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Positive Solutions for Systems of Three-Point Nonlinear Boundary Value Problems on Time Scales

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GJSFR-F Classification : *MSC 2010: 34B15, 39B10, 34B18*

POSITIVE SOLUTIONS FOR SYSTEMS OF THREE-POINT NONLINEAR BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Positive Solutions for Systems of Three-Point Nonlinear Boundary Value Problems on Time Scales

A Kameswara Rao [α] & S. Nageswara Rao [σ]

Abstract - Values of λ are determined for which there exist positive solutions of the system of dynamic equations, $u^{\Delta\Delta}(t)+\lambda p(t)f(v(\sigma(t)))=0,\ v^{\Delta\Delta}(t)+\lambda q(t)g(u(\sigma(t)))=0,$ for $t\in[a,b]_{\mathbb{T}}$ Satisfying the three - point boundary conditions, $\alpha u(a)-\beta u^{\Delta}(a)=0,\ u(\sigma^2(b))-\delta u(\eta)=0,\ \alpha v(a)-\beta v^{\Delta}(a)=0,\ v(\sigma^2(b))-\delta v(\eta)=0,$ where $\mathbb{T}$ is a time scales. A Guo-Krasnosel'skii fixed point theorem is applied.

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I. Introduction

Let $\mathbb{T}$ be a time scale with $a,\sigma^2(b)\in\mathbb{T}$. Given an interval J of $\mathbb{R}$, we will use the interval notation

$$J_{\mathbb{T}}=J\cap\mathbb{T}.$$

We are concerned with determining intervals of the parameter λ (eigenvalues) for which there exist positive solutions for the system of dynamic equations,

$$\begin{aligned} u^{\Delta\Delta}(t)+\lambda p(t)f(v(\sigma(t)))=0,\ \ t\in[a,b]_{\mathbb{T}},\\ v^{\Delta\Delta}(t)+\lambda q(t)g(u(\sigma(t)))=0,\ \ t\in[a,b]_{\mathbb{T}}, \end{aligned} \tag{1.1}$$

satisfying the boundary conditions

$$\begin{aligned} \alpha u(a)-\beta u^{\Delta}(a)=0,\quad u(\sigma^2(b))-\delta u(\eta)=0,\\ \alpha v(a)-\beta v^{\Delta}(a)=0,\quad v(\sigma^2(b))-\delta v(\eta)=0, \end{aligned} \tag{1.2}$$

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where $\alpha, \beta \geq 0$, $\alpha + \beta > 0$, $\lambda > 0$, $0 < \delta < 1$, $\eta \in [a, \sigma^2(b)]$, and

(A1) $f, g \in C([0, \infty), [0, \infty))$,

(A2) $p, q \in C([a, \sigma(b)]_{\mathbb{T}}, [0, \infty))$, and each does not vanish identically on any closed subinterval of $[a, \sigma(b)]_{\mathbb{T}}$,

(A3) All of

$$f_0 := \lim_{x \to 0^+} \frac{f(x)}{x}, \quad g_0 := \lim_{x \to 0^+} \frac{g(x)}{x},$$

$$f_\infty := \lim_{x \to \infty} \frac{f(x)}{x}, \quad g_\infty := \lim_{x \to \infty} \frac{g(x)}{x}$$

exist as positive real numbers.

On a larger scale, there has been a great deal of activity in studying positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from a theoretical sense [9, 10, 12, 15] and as applications for which only positive solutions are meaningful. These considerations are caste primarily for scalar problems, but good attention has been given to boundary value problems for systems of differential equations [13, 14, 19, 21, 22].

The main tool in this paper is an application of the Guo-Krasnoselskii fixed point theorem for operators leaving a Banach space cone invariant [10]. A Green's function plays a fundamental role in defining an appropriate operator on a suitable cone.

II. Some Preliminaries

In this section, we state some preliminary lemmas and the well-known Guo-Krasnosel'skii fixed point theorem.

