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A New Self-Adjusting Numerical Integrator for the Numerical Solutions of Ordinary Differential Equations

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A NEW SELF-ADJUSTING NUMERICAL INTEGRATOR FOR THE NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

Strictly as per the compliance and regulations of :

A New Self-Adjusting Numerical Integrator for the Numerical Solutions of Ordinary Differential Equations

O. O.A. Enoch [α] & A. A. Olatunji [σ]

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I. Introduction

Authors like Lambert and Shaw (1965) [1, 15] considered a class of formulae for the numerical solution of

$$y' = f(x, y); y(x) = y \tag{1}$$

in which the underlying interpolant was a rational function, which was in contrast with the classical formulae. The numerical methods that resulted from the works of the above mentioned authors afforded an improved numerical solution which was closed to a singularity of the theoretical solution of (1), since they locally represented the numerical solution of (1) by an interpolant which can possess a simple pole.

II. Determination of the Undetermined Coefficients

The Interpolant considered in this work is presented as:

$$F(x_n) = \sum_{J=0}^{L} a_j x_n^j + b\,|A + x_n|^N, N \notin \{0,1,2,\ldots,L\} \tag{2}$$

where a_n, b, A and N are real, L is a positive integers.
Assuming that

$$F(x_n) = y_n \text{ and } F(x_{n+1}) = y_{n+1}; x_{n+1} = x_n + h \text{ for which } x_n = a + nh$$

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Ref.
1. C. W. Gear.(1971) Numerical Initial Value Problems in Ordinary Differential Equations. Prentice Hall, Englewood Cliffs, NJ.

$$F(x_{n+1}) - F(x_n) = y_{n+1} - y_n \quad (3)$$

Let $f^{(i)}$ denotes the i^{th} total derivative of f(x,y) with respect to x such that

$$F^{(1)}(x_n) = f(x_n, y_n) = f_n \text{ and} \quad (4)$$

$$F^{(2)}(x_n) = f^{(1)}(x_n, y_n) = f_n^{(1)} \quad (5)$$

$$F^{(m)}(x_n) = f^{(m-1)}(x_n, y_n) = f_n^{(m-1)} \quad (6)$$

It follows thus;

$$y_{n+1} - y_n = \sum_{j=0}^{L} a_j \left[x_{n+1}^j - x_n^j \right] + b\left[(A + x_{n+1})^N - (A + x_n)^N \right] \quad (7)$$

The above expressions hold provided all the derivatives concerned exist.
Elimination of the undetermined coefficients from (7) then gives the required algorithm:

When L = 1 (i.e. the polynomial $P_j(x)$ is linear)

$$P_j(x) = \sum_{j=0}^{1} a_j x^j = a_0 x_0 + a_1 x_1 = a_0 + a_1 x \quad (8)$$

$$\mathrm{F}(x_n) = a_0 + a_1 x_n + b(A + x_n)^N \quad (9)$$

$$\mathrm{F}(x_{n+1}) = a_0 + a_1 x_{n+1} + b(A + x_{n+1})^N \quad (10)$$

$$\text{Let } y_n = F(x_n) \text{ and } y_{n+1} = F(x_{n+1}) \quad (11)$$

$$\Rightarrow \mathrm{F}(x_{n+1}) - F(x_n) = y_{n+1} - y_n \quad (12)$$

$$y_{n+1} - y_n = a_1(x_{n+1} - x_n) + b\left[(A + x_{n+1})^N - (A + x_n)^N \right] \quad (13)$$

$$y_{n+1} - y_n = a_1 h + b\left[(A + x_n + h)^N - (A + x_n)^N \right] \quad (14)$$

