Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 11 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Some Integrals Pertaining Biorthogonal Polynomials and Certain Product of Special Functions

By Poonia, M.S

NIMS University, India

Abstract - The main object of this paper is to obtain some integrals and series relation pertaining to biorthogonal polynomials, Fox's H-function, the general class of polynomials and the H-function of several complex variables.

Keywords : Fox's H-function, the general class of polynomials and the H-function of complex variables.

GJSFR-F Classification : MSC 2010: 14K20

SOME INTEGRALS PERTAINING BIORTHOGONAL POLYNOMIALS AND CERTAIN PRODUCT OF SPECIAL FUNCTIONS

Strictly as per the compliance and regulations of :

Ref.
7. Prabhakar, T.R. and Tomar, R.C., Some integrals and series relations for biorthogonal polynomials suggested by the Legendre polynomials, Indian J. pure appl. Math., 11(7), (1980), 863-869.

Some Integrals Pertaining Biorthogonal Polynomials and Certain Product of Special Functions

Poonia, M.S

Abstract -The main object of this paper is to obtain some integrals and series relation pertaining to biorthogonal polynomials, Fox's H-function, the general class of polynomials and the H-function of several complex variables.

Keywords : Fox's H-function, the general class of polynomials and the H-function of complex variables.

I. Introduction

Integrals with Fox's H-function, the general class of polynomials and the H-function of complex variables were studied by many authors.
Prabhakar and Tomar [7] have given a biorthogonal pair of polynomial sets

$$U_n(x,k) \text{ and } V_n(x,k)$$

where

$$U_n(x,k) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\left(\frac{j+1}{k}\right)_n}{(1/k)_n} \left(\frac{1-x)}{2}\right)^j, \tag{1}$$

and

$$V_n(x,k) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(1+n)_{kj}}{(1)_{kj}} \left(\frac{1-x}{2}\right)^{kj} \tag{2}$$

The general multivariable polynomials defined by Srivastava ([12], p.185, eq. (7)) is represented in the following manner:

$$S^{p_1,\ldots,p_s}_{q_1,\ldots,q_s}[x_1,\ldots,x_s] = \sum_{k_1=0}^{[q_1/p_1]} \cdots \sum_{k_s=0}^{[q_s/p_s]} \frac{(-q_1)_{p_1k_1} \ldots (-q_s)_{p_sk_s}}{k_1! \ldots k_s!}$$

Author : NIMS University, Jaipur, Rajsthan, India.

$$. \, L[q_1,k_1;...;q_s k_s] x_1^{k_1} ... x_s^{k_s}$$

where $$q_m, p_m \, (m=1,...,s) \tag{3}$$

are non-zero arbitrary positive integers. The coefficients $L[q_1,k_1;...;q_s,k_s]$ being arbitrary constants real or complex.

Taking s = 1, the equation (3) reduces to the well known general class of polynomials $S_q^p[x]$ due to Srivastava ([13], p.158, eq. (1.1)].

II. Main Integrals

The following integrals concerning the biorthogonal polynomials with certain products of special functions have been derived in the paper.

a) First Integral

$$\int_0^{\pi/2} \cos 2u\theta (\sin\theta)^v \, U_n(1-2x \sin^{2h}\theta; k)$$

$$. \, H_{P_1,Q_1}^{M_1,N_1}\left[z(\sin\theta)^{2\rho_1'} \middle| \begin{matrix} (a_{P_1}, e_{P_1}) \\ (b_{Q_1}, f_{Q_1}) \end{matrix} \right] {}_{P_2}M_{Q_2}^{\alpha'}[y(\sin\theta)^{2\rho_2'}]$$

$$H(z_1(\sin\theta)^{2\sigma_1}, ..., z_r(\sin\theta)^{2\sigma_r} \; . \; S_{q_1,...,q_s}^{p_1,...,p_s}[x_1(\sin\theta)^{2\rho_1} ... x_s(\sin\theta)^{2\rho_s}] \, d\theta$$

$$= \sum_{\tau_1=1}^{M_1} \sum_{\tau_2=0}^{n} \sum_{\substack{s'=0 \\ s''=0}}^{\infty} \sum_{k_1=0}^{[q_1/p_1]} \cdots \sum_{k_s=0}^{[q_s/p_s]} (-1)^{\tau_2} \binom{n}{\tau_2} \frac{\left(\frac{\tau_2+1}{k}\right) x^{\tau_2}}{(1/k)_n}$$

$$. \, \frac{(-1)^{s'} z^{\eta_{s'}} \phi(\eta_{s'}) \Gamma\left(\frac{1}{2} \pm u\right)}{\tau_1! f_{\tau_1} \, s'! \, 2^{v+2h\tau_2+2\rho_1'\eta_{s'}} + 2\rho_2' s'' + \sum_{i=1}^{s} 2\rho_i}$$

$$. \, \frac{(a_1)_{s''} ... (a P_2)_{s''} \, y^{s''}}{(b_1)_{s''} ... (b_{Q_2})_{s''} \, \Gamma(\alpha' s''+1)} \frac{(-q_1)_{p_1 k_1} ... (-q_s) p_s k_s}{k_1! \qquad k_s!}$$