Let $G(t,s)$ be the Green's function for the boundary value problem

$$-y^{\Delta\Delta}(t) = 0, \tag{2.1}$$

$$\alpha y(a) - \beta y^{\Delta}(a) = 0, \qquad y(\sigma^2(b)) - \delta y(\eta) = 0, \tag{2.2}$$

which is given by

$$G(t,s) = \frac{1}{d} \begin{cases} G_1(t,s): & a \leq s \leq \eta \\ G_2(t,s): & \eta \leq \sigma(s) \leq \sigma^2(b) \end{cases}$$

where

$$G_1(t,s) = \begin{cases} [\beta + \alpha(\sigma(s) - a)]\,[\sigma^2(b) - \delta\eta - t(1-\delta)], & \sigma(s) \leq t \\ [\beta + \alpha(t - a)]\,[\sigma^2(b) - \delta\eta - \sigma(s)(1-\delta)], & t \leq s \end{cases}$$

Ref.

[9] L. H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.*, **120**(1994), No. 3, 743-748.

$$G_2(t,s)=\begin{cases}[\beta+\alpha(\sigma(s)-a)]\,(\sigma^2(b)-t)+(t-\sigma(s))\,(\eta+\beta-\alpha a)\,\delta, & \sigma(s)\le t\\ [\beta+\alpha(t-a)]\,(\sigma^2(b)-\sigma(s))\,, & t\le s\end{cases}$$

and

$$d:=\beta(1-\delta)+\alpha(\sigma^2(b)-a-\delta(\eta-a)).$$

Notes

Lemma 2.1 *For $h(t)\in C[a,\sigma^2(b)]_{\mathbb{T}}$, the BVP*

$$-y^{\Delta\Delta}(t)=h(t),\quad t\in[a,b]_{\mathbb{T}}, \tag{2.3}$$

$$\alpha y(a)-\beta y^{\Delta}(a)=0,\quad y(\sigma^2(b))-\delta y(\eta)=0, \tag{2.4}$$

has a unique solution

$$y(t)=\frac{\beta+\alpha(t-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))h(s)\Delta s$$

$$-\frac{\delta(\beta+\alpha(t-a))}{d}\int_a^{\eta}(\eta-\sigma(s))h(s)\Delta s-\int_a^t(t-\sigma(s))h(s)\Delta s. \tag{2.5}$$

From (2.5) obviously we have that

$$y(t)\le\frac{\beta+\alpha(t-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))h(s)\Delta s, \tag{2.6}$$

and

$$y(\eta)\ge\frac{\beta+\alpha(\eta-a)}{d}\int_\eta^{\sigma(b)}(\sigma^2(b)-\sigma(s))h(s)\Delta s. \tag{2.7}$$

Lemma 2.2 *Let $0<\delta<1$. If $h(t)\in C[a,\sigma^2(b)]_{\mathbb{T}}$, and $h\ge 0$, then the unique solution y of the problem* (2.3), (2.4) *satisfies*

$$y(t)\ge 0,\quad t\in(a,\sigma^2(b))_{\mathbb{T}}.$$

Proof: From the fact that $y^{\Delta\Delta}(t)=-h(t)\le 0$, we know that the graph of $y(t)$ is concave down on $[a,\sigma^2(b)]_{\mathbb{T}}$ and $y^{\Delta}(t)$ is monotone decreasing. Thus $y^{\Delta}(t)\le y^{\Delta}(a)=\frac{\alpha}{\beta}y(a)$, where $\beta\neq 0$.

Case 1. If $y(a)<0$, then $y^{\Delta}(t)<0$ for $[a,\sigma^2(b)]_{\mathbb{T}}$. Thus y is a monotone decreasing function, that is $y(t)\ge y(\sigma^2(b))$.

1. If $y(\sigma^2(b))\ge 0$, then $y(t)>0$. So this contradicts the assertion $y(t)$ is a monotone decreasing function.

2. If $y(\sigma^2(b))<0$, then we have that

$$y(\eta) = \frac{1}{\delta} y(\sigma^2(b)) < 0,$$

$$y(\sigma^2(b)) = \delta y(\eta) \geq y(\eta),$$

which contradicts the assertion that $y(t)$ is monotone decreasing.

Case 2. If $y(a) \geq 0$, then $y^{\Delta}(a) \geq 0$. So $y(t)$ is a monotone increasing on $[a, a+\epsilon]$.

