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Solution of Einstein's Field Equations for Mixed Potential of a Radiating Star

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Solution of Einstein's Field Equations for Mixed Potential of a Radiating Star

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Abstract- We have studied the behaviour of a radiating star when the interior expanding, shearing fluid particles are traveling in geodesic motion. A systematic approach enables us to write the junction condition as a Riccati equation. In this article we obtained two new solutions in terms of elementary functions with assuming a separation of variables and also have discussed the physical significance of these solutions.

Keywords : Radiating star, Vaidya exterior equation, Einstein's field equations, Exact solutions, Variable separation.

I. Introduction

The interior space-time of the collapsing radiating star should match to the exterior space-time described by the Vaidya solution in 1951. To obtain realistic analytic solutions, different authors constructed different models. De Oliviera et al (1985) proposed a radiating model of an initial interior static configuration leading to slow gravitational collapse. Herrera et al (2004) proposed a relativistic radiating model with a vanishing Weyl-tensor, in a first order approximation, without solving the junction condition exactly. Then Maharaja and Govender (2005) & Herrera et al (2006) solved the relevant junction condition exactly, and generated classes of solutions in terms of elementary functions which contain the Friedmann dust solution as special case. The first exact solution, with nonzero shear was obtained by Naidu et al (2006) in 2006, considering geodesic motion of fluid particles; later in 2008, Rajah and Maharaja (2008) obtained two classes of nonsingular solutions by assuming that the gravitational function Y(r,t) is a separable function and solving a Riccati equation. Recently S.

Thirukanesh and Maharaj (2010) demonstrate and obtained exact solutions systematically without assuming separable forms and not fixing the temporal evolution of the model. We further extended it and obtained two new solutions by assuming that the gravitational potential $Y(t, r)$ and $B\,(r,\ t)$ is a separable function.

II. The Model

In general relativity, the form for the interior space time of a spherically symmetric collapsing star with nonzero shear when the fluid trajectories are geodesies is given by the line metric.

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$$ds^2 = -dt^2 + B^2dr^2 + Y^2(d\theta^2 + sin^2\theta d\phi^2) \quad (1)$$

Here B and Y are functions of both the temporal coordinate t and radial coordinate *r*. The fluid four − velocity vector u is given by $u^a = \delta_0^a$ which is comoving. For the line element (1), the four acceleration $\dot{u}^a$, the expansion scalar Θ , and the magnitude of the shear scalar are given by

$$\dot{u}^a = 0 \quad (2a)$$

$$\Theta = \frac{\boldsymbol{B}}{B} + 2\,\frac{Y}{Y} \quad (2b)$$

$$\xi = \frac{1}{3}\left(\frac{\dot{y}}{y} - \frac{\dot{B}}{B}\right) \quad (2c)$$

respectively, and dots denote the differentiation with respect to *t*. The energy momentum tensor for the interior matter distribution is described by

$$T_{ab} = (p + \rho)\, u_a u_b + p g_{ab} + \pi_{ab} + q_a u_b + q_a u_a \quad (3)$$

where *p* is the isotropic pressure, *p* is the energy density of the fluid, π_{ab} is the stress tensor, and q_a is the heat flux vector. The stress tensor has the from

$$\pi_{ab} = (P_r - P_t)\,(n_a n_b - \frac{1}{3} h_{ab}) \quad (4)$$

Where P_r is the radial pressure, and P_t is the tangential pressure and n is a unit radial vector given by $n^a = (\frac{1}{3})\,\delta_1^a$. The isotropic pressure is given by

$$P = \frac{1}{3}\,(p_r + 2p_t) \quad (5)$$

In terms of the radial pressure and the tangential pressure, for the line element (1) and matter distribution (3) the Einstein field equations becomes

$$\rho = 2\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{1}{Y^2} + \frac{\dot{Y}^2}{Y^2} - \frac{1}{B^2}\left(2\frac{Y''}{Y} + \frac{Y'^2}{Y^2} - 2\frac{B'Y'}{BY}\right) \quad (6a)$$

$$P_r = -\frac{2\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} - \frac{1}{Y^2} + \frac{1}{B^2}\frac{Y'^2}{Y^2} \quad (6b)$$

