f^{-1}(B)$ for all $B \in F(S)$.

Then Φ and Ψ are well-defined and are inverses of each other, thus giving the one to one correspondence. It can easily be verified that this correspondence preserves the order too.

Theorem 3.7. *Let* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be an ILFSR and* C *be any ideal of* R*. Let*

$$D = \{\langle [x], \mu_D[x], \nu_D[x]\rangle / [x] \in R/C\}$$

be an ILFS of R/C*, where*

$$\mu_D[x] = \vee\{\mu_B(z)/z \in [x]\}, \quad \nu_D[x] = \wedge\{\nu_B(z)/z \in [x]\}, \textit{ for all } x \in R,$$

and $[x] = x + C$*. Then* D *is an ILFSR of* R/C*.*

Proof. Let $x, y \in R$. Then

$$\begin{aligned}
\nu_D([x]-[y]) &= \nu_D([x-y]) \\
&= \wedge\{\nu_B(x-y+z)/z \in C\} \\
&\le \wedge\{\nu_B(x-y+a-b)/a, b \in C\} \\
&= \wedge\{\nu_B((x+a)-(y+b))/a, b \in C\} \\
&\le (\wedge\{\nu_B(x+a)/a \in C\}) \vee (\wedge\{\nu_B(y+b)/b \in C\}) \\
&= \nu_D[x] \vee \nu_D[y].
\end{aligned}$$

Also

$$\begin{aligned}
\nu_D([x][y]) &= \nu_D([xy]) \\
&= \wedge\{\nu_B(xy+z)/z \in C\} \\
&\le \wedge\{\nu_B(xy+(xv+uy+uv))/u, v \in C\} \\
&= \wedge\{\nu_B(x(y+v)+u(y+v))/u, v \in C\} \\
&= \wedge\{\nu_B((y+v)(x+u))/u, v \in C\} \\
&\le \wedge\{(\nu_B(y+v)) \vee (\nu_B(x+u))/u, v \in C\} \\
&= (\wedge\{\nu_B(x+u)/u \in C\}) \vee (\wedge\{\nu_B(y+v)/v \in C\}) \\
&= \nu_D[x] \vee \nu_D[y].
\end{aligned}$$

Hence D is an ILFSR of R/C.

The ILFSR, D={<[x], μ D[x], ν[x]>/[x] belongs to R/C} is called the Intuitionistic L-fuzzy Quotient Subring of B relative to C and denoted as B/C and is abbreviated as ILFQSR.

Notes

IV. INTUITIONISTIC L-FUZZY IDEAL OF AN INTUITIONISTIC L-FUZZY SUBRING

Notes

In this section we define an ILFI of an ILFSR and some elementary results are obtained. Also we discuss the ILFI of an ILFSR under an epimorphism.

Definition 4.1. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ be an ILFS. Let $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ be an ILFSR with $A \subseteq B$. Then A is called an ILFI of B if for all $x, y \in R$,*

(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$

(ii) $\mu_A(xy) \geq (\mu_B(x) \wedge \mu_A(y)) \vee (\mu_A(x) \wedge \mu_B(y))$

(iii) $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$

(iv) $\nu_A(xy) \leq (\nu_B(x) \vee \nu_A(y)) \wedge (\nu_A(x) \vee \nu_B(y))$

Since R is commutative, $\nu_A(xy) \leq (\nu_B(x) \vee \nu_A(y)) \wedge (\nu_A(x) \vee \nu_B(y))$ for all $x, y \in R$ if and only if $\nu_A(xy) \leq \nu_B(x) \vee \nu_A(y)$, for all $x, y \in R$.

Definition 4.2. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ be an ILFI. Then $A^* = \{x \in R / \mu_A(x) > 0, \nu_A(x) = 0\}$ is an ideal of R, if L is regular.*

The proof of the following result is direct.