1. If $y(\sigma^2(b)) \geq 0$, then $y(t) \geq 0$ on $[a, \sigma^2(b)]_{\mathbb{T}}$.
2. If $y(\sigma^2(b)) < 0$, then we have that

$$y(\eta) = \frac{1}{\delta} y(\sigma^2(b)) < 0,$$

$$y(\sigma^2(b)) = \delta y(\eta) \geq y(\eta),$$

which contradicts the assertion that the graph of $y(t)$ is concave down on $(a, \sigma^2(b))_{\mathbb{T}}$.

If $\beta = 0$, from the boundary conditions we obtain $y(a) = 0$.

1. If $y(\sigma^2(b)) \geq 0$, then the concavity of y implies that $y(t) \geq 0$ for $t \in [a, \sigma^2(b)]_{\mathbb{T}}$.
2. If $y(\sigma^2(b)) < 0$, then

$$y(\eta) = \frac{1}{\delta} y(\sigma^2(b)) < 0,$$

$$y(\sigma^2(b)) = \delta y(\eta) \geq y(\eta).$$

This contradicts with the concavity of y.

Lemma 2.3 *If $y^{\Delta\Delta}(t) \leq 0$, then $\frac{y(\sigma^2(b))}{\sigma^2(b)} \leq \frac{y(t)}{t} \leq \frac{y(\eta)}{\eta}$ for all $t \in [\eta, \sigma^2(b)]_{\mathbb{T}}$.*

Proof: Let $h(t) := y(t) - \frac{t}{\sigma^2(b) - a} y(\sigma^2(b))$. Thus, we have $h(\eta) > 0$ and $h(\sigma^2(b)) = 0$. Since $h^{\Delta\Delta}(t) \leq 0$ then $h(t) \geq 0$ on $[\eta, \sigma^2(b)]_{\mathbb{T}}$. So $\frac{y(\sigma^2(b))}{\sigma^2(b)} \leq \frac{y(t)}{t}$. For the function $h(t)$, since $h(\eta) > 0$, $h(\sigma^2(b)) = 0$ and $h^{\Delta\Delta}(t) \leq 0$ then the function $h(t)$ is decreasing on $[\eta, \sigma^2(b)]_{\mathbb{T}}$. So $\frac{y(t)}{t} \leq \frac{y(\eta)}{\eta}$ for all $t \in [\eta, \sigma^2(b)]_{\mathbb{T}}$.

Lemma 2.4 *Let $0 < \delta < 1$. If $h(t) \in C[a, \sigma^2(b)]_{\mathbb{T}}$, and $h \geq 0$, then the unique solution y of the problem* (2.3), (2.4) *satisfies*

$$\inf_{t \in [\eta, \sigma^2(b)]_{\mathbb{T}}} y(t) \geq \gamma \|y\|,$$

where

$$\gamma := \min \left\{ \frac{\delta(\sigma^2(b) - \eta)}{\sigma^2(b) - \delta\eta - a(1-\delta)}, \frac{\delta\eta}{\sigma^2(b)} \right\}.$$

Notes

Proof: By the second boundary condition we know that $y(\eta) \geq y(\sigma^2(b))$. Pick $t_0 \in (a, \sigma^2(b))_{\mathbb{T}}$ such that $y(t_0) = \|y\|$. If $t_0 < \eta < \sigma^2(b)$, then

$$\min_{t\in[\eta,\sigma^2(b)]_{\mathbb{T}}} y(t) = y(\sigma^2(b)),$$

and

$$\frac{y(\sigma^2(b)) - y(\eta)}{\sigma^2(b) - \eta} \leq \frac{y(\eta) - y(t_0)}{\eta - t_0}.$$

Therefore

$$\min_{t\in[\eta,\sigma^2(b)]_{\mathbb{T}}} y(t) \geq \frac{\delta(\sigma^2(b) - \eta)}{\sigma^2(b) - \delta\eta - a(1-\delta)}\|y\|.$$

If $\eta \leq t_0 < \sigma^2(b)$, again we have $y(\sigma^2(b)) = \min_{t\in[\eta,\sigma^2(b)]_{\mathbb{T}}} y(t)$. From Lemma 2.3, we have $\frac{y(\eta)}{\eta} \geq \frac{y(t_0)}{t_0}$. Combining with the boundary condition $\delta y(\eta) = y(\sigma^2(b))$, we conclude that

$$\frac{y(\sigma^2(b))}{\delta\eta} \geq \frac{y(t_0)}{t_0} \geq \frac{y(t_0)}{\sigma^2(b)} = \frac{\|y\|}{\sigma^2(b)}.$$