Differentiate $\mathrm{F}(x_n) = a_0 + a_1 x_n + b(A + x_n)^N$ to eliminate the undetermined coefficients

$$a_1 = f_n - \left[Nb(A + x_n)^{N-1} \right] \quad (15)$$

$$b = \frac{f_n^{(1)}}{N(N-1)(A + x_n)^{N-2}} \quad (16)$$

Notes

Therefore

$$y_{n+1}-y_n=hf_n+\frac{(A+x_n)^2}{N(N-1)}\left[\left(1+\frac{h}{A+x_n}\right)^N-1-\frac{Nh}{A+x_n}\right]f_n^{(1)}$$

Notes

Let us introduce $\frac{N(A+x_n)}{N(A+x_n)}$ to the third term in the bracket to have;

$$\Rightarrow\quad hf_n+\left[\frac{(A+x_n)^2}{N(N-1)}\left(1+\frac{h}{A+x_n}\right)^N-\frac{(A+x_n)^2}{N(N-1)}-\frac{N(A+x_n)(A+x_n)h}{N(N-1)(A+x_n)}\right]f_n^{(1)} \tag{17}$$

$$\Rightarrow\quad y_{n+1}=y_n+hf_n+\frac{(A+x_n)^2 f_n^{(1)}}{N(N-1)}\left[\left(1+\frac{h}{A+x_n}\right)^N-1-\frac{Nh}{A+x_n}\right] \tag{18}$$

When L=2 (i.e. the polynomial $P_j(x)$ is a quadratic):

$$P_j(x)=\sum_{j=0}^{2}a_jx^j=a_0x^0+a_1x^1+a_2x^2=a_0+a_1x+a_2x^2 \tag{19}$$

$$\mathrm{F}(x_n)=a_0+a_1x_n+a_2x_n^2+b(A+x_n)^N \tag{20}$$

By applying the above assumptions, one obtains the undetermined coefficients as;

$$b=\frac{(A+x_n)^3 f_n^{(2)}}{N(N-1)(N-2)(A+x_n)^N} \qquad a_2=\frac{1}{2}\left[f_n^{(1)}-\frac{(A+x_n)}{(N-2)}f_n^{(2)}\right] \tag{22}$$

$$a_1=f_n-\left\{x_nf_n^{(1)}-x_n\frac{(A+x_n)f_n^{(2)}}{(N-2)}+\frac{(A+x_n)^3f_n^{(2)}}{(N-1)(N-1)}\right\} \tag{23}$$

Thus

$$y_{n+1}-y_n=hf_n+\frac{h^2}{2}f_n^{(1)}+\frac{(A+x_n)^3f_n^{(2)}}{N(N-1)(N-2)}\left[\left(1+\frac{h}{A+x_n}\right)^N-1-\left(Nh+\frac{N(N-1)}{2}\right)\left(\frac{h}{A+x_n}\right)^2\right]$$

Let us introduce $\frac{N(A+x_n)}{N(A+x_n)}$ to the third term in the bracket to have;

$$hf_n+\frac{h^2}{2}f_n^{(1)}+\frac{(A+x_n)^3f_n^{(2)}}{N(N-1)(N-2)}\left[\left(1+\frac{h}{A+x_n}\right)^N-1-\left(Nh+\frac{N(N-1)}{2}\left(\frac{h}{A+x_n}\right)\right)\right] \tag{24}$$

To generalize this integrator, we let

$$\mathrm{F(x)} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + b(A+x)^N \quad (25)$$

$$F(x_n) = a_0 + a_1 x_n^1 + a_2 x_n^2 + a_3 x_n^3 + \cdots + a_n x_n^n + b(A+x_n)^N \quad (26)$$

Let $(\mathrm{A}+x_n) = \emptyset_n and (\mathrm{A}+x_{n+1}) = \phi_{n+1}$ (27)

$$F(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + \cdots + a_n x_n^N + b\emptyset_n^N \quad (28)$$

And $F(x_{n+1}) = a_0 + a_1 x_n^1 + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + \cdots + a_n x_{n+1}^n + b\emptyset_{n+1}^N$ (27)

It follows (3) that

$$y_n = a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + \cdots + a_n x_n^n + b[\phi(n)]^N \quad (29)$$