Theorem 4.3. *Let $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ be an ILFSR and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ be an ILFI of B . If L is regular, then $A^* = \{x \in R / \mu_A(x) > 0, \nu_A(x) = 0\}$ is an ideal of $B^* = \{x \in R / \mu_B(x) > 0, \nu_B(x) = 0\}$.*

Theorem 4.4. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ be an ILFI of an ILFSR $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$. Then A is an ILFSR.*

Proof. For $x, y \in R$

$$\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y).$$

For $x, y \in R$

$$\begin{aligned}\nu_A(xy) &\leq (\nu_B(x) \vee \nu_A(y)) \wedge (\nu_A(x) \vee \nu_B(y)) \\ &\leq (\nu_A(x) \vee \nu_A(y)) \wedge (\nu_A(x) \vee \nu_A(y)) \\ &= \nu_A(x) \vee \nu_A(y).\end{aligned}$$

Hence A is an ILFSR.

Theorem 4.5. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ be an ILFI of R and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ be an ILFSR. Then $A \cap B$ is an ILFI of B.*

Proof. Clearly $A \cap B \subseteq B$ and $A \cap B$ is an ILFSR of R. For $x, y \in R$,

$$\begin{aligned}(\nu_A \cup \nu_B)(xy) &= \nu_A(xy) \vee \nu_B(xy)\\ &\leq [\nu_A(x) \wedge \nu_A(y)] \vee [\nu_B(x) \vee \nu_B(y)]\\ &= (\nu_A(x) \vee [\nu_B(x) \vee \nu_B(y)]) \wedge (\nu_A(y) \vee [\nu_B(x) \vee \nu_B(y)])\\ &= ([\nu_A(x) \vee \nu_B(x)] \vee \nu_B(y)) \wedge ([\nu_A(y) \vee \nu_B(y)] \vee \nu_B(x))\\ &\leq \nu_B(x) \vee [\nu_A(y) \vee \nu_B(y)]\\ &= \nu_B(x) \vee (\nu_A \cup \nu_B)(y).\end{aligned}$$

Therefore $A \cap B$ is an ILFI of B.

Theorem 4.6. *Let* $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle / x \in R\}$ *be an ILFSR and* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be two ILFI of* C. *Then* $A \cap B$ *is an ILFI of* C.

Proof. Clearly $A \cap B \subseteq C$ and $A \cap B$ is an ILFSR. For $x, y \in R$,

$$\begin{aligned}(\nu_A \cup \nu_B)(xy) &= \nu_A(xy) \vee \nu_B(xy)\\ &\leq (\nu_C(x) \vee \nu_A(y)) \vee (\nu_C(x) \vee \nu_B(y))\\ &= \nu_C(x) \vee (\nu_A(y) \vee \nu_B(y))\\ &= \nu_C(x) \vee (\nu_A \cup \nu_B)(y).\end{aligned}$$

Therefore $A \cap B$ is an ILFI of C.

Theorem 4.7. *Let* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be an ILFSR.* *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be an ILFI of* B. *Let* $f : R \to S$ *be an onto homomorphism. Then* $f(A)$ *is an ILFI of* $f(B)$.

Proof. Clearly $f(A)$ and $f(B)$ are ILFSR of S and $f(A) \subseteq f(B)$. Now for all $x, y \in S$,

$$\begin{aligned}f(\nu_A)(xy) &= \wedge\{\nu_A(w) : w \in R, f(w) = xy\}\\ &\leq \wedge\{\nu_A(uv) : u, v \in R, f(u) = x, f(v) = y\}\\ &\leq \wedge\{\nu_B(u) \vee \nu_A(v) / f(u) = x, f(v) = y, u, v \in R\}\\ &= (\wedge\{\nu_B(u) / u \in R, f(u) = x\}) \vee (\wedge\{\nu_A(v) / v \in R, f(v) = y\})\\ &= f(\nu_B)(x) \vee f(\nu_A)(y).\end{aligned}$$

Therefore $f(A)$ is an ILFI of $f(B)$.