This is

$$\min_{t\in[\eta,\sigma^2(b)]_{\mathbb{T}}} y(t) \geq \frac{\delta\eta}{\sigma^2(b)}\|y\|.$$

We note that a pair $(u(t), v(t))$ is a solution of the eigenvalue problem (1.1), (1.2) if, and only if,

$$u(t) = \lambda\int_a^{\sigma(b)} G(t,s)p(s)f\left(\lambda\int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\right)\Delta s,\ a \leq t \leq \sigma^2(b),$$

where

$$v(t) = \lambda\int_a^{\sigma(b)} G(t,s)q(s)g(u(\sigma(s)))\Delta s, \quad a \leq t \leq \sigma^2(b).$$

Values of λ for which there are positive solutions (positive with respect to a cone) of (1.1), (1.2) will be determined via applications of the following fixed point-theorem [17].

Theorem 2.5 (***Krasnosel'skii***) *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that, either

(i) $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2,$ *or*

Ref.

[17] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff Ltd, Groningen, The Netherlands (1964).

(ii) $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2.$

Then, T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

III. Positive Solutions in a Cone

In this section, we apply Theorem 2.5 to obtain solutions in a cone (i.e., positive solutions) of (1.1), (1.2).

For our construction, let $\mathcal{B} = \{x : [a, \sigma^2(b)]_{\mathbb{T}} \to \mathbb{R}\}$ with supremum norm $\|x\| = \sup\{| x(t) | : t \in [a, \sigma^2(b)]_{\mathbb{T}}\}$ and define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{ x \in \mathcal{B} \mid x(t) \geq 0 \text{ on } [a, \sigma^2(b)]_{\mathbb{T}}, \text{ and } \min_{t \in [\eta, \sigma^2(b)]_{\mathbb{T}}} x(t) \geq \gamma \|x\| \right\}.$$

For our first result, define positive numbers L_1 and L_2 by

$$L_1 := \max\left\{ \left[\gamma \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s)) p(s) \Delta s f_\infty \right]^{-1}, \left[\gamma \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s)) q(s) \Delta s g_\infty \right]^{-1} \right\},$$

and

$$L_2 := \min\left\{ \left[\frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(s)) p(s) \Delta s f_0 \right]^{-1}, \left[\frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(s)) q(s) \Delta s g_0 \right]^{-1} \right\}.$$

Theorem 3.1 *Assume that conditions $(A1) - (A3)$ are satisfied. Then, for each λ satisfying*

$$L_1 < \lambda < L_2, \tag{3.1}$$

there exists a pair (u, v) satisfying (1.1), (1.2) such that $u(x) > 0$ and $v(x) > 0$ on $(a, \sigma^2(b))_{\mathbb{T}}$.

Proof: Let λ be as in (3.1). And let $\epsilon > 0$ be chosen such that

$$\max\left\{ \left[\gamma \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s)) p(s) \Delta s (f_\infty - \epsilon) \right]^{-1}, \right.$$

Notes

$$\left[\gamma\frac{\beta+\alpha(\eta-a)}{d}\int_{\eta}^{\sigma(b)}(\sigma^2(b)-\sigma(s))q(s)\Delta s(g_\infty-\epsilon)\right]^{-1}\Bigg\}\le\lambda,$$

and

Notes

$$\lambda\le\min\left\{\left[\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)\Delta s(f_0+\epsilon)\right]^{-1},\right.$$

$$\left.\left[\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))q(s)\Delta s(g_0+\epsilon)\right]^{-1}\right\},$$

Define an integral operator $T:\mathcal{P}\to\mathcal{B}$ by

$$Tu(t):=\lambda\int_a^{\sigma(b)}G(t,s)p(s)f\left(\lambda\int_a^{\sigma(b)}G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\right)\Delta s,\ u\in\mathcal{P}. \tag{3.2}$$

We seek suitable fixed points of T in the cone $\mathcal{P}$.

By Lemma 2.4, $T\mathcal{P}\subset\mathcal{P}$. In addition, standard arguments show that T is completely continuous.

