Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 12 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Proof of 'J is a Radical Class' Using Amitsur Theorem

By Raju Chowdhury, Dewan Ferdous Wahid & Md. Anowar Hossain
Stamford University Bangladesh, Dhaka

Abstract - The aim of this paper is to study radical class of rings, right quasi-regular rings and finally, to prove that J , the class of all right quasi-regular rings is a radical class. Amitsur gives a theorem of radical class for the sufficient condition that a class of rings would be a radical class. This paper represents, the proof of, J is a radical class using the theorem of radical class given by Amitsur.

Keywords : *Ring, Ideal, Radical Class, Right quasi-regular ring.*

GJSFR-F Classification : *MSC 2010: 16D60, 16N80*

PROOF OF J IS A RADICAL CLASSUSING AMITSUR THEOREM

Strictly as per the compliance and regulations of :

© 2012. Raju Chowdhury, Dewan Ferdous Wahid & Md. Anowar Hossain. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Ref.

Proof of 'J is a Radical Class' Using Amitsur Theorem

Raju Chowdhury [α], Dewan Ferdous Wahid [σ] & Md. Anowar Hossain [ρ]

Abstract - The aim of this paper is to study radical class of rings, right quasi-regular rings and finally, to prove that J, the class of all right quasi-regular rings is a radical class. Amitsur gives a theorem of radical class for the sufficient condition that a class of rings would be a radical class. This paper represents, the proof of, J is a radical class using the theorem of radical class given by Amitsur.

Keywords : *Ring, Ideal, Radical Class, Right quasi-regular ring.*

I. Introduction

The concept of a radical was introduced by J. H. M. Wedderburn [10] in 1908, for the determination of structures of algebras and later on various radicals have been proposed by Artin [14], Baer [11], Jacobson [9], Brown-McCoy [12], Levitzki [7] etc. for the study of rings in the forties. The general theory of radicals was initiated by Kurosh [6] (1953) and Amitsur [1] in the early fifties. Andrunakievic [4], Sulinski [15], Divinsky [8] and many others have followed up the works of Kurosh and Amitsur.

Radical properties based on the notion of nilpotence do not seem to yield fruitful results for rings without chain conditions. The notion of quasi-regularity was introduced by Perlis [16]. In 1945, Jacobson [9] used it and the significant "chainless" results were obtained.

In this paper, the general ring theory covering elementary definition of rings and its ideals, homomorphism, theorem related to homomorphism and some definitions related to radical class has been discussed in preliminaries. Also, we will introduce radical class of rings and some theorems related to radical class. Amitsur gives a theorem of radical class, which works as a sufficient condition of a class of rings that would be a radical class. We will know about this theorem and also right quasi-regular ring, right quasi-regular right ideal and some lemmas related to right quasi regular rings. Finally, we will prove that J, the class of all right quasi-regular rings is a radical class. It has already been proved by using the definition of radical class. But, here we will prove this using Amitsur theorem of radical class.

1. S. Amitsur, "A general theory of radicals I", Amer. J. Math. 74,774, 1952.

Author α : Lecturer (Mathematics), Department of Natural Science, Stamford University Bangladesh, Dhaka-1217.
E-mail : rajumath@stamforduniversity.edu.bd
Author σ : Lecturer (Mathematics), Department of Natural Science, Stamford University Bangladesh, Dhaka-1217.
E-mail : dfwahid@stamforduniversity.edu.bd
Author ρ : Lecturer (Mathematics), Department of Natural Science, Stamford University Bangladesh, Dhaka-1217.
E-mail : hossain_anowar45@yahoo.com

© 2012 Global Journals Inc. (US)

II. Preliminaries

2.1. Definition

A ***ring*** is an ordered triple $(R, +, \cdot)$ such that R is a nonempty set and $+$ and $\cdot$ are two binary operations on R satisfying the following axioms:

a) R is an additive abelian group. i.e.

i) $a + b \in R$ for all $a, b \in R$ [closure law]

ii) $(a + b) + c = a + (b + c)$ for all $a, b, c \in R$. [associative law]

iii) there exists an element $0 \in R$ such that $a + 0 = 0 + a = a$, for all $a \in R$. [identity law]

iv) for every non-zero element $a \in R$ there exists an element $-a \in R$ such that $a + (-a) = (-a) + a = 0$. [inverse law]

v) $a + b = b + a$ for all $a, b \in R$. [commutative law]

b) $(R, \cdot)$ is a semi group. i.e.