$$P_t = -\left(\frac{\ddot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y}\right) + \frac{1}{B^2}\left(\frac{Y''}{Y} - \frac{B'Y'}{BY}\right) \quad (6c)$$

$$q = -\frac{2}{B^2}\left(\frac{\dot{B}Y'}{BY} - \frac{\dot{Y}'}{Y}\right) \quad (6d)$$

where the heat flux q^a = (o, q, o, o) is radially directed and primes denote the differentiation with respect to *r*. These equation describe the gravitational interactions of a shearing matter distribution with heat flux and anisotropic pressure for particles travelling along geodesics from (6*a*) – (6*d*), we observe that if the gravitational potentials *B*(*t*, *r*) and *y* (*t*, *r*) are specified, then the expressions for the matter variables p, p_r , p_t and *q* follow by simple substitution.

The vaidya exterior space-time of radiating star is given by

$$ds^2 = -(1 - \frac{2m(v)}{R})\, dv^2 - 2dvdR + R^2\,(d\theta^2 + sin^2\theta\, d\,\phi^2) \quad (7)$$

where *m*(*v*) denoted the mass of the fluid as measured by an observer at infinity. The matching of the interior space-time (1) with the exterior spacetime (7) generates the set if junction conditions on the hyper surface Σ given by

Notes

$$dt = \left(1 - \frac{2m}{R_\Sigma} + 2\frac{dR_\Sigma}{dv}\right)^{1/2} dv \tag{8a}$$

$$Y(R_\Sigma, t) = R_\Sigma(v) \tag{8b}$$

$$m(v)_\Sigma = \left[\frac{Y}{2}\left(1 + \dot{Y}^2 - \frac{Y'^2}{B^2}\right)\right]_\Sigma \tag{8c}$$

Notes

$$(P_r)_\Sigma = (qB)_\Sigma \tag{8d}$$

The nonvanishing of the radial pressure at the boundary Σ is reflected in equation (8*d*). Equation (8*d*) is an additional constraint which has to be satisfied together with the system of equations (6*a*)- 6*d*). On substituting (6*b*) and (6*d*) in (8*d*) we obtain

$$2Y\ddot{Y} + \dot{Y}^2 - \frac{Y'^2}{B^2} + \frac{2}{B}Y\dot{Y}' - \frac{2\dot{B}}{B^2}YY' + 1 = 0 \tag{9}$$

which has to satisfied on Σ. Equation (9) governs the gravitational behaviour of the radiating anisotropic star with nonzero shear and no acceleration. As equation (9) is highly nonlinear, it is difficult to solve without some simplifying assumption. This equation comprises two unknown functions $B(t, r)$ and $Y(t, r)$.

III. Exact Solutions

For convenience rewrite equation (9) in the form of the Riccati equation in the gravitational potential B as follows

$$\dot{B} = \left[\frac{\ddot{Y}}{Y'} + \frac{\dot{Y}^2}{2YY'} + \frac{1}{2YY'}\right]B^2 + \frac{\dot{Y}'}{Y'}B - \frac{Y'}{2Y} \tag{10}$$

Equation (10) was analyzed by Nogueira and Chan (2004) who obtained approximate solutions using numerical techniques. To describe properly the physical features of a radiating relativistic star exact solutions are necessary, preferably written in terms of elementary functions. An exact solution was found by Naidu et al (2006), which was singular at the stellar centre.

The Riccati equation (10), which has to be satisfied on the stellar boundary Σ, is highly nonlinear and difficult to solve. In 2008, Rajah and Maharaj obtained solutions by assuming that the gravitational potential $Y(t, r)$ is a separable function and specifying the temporal evolution of the model. Later in 2010, Thirukanesh and Maharaj, demonstrate that it is possible to find another exact solutions systematically without assuming separable forms for $Y(t, r)$ and not fixing the temporal evolution of the model a priori by introduce the transformation

$$B = ZY' \tag{11}$$

Then equation (10) becomes

$$\dot{Z} = \frac{1}{2Y}[FZ^2 - 1] \tag{12}$$

where set $F = 2Y\ddot{Y} + \dot{Y}^2 + 1$

Observe that equation (12) becomes a separable equation in Z and t, and therefore integrable, let F be a constant or a function of r only. In other word, (12) is integrable as long as F is independent of t.