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue XI Version I Year 2012

$$. \, L[q_1,k_1;...;q_s,k_s]x_1^{k_1}...x_s^{k_s}$$

$$. \, H^{0,\lambda+1 \; :(u',v');...;(u^{(r)},v^{(r)})}_{A+1,C+2:(B',D');...;B^{(r)},D^{(r)}} \left[\begin{matrix} [-V-2h\tau_2-2\rho_1'\eta_{s'}-2\rho_2's''-\sum_{\xi=1}^{s} 2\rho_\xi k_\xi : 2\sigma_1,...,2\sigma_r], \\ [(c):\psi',...,\psi^{(r)}], \end{matrix} \right.$$

$$\left. \begin{matrix} [(a):\theta',...,\theta^{(r)}], \\ [-\frac{v}{2}\pm\frac{u}{2}-h\tau_2-\rho_1'\eta_{s'}-\rho_2's''-\sum_{\xi=1}^{s} 2\rho_\xi k_\xi:\sigma_1,...,\sigma_s] \end{matrix} \begin{matrix} [(b'):\phi'];...;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];...;[(d^{(r)}):\delta^{(r)}]; \end{matrix} z_1 2^{-\sigma_1},...,z_r 2^{-\sigma_r} \right], \quad (4)$$

$$\text{where } u=0,1,2,...; \mathrm{Re}\left(V+2\rho_1'+2\sum_{i=1}^{r}\sigma_i \frac{d^{(i)}}{\delta_j^{(i)}} \right) > 0, j'=1,...,M; j=1,...,u^{(i)}$$

$$T_i>0, |\arg(z_i)| < \frac{1}{2}T_i\pi, T'>0, |\arg z| \frac{1}{2}T'\pi,$$

$\rho_1'>0, \rho_2'>0, \rho_1,...,\rho_s>0, p_2<Q_2, |y|<1, p_m \, (m=1 \text{ to } s)$ are non-zero arbitrary positive integers and the coefficients L $[q_1,k_1;...; q_s, k_s]$ are arbitrary constants, real or complex.

b) Second Integral

$$\int_0^{\pi/2} \cos 2u\theta(\cos\theta)^v U_n(1-2x\cos^{2h}\theta;k)$$

$$. \, H^{M_1,N_1}_{P_1,Q_1}\left[z(\cos\theta)^{2\rho_1'} \middle| \begin{matrix} (a_{P_1},e_{P_1}) \\ (b_{Q_1},f_{Q_1}) \end{matrix} \right]$$

$$. \, H(z_1(\cos\theta)^{2\sigma_1},...,z_r(\cos\theta)^{2\sigma_r} \quad {}_{P_2}M^{\alpha'}_{Q_2}[y(\cos\theta)^{2\rho_2'}]$$

$$. \, S^{p_1,...,p_s}_{q_1,...,q_s}[x_1(\cos\theta)^{2\rho_1}...x_s(\cos\theta)^{2\rho_s}] \, d\theta$$

Notes

$$= \sum_{\tau_2=0}^{n} \sum_{\tau=1}^{M_1} \sum_{s',s''=0}^{\infty} \prod_{m=1}^{s} \left[\sum_{m}^{[q_m/p_m]} \frac{(-q_m)_{p_m k_m} x_m^{k_m}}{k_m !} \right] (-1)^{\tau_2} \binom{n}{\tau_2} \frac{\left(\frac{\tau_2+1}{k}\right)_n x^{\tau_2}}{(1/k)_n}$$

$$. \frac{(-1)^{s'} z^{\eta_{s'}} \phi(\eta_{s'})}{f_{\tau_1} s'!} \frac{\pi \Gamma(u+1)}{2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2's''+\sum_{i=1}^{s} 2\rho_i k_i+1}}$$

$$. H^{0,\lambda\ :(u',v');...;(u^{(r)},v^{(r)})}_{A,C+2:(B',D');...;(B^{(r)},D^{(r)})} \left[\begin{array}{l} [-------],[------------------]: \\ [(c):\psi',...,\psi^{(r)}], \left[-\frac{v}{2} \pm \frac{u}{2} - h\,\tau_2 - \rho_1'\eta_{s'} - \rho_2's'' - \sum_{i=1}^{s} \rho_i k_i : \sigma_1,...,\sigma_r \right]: \end{array} \right.$$

$$\left. \begin{array}{l} [(b'):\phi'];...;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];...;[(d^{(r)}):\delta^{(r)}]; \end{array} z_1 2^{-2\sigma_1},...,z_r 2^{-2\sigma_s} \right], \qquad (5)$$

where $u = 0,1,2,...,$ $\mathrm{Re}\left(V + 2\rho_1' \frac{b_{j'}}{f_j'} + 2\sum_{i=1}^{r} \sigma_i \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > 0, j' = 1,...,Q_2; j = 1,...,u^{(i)};$

$$T_i > 0, |\arg(z_i)| < \frac{1}{2} T_i \pi, T' > 0, |\arg(z)| < \frac{1}{2} T' \pi, \rho_1,...,\rho_s > 0, P_2 < Q_2, |y| < 1, h > 0,$$

$p_m (m = 1,...,s)$ are non-zero arbitrary positive integers and the coefficients $L[q_1, k_1,...,q_s, k_s]$ are arbitrary constants, real or complex.