And so

$$y_{n+1} = a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + \cdots + a_n x_{n+1}^n + b[\phi(x_{n+1})]^N \quad (30)$$

Subtraction equation (29) from (30) we have

$$y_{n+1} = y_n + a_1(x_{n+1} - x_n) + a_2(x_{n+1}^2 - x_n{}^2) + \ldots + a_n(x_{n+1}^i - x_n^i) + b[\phi(x_{n+1})]^N - [\phi(x_n)]^N \quad (31)$$

Since the mesh size is defined as $x_t = a + th$ and Continuing untox_t^n;

$x_t^n = (a+th)^n usingbinomialexpansion$

We obtain

$$x_{t+1}^n - x_t{}^n = na^{n-1}h + n(n-1)a^{n-2}th^2 + \frac{n(n-1)a^{n-2}h^2}{2!} + \frac{3n(n-1)(n-2)a^{n-3}t^3h^2}{3!} + \frac{3n(n-1)(n-2)a^{n-3}th^3}{3!}$$

$$+ \frac{(n-1)(n-2)a^{n-3}h^3}{3!} \quad (36)$$

Thus, one obtains:

$$y_{t+1} - y_t = a_0 + a_1 h + a_2 \left(2ah + h^2(1+2t)\right)$$

$$+ a_3(3a^2h + 3a^2h(1+2t) + h^3(3t^3 + 3t + 1)) + \cdots + a_n(x_{t+1}^n - x_t{}^n) \quad (37)$$

Also with the generalized interpolant;

$$F(x_t) = a_0 + a_1 x_t + a_2 x_t^2 + a_3 x_t^3 + \cdots + a_n x_t^n + b[\phi(x_t)]^N \quad (38)$$

Notes

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This can be written as;

$$F(x) = \sum_{i=0}^{n} a_i\, x_t^i + b[\phi(x_t)]^N \tag{39}$$

By differentiating 6.1.29 nth times, one obtains;

$$F^{1}(x_t) = a_1 + 2a_2x_{2t}^2 + 3a_3x_t^2 + \cdots + n\, a_n x_t^{n-1} + bN[\phi(x_t)]^{N-1} = f_t \tag{40}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$F^{(n-1)} = (n-1)!\, a_{n-1} + n!\; a_n\, x_t + \cdots + n(n-1)(n-2) \ldots (n-[(n-1)-1])a_n x_t^{n-(n-1)}$$
$$+bN\,(N-1)(N-2)\; \ldots\; (N-[(n-1)-1])\; \phi\,(x_t)^{N-(n-1)} = f_t^{(n-1)-1} \tag{41}$$

$$F^{n} = n!\, a_n + bN\,(N-1)(N-2)\; \ldots\; (N-[(n-1)-1])\; \phi\,(x_t)^{N-n} = f_t^{(n-1)-1} \tag{42}$$

$$F^{n} = n(n+1)(n-2) \ldots (n-[(n-1)])\quad a_n \quad + \cdots$$
$$+bN\,(N-1)(N-2)\; \ldots\; (N-n)\; \phi\,(x_t)^{N-(n+1)} = f_t^{(n-1)-1} \tag{43}$$

$$f_t^{(n)} = bN\,(N-1)(N-2)(N-3)\; \ldots\; (N-n)[A+x_t]^{N-(n+1)} \tag{44}$$

Thus, the undetermined coefficients are obtained asfollows:

$$b = \frac{[A+x_t]^{n+1} f_t^{(n)}}{N\,(N-1)(N-2)(N-3)\; \ldots\; (N-n)[A+x_t]^N} \tag{45}$$

$$a_n = \frac{1}{n!}\left[f_t^{(n-1)} - \frac{[A+x_t]}{(N-n)} f_t^{(n)}\right] \tag{46}$$

