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Linear Canonical Transforms On the Zemanian Spaces

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Abstract - The objective of this paper is to make the theory of Linear canonical transform mathematically rigorous. Here we define the linear canonical transform on the Zemanian space $\mathcal{SS}$, the space of functions of rapid descent, and prove some results. We have also discussed some operation formulae for linear canonical transform on this space.

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I. Introduction

The linear canonical transform is a much more general integral transform which has the Fourier transform, Fractional Fourier transform, Fresnel transform etc as its special cases. As it is well known that fractional Fourier transform (FrFT) is the powerful mathematical tool and is widely used for spectral analysis, signal processing, optical system analysis etc.[6], the LCT which is the generalization of FrFT has obviously more ability and potential due to its three parameters. Especially it is used to analyze optical systems with prisms or shifted lenses.

Since Namias [5] had introduced Fractional Fourier transform, many mathematicians had studied it thoroughly e.g. Almeida [2] studied its convolution and product, Akay [1] discussed unitary operators on it, where as Kerr [4] extended it to the spaces of distributions.

As compared to this, Linear Canonical transform is less attended integral transform. We have studied its analytic properties [3] and here we want to study it in some Zemanian spaces.

Definition : The linear canonical transform (LCT) is a four parameter (a, b, c, d), class of integral transform given by,

$$F_A(u) = [LCT f(t)](u) = \frac{1}{\sqrt{2\pi i b}}\, e^{\frac{id}{2b}u^2} \int_{-\infty}^{\infty} e^{\frac{ia}{2b}t^2}\, e^{\frac{-i}{b}ut} f(t)dt \quad \text{for } b \neq 0$$

$$= \sqrt{d}\, e^{\frac{i}{2}cdu^2} f(du) \qquad \text{for } b = 0 \qquad (1)$$

It is characterized by a general 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc = 1$, moreover the entries a, b, c, d are real or complex but for this paper we assume that they are real parameters.

Inverse Linear canonical transform is given by

$$[F_A^{-1}(u)]\,(t) = f(t) = \int_{-\infty}^{\infty} F_A(u)\, K_A^{-1}\,(t,u)du \qquad (2)$$

where $K_A^{-1}(t,u)$ is the Hermitian conjugate of $K_A(t,u)$ that is $K_A^{-1}(t,u) = K_A^*\,(u,t)$.

This paper attempts to provide the necessary mathematical framework for extending LCT to the spaces of generalized functions. We begin by recalling the definition of the space $\mathcal{S}$, the space of functions of rapid descent and some properties of the Fourier transform on it.

Section 2 is devoted to define LCT on the space $\mathcal{S}$ and prove the basic properties of the concerned transform. In section 3 some operational formulae are developed. Application of the

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theory is discussed by solving one example in section 4 and finally we conclude in the last section.

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Notations as per Zemanian [7].

II. Preliminaries

In this section we note some of the results which will be used later on.

Let $\mathcal{S}$ be the vector space of all smooth functions φ such that,

$\gamma_{m,n}(\varphi) = \sup_{t \epsilon R} \left| t^m \, \varphi^{(n)}(t) \right| < \infty$, for all $m, n \in \{0,1,2,\dots\}$.

Clearly $\mathcal{S}$ is equipped with the topology generated by the collection of seminorms $\{\gamma_{m,n}\}_{m,n=0}^{\infty}$ also $\mathcal{S}$ is a Frechet space.

Fourier transform defined on $\mathcal{S}$

$$\mathcal{F}\big(\varphi(t)\big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itw} \; \varphi(t)\, dt, \qquad \varphi \, \epsilon \, \mathcal{S}, w \in R \tag{3}$$

Some useful results as in [3] :

a) $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a homeomorphism.

b) $(\mathcal{F}^{-1}(\varphi(t))(w) = (\mathcal{F}\varphi(t))(-x) \qquad \varphi \, \epsilon \, \mathcal{S}, w \in R.$

c) If $f(t) = e^{\frac{1}{2}At^2}, A \, \epsilon \, C$ then for $n = 0, 1, 2, \dots.$,

$$f^{(2n)}(t) = e^{\frac{1}{2}At^2} \sum_{r=0}^{n} \alpha_r \; A^{2n-r} \; x^{2n-2r}$$

$$f^{(2n+1)}(t) = e^{\frac{1}{2}At^2} \sum_{r=0}^{n} \beta_r \; A^{2n+1-r} \; x^{2n+1-2r}$$

where the constants α_r and β_r are independent of A.