Proof of (4)

Expressing the polynomials *Un* as given (1), Fox's H-function in series , the generalized multivariable polynomials by (3), M-series and the H-function of several complex variables in Mellin - Barnes contour integral by, changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and then evaluating the resulting integral with the help of the following result,

$$\int_0^{\pi/2} \cos 2u\theta (\sin\theta)^v \, d\theta = \frac{\Gamma(v+1)\Gamma(\frac{1}{2} \pm u)}{2^{v+1} \Gamma\left(\frac{v}{2} \pm \frac{u}{2} + 1\right)} \qquad (6)$$

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue XI Version I Year 2012

where u = 0,1,2,..., and Re(v) > 0.

Finally interpreting the result thus obtained with the help of (1.2.1), we arrive at the required result (2.3.1).

The integrals from (2.3.2) to (2.3.4) can also be obtained in the similar manner with the help of the appropriate integral (2.3.5) and the following result

Notes

$$\int_0^{\pi/2} \cos u\theta (\cos\theta)^v \, d\theta = \frac{\pi\Gamma(u+1)}{2^{v+1}\Gamma\left(\frac{v}{2}\pm\frac{u}{2}+1\right)} \quad (7)$$

where u = 0,1,2,..., and Re (v) > 0.

III. Special Cases

(i) Putting $\lambda = A, u^{(i)} = 1, v^{(i)} = B^{(i)}, D^{(i)} = D^{(i)} + 1, \forall i = 1,...,r$ in (4), we find

$$\int_0^{\pi/2} \cos 2u\theta (\sin\theta)^v U_n(1-2x\sin^{2h}\theta;k)$$

$$. H^{M_1,N_1}_{P_1,Q_1}\left[z(\sin\theta)^{2\rho_1'}\left|\begin{matrix}(a_{P_1},e_{P_1})\\(b_{Q_1},f_{Q_1})\end{matrix}\right.\right] S^{p_1,...,p_s}_{q_1,...,q_s}[x_1(\sin\theta)^{2\rho_1}...x_s(\sin\theta)^{2\rho_s}]$$

$$. {}_{P_2}M^{\alpha'}_{Q_2}[y(\sin\theta)^{2\rho_2'}]$$

$$. F^{A:B';...;B^{(r)}}_{B:D';...;D^{(r)}}\left(\begin{matrix}[1-(a):\theta',...,\theta^{(r)}]:[1-(b'):\phi'];...;[1-(b^{(r)}):\phi^{(r)}];\\ [1-(c):\psi',...,\psi^{(r)}]:[1-(d'):\delta'];...;[1-(d^{(r)}):\delta^{(r)}];\end{matrix} -z_1(\sin\theta)^{2\sigma_1},...,-z_r(\sin\theta)^{2\sigma_r}\right)d\theta$$

$$= \sum_{\tau_2=0}^{n}\sum_{\tau_1=1}^{M_1}\sum_{s',s''=0}^{\infty}\prod_{m=1}^{s}\left[\sum_{m}^{[q_m/p_m]}\frac{(-q_m)_{p_m k_m} x_m^{k_m}}{k_m!}\right](-1)^{\tau_2}\binom{n}{\tau_2}\frac{\left(\frac{\tau_2+1}{k}\right)_n}{(1/k)_n}x^{\tau_2}.$$

$$\frac{(-1)^{s'} z^{\eta_{s'}}\phi(\eta_{s'})}{f_{\tau_1} s'!} \frac{(a_1)_{s''}...(a_{P_2})_{s''} y^{s''}\Gamma(\frac{1}{2}\pm u)}{(b_1)_{s''}...(b_{Q_2})_{s''}\Gamma(\alpha's''+1)\, 2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2's''+\sum_{i=1}^{s}2\rho_i k_i}}$$

$$. F^{A+1:B';\dots;B^{(r)}}_{C+1:D';\dots;D^{(r)}}\left[\begin{array}{l}[-v-2h\tau_2-2\rho_1'\eta_{s'}-2\rho_2's''-\sum_{\xi=1}^{s}\rho_i k_i:2\sigma_1,\dots,2\sigma_r],\\ [(c):\psi',\dots,\psi^{(r)}],\end{array}\right.$$

$$\left.\begin{array}{l}[1-(a):\theta',\dots,\theta^{(r)}],\\ [-\frac{v}{2}\pm\frac{u}{2}-hk\tau_2-\rho_1'\eta_{s'}-\rho_2's''-\sum_{i=1}^{s}\rho_i k_i:\sigma_1,\dots,\sigma_r]\end{array}\begin{array}{l}[1-(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}];\\ [1-(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{array} z_1 2^{-2\sigma_1},\dots,z_r 2^{-2\sigma_r}\right],$$

(8)

provided that $u=0,1,2,\dots,\mathrm{Re}\left(v+2\rho_1'\frac{b_{j'}}{f_{j'}}\right)>0, j'=1,\dots,Q_2,|\arg(z)|<\frac{1}{2}T'\pi,T>0$ and the series on the right of (8) is absolutely convergent.

