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New Results on Q-Product Identities Based on Ramanujan's Findings

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Keywords : *Generating functions, triple product identities.*

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Ref.

2. B.C. Berndt; *What is a q-series?*, preprint.
3. B.C. Berndt; *Ramanujan's notebook Part III*, Springer-Verlag, New York, 1991.

New Results on q-Product Identities Based on Ramanujan's Findings

M.P. Chaudhary

Abstract - In this paper author has established four q-product identities by using elementary method. These identities are new and not available in the literature of special functions.

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I. Introduction

For $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n) \tag{1.1}$$

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1 - aq^{(n-1)}) \tag{1.2}$$

$$(a_1, a_2, a_3, ..., a_k; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty...(a_k;q)_\infty \tag{1.3}$$

Ramanujan [2, p.1(1.2)]has defined general theta function, as

$$f(a,b) = \sum_{-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \quad ; \quad |ab| < 1, \tag{1.4}$$

Jacobi's triple product identity [3,p.35] is given, as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{1.5}$$

Special cases of Jacobi's triple products identity are given, as

$$\phi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty \tag{1.6}$$

$$(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.7}$$

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.8}$$

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Equation (1.8) is known as Euler's pentagonal number theorem. Euler's another well known identity is as

$$(q;q^2)_\infty^{-1} = (-q;q)_\infty \tag{1.9}$$

Throughout this paper we use the following representations

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty = (q^a,q^b,q^c\cdots q^t;q^n)_\infty \tag{1.10}$$

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty = (q^a,q^b,q^c\cdots q^t;q^n)_\infty \tag{1.11}$$

$$(-q^a;q^n)_\infty(-q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty = (-q^a,-q^b,q^c\cdots q^t;q^n)_\infty \tag{1.12}$$

Notes

Now we can have following q-products identities, as

$$(q^2;q^2)_\infty = \prod_{n=0}^{\infty}(1-q^{2n+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{2(4n)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+1)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+2)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+3)+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{8n+2}) \times \prod_{n=0}^{\infty}(1-q^{8n+4}) \times \prod_{n=0}^{\infty}(1-q^{8n+6}) \times \prod_{n=0}^{\infty}(1-q^{8n+8})$$

or,

$$(q^2;q^2)_\infty = (q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty = (q^2,q^4,q^6,q^8;q^8)_\infty \tag{1.13}$$

$$(q^4;q^4)_\infty = \prod_{n=0}^{\infty}(1-q^{4n+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{4(3n)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+1)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+2)+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{12n+4}) \times \prod_{n=0}^{\infty}(1-q^{12n+8}) \times \prod_{n=0}^{\infty}(1-q^{12n+12})$$

or,

$$(q^4;q^4)_\infty = (q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty = (q^4,q^8,q^{12};q^{12})_\infty \tag{1.14}$$

$$(q^4;q^{12})_\infty = \prod_{n=0}^{\infty}(1-q^{12n+4}) = \prod_{n=0}^{\infty}(1-q^{12(5n)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+1)+4}) \times$$

$$\times \prod_{n=0}^{\infty}(1-q^{12(5n+2)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+3)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+4)+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{60n+4}) \times \prod_{n=0}^{\infty}(1-q^{60n+16}) \times \prod_{n=0}^{\infty}(1-q^{60n+28}) \times \prod_{n=0}^{\infty}(1-q^{60n+40}) \times \prod_{n=0}^{\infty}(1-q^{60n+52})$$

or,

$$(q^4;q^{12})_\infty = (q^4;q^{60})_\infty(q^{16};q^{60})_\infty(q^{28};q^{60})_\infty(q^{40};q^{60})_\infty(q^{52};q^{60})_\infty$$

$$= (q^4,q^{16},q^{28},q^{40},q^{52};q^{60})_\infty \tag{1.15}$$

Similarly we can compute following as

$$(q^5;q^5)_\infty = (q^5;q^{15})_\infty (q^{10};q^{15})_\infty (q^{15};q^{15})_\infty \tag{1.16}$$

$$(q^6;q^6)_\infty = (q^6;q^{24})_\infty (q^{12};q^{24})_\infty (q^{18};q^{24})_\infty (q^{24};q^{24})_\infty = (q^6,q^{12},q^{18},q^{24};q^{24})_\infty \tag{1.17}$$

$$\begin{aligned}(q^6;q^{12})_\infty &= (q^6;q^{60})_\infty (q^{18};q^{60})_\infty (q^{30};q^{60})_\infty (q^{42};q^{60})_\infty (q^{54};q^{60})_\infty \\ &= (q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty \end{aligned} \tag{1.18}$$

The outline of this paper is as follows. In sections 2, some recent results obtained by the author [1], and also some well known results are recorded in [6;7], those are useful to the rest of the paper. In section 3, we state and prove four q-product identities, which are new and not recorded in the literature of special functions.