III. Linear Canonical Transform on $\mathcal{S}$ Spaces

For each $f \, \epsilon \, \mathcal{S}$, the Linear canonical transform of f as given by,

$$F_A(u) = [LCT_A f(t)]\,(u) = \int_{-\infty}^{\infty} f(t)\; K_A\,(t,u)\, dt$$

where $K_A(t,u) = \frac{1}{\sqrt{2\pi i b}} \, e^{\frac{i}{2b}(at^2 + du^2)} \; e^{\frac{-i}{b}ut}$ for $b \neq 0$

$= \sqrt{d}\, e^{\frac{i}{2} c d u^2} \, f(du)$ for $b = 0$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and a, b, c, d are real.

a) Proposition : $F_A(u) = \frac{1}{\sqrt{ib}} e^{\frac{id}{2b}u^2} [\mathcal{F}\left(e^{\frac{ia}{2b}t^2} f(t)\right)](\frac{u}{b})$ where $\mathcal{F}$ is the Fourier transform as explained above.

Proof : The result is very clear from the definition (1) and (2).
Also we note that

b) Proposition : $F_A(u)$, Linear canonical transform of $f(t)$, is the homeomorphism on $\mathcal{S}$ with inverse as given by (3).

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Proof : We know that $\mathcal{F}$, Fourier transform is homeomorphism on $\mathcal{S}$ [7]. Also the function $h(u) = e^{\frac{id}{2b}u^2} g(u)$, for $g(u) \epsilon \mathcal{S}$, is a homeomorphism on $\mathcal{S}$. Therefore by 2.1, Linear canonical transform of $f(t)$, is the homeomorphism on $\mathcal{S}$.

Now it remains to show that (2) represents the inverse under the space $\mathcal{S}$.

$$[LCT_A^{-1}\, LCT_A\, f(t)](v) = \int_{-\infty}^{\infty} f(t)\; K_A(t,u) \int_{-\infty}^{\infty} K_A^{-1}(u,v)\, dt\, du$$

$$= C_1 e^{\frac{-ia}{2\,b}v^2} \int_{-\infty}^{\infty} e^{\frac{i\,d}{2b}u^2} \int_{-\infty}^{\infty} C_1^* \, e^{\frac{-ivu}{b}} \; e^{\frac{i\,a}{2b}t^2} e^{\frac{-id}{2\,b}u^2} e^{\frac{iut}{b}}\, dt\, du$$

where $C_1 = \frac{1}{\sqrt{2\pi i b}}$ and C_1^* is its conjugate.

$$= \frac{1}{2\pi|b|}\; e^{\frac{-ia}{2\,b}v^2} \; \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty}[e^{\frac{i\,a}{2b}t^2}\; f(t)]\, e^{\frac{iut}{b}}\; dt\}e^{\frac{-iuv}{b}}\; du$$

$$= \; e^{\frac{-ia}{2\,b}v^2}[\,\mathcal{F}^{-1}\mathcal{F}\{e^{\frac{i\,a}{2b}t^2}\; f(t)\}]\,(v)$$

$$= \; f(v)$$

Hence the result follows.

c) Theorem : For each $f \,\epsilon\, \mathcal{S}$, $\{LCT_A f(t)\}(u)$ converges to $f(t)$ with respect to the topology of $\mathcal{S}$ as $b \rightarrow 0^+$ and $a \rightarrow 1$.

Proof : Since Fourier transform is a homeomorphism on $\mathcal{S}$, equivalently we prove that, $LCT_A\mathcal{F}f$ converges to $\mathcal{F}f$ as $b \rightarrow 0^+$ in $\mathcal{S}$ where $\mathcal{F}$ denotes the Fourier transform of f.

$$(F_A\mathcal{F}f)(v) = \; [LCT_A\mathcal{F}f](v) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi i b}}\; e^{\frac{i}{2b}(au^2+dv^2)}\;\; e^{\frac{-i}{b}uv}\;\; \{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-itu}\; f(t)dt\}\, du$$

$$= \frac{1}{\sqrt{2\pi i b}}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{id\,v^2}{2b}} \left\{\sqrt{\frac{2\pi i b}{a}}\;\; e^{\frac{i}{2a}\left(\frac{v}{b}+t\right)^2}\right\} f(t)dt$$