Now, from the definitions of f_0 and g_0, there exists $H_1>0$ such that

$$f(x)\le(f_0+\epsilon)x\ \text{ and }\ g(x)\le(g_0+\epsilon)x,\quad 0<x\le H_1.$$

Let $u\in\mathcal{P}$ with $\|u\|=H_1$. We first have from (2.6) and choice of ϵ,

$$\begin{aligned}
&\lambda\int_a^{\sigma(b)}G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\\
&\le\lambda\frac{\beta+\alpha(t-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)g(u(\sigma(r)))\Delta r\\
&\le\lambda\frac{\beta+\alpha(t-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)(g_0+\epsilon)u(r)\Delta r\\
&\le\lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)\Delta r(g_0+\epsilon)\|u\|\\
&\le\|u\|\\
&=H_1.
\end{aligned}$$

As a consequence, we next have from (2.6) and choice of ϵ,

$$Tu(t)=\lambda\int_a^{\sigma(b)}G(t,s)p(s)f\left(\lambda\int_a^{\sigma(b)}G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\right)\Delta s$$

$$\leq \lambda\frac{\beta+\alpha(t-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)$$
$$f\left(\lambda\int_a^{\sigma(b)}G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\right)\Delta s$$
$$\leq \lambda\frac{\beta+\alpha(t-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)$$
$$(f_0+\epsilon)\lambda\int_a^{\sigma(b)}G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Delta s$$
$$\leq \lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)(f_0+\epsilon)H_1\Delta s$$
$$\leq H_1$$
$$= \|u\|.$$

So, $\|Tu\|\leq\|u\|$. If we set

$$\Omega_1=\{x\in\mathcal{B}:\|x\|<H_1\},$$

then

$$\|Tu\|\leq\|u\|,\quad \text{for}\quad u\in\mathcal{P}\cap\partial\Omega_1. \tag{3.3}$$

Next, from the definitions of f_∞ and g_∞, there exists $\overline{H}_2>0$ such that

$$f(x)\geq(f_\infty-\epsilon)x \text{ and } g(x)\geq(g_\infty-\epsilon)x,\quad x\geq\overline{H}_2.$$

Let

$$H_2=\max\left\{2H_1,\frac{\overline{H}_2}{\gamma}\right\}.$$

Let $u\in\mathcal{P}$ and $\|u\|=H_2$. Then,

$$\min_{t\in[\eta,\sigma^2(b)]_{\mathbb{T}}}u(t)\geq\gamma\|u\|\geq\overline{H}_2.$$

Consequently, from (2.7) and choice of ϵ,

$$\lambda\int_a^{\sigma(b)}G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r$$
$$\geq \lambda\frac{\beta+\alpha(\eta-a)}{d}\int_\eta^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)g(u(\sigma(r)))\Delta r$$
$$\geq \lambda\frac{\beta+\alpha(\eta-a)}{d}\int_\eta^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)(g_\infty-\epsilon)u(r)\Delta r$$

Notes

$$\begin{aligned}
&\geq \lambda \frac{\beta+\alpha(\eta-a)}{d} \int_{\eta}^{\sigma(b)} (\sigma^2(b)-\sigma(r))q(r)(g_\infty-\epsilon)\Delta r\gamma\|u\| \\
&\geq \|u\| \\
&= H_2.
\end{aligned}$$

Notes

And so, we have from (2.7) and choice of ϵ,

$$\begin{aligned}
Tu(\eta) \geq\ & \lambda \frac{\beta+\alpha(\eta-a)}{d} \int_{\eta}^{\sigma(b)} (\sigma^2(b)-\sigma(s))p(s) \\
& f\left(\lambda \int_{\eta}^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\right)\Delta s \\
\geq\ & \lambda \frac{\beta+\alpha(\eta-a)}{d} \int_{\eta}^{\sigma(b)} (\sigma^2(b)-\sigma(s))p(s) \\
& (f_\infty-\epsilon)\lambda \int_{\eta}^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Delta s \\
\geq\ & \lambda \frac{\beta+\alpha(\eta-a)}{d} \int_{\eta}^{\sigma(b)} (\sigma^2(b)-\sigma(s))p(s)(f_\infty-\epsilon)H_2\Delta s \\
\geq\ & \lambda\gamma \frac{\beta+\alpha(\eta-a)}{d} \int_{\eta}^{\sigma(b)} (\sigma^2(b)-\sigma(s))p(s)(f_\infty-\epsilon)H_2\Delta s \\
\geq\ & H_2 \\
=\ & \|u\|.
\end{aligned}$$