i) $a \cdot b \in R$ for all $a, b \in R$. [closure law]

ii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$ [associative law]

c) Distributive laws are true in R. i.e. for all $a, b, c \in R$,

i) $a \cdot (b + c) = a \cdot b + a \cdot c$

ii) $(a + b) \cdot c = a \cdot c + b \cdot c$

Example

i) $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ are rings.

ii) The residue class of modulo 6,

$\mathbb{Z}_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ is a ring.

iii) $[x]$, the set of all polynomials in x with real coefficients, is a ring.

2.2. Definition

A non-empty subset I of a ring R is called a ***left (right) ideal*** of R if

i) I is an additive subgroup of R

ii) $\forall\, r \in R$ and $\forall\, i \in I$, $ri \in I$, $(ir \in I)$.

2.3. Definition

A non-empty subset I of a ring R is called an ***ideal*** of R if I is both a left ideal and a right ideal of R. For a commutative ring all left and right ideals are ideals.

Example:

1. $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$.

2. The set of integers $\mathbb{Z}$ is only a subring but not an ideal of the ring of rational numbers $\mathbb{Q}$. As $3 \in \mathbb{Z}$, $\frac{2}{5} \in \mathbb{Q}$ but $3 \cdot \frac{2}{5} = \frac{6}{5} \notin \mathbb{Z}$.

2.4. Definition

Let R be a ring and I be an ideal of R then the ***quotient ring*** or ***factor ring*** $\frac{R}{I}$ is the set $\{ r + I : r \in R \}$, where addition and multiplication of two elements $r_1 + I,\ r_2 + I \in \frac{R}{I}$ are given by

i) $(r_1 + I) + (r_2 + I) = (r_1 + r_2) + I$.

ii) $(r_1 + I)(r_2 + I) = r_1 r_2 + I$.

Example:

$\frac{\mathbb{Z}}{2\mathbb{Z}}$ is a quotient ring.

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue XII Version I Year 2012

© 2012 Global Journals Inc. (US)

2.5. Definition

Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be two rings. A mapping $f: R \to R'$ is called a ring ***homomorphism*** if $\forall\ a, b \in R$

i) $f(a+b) = f(a) + f(b)$ and

ii) $f(ab) = f(a)\ f(b)$.

Notes

2.5.1. Theorem

Every factor ring of a ring is the homomorphic image of that ring.

Proof: Let R be a ring and I be an ideal of R, then we have to show that $\frac{R}{I}$ is a homomorphic image of R. Let us define a map $f: R \to \frac{R}{I}$ by $f(r) = r + I$ for all $r \in R$. We need to show that f is a onto homomorphism.

Clearly f is well defined.

Now $f(r_1 + r_2) = (r_1 + r_2) + I = (r_1 + I) + (r_2 + I) = f(r_1) + f(r_2)$

and $f(r_1 r_2) = r_1 r_2 + I = (r_1 + I)\ (r_2 + I) = f(r_1)\ f(r_2)$.

Thus f is a homomorphism.

Let $r + I \in \frac{R}{I}$ where $r \in R$. Then by definition, $f(r) = r + I$. i.e. $r + I = f(r)$. This implies that every element of $\frac{R}{I}$ is the image of some element in R. Thus f is onto. Hence the theorem is proved.

2.6. Definition

A ring R is said to have the ***ascending chain condition (A.C.C.)*** on left (right) ideals, if every ascending sequence of left (right) ideals $L_1 \subseteq L_2 \subseteq L_3 \subseteq \ldots\ldots \subseteq L_n \subseteq \ldots\ldots$, terminates after a finite number of steps, i.e. there exists a positive integer n such that $L_n = L_{n+1} = \ldots\ldots\ldots$.

2.7. Definition

A ring R is said to have the ***descending chain condition (D.C.C.)*** on left (right) ideals, if every descending sequence of left (right) ideals $R \supseteq L_1 \supseteq L_2 \supseteq L_3 \supseteq \ldots\ldots \supseteq L_n \supseteq \ldots\ldots\ldots$, terminates after a finite number of steps, i.e. there exists a positive integer n such that $L_n = L_{n+1} = \ldots\ldots\ldots$.