Using $\int_{-\infty}^{\infty} e^{iau^2}\, e^{ibu}\; du = \sqrt{\frac{\pi i}{a}}\; e^{\frac{-ib^2}{4a}}$ if $a > 0$.

$$= \frac{1}{\sqrt{2\pi a}}\int_{-\infty}^{\infty} e^{\frac{id\,v^2}{2b}}\;\; e^{\frac{-ib}{2a}\,(\frac{v^2}{b^2}+\frac{2}{b}vt+t^2)}\; f(t)dt$$

$$= \frac{1}{\sqrt{2\pi a}} e^{\frac{i(d-\frac{1}{a})v^2}{2b}} \int_{-\infty}^{\infty} e^{\frac{-ib}{2a}t^2}\; f(t)\; e^{-\frac{i}{a}vt}\; dt \qquad (1)$$

$$\therefore\; v^m\,(\,D_v^n\; F_A\mathcal{F}f)(v) = \;\; \frac{v^m}{\sqrt{2\pi a}}\sum_{k=0}^{n}\binom{n}{k}\; D_v^k\; e^{\frac{i(d-\frac{1}{a})v^2}{2b}}\;\; D_v^{n-k}\int_{-\infty}^{\infty} e^{\frac{-ib}{2a}t^2}\; f(t)\; e^{-\frac{i}{a}vt}\; dt$$

$$= \frac{1}{\sqrt{2\pi a}}\; \sum_{k=0}^{n}\sum_{r=0}^{[\frac{k}{2}]}\binom{n}{k}\, a_r\;\; (\frac{i}{2b}\,(d-\frac{1}{a}))^{k-r} e^{\frac{i(d-\frac{1}{a})v^2}{2b}}\; v^{m+k-2r}\int_{-\infty}^{\infty} e^{\frac{-ib}{2a}t^2}\; f(t)(\frac{-it}{a})^{n-k}\; e^{-\frac{i}{a}vt}dt$$

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$$= \frac{1}{\sqrt{2\pi a}} \sum_{k=0}^{n} \sum_{r=0}^{[\frac{k}{2}]} \binom{n}{k} a_r \ (\frac{i}{2b}(d-\frac{1}{a}))^{k-r} e^{\frac{i(d-\frac{1}{a})v^2}{2b}} \int_{-\infty}^{\infty} e^{\frac{-ib}{2a}t^2} f(t)(\frac{-it}{a})^{n-k} v^{m+k-2r} e^{-\frac{i}{a}vt} dt \qquad (2)$$

Now,

$$D_t^{m+k-2r}\left(e^{\frac{-ivt}{a}}\right) = (\frac{-iv}{a})^{m+k-2r} \ e^{\frac{-ivt}{a}}$$

Hence (3) becomes,

$$= \frac{1}{\sqrt{2\pi a}} \sum_{k=0}^{n} \sum_{r=0}^{[\frac{k}{2}]} \binom{n}{k} a_r \ (\frac{i}{2b}(d\frac{1}{a}))^{k-r} e^{\frac{i(d-\frac{1}{a})v^2}{2b}} \int_{-\infty}^{\infty} e^{\frac{-ib}{2a}t^2} f(t)(\frac{-it}{a})^{n-k}$$

$$(ia)^{m+k-2r} D_t^{m+k-2r} e^{-\frac{i}{a}vt} dt$$

$$= \frac{1}{\sqrt{2\pi a}} \sum_{k=0}^{n} \sum_{r=0}^{[\frac{k}{2}]} \binom{n}{k} a_r \ (\frac{i}{2b}(d\frac{1}{a}))^{k-r} (ia)^{m+2k-2r-n} e^{\frac{i(d-\frac{1}{a})v^2}{2b}} \int_{-\infty}^{\infty} e^{-\frac{i}{a}vt} D_t^{m+k-2r}\{e^{\frac{-ib}{2a}t^2} f(t)(t)^{n-k}\} dt$$

by [7 p 49]

$$= \frac{1}{\sqrt{2\pi a}} \sum_{k=0}^{n} \sum_{r=0}^{[\frac{k}{2}]} \binom{n}{k} a_r \ (\frac{i}{2b}(d\frac{1}{a}))^{k-r} (ia)^{m+2k-2r-n} e^{\frac{i(d-\frac{1}{a})v^2}{2b}} \int_{-\infty}^{\infty} e^{-\frac{i}{a}vt}$$