(ii) Taking r = 2, the result in (8) reduces to the following integral

$$\int_0^{\pi/2}\cos 2u\theta(\sin\theta)^v U_n(1-2x\sin^{2h}\theta;k)$$

$$. H^{M_1,N_1}_{P_1,Q_1}\left[z(\sin\theta)^{2\rho_1'}\left|\begin{array}{l}(a_{P_1},e_{P_1})\\(b_{Q_1},f_{Q_1})\end{array}\right.\right] S^{p_1,\dots,p_s}_{q_1,\dots,q_s}[x_1(\sin\theta)^{2\rho_1}\dots x_s(\sin\theta)^{2\rho_s}]$$

$$S^{A:B';B''}_{B:D';D''}\left(\begin{array}{l}[1-(a):\theta',\dots,\theta'']:[1-(b'):\phi'];\dots;[1-(b''):\phi''];\\ [1-(c):\psi',\dots,\psi'']:[1-(d'):\delta'];\dots;[1-(d''):\delta''];\end{array}-z_1(\sin\theta)^{2\sigma_1},-z_2(\sin\theta)^{2\sigma_2}\right)d\theta$$

$$. {}_{P_2}M^{\alpha'}_{Q_2}[y(\sin\theta)^{2\rho_1'}]\,d\theta$$

$$=\sum_{\tau_2=0}^{n}\sum_{\tau_1=1}^{M_1}\sum_{s',s''=0}^{\infty}\prod_{m=1}^{s}\left[\sum_{m}^{[q_m/p_m]}\frac{(-q_m)_{p_m k_m}\,x_m^{k_m}}{k_m!}\right](-1)^{\tau_2}\binom{n}{\tau_2}\frac{\left(\frac{\tau_2+1}{k}\right)_n}{(1/k)_n}x^{\tau_2}.$$

$$\frac{(-1)^{s'}z^{\eta_{s'}}\phi(\eta_{s'})}{f_{\tau_1}s'!}\;\frac{(a_1)_{s''}\dots(a_{P_2})_{s''}y^{s''}\Gamma(\frac{1}{2}\pm u)}{2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2's''+\sum_{i=1}^{s}2\rho_i k_i+1}}$$

Global Journal of Science Frontier Research (F) Volume XII Issue XI Version I Year 2012

Notes

$$S^{A+1:B';...;B''}_{C+1:D';...;D''}\left(\begin{matrix} [-v-2h\tau_2-2\rho_1^{'}\eta_{s'}-2\rho_2^{'}s''-\sum_{i=1}^{s}\rho_i k_i:2\sigma_1,2\sigma_2], \\ [1-(c):\psi',...,\psi^{(r)}], \end{matrix} \right.$$

$$\left. \begin{matrix} [1-(a):\theta'\theta''], & [1-(b'):\phi'];[(b''):\phi'']; \\ [-\frac{v}{2}\pm\frac{u}{2}-hk\tau_2-\rho_1^{'}\eta_{s'}-\rho_2^{'}s''-\sum_{i=1}^{s}\rho_i k_i:\sigma_1,\sigma_2],[1-(d'):\delta'];[(d''):\delta'']; \end{matrix} -z_1 2^{-2\sigma_1},-z_2 2^{-2\sigma_2} \right) \quad (9)$$

where $u=0,1,2,...,\operatorname{Re}\left(v+2\rho_1^{'}\frac{b_j}{f_j}\right)>0, j=1,...,M; |\arg(z)|<\frac{1}{2}T^{'}\pi, T^{'}>0$, and the series on the right of (9) converges absolutely.

(iii) Letting $\lambda = A = C = 0$ in (4), we get

$$\int_0^{\pi/2} \cos 2u\theta(\sin\theta)^v U_n(1-2x\sin^{2h}\theta;k)$$

$$.\, H^{M_1,N_1}_{P_1,Q_1}\left[z(\sin\theta)^{2\rho_1^{'}} \middle| \begin{matrix} (a_{P_1},e_{P_1}) \\ (b_{Q_1},f_{Q_1}) \end{matrix} \right] {}_{P_2}M^{\alpha'}_{Q_2}[y(\sin\theta)^{2\rho_2^{'}}]$$

$$.\, S^{p_1,...,p_s}_{q_1,...,q_s}(x_1(\sin\theta)^{2\rho_1}...x_s(\sin\theta)^{2\rho_s})$$

$$.\, \prod_{i=1}^{r}\left\{ H^{u^{(i)},v^{(i)}}_{B^{(i)},D^{(i)}}\left[z_1(\sin\theta)^{2\sigma_1} \middle| \begin{matrix} [(b^{(i)}):(\phi^{(i)})] \\ [(d^{(i)}):(\delta^{i})] \end{matrix} \right] \right\} d\theta$$

$$= \sum_{\tau_2=0}^{n}\sum_{\tau_1=1}^{M_1}\sum_{s',s''=0}^{\infty}\prod_{m=1}^{s}\left[\sum_{m}^{[q_m/p_m]} \frac{(-q_m)_{p_m k_m} x_m^{k_m}}{k_m!} \right] (-1)^{\tau_2}\binom{n}{\tau_2}\frac{\left(\frac{\tau_2+1}{k}\right)_n}{(1/k)_n} x^{\tau_2}.$$

$$\frac{(-1)^{s'} z^{\eta_{s'}} \phi(\eta_{s'})}{f_{\tau_1} s'!} \frac{\Gamma(\frac{1}{2}\pm u)}{2^{v+2h\tau_2+2\rho_1^{'}\eta_{s'}+2\rho_2^{'}s''+\sum_{i=1}^{s}2\rho_i k_i+1}} . \frac{(a_1)_{s''}...(a_{P_2})_{s''} y^{s''}}{(b_1)_{s''}...(b_{Q_2})_{s''}\gamma(\alpha's''+1)}$$