Hence, $\|Tu\| \geq \|u\|$. So, if we set

$$\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_2\},$$

then

$$\|Tu\| \geq \|u\|, \quad \text{for} \quad u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.4}$$

Applying Theorem 2.5 to (3.3) and (3.4), we obtain that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$. As such, and with v being defined by

$$v(t) = \lambda \int_{a}^{\sigma(b)} G(t,s)q(s)g(u(\sigma(s)))\Delta s,$$

the pair (u,v) is a desired solution of (1.1), (1.2) for the given λ. The proof is complete.

Prior to our next result, we define positive numbers L_3 and L_4 by

$$\begin{aligned}
L_3 := \max\Bigg\{ & \left[\gamma \frac{\beta+\alpha(\eta-a)}{d} \int_{\eta}^{\sigma(b)} (\sigma^2(b)-\sigma(s))p(s)\Delta s f_0\right]^{-1}, \\
& \left[\gamma \frac{\beta+\alpha(\eta-a)}{d} \int_{\eta}^{\sigma(b)} (\sigma^2(b)-\sigma(s))q(s)\Delta s g_0\right]^{-1} \Bigg\},
\end{aligned}$$

and

$$L_4 := \min\left\{ \left[\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)\Delta s f_\infty\right]^{-1}, \left[\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))q(s)\Delta s g_\infty\right]^{-1}\right\}.$$

Theorem 3.2 *Assume that conditions $(A1)-(A4)$ are satisfied. Then, for each λ satisfying*

$$L_3 < \lambda < L_4, \tag{3.5}$$

there exists a pair (u, v) satisfying (1.1), (1.2) *such that $u(x) > 0$ and $v(x) > 0$ on $(a, \sigma^2(b))_{\mathbb{T}}$.*

Proof: Let λ be as in (3.5). And let $\epsilon > 0$ be chosen such that

$$\max\left\{ \left[\gamma\frac{\beta+\alpha(\eta-a)}{d}\int_\eta^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)\Delta s(f_0-\epsilon)\right]^{-1}, \left[\gamma\frac{\beta+\alpha(\eta-a)}{d}\int_\eta^{\sigma(b)}(\sigma^2(b)-\sigma(s))q(s)\Delta s(g_0-\epsilon)\right]^{-1}\right\} \le \lambda,$$

and

$$\lambda \le \min\left\{ \left[\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)\Delta s(f_\infty+\epsilon)\right]^{-1}, \left[\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))q(s)\Delta s(g_\infty+\epsilon)\right]^{-1}\right\}.$$

Let T be the cone preserving, completely continuous operator that was defined by (3.2).

From the definitions of f_0 and g_0, there exists $H_3 > 0$ such that

$$f(x) \ge (f_0-\epsilon)x \quad\text{and}\quad g(x) \ge (g_0-\epsilon)x, \qquad 0 < x \le H_3.$$

Also, from the definition of g_0 it follows that $g(0) = 0$ and so there exists $0 < H_3 < \overline{H}_3$ such that

$$\lambda g(x) \le \frac{\overline{H}_3}{\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)\Delta r}, \quad 0 \le x \le H_3.$$

Notes

Notes

Choose $u \in \mathcal{P}$ with $\|u\| = H_3$. Then

$$\begin{aligned}
&\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r \\
&\quad \le \lambda \frac{\beta + \alpha(t-a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(r))q(r)g(u(\sigma(r)))\Delta r \\
&\quad \le \lambda \frac{\beta + \alpha(\sigma^2(b)-a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(r))q(r)g(u(\sigma(r)))\Delta r \\
&\quad \le \frac{\frac{\beta+\alpha(\sigma^2(b)-a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(r))q(r)\overline{H}_3\Delta r}{\frac{\beta+\alpha(\sigma^2(b)-a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(s))q(s)\Delta s} \\
&\quad \le \overline{H}_3.
\end{aligned}$$