III. Radical Class of Rings

3.1. Definition

Let $\mathfrak{R}$ be a nonempty class of rings with a certain property. A ring A is said to be an ***$\mathfrak{R}$-ring*** if $A \in \mathfrak{R}$.

Example:

Let $\mathfrak{R}$ be the class of all nilpotent ring and A be an idempotent ring. Then A is not nilpotent ring and hence $A \notin \mathfrak{R}$. Therefore A is not an $\mathfrak{R}$-ring.

3.2. Definition

An ideal I of a ring A is said to be an $\mathfrak{R}$-ideal if I is an $\mathfrak{R}$-ring. i.e. $I \in \mathfrak{R}$.

Example:

Let $\mathfrak{R}$ be the class of all nilpotent ring and I be an ideal of a nilpotent ring A. Then $I \in \mathfrak{R}$. Therefore I is an $\mathfrak{R}$-ideal.

© 2012 Global Journals Inc. (US)

3.3. Definition

A ring A is said to be $\mathfrak{R}$-semi-simple if A has no non-zero $\mathfrak{R}$-ideal.

3.4. Definition

Let $\mathfrak{R}$ be a non-empty class of rings with a certain property. Then $\mathfrak{R}$ is said to be a ***radical property*** or ***radical class*** if the following conditions are hold:

A) $\mathfrak{R}$ is homomorphically closed. i.e. every homomorphic image of an $\mathfrak{R}$-ring A is an $\mathfrak{R}$-ring. i.e. if $A\in\mathfrak{R}$ and $I\lhd A$, then $\frac{A}{I}\in\mathfrak{R}$.

B) Every ring $A\in\mathfrak{R}$ contains a non-zero $\mathfrak{R}$-ideal $R(A)$ which contains every other $\mathfrak{R}$-ideals of A.

C) $\frac{A}{R(A)}$ has no non-zero $\mathfrak{R}$-ideal. i.e. $\frac{A}{R(A)}$ is $\mathfrak{R}$-is semi-simple.

A radical class is simply called a radical.

3.5. Definition

Let $\mathfrak{R}$ be a radical class. The $\mathfrak{R}$-ideal $R(A)$ of a ring A is called the **$\mathfrak{R}$ - *radical*** of the ring A.

3.6. Definition

Let $\mathfrak{R}$-be a radical class. Then a ring A is said to be an ***$\mathfrak{R}$-radical ring*** if $R(A) = A$, where $R(A)$ is the radical of A.

3.7. Definition

Let $\mathfrak{R}$ be a radical class. Then a ring A is said to be an $\mathfrak{R}$-semi-simple ring if $R(A) = 0$, where $R(A)$ is the radical of A.

0 is the only ring which is both an $\mathfrak{R}$-radical ring and an $\mathfrak{R}$-semi-simple ring.

3.7.1. Theorem [8]

Let $\mathfrak{R}$ be a non-empty class $\mathfrak{R}$ of rings. Then $\mathfrak{R}$ is said to be a radical class if and only if

A) $\mathfrak{R}$ is homomorphically closed.

D) If every non-zero homomorphic image of a ring A contains a non-zero $\mathfrak{R}$-ideal, then A is in $\mathfrak{R}$. i.e.$\forall$ $I\lhd A$, if $\frac{A}{I} \supset \frac{B}{I} \in \mathfrak{R}$ then $A\in\mathfrak{R}$, where $B\lhd A$.

This theorem is known as Kurosh's Theorem.

3.7.2.1. Lemma (Zorn's Lemma)

Let A be a nonempty partially ordered set in which every totally ordered subset has an upper bound in A. Then A contains at least one maximal element.

3.7.2. Theorem (Amitsur) [3]

Let $\mathfrak{R}$ be a nonempty class of rings. Then $\mathfrak{R}$ is a radical class if and only if

A′) $\mathfrak{R}$ is homomorphically closed.

B′) For any ring A and an ideal I of A if both I and $\frac{A}{I}$ is in $\mathfrak{R}$, then $A\in\mathfrak{R}$. i.e. $\mathfrak{R}$ is closed under extension.