$$. . H^{0,1:(u',v');...;(u^{(r)},v^{(r)})}_{1,2:(B',D');...;(B^{(r)},D^{(r)})}\left[\begin{matrix}[-v-2h\tau_2-2\rho_1'\eta_{s'}-2\rho_2's''-\sum_{i=1}^{s}\rho_i k_i:2\sigma_1,...,2\sigma_r],\\ [(c):\psi',...,\psi^{(r)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[1-(a):\theta',...,\theta^{(r)}], & [(b'):\phi'];...;[(b^{(r)}):\phi^{(r)}];\\ [-\frac{v}{2}\pm\frac{u}{2}-hk\tau_2-\rho_1'\eta_{s'}-\rho_2's''-\sum_{i=1}^{s}\rho_i k_i:\sigma_1,\sigma_2...\sigma_r],[1-(d'):\delta'];...;[(d^{(r)}):\delta^{(r)}]; & \end{matrix} z_1 2^{-2\sigma_1},...,z_2 2^{-2\sigma_2}\right), \tag{10}$$

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue XI Version I Year 2012

valid under the same conditions as obtainable from result (4).

(iv) Setting r = 2 in equation (4), we find

$$\int_0^{\pi/2} \cos 2u\theta(\sin\theta)^v\, U_n(1-2x\sin^{2h}\theta;k)$$

$$. H^{M_1,N_1}_{P_1,Q_1}\left[z(\sin\theta)^{2\rho_1'}\left|\begin{matrix}(a_{P_1},e_{P_1})\\(b_{Q_1},f_{Q_1})\end{matrix}\right.\right]\ {}_{P_2}M^{\alpha'}_{Q_2}[y(\sin\theta)^{2\rho_2'}]$$

$$.S^{p_1,...,p_s}_{q_1,...,q_s}(x_1(\sin\theta)^{2\rho_1}...x_s(\sin\theta)^{2\rho_s})$$

$$. H^{0,\lambda:(u',v');(u'',v'')}_{A,C:(B',D');(B'',D'')}\left[\begin{matrix}[(a):\theta',\theta'']:[(b'):\phi'];[(b''):\phi''];\\ [(c):\psi',\psi'']:[(d'):\delta'];[(d''):\delta''];\end{matrix} z_1(\sin\theta)^{2\sigma_1},z_2(\sin\theta)^{2\sigma_2}\right]d\theta$$

$$=\sum_{\tau_2=0}^{n}\sum_{\tau_1=1}^{M_1}\sum_{s',s''=0}^{\infty}\prod_{m=1}^{s}\left[\sum_{m}^{[q_m/p_m]}\frac{(-q_m)_{p_m k_m}\,x_m^{k_m}}{k_m!}\right](-1)^{\tau_2}\binom{n}{\tau_2}\frac{\left(\frac{\tau_2+1}{k}\right)_n}{(1/k)_n}x^{\tau_2}. \qquad .$$

$$\frac{(-1)^{s'}z^{\eta_s'}\phi(\eta_s')}{f_{\tau_1}\,s'!}\ \frac{\Gamma(\frac{1}{2}\pm u)}{2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2's''+\sum_{i=1}^{s}2\rho_i k_i+1}}.\frac{(a_1)_{s''}...(a_{P_2})_{s''}\,y^{s''}}{(b_1)_{s''}...(b_{Q_2})_{s''}\gamma(\alpha's''+1)}$$

$$. . H^{0,\lambda+1:(u',v');u'',v''}_{A+1,C+2:(B',D');B'',D''}\left[\begin{array}{l}[-v-2h\tau_2-2\rho_1'\eta_{s'}-2\rho_2's''-\sum_{i=1}^{s}:\rho_i k_i:2\sigma_1,2\sigma_2],\\ [(c):\psi',\psi''],\end{array}\right.$$

Notes

$$\left.\begin{array}{ll}[(a):\theta',...,\theta''], & [(b'):\phi'];[(b''):\phi''] \\ [-\frac{v}{2}\pm\frac{u}{2}-hk\tau_2-\rho_1'\eta_{s'}-\rho_2's''-\sum_{i=1}^{s}\rho_i k_i:2\rho_\xi k_\xi:\sigma_1,\sigma_2],[(d'):\delta'];[(d''):\delta''];\end{array} -z_1 2^{-2\sigma_1},-z_2 2^{-2\sigma_2}\right], \quad (11)$$

where $u=0,1,2,...,\ \mathrm{Re}\left(v+2\rho_1'\frac{b_{j'}}{f_{j'}}+2\sigma_1\frac{d_j'}{\delta_j'}+2\sigma_2\frac{d_j''}{\delta_j''}\right)>0,\ j=1,...,M_1,\ j'=1,...,u',$

$j''=1,...,u'',T_1,T_2>0,|\arg(z_1)|<\frac{1}{2}T_1\pi,|\arg(z_2)|<\frac{1}{2}T_2\pi;|\arg(z)|<\frac{1}{2}\pi T',T>0,$

$\rho_m>0\ (m=1,...,s),\ P_2<Q_2,|y|<1,h>0,p_m\ (m=1,...,s)$ are positive coefficients and $L(q_1k_1,...,q_sk_s]$ are arbitrary constants, real or complex.