$$\sum_{j=0}^{m+k-2r} \binom{m+k-2r}{j} D_t^j \ e^{\frac{-ib}{2a}t^2} D_t^{m+k-2r-j}\left(t^{n-k} f(t)\right) dt$$

$$= \frac{1}{\sqrt{2\pi a}} \sum_{k=0}^{n} \sum_{r=0}^{[\frac{k}{2}]} \binom{n}{k} a_r \ (\frac{i}{2b}(d-\frac{1}{a}))^{k-r} (ia)^{m+2k-2r-n} e^{\frac{i(d-\frac{1}{a})v^2}{2b}} \int_{-\infty}^{\infty} e^{-\frac{i}{a}vt}$$

$$\sum_{j=0}^{m+k-2r} \sum_{l=0}^{[\frac{j}{2}]} b_r \binom{m+k-2r}{j} (\frac{b}{ia})^{j-l} \ t^{j-2l} D_t^{m+k-2r-j}\left(t^{n-k} f(t)\right) dt$$

$$= \sum_{k=0}^{n} \sum_{r=0}^{[\frac{k}{2}]} \sum_{j=0}^{m+k-2r} \sum_{l=0}^{[\frac{j}{2}]} \binom{n}{k} \binom{m+k-2r}{j} a_r b_r \ (\frac{i}{2b}(d-\frac{1}{a}))^{k-r} (ia)^{m+2k-2r-n} \ (\frac{b}{ia})^{j-l}$$

$$\int_{-\infty}^{\infty} e^{-\frac{i}{a}vt} \ t^{j-2l} D_t^{m+k-2r-j}\left(t^{n-k} f(t)\right) dt$$

$$= \sum_{k=0}^{n} \sum_{r=0}^{[\frac{k}{2}]} \sum_{j=0}^{m+k-2r} \sum_{l=0}^{[\frac{j}{2}]} \ G_{j,k,r,l}$$

where, $G_{j,k,r,l} = \binom{n}{k} \binom{m+k-2r}{j} a_r b_r \left(\frac{i}{2b}\left(d-\frac{1}{a}\right)\right)^{k-r} \ (ia)^{m+2k-2r-n} \ \left(\frac{b}{ia}\right)^{j-l}$

$$\int_{-\infty}^{\infty} e^{-\frac{i}{a}vt} \ t^{j-2l} D_t^{m+k-2r-j}\left(t^{n-k} f(t)\right) dt$$

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$\therefore \left| G_{j,k,r,l} \right|$

$$= \left| \binom{n}{k} \binom{m+k-2r}{j} a_r b_r \left(\frac{i}{2b} \left(-\frac{1}{a} \right) \right)^{k-r} (ia)^{m+2k-2r-n} \left(\frac{b}{ia} \right)^{j-l} \int_{-\infty}^{\infty} e^{-\frac{i}{a}vt}\, t^{j-2l} D_t^{m+k-2r-j} \left(t^{n-k} f(t) \right) dt \right|$$

Since $\left| e^{-\frac{i}{a}vt} \right| = 1$

$\left| G_{j,k,r,l} \right| =$

$$\left| \binom{n}{k} \binom{m+k-2r}{j} a_r b_r \left(\frac{i}{2b} \left(d - \frac{1}{a} \right) \right)^{k-r} (ia)^{m+2k-2r-n} \left(\frac{b}{ia} \right)^{j-l} \int_{-\infty}^{\infty} t^{j-2l} D_t^{m+k-2r-j} \left(t^{n-k} f(t) \right) dt \right|$$

$$= \left| \binom{n}{k} \binom{m+k-2r}{j} a_r b_r \left(\frac{i}{2b} \left(\frac{bc}{a} \right) \right)^{k-r} (ia)^{m+2k-2r-n} \left(\frac{b}{ia} \right)^{j-l} \int_{-\infty}^{\infty} t^{j-2l} D_t^{m+k-2r-j} \left(t^{n-k} f(t) \right) dt \right|$$

$$= \left| \binom{n}{k} \binom{m+k-2r}{j} a_r b_r \left(\frac{i}{2} \left(\frac{c}{a} \right) \right)^{k-r} (ia)^{m+2k-2r-n} \left(\frac{b}{ia} \right)^{j-l} \int_{-\infty}^{\infty} t^{j-2l} D_t^{m+k-2r-j} \left(t^{n-k} f(t) \right) dt \right|$$