C′) If $I_1\subseteq I_2\subseteq I_3\subseteq$ ………, is an ascending chain of $\mathfrak{R}$-ideals of a ring A, then $\bigcup_\alpha I_\alpha$ is an $\mathfrak{R}$-ideal.

IV. RIGHT QUASI-REGULAR RINGS

4.1. Definition

Let R be a ring and $x\in R$. Then x is called ***right quasi-regular*** if there exists an

Ref.

3. S. Tumurbat & R. Wisbauer, "Radicals With The α-Amitsur Property", W. Sc. P. C., Vol. 7, No. 3,347–361, 2008.

© 2012 Global Journals Inc. (US)

element $y \in R$ such that $x + y + xy = 0$.

We often write $x + y + xy$ by $x \circ y$. When $x \circ y = 0$ then the element y is called ***right quasi-inverse*** of x.

4.2. Definition

A ring R is said to be ***right quasi-regular*** if every element in R is right quasi-regular.

4.2.1. Lemma

If R is a ring with 1, then $(1+x)$ has right inverse $(1 + y)$ iff x is right quasi-regular.

Proof: Let us consider a ring with unity element 1. Let $(1 + y)$ be the right inverse of $(1 + x)$. Then we have,

$(1 + x)\ (1 + y) = 1$

$\Rightarrow 1 + y + x + xy = 1$

$\Rightarrow x + y + xy = 0$

$\Rightarrow x$ is right quasi-regular.

Conversely, let x be right quasi-regular. Then there is a right quasi-inverse y such that $x + y + xy = 0$

$\Rightarrow 1 + y + x + xy = 1$

$\Rightarrow 1(1 + y) + x(1 + y) = 1$

$\Rightarrow (1 + x)\ (1 + y) = 1$

i.e. $(1 + y)$ is right inverse of $(1 + x)$.

4.2.2. Lemma

Let R be a ring. Then for any element x in R, x is a right quasi-regular if and only if $\{r + xr\} = R, \forall\ r \in R$.

Proof: Let R be a ring and $x \in R$. Consider $\{r + xr\}$, the set of all elements $r + xr$, $\forall\ r \in R$. Then $\{r + xr\}$ is clearly a right ideal of R. Now suppose that $\{r + xr\} = R$. We are to show that x is right quasi-regular. Since $\{r + xr\} = R$, then $x = r + xr$ for some r in R. This implies that $x + (-r) + x\ (-r) = 0$. This implies that x is right quasi-regular for some $r \in R$.

Conversely, suppose that x is right quasi-regular element of R. We have to show that $R = \{r + xr\}$. Since x is right quasi-regular then $\exists$ an element $y \in R$ such that $x + y + xy = 0 \Rightarrow x = (-y) + x\ (-y) \in \{r + xr\}$. Then $xr \in \{r + xr\}$ and therefore $r \in \{r + xr\}$ for every $r \in R$. Hence $\{r + xr\} = R$.

4.3. Definition

Let R be a ring and I be a right ideal of R. Then I is called a right quasi-regular right ideal if every element of I is right quasi-regular.

4.3.1. Lemma[8]

If x is right quasi-regular and if y belongs to a right quasi-regular right ideal I, then $x + y$ is right quasi-regular.

Proof: Since x is right quasi-regular, then there exists an element x' such that, $x + x' + xx' = 0$. Now, consider the element $y + yx'$. Then $y + yx'$ is in I and thus is right quasi-regular. Let z be the right quasi-inverse of $y + yx'$ then, $(y + yx') + z + (y + yx')z = 0$.

Ref.

8. N. Divinsky, "Rings and Radicals (University of Toronto press)", 1965.

© 2012 Global Journals Inc. (US)

Now we will show that $x' + z + x'z$ is a right quasi-inverse of $x + y$.

Therefore, $(x + y) + (x' + z + x'z) + (x + y)(x' + z + x'z)$

$= x + y + x' + z + x'z + xx' + xz + xx'z + yx' + yz + yx'z$

$= (x + x' + xx') + (y + yx') + z + (y + yx')z + (x + x' + xx')z$

$= 0$

Hence $x + y$ is right quasi-regular.

4.3.2. Lemma

The sum of two right quasi-regular right ideals of a ring is also a right quasi-regular right ideal.