(v) Taking $\lambda=A,U^{(i)}=1,v^{(i)}=B^{(i)},D^{(i)}=D^{(i)}+1\,\forall i=1,...,r$ in (5), we get

$$\int_0^{\pi/2}\cos 2u\theta(\cos\theta)^v\,U_n(1-2x\cos^{2h}\theta;k)$$

$$.\,H^{M_1,N_1}_{P_1,Q_1}\left[z(\cos\theta)^{2\rho_1'}\left|\begin{array}{l}(a_{P_1},e_{P_1})\\(b_{Q_1},f_{Q_1})\end{array}\right.\right]\ {}_{P_2}\overset{\alpha'}{M}_{Q_2}[y(\cos\theta)^{2\rho_2'}]$$

$$.\,S^{p_1,...,p_s}_{q_1,...,q_s}(x_1(\cos\theta)^{2\rho_1}...x_s(\cos\theta)^{2\rho_s})$$

$$F^{A:B';...;B^{(r)}}_{B:D';...;D^{(r)}}\left(\begin{array}{l}[1-(a):\theta',...,\theta^{(r)}]:[1-(b'):\phi'];...;[1-(b^{(r)}):\phi^{(r)}];\\ [1-(c):\psi',...,\psi^{(r)}]:[1-(d'):\delta'];...;[1-(d^{(r)}):\delta^{(r)}];\end{array}-z_1(\cos\theta)^{2\sigma_1},...,-z_r(\cos\theta)^{2\sigma_r}\right)d\theta$$

$$= \sum_{\tau_2=0}^{n} \sum_{\tau_1=1}^{M_1} \sum_{s',s''=0}^{\infty} \prod_{m=1}^{s} \left[\sum_{m}^{[q_m/p_m]} \frac{(-q_m)_{p_m k_m} x_m^{k_m}}{k_m!} \right] (-1)^{\tau_2} \binom{n}{\tau_2} \frac{\left(\frac{\tau_2+1}{k}\right)_n}{(1/k)_n} x^{\tau_2} \quad .$$

$$\frac{(-1)^{s'} z^{\eta_{s'}} \phi(\eta_{s'})}{f_{\tau_1} s'!} \frac{(a_1)_{s''} \ldots (a_{P_2})_{s''} y^{s''}}{(b_1)_{s''} \ldots (b_{Q_2})_{s''} \Gamma(\alpha' s''+1)} \frac{\pi\Gamma(u+1)}{2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2' s''+\sum_{i=1}^{s} 2\rho_i k_i+1}}$$

$$. F^{A\ :B';\ldots;B^{(r)}}_{C+2:D';\ldots;D^{(r)}} \left[\begin{matrix} [1-(a):\theta',\ldots,\theta^{(r)}]:[-----------------------], \\ [1-(c):\psi',\ldots,\psi^{(r)}]:\left[-\frac{v}{2} \pm \frac{u}{2} - hk\tau_2 - \rho_1' \eta_{s'} - \rho_2' s'' - \sum_{i=1}^{s} \rho_i k_i \right], \end{matrix} \right.$$

$$\left. \begin{matrix} [1-(b'):\phi'];\ldots;[1-(b^{(r)}):\phi^{(r)}]; \\ [1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}]; \end{matrix} -z_1 e^{-2\sigma_1},\ldots,-z_r e^{-2\sigma_r} \right], \qquad (12)$$

Provided $u = 0,1,2,\ldots, \mathrm{Re}\left(v + 2\rho_1' \frac{b_j}{f_j}\right) > 0, j = 1,\ldots,M_1, |\arg(z)| < \frac{1}{2} T'\pi, T' > 0,$

$|y| < 1, P_2 < Q_2$ and the series on the right of (12) converges absolutely.

(vi) Putting r = 2 in (12), we obtain

$$\int_0^{\pi/2} \cos 2u\theta (\cos\theta)^v U_n(1 - 2x\cos^{2h}\theta; k)$$

$$. H^{M_1,N_1}_{P_1,Q_1}\left[z(\cos\theta)^{2\rho_1'} \middle| \begin{matrix} (a_{P_1}, e_{P_1}) \\ (b_{Q_1}, f_{Q_1}) \end{matrix} \right] {}_{P_2}M^{\alpha'}_{Q_2} [y(\cos\theta)^{2\rho_2'}]$$

$$. S^{p_1,\ldots,p_s}_{q_1,\ldots,q_s}(x_1(\cos\theta)^{2\rho_1} \ldots x_s(\cos\theta)^{2\rho_s})$$

Notes

Notes

$$S^{A:B';B''}_{B:D';D''}\left(\begin{matrix}[1-(a):\theta',\theta'']:[1-(b'):\phi'];[1-(b''):\phi''];\\ [1-(c):\psi',\psi'']:[1-(d'):\delta'];[1-(d''):\delta''];\end{matrix} -z_1(\cos\theta)^{2\sigma_1}, -z_2(\cos\theta)^{2\sigma_2}\right)d\theta$$

$$=\sum_{\tau_2=0}^{n}\sum_{\tau_1=1}^{M_1}\sum_{s',s''=0}^{\infty}(-1)^{\tau_2}\binom{n}{\tau_2}\frac{\left(\frac{\tau_2+1}{k}\right)_n}{(1/k)_n}x^{\tau_2}$$

$$.\frac{(-1)^{s'}z^{\eta'_s}\phi(\eta'_s)}{f_{\tau_1}s'!}\frac{\pi\Gamma(u+1)}{2^{v+2h\tau_2+2\rho'_1\eta_{s'}+2\rho'_2 s''+\sum_{i=1}^{s}2\rho_i k_i+1}}\frac{(a_1)_{s''}\ldots(a_{P_2})_{s''}\,y^{s''}}{(b_1)_{s''}\ldots(b_{Q_2})_{s''}\,\Gamma(\alpha's''+1)}$$