$$a_{n-1} = \frac{1}{(n-1)!}\left(f_t^{(n-2)} - x_t f_t^{(n-1)} - \left[\frac{(N-n+2)^{(A+x_t)^2}}{(N-n)(N-(n-1)} - \frac{x_t(A+x_t)}{(N-n)}\right] f_t^{(n)}\right) \tag{47}$$

$$a_{n-2} = \frac{1}{(n-2)!}\left[\begin{array}{c} f_t^{(n-3)} - x_t\, f_t^{(n-2)} + x_t^2 f_t^{(n-1)} \\ +f_t^{(n)}\left\{\frac{x_t\,(A+x_t)^2}{(N-(n-1))(N-n)} - \frac{x_t^{2}(A+x_t)}{(N-n)}\frac{(A+x_t)^3}{(N-(n-2))\,(N-(n-1))(N-n)}\right\}\end{array}\right] \tag{48}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_5 = \frac{1}{5!}\left[\begin{array}{c} f_t^{(4)} - 720\, a_6 x_t - \cdots - n(n-1) \ldots (n-4)a_n\, x^{n-5} \\ -bN(N-1) \ldots (n-4)[A+x_t]^{N-5}\end{array}\right] \tag{49}$$

$$a_4 = \frac{1}{4!}\begin{bmatrix} f_t^{(3)} - 120\, a_5 x_t - \cdots - n(n-1)\ldots(n-4)a_n\, x^{n-4} \\ -bN(N-1)\ldots(N-3)[A+x_t]^{N-4} \end{bmatrix} \quad (50)$$

$$a_3 = \frac{1}{3!}\begin{bmatrix} f_t^{(2)} - 24\, a_4 x_t - \cdots - n\,(n-1)(n-2)a_n\, x^{n-3} \\ -bN\,(N-1)(N-2)[A+x_t]^{N-3} \end{bmatrix} \quad (51)$$

$$a_2 = \frac{1}{2!}\left[f_t^{(1)} - 6\, a_3 x_t - \cdots - n\,(n-1)a_n\, x_t^{\,n-2} - bN\,(N-1)[A+x_t]^{N-2}\right] \quad (52)$$

$$a_1 = [f_t - 2a_2 x_t - 3a_3\, x_t^{\,2} - \cdots - n\, a_n\, x_t^{\,n-1} - bN\,[A+x_t]^{N-1}] \quad (53)$$

Notes

In all, by substituting the undetermined coefficients appropriately, one obtains;

$$y_{n+1} - y_n = \sum_{K=1}^{L} \frac{h^k}{k!} f_n^{(k-1)} + \frac{(A+x_n)^{L-1}}{\alpha_L^N} f_n^L \left[\left(1+\frac{h}{A+x_n}\right)^N - 1 - \sum_{K=1}^{L=N} \frac{K-1}{K!}\left(\frac{h}{A+X_n}\right)\right]$$

Prove of Convergence for the Scheme

According to Henrici (1962): we define any algorithm for solving a differentialequation in which the approximation y_{t+1} to the solution at the x_{t+1} can be calculated if only x_t, y_t *and* h are known as a ONE-STEP METHOD. We proceed to establish that our numerical algorithm is one step methods. From (2), the numerical integrator generated is given by (). If we expand $\left(\left[1+\frac{h}{A+x_n}\right]\right)^N$ by binomial expansion and taking N as a real, we shall have

$$= h\left\{\frac{1}{h} + \frac{N}{A+x_n} + \sum_{i=1}^{\infty} \frac{N!}{(N-(i+1))!}\left(\frac{h^i}{(i+1)!\,(A+x_n)^{(i+1)}}\right)\right\}$$

This implies

$$y_{n+1} = y_n + \mathrm{h}\left(\left(\sum_{K=1}^{L} \frac{h^{K-1}}{K!} f_n^{(k-1)}\right) + \frac{(A+x_n)}{\alpha_L^N} f_n^{(L)} \left\{\frac{N}{A+x_n} + \beta - \sum_{K=1}^{L} \Psi\left(\frac{h^{K-1}}{(A+x)^K}\right)\right\}\right) \quad (56)$$