II. Preliminaries

In [1], following identities are being established

$$(q;q^2)_\infty = (q,q^3,q^5;q^6)_\infty \tag{2.1}$$

$$\left[\frac{(-q;q^2)_\infty^8 - (q;q^2)_\infty^8}{q}\right]^{\frac{1}{4}} = \frac{2}{\left[(q^2;q^4)_\infty\right]^2} \tag{2.2}$$

$$\frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty} = (q,-q;q^2)_\infty \tag{2.3}$$

$$(q^2;q^2)_\infty = (q^2;q^4)_\infty (q^4;q^4)_\infty \tag{2.4}$$

In Ramanujan's notebook [7, p.107], Chapter IX, Entry $7(iii)$ is recorded as

$$\phi(q) + \phi(-q) = \frac{1}{4}\phi(q^2) \tag{2.5}$$

In Ramanujan's notebook [7, p.198], Chapter XVI, following entries are recorded as

Entry $24(i)$:

$$\frac{f(q)}{f(-q)} = \frac{\psi(q)}{\psi(-q)} = \frac{\chi(q)}{\chi(-q)} = \sqrt{\frac{\phi(q)}{\phi(-q)}} \tag{2.6}$$

where $\chi(q)$ is given in [7, p.197], Chapter XVI, Entry $22(iv)$, as

$$\chi(q) = \prod(q,q^2) = (1+q)(1+q^3)(1+q^5)(1+q^7)\ \textit{and constant} \tag{2.7}$$

Entry $24(ii)$:

$$f^3(-q) = \phi^2(-q)\psi(q) = 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - \ \textit{and constant} \tag{2.8}$$

Entry $24(iii)$:

$$\chi(q) = \frac{f(q)}{f(-q^2)} = \sqrt[3]{\frac{\phi(q)}{\psi(-q)}} = \frac{\phi(q)}{f(q)} = \frac{f(-q^2)}{\psi(-q)} \tag{2.9}$$

where $\chi(q)$ is given by (2.7)

Ref.

1. M.P. Chaudhary; *Development on q-product identities*, preprint.
7. S. Ramanujan; *Notebooks (Volume II)*, Tata Institute of Fundamental Research, Bombay, 1957.

Entry 24(iv) :

$$f^3(-q^2) = \phi(-q)\psi^2(x) \tag{2.10}$$

and

$$\chi(q)\chi(-q) = \chi(-q^2) \tag{2.11}$$

where $\chi(q)$ is given by (2.7)

III. Main Results

Notes

In this section, we established following new results with the help of $\psi(.)$ and $\phi(.)$ functions or in more general language we can say that by using the properties of Jacobi's triple product identity as $\psi(.)$ and $\phi(.)$ functions are its special cases. These results are not recorded in the literature of special functions

$$(-q^2;q^4)_\infty = 2(-q,-q;q^2)_\infty^{\frac{1}{2}}[(-q;q^2)_\infty^2 + (q;q^2)_\infty^2]^{\frac{1}{2}} \tag{3.1}$$

$$(-q;q^2)_\infty(q;q)_\infty = (q;q^2)_\infty(-q;-q)_\infty \tag{3.2}$$

$$(q;q)_\infty = (q;q^2)_\infty(q^2;q^2)_\infty = (q,q^2;q^2)_\infty \tag{3.3}$$

$$(-q;-q)_\infty = (-q;q^2)_\infty(q^2;q^2)_\infty \tag{3.4}$$

Proof of (3.1): By substituting, $q=-q$ and $q=q^2$ respectively in (1.6), we have

$$\phi(-q) = (q;q^2)_\infty^2(q^2;q^2)_\infty \;;\; \phi(q^2) = (-q^2;q^4)_\infty^2(q^4;q^4)_\infty$$

by substituting the values $\phi(-q)$, $\phi(q^2)$, and employing (1.6) in (2.5), we get

$$(q^2;q^2)_\infty[(-q;q^2)_\infty^2 + (q;q^2)_\infty^2] = \frac{1}{4}(-q^2;q^4)_\infty^2(q^4;q^4)_\infty$$

further using (2.3), and after simplification, we get

$$(-q^2;q^4)_\infty = 2(-q,-q;q^2)_\infty^{\frac{1}{2}}[(-q;q^2)_\infty^2 + (q;q^2)_\infty^2]^{\frac{1}{2}}$$

which established (3.1)

Proof of (3.2): By substituting, $q=-q$ in (1.7) and (1.8) respectively, we have

$$\psi(-q) = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \;;\; f(q) = (-q;-q)_\infty$$

by substituting the values of $f(q)$ and $\psi(-q)$, and employing (1.7) and (1.8), in first and second part of (2.6), after little simplification, we get

$$\frac{(-q;-q)_\infty}{(q;q)_\infty} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}$$

which can also be written as

$$(-q;q^2)_\infty(q;q)_\infty = (q;q^2)_\infty(-q;-q)_\infty$$

which established (3.2)

Note: We verified that the result (3.2), can also be proved by taking any other two parts of (2.6).

Proof of (3.3): By (1.6) and (1.8) respectively, we have

$$\phi^2(-q) = (q;q^2)^4_\infty (q^2;q^2)^2_\infty \; ; \; f^3(-q) = (q;q)^3_\infty$$

Notes

by substituting the values of $\phi^2(-q)$ and $f^3(-q)$, and employing (1.7), in first and second part of (2.8), after little simplification, we get

$$(q;q)_\infty = (q;q^2)_\infty (q^2;q^2)_\infty = (q,q^2;q^2)_\infty$$

which established (3.3)

Note: If we put $q = q^2$ in (3.3), then we find (2.4) a result already proved by the author in [1].

Proof of (3.4): By (1.7) and (1.8) respectively, we have

$$(-q) = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \; ; f^3(q) = (-q;-q)^3_\infty \; ; \; f^3(-q^2) = (q^2;q^2)^3_\infty$$

by substituting the values of $(-q)$, $f^3(q)$, $f^3(-q^2)$ and employing (1.6), in second and third part of (2.9), after little simplification, we get

$$(-q;-q)_\infty = (-q;q^2)_\infty (q^2;q^2)_\infty$$

which established (3.4)

Note: We verified that the result (3.4), can also be proved by taking any other two parts of (2.9).

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Notes

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