$$.\prod_{m=1}^{s}\left[\sum_{m}^{[q_m/p_m]}\frac{(-q_m)_{p_m k_m}\,x_m^{k_m}}{k_m!}\right]$$

$$.S^{A\ :B';B''}_{C+2:D';D''}\left[\begin{matrix}[1-(a):\theta',\ldots,\theta'']:[-----------------------------],\\ [1-(c):\psi',\ldots,\psi'']:\left[-\frac{v}{2}\pm\frac{u}{2}-hk\tau_2-\rho'_1\eta_{s'}-\rho'_2 s''-\sum_{i=1}^{s}\rho_i k_i:\sigma_1,\sigma_2\right],\end{matrix}\right.$$

$$\left.\begin{matrix}[1-(b'):\phi'];[1-(b''):\phi''];\\ [1-(d'):\delta'];[1-(d''):\delta''];\end{matrix} -z_1 2^{-2\sigma_1}, -z_2 2^{-2\sigma_2}\right], \tag{13}$$

provided that $u=0,1,2,\ldots,\mathrm{Re}\left(v+2\rho'_1\frac{b_j}{f_j}\right)>0, j=1,\ldots,M_1, |\arg(z)|<\frac{1}{2}T'\pi, T'>0,$

and the series on the right of (13) is absolutely convergent.

(vii) Letting $\lambda = A = C = 0$ in (5), we have

$$\int_0^{\pi/2}\cos 2u\theta(\cos\theta)^{v}\,U_n(1-2x\cos^{2h}\theta;k)$$

$$. H^{M_1,N_1}_{P_1,Q_1}\left[z(\cos\theta)^{2\rho_1'} \middle| \begin{matrix} (a_{P_1}, e_{P_1}) \\ (b_{Q_1}, f_{Q_1}) \end{matrix} \right] {}_{P_2}\overset{\alpha'}{M}_{Q_2}[y(\cos\theta)^{2\rho_2'}]$$

$$. S^{p_1,\dots,p_s}_{q_1,\dots,q_s}(x_1(\cos\theta)^{2\rho_1}\dots x_s(\cos\theta)^{2\rho_s})$$

$$\prod_{i=1}^{r}\left\{ H^{u^{(i)},v^{(i)}}_{B^{(i)},D^{(i)}}\left[z_1(\cos\theta)^{2\sigma_1} \middle| \begin{matrix} [(b^{(i)}):(\phi^{(i)})] \\ [(d^{(i)}):(\delta^{i)})] \end{matrix} \right]\right\} d\theta$$

$$= \sum_{\tau_2=0}^{n}\sum_{\tau_1=1}^{M_1}\sum_{s',s''=0}^{\infty}\prod_{m=1}^{s}\left[\sum_{m}^{[q_m/p_m]} \frac{(-q_m)_{p_m k_m}\, x_m^{k_m}}{k_m!}\right](-1)^{\tau_2}\frac{\left(\frac{\tau_2+1}{k}\right)_n x^{\tau_2}\binom{n}{\tau_2}}{(1/k)_n}.$$

$$\frac{(-1)^{s'} z^{\eta_{s'}}\phi(\eta_{s'})}{f_{\tau_1}\, s'!}\;\frac{\Gamma(u+1)}{2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2's''+\sum_{i=1}^{s}2\rho_i k_i+1}}\;\frac{(a_1)_{s''}\dots(a_{P_2})_{s''}\, y^{s''}}{(b_1)_{s''}\dots(b_{Q_2})_{s''}\,\Gamma(\alpha's''+1)}$$

$$H^{0,0:(u',v');\dots;(u^{(r)},v^{(r)})}_{0,2\,:(B',D');\dots;(B^{(r)},D^{(r)})}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]:[\text{------------------------------}]: \\ [(c):\psi',\dots,\psi^{(r)}],\left[-\frac{v}{2}\pm\frac{u}{2}-h\,\tau_2-\rho_1'\eta_{s'}-\rho_2's''-\sum_{i=1}^{s}\rho_i k_i:\sigma_1,\dots,\sigma_r\right],\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; z_1 z^{-2\sigma_1},\dots,z_r z^{-2\sigma_r}\right], \qquad (14)$$

valid under the same conditions as stated for (5).

(viii) Putting r = 2 in (5), we get

$$\int_0^{\pi/2}\cos 2u\theta(\cos\theta)^{v}\,U_n(1-2x\cos^{2h}\theta;k)$$

Notes

Notes

$$. H^{M_1,N_1}_{P_1,Q_1}\left[z(\cos\theta)^{2\rho_1'}\middle|\begin{matrix}(a_{P_1},e_{P_1})\\(b_{Q_1},f_{Q_1})\end{matrix}\right] {}_{P_2}M^{\alpha'}_{Q_2}[y(\cos\theta)^{2\rho_2'}]$$

$$.S^{p_1,\dots,p_s}_{q_1,\dots,q_s}(x_1(\cos\theta)^{2\rho_1}\dots x_s(\cos\theta)^{2\rho_s})$$