Thus

$$y_{n+1} = y_n + h\left\{\sum_{k=1}^{L}\left(G f_n^{(k-1)} + \gamma f_n^{(L)}\right)\right\} \quad (57)$$

$$y_{n+1} = y_n + h\theta(x_n, y_n; h) \quad (58)$$

$$\phi(x_n, y_n; h) = \sum_{k=1}^{L}\left(G f_{(x_n,y_n)}^{(k-1)} + \gamma f_{(x_n,y_n)}^{(L)}\right) \quad (59)$$

where

$$G = \frac{h^{K-1}}{K!}\ ;\ \gamma = \frac{(A+x_n)^1}{\alpha_L^N}\left\{\left(\frac{N!}{(A+x_n)^1}\right) + \beta - \sum_{K=1}^{L}\Psi\left(\frac{h^{K-1}}{(A+x)^K}\right)\right\}\ ;\ \Psi = \frac{\alpha_{K-1}^N}{K!}\ ;\beta = \sum_{i=1}^{\infty}\frac{N!}{(N-(i+1))!}\left(\frac{h^i}{(i+1)!(A+x_n)^{(i+1)}}\right)$$

where $\vartheta(x_t, y_t; h)$ is called the increment function.

Derivationofthelocation andnatureofthepointofsingularity

To derive A(n) and N(n), we make use of the Taylor series expansion of (55). This gives the following expression for the truncation error:

$$T.E = y_{n+1} - y(x_{n+1}) \tag{63}$$

$$T.E = \sum_{q=1}^{\infty}[-f_n^{(L+q)} + \frac{\alpha_{q-1}^{N-L-1}}{(A+x_n)^q} f_n^{(L)}]\frac{h^{L+q+1}}{(L+q+1)!} \tag{64}$$

$$T_q = -f_n^{(L+q)} + \frac{\alpha_{q-1}^{N-L-1}}{(A+x_n)^q} f_n^{(L)}$$

The values of the parameters A(n) and N(n) are now chosen to satisfy

$$T_1 = T_2 = 0$$

So that :

$$T.E_1 = -f_n^{(L+1)} + \frac{\alpha_0^{N-L-1}}{(A+x_n)^0} f_n^{(L)} = 0 \tag{65}$$

$$T.E_2 = -f_n^{(L+2)} + \frac{\alpha_1^{N-L-1}}{(A+x_n)^2} f_n^{(L)} = 0 \tag{66}$$

$$\frac{-(A+x_n)^1 f_n^{(L+1)} + \alpha_0^{N-L-1} f_n^{(l)}}{(A+x_n)^1} = 0 \tag{67}$$

It can be shown that;

$$-Af_n^{(L+1)} = x_n f_n^{(L+1)} - \alpha_0^{N-L-1} f_n^{(L)} \tag{68}$$

$$-A(n) = x_n - \frac{\alpha_0^{N-L-1} f_n^{(L)}}{f_n^{(L+1)}} \tag{69}$$

$$x_n^2 f_n^{(L+2)} - \frac{4x_n\alpha_0^{N-L-1} f_n^{(L)} f_n^{(L+2)}}{f_n^{(L+1)}} + \left(\frac{\alpha_0^{N-L-1} f_n^{(L)}}{f_n^{(L+1)}}\right)^2 f_n^{(L+2)} = -\alpha_1^{N-L-1} f_n^{(L)} \tag{70}$$