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue XIV Version I Year 2012

Theorem 4.8. *Let $f : R \to S$ be an onto homomorphism.*
Let $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in S\}$ be an ILFSR of S and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in S\}$ be an ILFI of B. Then $f^{-1}(A)$ is an ILFI of $f^{-1}(B)$.

Proof. Clearly $f^{-1}(A)$ and $f^{-1}(B)$ are ILFSR of R and $f^{-1}(A) \subseteq f^{-1}(B)$. Now

$$\begin{aligned} f^{-1}(\nu_A)(xy) &= \nu_A(f(xy)) \\ &= \nu_A(f(x)f(y)) \\ &\leq \nu_B(f(x)) \vee \nu_A(f(y)) \\ &= f^{-1}(\nu_B)(x) \vee f^{-1}(\nu_A)(y). \end{aligned}$$

Therefore $f^{-1}(A)$ is an ILFI of $f^{-1}(B)$.

Notes

V. Isomorphism Theorems for ILFSR

Here we define homomorphism and isomorphism of an ILFSR. The fundamental theorem of ring homomorphism and the Third Isomorphism Theorem for rings are established for ILFSR.

Definition 5.1. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ be an ILFSR of R and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in S\}$ be an ILFSR of S.*

(1) A weak homomorphism from A into B is an epimorphism f of R onto S such that $f(A) \subseteq B$. If f is a weak homomorphism of A into B, then A is said to be weakly homomorphic to B and written as $A \overset{f}{\sim} B$ or $A \sim B$.

(2) A weak isomorphism from A into B is a weak homomorphism f from A into B which is also an isomorphism of R onto S. If f is a weak isomorphism from A into B, then A is said to be weakly isomorphic to B and written as $A \overset{f}{\simeq} B$ or $A \simeq B$.

(3) A homomorphism from A onto B is a weak homomorphism f from A onto B such that $f(A) = B$. If f is a homomorphism of A onto B, then A is said to be homomorphic to B and written as $A \overset{f}{\approx} B$ or $A \approx B$.

(4) An isomorphism from A onto B is a weak isomorphism f from A into B such that $f(A) = B$. If f is an isomorphism from A onto B, then A is said to be isomorphic to B and written as $A \overset{f}{\cong} B$ or $A \cong B$.

Let $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ be an ILFSR of R.
Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ be an ILFI of B. Assume that L is regular. Then it is clear that A^* is an ideal of B^* and B/B^* is an ILFSR of B^*. Thus we can consider the ILFQSR of B/B^* relative to A^*. This ILFQSR is denoted as B/A.

Theorem 5.2. *Let* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be an ILFSR and* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be an ILFI of* B*. Suppose that* L *is regular. Then* $B/B^* \approx B/A$*.*

Proof. Let f be the natural homomorphism from B^* onto B^*/A^*. Then

$$\begin{aligned} f(\nu_B/\nu_{B^*})([y]) &= \wedge\{(\nu_B/\nu_{B^*})(x)/x \in B^*, f(x) = [y]\} \\ &= \wedge\{\nu_B(z)/z \in [y]\} \\ &= (\nu_B/\nu_A)([y]) \end{aligned}$$

for all $y \in B^*$ where $[y] = y + A^*$.

Therefore $B/B^* \overset{f}{\approx} B/A$.

The result of the following theorems are proved for membership and non membership functions.

Theorem 5.3. *Let* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *and* $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle / x \in S\}$ *be an ILFSR of the rings* R *and* S *such that* $B \approx C$*. Suppose that* L *is regular. Then there exists an ILFI* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *of* B *such that* $B/A \cong C/C^*$*.*

Proof. Since $B \approx C$, there exists an epimorphism f of R onto S such that $f(B) = C$. Define an ILFS, $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ as follows:

$$\mu_A(x) = \begin{cases} \mu_B(x), & x \in Ker\ f \\ 0, & \text{otherwise, for all } x \in R \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0 & x \in Ker\ f \\ \nu_B(x), & \text{otherwise, for all } x \in R \end{cases}$$