Proof: Let I_1 and I_2 be two right quasi-regular right ideals of a ring R. We have to show that $I_1 + I_2$ is also a right quasi-regular right ideal of R. Let $p \in I_1 + I_2$ then $p = x + y$ for some $x \in I_1$ and $y \in I_2$. Since x is right quasi-regular and $y \in I_2$ then we have $x + y$ is also right quasi-regular (by Lemma 4.3.1). i.e. p is right quasi-regular.

Hence every element of $I_1 + I_2$ is right quasi-regular.

Hence $I_1 + I_2$ is right quasi-regular right ideal of R.

4.3.3. Lemma

The sum of any finite number of right quasi-regular right ideals of a ring is again a right quasi-regular right ideal.

Proof: Let $I_1, I_2 \ldots, I_n$ are right quasi-regular right ideals of a ring R. We have to show that $I_1 + I_2 + \ldots\ldots + I_n$ is right quasi-regular right ideal. We shall prove this by the method of induction on n.

If $n = 1$ then the proof is obvious. Now suppose $n = 2$, then, $I_1 + I_2$ is right quasi-regular right ideal (by Lemma 4.3.1).

Now, let $I = I_1 + I_2 + \ldots\ldots + I_{n-1}$ a right quasi-regular right ideal of R. We show that $I + I_n$ is right quasi-regular right ideal of R.

Let $p \in I + I_n$ then $p = x' + y'$ for some $x' \in I$ and $y' \in I_n$. Then x' is right quasi-regular and y' belongs to a right quasi-regular right ideal I_n. Therefore $x' + y'$ is right quasi-regular (by Lemma 4.3.1). Hence $I + I_n$ is right quasi-regular right ideal. i.e. $I_1 + I_2 + \ldots\ldots + I_{n-1} + I_n$ is a right quasi-regular right ideal of R.

4.3.4. Lemma

Sum (Union) of all right quasi-regular right ideals of a ring R is a right quasi-regular right ideal of R.

4.3.5. Lemma[8]

$J(R)$, the sum of all right quasi-regular right ideals of a ring R is a two sided ideal of R.

Proof: Let x be any element in $J(R)$ and r any element of R. We have to show that rx is in $J(R)$ i.e. $J(R)$ is a left ideal. We know that $J(R)$ is a right quasi-regular right ideal. Hence $xr \in J(R)$ is a right quasi-regular. Then there exists an element w such that

$xr + w + xrw = 0$. Then

$rx + (-rx - rwx) + rx(-rx - rwx)$

$= rx - rx - rwx - rx \cdot rx - rx \cdot rwx$

$= -r(w + xr + xrw)x$

$= -r \cdot 0 \cdot x$

$= 0$

Ref.

8. N. Divinsky, "Rings and Radicals (University of Toronto press)", 1965.

© 2012 Global Journals Inc. (US)

Therefore, rx is right quasi-regular.

Next consider the right ideal generated by rx. This is the set of all $rxi + rxs$, where i is an integer and s is in R. The element $xi + xs$ is in $J(R)$ and, as above, $r(xi + xs)$ is right quasi-regular. Therefore, $\{rxi + rxs\}$ is a right quasi-regular right ideal. It is thus in $J(R)$ and then, in particular, rx is in $J(R)$. Therefore $J(R)$ is a two-sided ideal of R.

4.3.6. Lemma

Every homomorphic image of a right quasi-regular ring R is right quasi-regular.

Proof: Let R be a right quasi-regular ring and I be any ideal of R, then we have to show that $\frac{R}{I}$ is right quasi-regular. Let $x \in \frac{R}{I}$ then $x = r + I$ for some $r \in R$.

Since R is right quasi-regular, then r is right quasi-regular. Then there exists an element $r' \in R$ such that $r + r' + rr' = 0$.

Now $(r + I) + (r' + I) + (r + I)(r' + I) = r + r' + I + rr' + I$

$= r + r' + rr' + I$

$= 0 + I$

$= I$

But I is the zero element of $\frac{R}{I}$. Therefore $r' + I$ is right quasi-inverse of $r + I$. Hence $r + I$ is right quasi-regular i.e. x is right quasi-regular. Therefore $\frac{R}{I}$ is right quasi-regular.