From the above, one obtains;

$$\left[x_n^2 - 2x_n\left(\frac{\alpha_0^{N-L-1} f_n^{(L)}}{f_n^{(L+1)}}\right) + \left(\frac{\alpha_0^{N-L-1} f_n^{(L)}}{f_n^{L+1}}\right)^2 + 2x_n^2 - \frac{2x_n\alpha_0^{N-L-1} f_n^{(L)}}{f_n^{L+1}}\right] f_n^{L+1} - x_n^2 f_n^{(L+2)} = -\alpha_0^{N-L-1} f_n^{(L)} \tag{71}$$

$$x_n^2\left(f_n^{(L+1)}\right)^2 f_n^{(L+2)} - \left(4x_n\alpha_0^{N-L-1} f_n^{(L+1)} - \left(\alpha_0^{N-L-1}\right)^2 f_n^{(L)}\right) f_n^{(L)} f_n^{(L+2)} = -\alpha_1^{N-L-1} f_n^{(L)}\left(f_n^{(L+2)}\right)^2 \tag{72}$$

$$\frac{(N-L-2)}{(N-L-1)^1} = \left(\frac{f_n^{(L)}}{(f_n^{(L+1)})^2}\right)^1 f_n^{(L+2)} \tag{73}$$

$$N(f_n^{(L+1)})^2 - (L+2)(f_n^{(L+1)})^2 = Nf_n^{(L)} f_n^{(L+2)} - (L+1) f_n^{(L)} f_n^{(L+2)} \tag{74}$$

This result to;

$$N(n)=\frac{L\left[(f_n^{(L+1)})^2-f_n^{(L)}f_n^{(L+2)}\right]+(f_n^{(L+1)})^2-f_n^{(L)}f_n^{(L+2)}}{\left[(f_n^{(L+1)})^2\,f_n^{(L)}\,f_n^{(L+2)}\right]} \tag{75}$$

$$N(n)=(L+1)\frac{\left[(f_n^{(L+1)})^2\right]}{\left[(f_n^{(L+1)})^2\,f_n^{(L)}\,f_n^{(L+2)}\right]} \tag{76}$$

Notes

Substitude (76) into (69) to obtain the value of A(n) as follow:

$$-A(n)=x_n-\left[[L+1]+\frac{\left[(f_n^{(L+1)})^2\right]}{\left[(f_n^{(L+1)})^2\,f_n^{(L)}\,f_n^{(L+2)}\right]}-L-1\right]\frac{f_n^{(L)}}{f_n^{(L+1)}} \tag{77}$$

This gives;

$$-A(n)=x_n-\frac{\left[f_n^{(L+1)}\right]\,f_n^{(L)}}{\left[(f_n^{(L+1)})^2-f_n^{(L)}\,f_n^{(L+2)}\right]} \tag{78}$$

In the above derivation,N(n) is the nature of singularity and A(n) is the location of singularity.

III. Convergence Theorem

Let the function Φ(x,y;h) be continuous (jointly as a function of its three arguments) in the region defined by x $x\epsilon[a,b]$, y ϵ (a,x) $0\leq h\leq h_0$, where $h_0>0$, and let there exist a constant L such that

$$\left|\Phi(x,y^*;h)-\Phi(x,y;h)\right|\leq L\left|y^*-y\right|, \tag{79}$$

for all (x,y;h) and (x,y*;h) in the region just defined. Then the relation $\Phi(x,y;0)=f(x,y)$ is a necessary and sufficient condition for theconvergence of the method defined by the incrementfunction, Φ. With the increment function deducted from the formula or scheme.

$$\phi(x_n,y_n^*;h)=\sum_{k=1}^{L}\left[Af^{(k-1)}_{(x_n,y_n^*)}\right]+Bf^{(L)}_{(x_n,y_n^*)}+Cf^{(L)}_{(x_n,y_n^*)}+\sum_{k=1}^{L}\left[Df^{(L)}_{(x_n,y_n^*)}\right] \tag{81}$$