Clearly A is an ILFSR and $A \subseteq B$. If $x \in \mathrm{Ker} f$ then

$$\begin{aligned} \mu_A(xy) &= \mu_B(xy) \\ &\geq \mu_B(x) \wedge \mu_B(y) \\ &\geq \mu_B(x) \wedge \mu_A(y) \end{aligned}$$

for all $y \in R$. If $x \in R \backslash \mathrm{Ker} f$, then $\mu_A(x) = 0$. Hence

$$\mu_A(xy) \geq \mu_B(x) \wedge \mu_A(y)$$

Notes

for all $y \in R$. If $x \in$Ker$f$, then $\nu_A(x) = 0$ and so

$$\nu_A(xy) \leq \nu_B(x) \vee \nu_A(y)$$

for all $y \in R$. If $x \in R\backslash$ Ker$f$, then

$$\begin{aligned} \nu_A(xy) &= \nu_B(xy) \\ &\leq \nu_B(x) \vee \nu_B(y) \\ &\leq \nu_B(x) \vee \nu_A(y) \end{aligned}$$

Notes

for all $y \in R$. Hence A is an ILFI of B.

Since $B \approx C$, $f(B^*) = C^*$. Let $g = f/B^*$. Then g is a homomorphism of B^* onto C^* and Ker$g = A^*$. Thus there exists an isomorphism h of B^*/A^* onto C^* such that $h([x]) = g(x)$ for all $x \in B^*$. For such an h, we have

$$\begin{aligned} h(\mu_B/\mu_A)(y) &= \vee\{(\mu_B/\mu_A)[x]/h([x]) = y, x \in B^*\} \\ &= \vee\{\vee\{\mu_B(z)/z \in [x]\}/g(x) = y, x \in B^*\} \\ &= \vee\{\mu_B(z)/z \in B^*, g(z) = y\} \\ &= \vee\{\mu_B(z)/z \in R, f(z) = y\} \\ &= f(\mu_B)(y) \\ &= \mu_C(y), \quad \text{for all } y \in C^*. \end{aligned}$$

and

$$\begin{aligned} h(\nu_B/\nu_A)(y) &= \wedge\{(\nu_B/\nu_A)[x]/h([x]) = y, x \in B^*\} \\ &= \wedge\{\wedge\{\nu_B(z)/z \in [x]\}/g(x) = y, x \in B^*\} \\ &= \wedge\{\nu_B(z)/z \in B^*, g(z) = y\} \\ &= \wedge\{\nu_B(z)/z \in R, f(z) = y\} \\ &= f(\nu_B)(y) \\ &= \nu_C(y), \text{ for all } y \in C^*. \end{aligned}$$

Therefore $B/A \overset{h}{\cong} C/C^*$.

Theorem 5.4. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in R\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle/x \in R\}$ *and* $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle/x \in R\}$ *be ILFSR. Let* A *be an ILFI of* B *and* A, B *be ILFI of* C*. Suppose that* L *is regular. Then*

$$(C/A)/(B/A) \cong C/B.$$

Global Journal of Science Frontier Research (F) Volume XII Issue XII Version I Year 2012

Proof. Clearly A^* is an ideal of B^* and A^*, B^* are ideals of C^*. By the Third Isomorphism Theorem for Rings,

$$(C^*/A^*)/(B^*/A^*) \overset{f}{\cong} C^*/B^*,$$

where f is given by

$$f(x + A^* + (B^*/A^*)) = x + B^* \text{ for all } x \in C^*.$$

Thus

$$\begin{aligned} f((\mu_C/\mu_A)/(\mu_B/\mu_A))(x + B^*) &= ((\mu_C/\mu_A)/(\mu_B/\mu_A))(x + A^* + (B^*/A^*)) \\ &= \vee\{(\mu_C/\mu_A)(y + A^*)/y \in C^*, y + A^* \in x + A^* + (B^*/A^*)\} \\ &= \vee\{\vee\{\mu_C(z)/z \in y + A^*\}/y \in C^*, y + A^* \in x + A^* + (B^*/A^*)\} \\ &= \vee\{(\mu_C(z)/z \in C^*, z + A^* \in x + A^* + (B^*/A^*)\} \\ &= \vee\{(\mu_C(z)/z \in x + A^* + (B^*/A^*)\} \\ &= \vee\{(\mu_C(z)/z \in C^*, f(z) \in x + B^*\} \\ &= (\mu_C/\mu_B)(x + B^*) \quad \text{for all } x \in C^*. \end{aligned}$$