For F = 1 S. Thirukanesh and S. D. Maharaj obtained the solution

$$ds^2 = -dt^2 + \frac{4}{9}\left[\frac{1+f(r)\exp\left[3[R_1(r)t+R_2(r)]^{1/3}/R_1\right]}{1-f(r)\exp\left[3[R_1(r)t+R_2(r)]^{1/3}/R_1\right]}\right]^2 \frac{[R_1't+R_2']^2}{[R_1t+R_2]^{2/3}}dr^2 + [R_1t+R_2]^{4/3}(d\theta^2+\sin^2\theta\, d\phi^2) \quad (13)$$

For $R_1 = R^{3/2}$, $R_2 = aR^{3/2}$ the line element (13) reduces to

$$ds^2 = -dt^2 + (t+a)^{4/3}\left\{R'^2\left[\frac{1+f(r)\exp\left[3[R_1(r)t+R_2(r)]^{1/3}/R_1\right]}{1-f(r)\exp\left[3[R_1(r)t+R_2(r)]^{1/3}/R_1\right]}\right]^2 dr^2 + R^2(d\theta^2+\sin^2\theta\, d\phi^2)\right\} \quad (14)$$

Which is the first category of the Rajah and Maharaja (2008) models for an anisotropic radiating star with shear. They match the line element (14) with the Naidu et al solution

$$ds^2 = -dt^2 + t^{4/3}\left\{R'^2\left[\frac{1+f(r)\exp\left[3t^{1/3}/r\right]}{1-f(r)\exp\left[3t^{1/3}/r\right]}\right]^2 dr^2 + R^2(d\theta^2+\sin^2\theta\, d\phi^2)\right\} \quad (15)$$

Again for F = 1 + $R^2_1(r)$ Thirukanesh & Maharaj found the line element

$$ds^2 = -dt^2 + \frac{1}{\sqrt{R_1^2+1}}\left[\frac{1+g(r)(R_1t+R_2)^{\sqrt{R_1^2+1}/R_1}}{1-g(r)(R_1t+R_2)^{\sqrt{R_1^2+1}/R_1}}\right]^2[R_1't+R_2']^2 + [R_1(r)t+R_2(r)]^2(d\theta^2+\sin^2\theta\, d\phi^2) \quad (16)$$

For $R_1 = R$, $R_2 = aR$ which is reduces to equation

$$ds^2 = -dt^2 + (t+a)^2\left\{\frac{R'^2}{R^2+1}\left[\frac{1+h(r)[t+a]^{\sqrt{R^2+1}/R}}{1-h(r)[t+a]^{\sqrt{R^2+1}/R}}\right]^2 dr^2 + R^2(d\theta^2+\sin^2\theta\, d\phi^2)\right\} \quad (17)$$

IV. New Solutions

Now we obtained solutions by assuming that the gravitational potential $Y(t,r)$ and $B(t,r)$ is a separable function and specifying the temporal evolution of the model Equation (9). Choose,

$Y=R(t)A(t)$ And $B=R(t)C(r)$

Now from equation (9) we have

$$2R(t)A(r).\ddot{R}(t)A(r) + \dot{R}^2(t)A^2(r) - \frac{R^2(t)A'^2(r)}{R^2(t)C^2(r)} + \frac{2R(t)A(r)\dot{R}(t)A'(r)}{R(t)C(r)}$$

$$-\frac{2\dot{R}(t)C(r)R(t)A(r)R(t)A'(r)}{R^2(t)C^2(r)} + 1 = 0 \;\Rightarrow 2R(t)\ddot{R}(t) + \dot{R}^2(t) = \frac{A'^2(r)}{C^2(r)A^2(r)} - \frac{1}{A^2(r)} = \gamma^2(say)$$

Notes

Case-1

For $\gamma = 0$, then we have

$$\therefore \quad R(t) = (a+bt)^{2/3} \text{ and } A(r) = \pm \int_0^r C(r')\,dr' + d$$

Therefore we have,

$$B = (a+bt)^{2/3}C(r) \quad and \quad Y = (a+bt)^{2/3}A(r)$$

Then we can calculate the metric (1) is

$$ds^2 = -dt^2 + (a+bt)^{4/3}\left[C^2(r)dr^2 + A^2(r)\left(d\theta^2 + \sin^2\theta\, d\Phi^2\right)\right]$$