Hence

$$\phi(x_n,y_n^*;h)-\phi(x_ny_n;h)=\sum_{k=1}^{L}\left[Af^{(k-1)}_{(x_n,y_n^*)}\right]+\sum_{k=1}^{L}\left[Af^{(k-1)}_{(x_n,y_n)}\right]+Bf^{(L)}_{(x_n,y_n^*)}-Bf^{(L)}_{(x_n,y_n)}+Cf^{(L)}_{(x_n,y_n^*)}-Cf^{(L)}_{(x_n,y_n)}+\sum_{k=1}^{L}Df^{(L)}_{(x_n,y_n^*)}-\sum_{k=1}^{L}Df^{(L)}_{(x_n,y_n)} \tag{82}$$

$$=\sum_{K=1}^{L}\left[A\left(f^{(k-1)}_{(x_n,y_n^*)}\right)-f^{(k-1)}_{(x_n,y_n)}\right]+B\left(f^{(L)}_{(x_n,y_n^*)}-f^{(L)}_{(x_n,y_n)}\right)+C\left(f^{(L)}_{(x_n,y_n^*)}-f^{(L)}_{(x_n,y_n)}\right)+\sum_{K=1}^{L}\left[D\left(f^{(L)}_{(x_n,y_n^*)}\right)-f^{(L)}_{(x_n,y_n)}\right] \tag{83}$$

Notes

Let y_t be defined as a point in the interior of the interval whose endpoints are y and y*, if we apply the mean value, we have

$$f(x_n, y_n^*) - f(x_n, y_n) = \frac{\partial f(x_n, y)}{\partial y_n}(y_n^* - y_n),\ f^{(1)}(x_n, y_n^*) - f^{(1)}(x_n, y_n) = \frac{\partial f^{(1)}(x_n, y)}{\partial y_n}(y_n^* - y_n), \dots,$$

$$f^{(L)}_{(x_n, y_n^*)} - f^{(L)}_{(x_n, y_n)} = \frac{\partial f^{(L)}_{(x_n, y)}}{\partial y_n}(y_n^* - y_n) \text{ And } f^{(k-1)}_{(x_n, y_n^*)} - f^{(k-1)}_{(x_n, y_n)} = \frac{\partial f^{(k-1)}_{(x_n, y)}}{\partial y_n}(y_n^* - y_n) \quad (84)$$

If we defined

$$L_1 = \sup_{1(x_n, \bar{l}_n)\in Dom} \frac{\partial f(x_n, \bar{y}_n)}{\partial y_n} \dots L_K = \sup_{(x_n, \bar{l}_n)\in Dom} \frac{\partial f^{(L)}_{(x_n, \bar{y}_n)}}{\partial y_n} \text{ and } L_L = \sup_{(x_n, \bar{l}_n)\in Dom} \frac{\partial f^{(L-1)}_{(x_n, \bar{y}_n)}}{\partial y_n}$$

Put equations

$$\phi(x_n, y_n^*; h) - \phi(x_n y_n; h) = \sum_{k=1}^{L} A\left(\frac{\partial f^{(k-1)}_{(x_n, y)}}{\partial y_n}(y_n^* - y_n)\right) + B\left(\frac{\partial f^{(L)}_{(x_n, y)}}{\partial y_n}(y_n^* - y_n)\right) + C\left(\frac{\partial f^{(L)}_{(x_n, y)}}{\partial y_n}(y_n^* - y_n)\right) + \sum_{k=1}^{L} D\left(\frac{\partial f^{(L)}_{(x_n, y)}}{\partial y_n}(y_n^* - y_n)\right)$$

$$= \phi(x_n, y_n^*; h) - \phi(x_n y_n; h)\left[L_L\left(B + C + \sum_{k=1}^{L} D\right) + L_k \sum_{k=1}^{L} A\right](y_n^* - y_n) \quad (85)$$

Taking the absolute value of both sides, we have

$$|\phi(x_n, y_n^*; h) - \phi(x_n y_n; h)| \le \left|L_L\left(B + C + \sum_{K=1}^{L} D\right) + L_K \sum_{K=1}^{L} A\right| |y_n^* - y_n| \quad (86)$$

$$\text{Let } K = \left|L_L\left(B + C + \sum_{K=1}^{L} D\right) + L_K \sum_{K=1}^{L} A\right| \quad (87)$$

$$\text{Thus } |\phi(x_n, y_n^*; h) - \phi(x_n y_n; h)| \le K|y_n^* - y_n| \quad (88)$$

which is the condition for convergence.