Now $\int_{-\infty}^{\infty} t^{j-2l} D_t^{m+k-2r-j} \left(t^{n-k} f(t) \right) dt$, is independent of the parameters $a, b, c, d,$ hence $\left| G_{j,k,r,l} \right| \to 0$ as $b \to 0$ and $a \to 1$ provided $j - l > 0$, which is always true.

$\therefore \left| v^m \left(D_v^n F_A \mathcal{F} f \right)(v) - v^m \frac{D_v^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ivt}\, dt \right| < \infty$, since $f \in \mathcal{S}$.

Hence the theorem is proved.

IV. Some Properties of Linear Canonical Transform in $\mathcal{S}$

a) **Theorem** : For each $f \in \mathcal{S}$, $F_A F_B = F_C$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $C = B\,A$.

Proof : Consider,

$$[F_B F_A f(t)](v) = LCT_B \frac{1}{\sqrt{2\pi i b}} \int_{-\infty}^{\infty} e^{\frac{i}{2b}(at^2 + du^2)}\, e^{\frac{-i}{b}ut}\, f(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi i b}} \frac{1}{\sqrt{2\pi i q}} \int_{-\infty}^{\infty} e^{\frac{i}{2q}(pu^2 + sv^2)}\, e^{\frac{-i}{q}uv} \left\{ \int_{-\infty}^{\infty} e^{\frac{i}{2b}(at^2 + du^2)}\, e^{\frac{-i}{b}ut}\, f(t)\, dt \right\} du$$

$$= \frac{1}{\sqrt{2\pi i b}} \frac{1}{\sqrt{2\pi i q}} \left[\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{p}{q} + \frac{d}{b}\right)u^2 - i\left(\frac{v}{q} + \frac{t}{b}\right)u}\, du \right] e^{\frac{iat^2}{2b}}\, e^{\frac{isv^2}{2q}}\, f(t) dt$$

$$= \frac{1}{\sqrt{2\pi i b}} \frac{1}{\sqrt{2\pi i q}} \int_{-\infty}^{\infty} \frac{\sqrt{i}\sqrt{2\pi bq}}{\sqrt{bp + dq}}\, e^{\frac{-i(bv + qt)^2}{2(bp + dq)}}\, e^{\frac{iat^2}{2b}}\, e^{\frac{isv^2}{2q}}\, f(t) dt$$

$$= \frac{1}{\sqrt{2\pi i (bp + dq)}} \int_{-\infty}^{\infty} e^{\frac{i}{2b}\left(a - \frac{q}{(bp + dq)}\right)t^2}\, e^{\frac{i}{2q}\left(s - \frac{b}{(bp + dq)}\right)v^2}\, e^{\frac{-ivt}{(bp + dq)}}\, f(t) dt$$

Now by simple calculations and using $ad - bc = 1$ and $ps - rq = 1$, it can be shown that the right hand side is nothing but,

$= [F_C f(t)](v).$

b) **Theorem :** For $f \in \mathcal{S}$, and $\tau \in R$,

$$F_A\, f(t-\tau) = e^{\frac{-iac\tau^2}{2}}\, e^{icu\tau}\, [F_A f(t)]\,(u - a\tau).$$

Proof : As $f \in \mathcal{S}$ implies $f(t-\tau) \in \mathcal{S}$, (by ex. 3, p 102 [2]) so that both sides of above equation are defined as elements of $\mathcal{S}$. We shall prove the result for $f \in \mathcal{D}$, the space of smooth functions of compact support, then the result will be clear from the continuity and denseness.

$$F_A\, f(t-\tau) = \frac{1}{\sqrt{2\pi i b}}\, e^{\frac{id}{2b}u^2} \int_{-\infty}^{\infty} e^{\frac{ia}{2b}t^2}\, e^{\frac{-i}{b}ut}\, f(t-\tau)dt$$

$$= \frac{1}{\sqrt{2\pi i b}}\, e^{\frac{id}{2b}u^2} \int_{-\infty}^{\infty} e^{\frac{ia}{2b}(t+\tau)^2}\, e^{\frac{-i}{b}u(t+\tau)}\, f(t)dt$$