Where $A(r) = \int_0^r C(r')dr' + d$

Here the expansion scalar Θ is non zero and if time(t) is increase this will be decrease. Again pressure is zero but density is nonzero i.e. the universe is gaseous.
Now we draw a graph (using MATLAB) for the scale factor

$$R(t) = (a+bt)^{2/3}$$

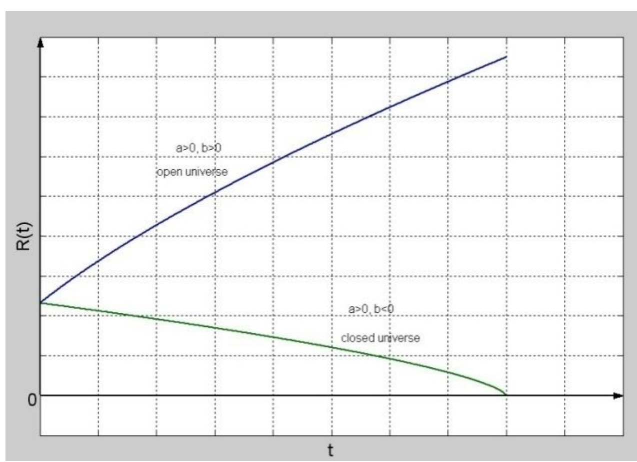

Figure 1 : This solution shows the universe is either open or closed.

Case-2

For $\gamma \neq 0$ we have

$$R(t) = \alpha + \gamma t \quad and \quad C(r) = \frac{A'(r)}{\sqrt{\gamma^2 A^2 + 1}}$$

$$\therefore B = (\alpha + \gamma t)C(r) \quad and \quad Y = (\alpha + \gamma t)A(r)$$

Therefore the metric (1) can be written as

$$ds^2 = -dt^2 + (\alpha + \gamma t)^2\left[C^2(r)dr^2 + A^2(r)\left(d\theta^2 + \sin^2\theta\, d\Phi^2\right)\right]$$

Where A(r) is the function of r only.

And $q = 0$; $P = 0$; $\rho = 0$; $\Theta = \frac{3\gamma}{(1+\gamma t)}$

which is not a collapse solution.

Consider, $A = r$. Then, $C(r) = \frac{1}{\sqrt{\gamma^2 r^2 + 1}}$

Therefore the metric (1) becomes

$$ds^2 = -dt^2 + (1+\gamma t)^2 \left[\frac{dr^2}{(\gamma^2 r^2 + 1)} + r^2\left(d\theta^2 + \sin^2\theta\, d\Phi^2\right)\right]$$

For

$R = (\propto + \gamma t)$, $A(r) = r$ and $C(r) = \frac{1}{\sqrt{1+\gamma^2 r^2}}$ gives

$B = (\propto + \gamma t)\frac{1}{\sqrt{1+\gamma^2 r^2}}$ and $Y = (\propto + \gamma t) r$

Therefore we have,

$$\therefore q = 0$$

$$\Theta = 3\frac{\gamma}{(\propto + \gamma t)}$$

$$P = \frac{1}{(\alpha + \gamma t)^2}\left[-7\gamma^2 + 3\gamma^2 r^2 - \frac{6}{(\alpha + \gamma t)^2} + 3\right]$$

$$\rho = \frac{4\gamma^2}{(\alpha + \gamma t)^2}$$

Here the expansion scalar Θ and density are non zero and both will be decrease if time is increase. Pressure is also non zero.

V. Conclusion

For the first case, the above solutions indicates that the space is very diluted as $P = 0$. So we may consider the solution for dust, the density and the expansion scalar Θ decreases when t is increases and tends to zero when $t \to \infty$ The density decreases rapidly than the expansion scalar Θ. This solution shows the universe is either open or closed. So the solutions are physically realistic.

For case-2 we see that for $3(\gamma^2 r^2 + 1) > 7\gamma^2 + \frac{6}{(\alpha+\gamma t)^2}$ both pressure p and density p decreases when r and t increases. Also it is clear that the density p decreases rapidly than the expansion scalar Θ. The form of the solution is Robertson Walker type solution for open universe. The pressure decreases slowly when both r and t increases but does not tends to zero when both r and t tends to infinity. The solution is physically realistic.

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