IV. Consistency

$$\phi(x_n, y_n; 0) = f(x_n, y_n) \quad (89)$$

If put $h = 0$

$$y_{n+1} = y_n + \sum_{K=1}^{L} \frac{0^k}{K!} f_n^{(k-1)} + \frac{(A + x_n)^{L+1}}{\alpha_L^N} f_n^{(L)} \{1^N + 0 - 1 - 0\} \quad (90)$$

$$y_{n+1} = y_n \Rightarrow f(x_n, y_n) \quad (91)$$

$$y_{n+1} = y_n + h\left\{\left(B + C + \sum_{k=1}^{L} D\right) f^{(L)}_{(x_n, y_n)} + \left(\sum_{K=1}^{L} A\right) f^{(K-1)}_{(x_n, y_n)}\right\} \quad (93)$$

$$l_{n+1} = l_n + h\left\{\left(B + C + \sum_{k=1}^{L} D\right) f^{(L)}_{(x_n, l_n)} + \left(\sum_{K=1}^{L} A\right) f^{(K-1)}_{(x_n, l_n)}\right\} \quad (97)$$

The application of mean value theorem and the subtraction of 4.6 and 4.6, one obtains;

$$y_{n+1}-l_{n+1}=y_n-l_n+h\left[\sup_{(x_n,\bar{l}_n)\in Dom}\frac{\partial f^{(L)}}{\partial l_n}(x_n,\bar{l}_n)\left(B+C+\sum_{K=1}^{L}D\right)+\sup_{(x_n,\bar{l}_n)\in Dom}\frac{\partial f^{(K-1)}}{\partial l_n}(x_n,\bar{l}_n)\left(\sum_{K=1}^{L}A\right)\right]$$

$$=y_n-l_n+h\left[\left(B+C+\sum_{K=1}^{L}D\right)L_L+\left(\sum_{K=1}^{L}A\right)L_{K-1}\right](y_n-l_n) \quad (98)$$

$$|y_{n+1}-l_{n+1}|\le|y_n-l_n|+|h||PL_L+ML_{K-1}||y_n-l_n| \quad (99)$$

If $1+hS=R$, $S=|PL_L+ML_{K-1}|$, $y_n=\lambda^*$ and $l_n=\lambda$

then, $$|y_{n+1}-l_{n+1}|\le R|\lambda^*-\lambda|\Rightarrow|y_{n+1}-l_{n+1}|\le[1+hS]|y_n-l_n|\Rightarrow|y_{n+1}-l_{n+1}|\le R|y_n-l_n| \quad (100)$$

V. Conclusion

If in (2), the parameter A is regarded as undetermined coefficients and eliminated in the same way as b *and* $a_p(p=0,1,\ldots L)$, another class of formulae would emerge, which is given as:

$$y_{n+1}-y_n=\frac{hf_{n+1}^{(N/(N-1))}-f_n^{(N/(N-1))}}{Nf_{n+1}^{(1/(N-1))}-f_n^{(1/(N-1))}},\ N\neq 0 \quad (60)$$

This shall be used to construct a subroutine called **GENFOR**, which shall be able to jump the point of singularity.

Ibijola, et (2004) constructed a one-step method, which was based on the non-linear interpolant:

$$F(x)=\frac{C}{1+ae^{\lambda x}}, \quad (61)$$

where C and a are real constants.

The resulting integrator is:

$$y_{n+1}=\frac{\lambda y_n^2}{\lambda y_n+(e^{\lambda x}-1)hy_n'}. \quad (62)$$

This is capable of skipping the point of singularity if the mesh size is carefully selected. This scheme can't give any information concerning the location and nature of singularity. However, it will be used for the construction of another subroutine called GENDOR, which could be preferred where GENFOR might not be strong enough to give a better approximation, hereafter, the programme retunes to (55) for a continuation after the point of singularity.

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