Then, by (2.7)

$$\begin{aligned}
Tu(\eta) &\ge \lambda \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s))p(s) \\
&\qquad f\left(\lambda \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(r))q(r)g(u(\sigma(r)))\Delta r\right)\Delta s \\
&\ge \lambda \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s))p(s) \\
&\qquad (f_0 - \epsilon)\lambda \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(r))q(r)g(u(\sigma(r)))\Delta r\Delta s \\
&\ge \lambda \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s))p(s) \\
&\qquad (f_0 - \epsilon)\lambda\gamma \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(r))q(r)(g_0 - \epsilon))\|u\|\Delta r\Delta s \\
&\ge \lambda \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s))p(s)(f_0 - \epsilon)\|u\|\Delta s \\
&\ge \lambda\gamma \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s))p(s)(f_0 - \epsilon)\|u\|\Delta s \\
&\ge \|u\|.
\end{aligned}$$

So, $\|Tu\| \ge \|u\|$. If we put

$$\Omega_3 = \{x \in \mathcal{B} : \|x\| < H_3\},$$

then

$$\|Tu\| \ge \|u\|, \quad \text{for} \quad u \in \mathcal{P} \cap \partial\Omega_3. \tag{3.6}$$

Next, by definitions of f_∞ and g_∞, there exists $\overline{H}_4$ such that

$$f(x) \leq (f_\infty + \epsilon)x \;\text{ and }\; g(x) \leq (g_\infty + \epsilon)x, \quad x \geq \overline{H}_4$$

Clearly, since g_∞ is assumed to be a positive real number, it follows that g is unbounded at ∞, and so, there exists $\widetilde{H}_4 > \max\{2H_3, \overline{H}_4\}$ such that $g(x) \leq g(\widetilde{H}_4)$, for $0 < x \leq \widetilde{H}_4$.

Set

$$f^*(t) = \sup_{a \leq s \leq t} f(s), \quad g^*(t) = \sup_{a \leq s \leq t} g(s), \text{ for } t \geq 0.$$

Clearly f^* and g^* are nondecreasing real valued functions for which it holds

$$\lim_{x\to\infty} \frac{f_i^*(x)}{x} = f_\infty, \quad \lim_{x\to\infty} \frac{g_i^*(x)}{x} = g_\infty.$$

Hence, there exists H_4 such that $f^*(x) \leq f^*(H_4)$, $g^*(x) \leq g^*(H_4)$, for $0 < x \leq H_4$.

Choosing $u \in \mathcal{P}$ with $\|u\| = H_4$, we have

$$\begin{aligned}
Tu(t) \leq{}& \lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)\\
&f\left(\lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)g(u(\sigma(r)))\Delta r\right)\Delta s\\
\leq{}& \lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)\\
&f^*\left(\lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)g(u(\sigma(r)))\Delta r\right)\Delta s\\
\leq{}& \lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)\\
&f^*\left(\lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)g^*(u(\sigma(r)))\Delta r\right)\Delta s\\
\leq{}& \lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)\\
&f^*\left(\lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)g^*(H_4)\Delta r\right)\Delta s\\
\leq{}& \lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)\\
&f^*\left(\lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)(g_\infty+\epsilon)H_4\Delta r\right)\Delta s\\
\leq{}& \lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)f^*(H_4)\Delta s
\end{aligned}$$

Notes

$$\leq \lambda \frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^\sigma (\sigma^2(b) - \sigma(s))p(s)(f_\infty + \epsilon)H_4 \Delta s$$
$$\leq H_4$$
$$= \|u\|,$$

and so $\|Tu\| \leq \|u\|$. For this case, if we let

Notes

$$\Omega_4 = \{x \in \mathcal{B} : \|x\| < H_4\},$$

then

$$\|Tu\| \leq \|u\|, \quad \text{for} \quad u \in \mathcal{P} \cap \partial\Omega_4. \tag{3.7}$$

Application of part (ii) of Theorem 2.5 yields a fixed point u of T belonging to $\mathcal{P} \cap (\overline{\Omega}_4 \backslash \Omega_3)$, which in turn yields a pair (u, v) satisfying (1.1), (1.2) for the chosen value of λ. The proof is complete.

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