and

$$\begin{aligned} f((\nu_C/\nu_A)/(\nu_B/\nu_A))(x + B^*) &= ((\nu_C/\nu_A)/(\nu_B/\nu_A))(x + A^* + (B^*/A^*)) \\ &= \wedge\{(\nu_C/\nu_A)(y + A^*)/y \in C^*, y + A^* \in x + A^* + (B^*/A^*)\} \\ &= \wedge\{\wedge\{\nu_C(z)/z \in y + A^*\}/y \in C^*, y + A^* \in x + A^* + (B^*/A^*)\} \\ &= \wedge\{(\nu_C(z)/z \in C^*, z + A^* \in x + A^* + (B^*/A^*)\} \\ &= \wedge\{(\nu_C(z)/z \in x + A^* + (B^*/A^*)\} \\ &= \wedge\{(\nu_C(z)/z \in C^*, f(z) \in x + B^*\} \\ &= (\nu_C/\nu_B)(x + B^*) \quad \text{for all } x \in C^*. \end{aligned}$$

Hence $(C/A)/(B/A) \overset{f}{\cong} C/B$.

Bibliography

[1] Adam Niewiadomski and Eulalia Szmidt, Handling Uncertainty in natural sentences via IFS, *Notes on Intuitionistic Fuzzy Sets*, 7 (2001), No.1, 22–29.

[2] Afshar Alam, Sadia Hussian and Yasir Ahmad, Applying Intuitionistic Fuzzy Approach to Reduce Search Domain in a Accidental Case, *IJACSA*, 1, No.4 (2010).

Notes

Notes

[3] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20(1) (1986), 87–96.

[4] K. T. Atanassov, New Operations defined over the Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 61 (1994), 137–142.

[5] K. T. Atanassov, Fuzzy sets, Theory and Applications, *Studies in fuzziness and soft computing*, 35, Physica-Verlag, Heidelberg, 1999.

[6] D. Boyadzhieva, B. Kolev and N. Netov, Intuitionistic fuzzy data warehouse and some analytical operations, IS, *5th IEEE International Conference* (2010), 331–336.

[7] D. Ćoker, An Introduction to Intuitionistic fuzzy topological spaces, *FSS*, 88 (1997), 81–89.

[8] Eulalia Szmidt and Januz Kaprzyk, Inter-Fuzzy set in some Medical Applications, *NIFS*, 7(2001), 4, 58–64.

[9] Y. B. Jun and K. H. Kim, Intuitionistic fuzzy ideals of semigroups, *Indian Journal of Pure Applied Math.*, 33 (2002) 4, 443–449.

[10] Magdaléna Renčová, An example of Applications of IFS to sociometry, *Bulgarian Academy of Sciences, Cybernetics and Technologies*, Vol 9, No.2 (2009).

[11] D. S. Malik and J. N. Mordeson, Extensions of fuzzy subrings and fuzzy ideals, *FSS*, 45 (1992), 245–251.

[12] D. S. Malik and J. N. Mordeson, Fuzzy Commutative Algebra, 1998, World Scientific.

[13] K. Meena and K. V. Thomas, Intuitionistic L-fuzzy Subrings, *IMF*, 6 (2011), No.52, 2561–2572.

[14] A. Rosenfeld, Fuzzy Groups, *J. Math. Anal. Appl.*, 35 (1971), 512–517.

[15] L. A. Zadeh, Fuzzy sets, *Information and Control*, 8 (1965), 338–353.