4.3.7. Lemma

Let R be a ring and I be an ideal of R. If I and $\frac{R}{I}$ are right quasi-regular then R is right quasi-regular.

Proof: Since $\frac{R}{I}$ is right quasi-regular, then for every $x \in \frac{R}{I}$, there exists $y \in \frac{R}{I}$ such that

$(x + I) + (y + I) + (x + I)(y + I) = I$

$\Rightarrow x + I + y + I + xy + I = I$

$\Rightarrow x + y + xy + I = I$

$\Rightarrow x + y + xy \in I$

Since I is right quasi-regular then there exists $w \in I$ such that

$x + y + xy + w + (x + y + xy)w = 0$

$\Rightarrow x + y + xy + w + xw + yw + xyw = 0$

$\Rightarrow x + (y + w + yw) + x(y + w + yw) = 0$

This implies that $y + w + yw$ is a right quasi-inverse of x and thus x is a right quasi-regular. Hence R is right quasi-regular.

V. Conclusions

From the above discussions, we can prove the following theorem.

5.1. Theorem

The class of all right quasi-regular rings is a radical class.

Proof: Let J be the class of all right quasi-regular rings. We shall prove this using Amitsur theorem.

By Lemma 4.3.6, A′) holds. i.e. J is homomorphically closed.

Notes

© 2012 Global Journals Inc. (US)

By Lemma 4.3.7, B′) holds. i.e. J is closed under extension.

To prove C′), let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots\ldots$ be an ascending chain of right quasi-regular right ideals of a ring R. We have to show that $\bigcup_\alpha I_\alpha$ is right quasi-regular. Let $x \in \bigcup_\alpha I_\alpha$ then $x \in I_\alpha$ for some α. Since each I_α is right quasi-regular right ideal then $\exists$ an element x' such that $x + x' + xx' = 0$. i.e. x is right quasi-regular. Hence every element of $\bigcup_\alpha I_\alpha$ is right quasi-regular. i.e. $\bigcup_\alpha I_\alpha$ is right quasi-regular. Hence J is a radical class.

Notes

References Références Referencias

1. S. Amitsur, "A general theory of radicals l", Amer. J. Math. 74,774, 1952.
2. S. Amitsur, "A general theory of radicals ll", Amer. J .Math., 76,100, 1954.
3. S. Tumurbat & R. Wisbauer, "Radicals With The α-Amitsur Property", W. Sc. P. C., Vol. 7, No. 3,347–361, 2008.
4. Andrunakievic, "Radicals of associative rings 1", Mat, Sbor, 44, (86) 179, 1958.
5. Kishor Pawar &Rajendra Deore, "A Note on Kurosh Amitsur Radical and Hoehnke Radical", Thai J. Math, V. 9 No. 3, 571–576, 2011.
6. A. Kurosh, "Radicals of rings and algebras", Mat. Sb. (N.S), 33, 13 – 26 (Russian), 1953.
7. J. Levitzki, "On the radical of a general ring", Bull. Am. Math. Soc.,49,462 – 466, 1943.
8. N. Divinsky, "Rings and Radicals (University of Toronto press)", 1965.
9. N. Jacobson, "The radical and semi-simplicity for arbitrary rings", Am. J. Math., 67, 300 – 20, 1945.
10. J. H. M. Wedderburn, "On hypercomplex numbers", Proc. London Math. Soc., 6, 77-117, 1908.
11. R. Bear, "Radical ideals", Am. J. Math., 65, 537 – 68, 1943.
12. B. Brown and N. H. McCoy, "The radical of a ring", Duke Math. J., 15, 495 – 9, 1947.
13. N. Jacobson, "The radical and semi-simplicity for arbitrary rings", Am. J. Math., 67, 300 – 20, 1945.
14. E. Artin, C. J. Nesbitt and R. M. Thrall, "Rings with minimum condition", University of Michigan Publications in Mathematics, no. 1, 1944.
15. Sulinski, "Certain questions in the general theory of radicals", Mat.Sb., 44,273-86, 1958.
16. S. Perlis, "A characterization of the radical of algebra", Bull.Am. Math. Soc.,48, 128 – 132, 1942.

© 2012 Global Journals Inc. (US)