$$. H^{0,\lambda:(u',v');(u'',v'')}_{A,C:(B',D');(B'',D'')}\left[\begin{matrix}[(a):\theta',\theta'']:[(b'):\phi'];[(b''):\phi''];\\ [(c):\psi',\psi'']:[(d'):\delta'];[(d''):\delta''];\end{matrix}\; z_1(\cos\theta)^{2\sigma_1}, z_2(\cos\theta)^{2\sigma_2})\right]d\theta$$

$$= \sum_{\tau_2=0}^{n}\sum_{\tau_1=1}^{M_1}\sum_{s',s''=0}^{\infty}(-1)^{\tau_2}\binom{n}{\tau_2}\frac{\left(\frac{\tau_2+1}{k}\right)_n x^{\tau_2}}{(1/k)_n}\frac{(-1)^{s'} z^{\eta_{s'}}\phi(\eta_{s'})}{f_{\tau_1} s'!}$$

$$\prod_{m=1}^{s}\left[\sum_{m}^{[q_m/p_m]}\frac{(-q_m)_{p_m k_m} x_m^{k_m}}{k_m!}\right]\frac{\pi\Gamma(u+1)}{2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2's''+\sum_{i=1}^{s}2\rho_i k_i+1}}$$

$$.\frac{(a_1)_{s''}\dots(a_{P_2})_{s''}\,y^{s''}}{(b_1)_{s''}\dots(b_{Q_2})_{s''}\,\Gamma(\alpha's''+1)}$$

$$H^{0,0:(u',v');\dots;(u'',v'')}_{0,2\,:(B',D');\dots;(B'',D'')}\left[\begin{matrix}[(a):\theta',\dots,\theta'']:[\text{------------------------------}]:\\ [(c):\psi',\dots,\psi''],\left[-\frac{v}{2}\pm\frac{u}{2}-h\,\tau_2-\rho_1'\eta_{s'}-\rho_2's''-\sum_{i=1}^{s}\rho_i k_i:\sigma_1,\dots,\sigma_2\right],\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\dots;[(b''):\phi''];\\ [(d'):\delta'];\dots;[(d''):\delta''];\end{matrix}\; z_1 2^{-2\sigma_1}, z_2 2^{-2\sigma_2}\right], \quad (15)$$

where $u=0,1,2,\dots,\ \mathrm{Re}\left(v+2\rho_1'\frac{b_{j'}}{f_{j'}}+2\sigma_1\frac{d_j'}{\delta_j'}+2\sigma_2\frac{d_{j''}''}{\delta_{j''}''}\right)>0,\ j=1,\dots,M_1;\ j'=1,\dots,u';$

$j''=1,...,u'';|\arg(z)|<\frac{1}{2}T'\pi,T',T_1,T_2>0,|y|<1,P_2<Q_2,h>0,\rho_1',\rho_2',\rho_m(m=1\text{ to }s),$

$(i=1,...,r)>0.$

Notes

References Références Referencias

1. **Bhatt, K.N.,** Integrals involving certain products of biorthogonal polynomials and the H-function, Vij. Anu. Pat., 30(2), (1987), 127-134.
2. **Braaksma, B.L.J.,** Asymptotic expansion and analytic continuations for a class of a Barnes integrals, Comp. Math. 15 (1963), 339-341.
3. **Chaurasia, V.B.L.,** Certain integral properties pertaining to special functions, Scientia, Series A, Math. Sc., 19 (2010), 1-6.
4. **Chaurasia, V.B.L. and Agnihotri, Mukesh,** The product of generalized polynomials sets pertaining to a class of convolution integral equations, Demo. Math. XLII (3), (2009), 505-512.
5. **Fox, C.,** The G and H-functions as symmetrical Fourier kernels, Trans. Amer. Math. Soc., 98 (1961), 395-429.
6. **Mittal, P.K. and Gupta, K.C.,** An integral involving generalized function of two variables, Proc. Ind. Acad. Sco. 75A (1972), 117-123.
7. **Prabhakar, T.R. and Tomar, R.C.,** Some integrals and series relations for biorthogonal polynomials suggested by the Legendre polynomials, Indian J. pure appl. Math., 11(7), (1980), 863-869.
8. **Srivastava, H.M. and Daoust, M.C.,** Certain generalized Neuman expansions associated with the Kampé de Fériet function, Ned. Akad. Wet. Proc. Ser. A 72 = Ind. Maths., 31 (1969), 449-457.
9. **Srivastavaa, H.M. and Daoust, M.C.,** Some generating functions for the Jacobi polynomials, Comment. Math. Univ. St. Paul. 20 (1971), 15-21.
10. **Srivastava, H.M. and Panda, R.,** Expansion theorem for the H-function of several complex variables, J. Reine Angew. Math. 288 (1976), 129-145.
11. **Srivastava, H.M. and Panda, R.,** Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine Angew. Math. 283/284, (1971), 265-274.
12. **Srivastava, H.M.,** A multilinear generating function for the Konhauser sets of biorthogonal polynomials suggested by Laguerre polynomials, Pacific J. Math., 117 (1985), 183-191.
13. A contour integral involving Fox's function, Indian J. Math., 14 (1972), 1-6.
14. **Skibinski, P.,** Some expansion theorems for the H-function, Ann. Polon. Math., 23 (1970), 125-138.
15. **Sharma, M.,** Fractional integration and fractional differentiation of the M-Series, Fractional Calculus Appl. Anal., 11(2) (2008), 188-191.