Simple calculations will show that right hand side is equal to,

$$= e^{\frac{-iac\tau^2}{2}}\, e^{icu\tau}\, [F_A f(t)]\,(u - a\tau)$$

c) **Theorem :** For $f \in \mathcal{S}$, and $n \in \mathbb{N}$,

$$[F_A\,(D^n f(t))]\,(u) = (-icu + a\frac{d}{du})^n\, F_A\,(u).$$

Proof : As $(D^n f(t)) \in \mathcal{S}$ if $f \in \mathcal{S}$, both sides of above equation are defined in $\mathcal{S}$.

We shall prove for $n = 1$ and it follows for all $n > 1$by induction.

Consider,

$$[F_A\,\big(Df(t)\big)(u) = \frac{1}{\sqrt{2\pi i b}}\, e^{\frac{id}{2b}u^2} \int_{-\infty}^{\infty} e^{\frac{ia}{2b}t^2}\, e^{\frac{-i}{b}ut}\, Df(t)dt$$

$$= \frac{-ia}{b}\, [F_A\,\big(tf(t)\big)](u) + \frac{iu}{b}\, [F_A\, f(t)](u)$$

and

$$\left(-icu + a\frac{d}{du}\right) F_A(u) = -icuF_A(u) + a\frac{d}{du}\frac{1}{\sqrt{2\pi i b}}\, e^{\frac{id}{2b}u^2} \int_{-\infty}^{\infty} e^{\frac{ia}{2b}t^2}\, e^{\frac{-i}{b}ut}\, f(t)dt$$

$$= -icuF_A(u) + \frac{iad}{b}u\, F_A\,(u) - \frac{ia}{b}\, [F_A(t\, f(t))](u)$$

$$= \frac{iu}{b}\, [F_A\, f(t)](u) + \frac{-ia}{b}\, [F_A\,\big(tf(t)\big)](u)$$

Hence the theorem.

d) **Theorem :** For $f \in \mathcal{S}$, and $k \in R$,

$$[F_A\,\left(e^{ikt} f(t)\right)(u) = e^{\frac{idk\,(2u-bk)}{2}} [F_A\big(f(t)\big)(u - bk).$$

Proof : The proof of this is simple and hence omitted.

V. Application

a) **Example** : If $f(t) = e^{\frac{-t^2}{2}}$ then $[F_A f(t)](u) = \frac{1}{\sqrt{a+ib}}\, e^{\frac{(c+id)u^2}{2(b-ai)}}$.

Sol. Clearly $e^{\frac{-t^2}{2}} \in \mathcal{S}$,

Now, $[F_A f(t)](u) = \frac{1}{\sqrt{2\pi ib}}\, e^{\frac{id}{2b}u^2} \int_{-\infty}^{\infty} e^{\frac{ia}{2b}t^2}\, e^{\frac{-i}{b}ut}\, e^{\frac{-t^2}{2}} dt$

$$= \frac{1}{\sqrt{2\pi ib}}\, e^{\frac{id}{2b}u^2} \int_{-\infty}^{\infty} e^{\frac{-(b-ia)}{2b}t^2}\, e^{\frac{-i}{b}ut}\, dt$$

$$= \frac{1}{\sqrt{a+ib}}\, e^{\frac{id}{2b}u^2}\, e^{\frac{-u^2}{2b(b-ai)}} = \frac{1}{\sqrt{a+ib}}\, e^{\frac{(c+id)u^2}{2(b-ai)}}.$$

b) **Example** : If $f(t) = \delta(t-\tau)$ then , $[F_A f(t)](u) = \frac{1}{\sqrt{2\pi ib}}\, e^{\frac{i}{2b}(a\tau^2+du^2)}\, e^{\frac{-i}{b}u\tau}$

Sol: Since $\delta(t-\tau) \in \mathcal{S}$ we can apply the above theory and by the definition of delta function the result is clear.

VI. Conclusion

In this paper we have confined ourselves to the space $\mathcal{S}$ next we propose to extend this theory to the spaces of generalized functions $\mathcal{S}'$ the space of distributions of slow growth. Another extension we plan is to study the linear canonical transform with complex entries. Then we shall use this theory to solve some partial differential equations.

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