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Application of Laplace Transform

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Abstract - The present discounted value equation in finance has a broad range of uses and may be applied to various areas of finance including corporate finance, banking finance and investment finance etc .The basic premise of present discounted value is the time value money .Not many analytic solutions exist for present discounted value problems but by using Laplace transform we can deduce some of the closed form solutions quite easily. In this note we show how present discounted value in finance related to Laplace transforms. Also we discus on the present value rules for the elementary functions and the general properties of the Laplace transform. And we will focus on the application of time derivative property using Laplace transforms to each present value rule.

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APPLICATION OF LAPLACE TRANSFORM

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Application of Laplace Transform

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Abstract - The present discounted value equation in finance has a broad range of uses and may be applied to various areas of finance including corporate finance, banking finance and investment finance etc .The basic premise of present discounted value is the time value money .Not many analytic solutions exist for present discounted value problems but by using Laplace transform we can deduce some of the closed form solutions quite easily. In this note we show how present discounted value in finance related to Laplace transforms. Also we discus on the present value rules for the elementary functions and the general properties of the Laplace transform. And we will focus on the application of time derivative property using Laplace transforms to each present value rule.

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I. Introduction

During the past few decades, methods based on integral transforms, in particular, the Laplace transforms, are being increasingly employed in mathematics, physics, mechanics and other engineering sciences. Laplace transforms have a wide variety of applications in the solution of differential, integral and difference equations. It is much less used in financial engineering. One reason is technical: not many people know that all that they need to do is to make simple calculations in the Laplace domain.

The outline of this note is as follows –

In section 1 we show the relation between present discounted value and Laplace transforms.

In section 2 we identified the present value rules for each of the cash flow.

In section 3 we discussed the general properties of Laplace transforms with present value rules.

In section 4 we show the application of time derivative property to each of the present value rules.

II. Relation Between Present Value and Laplace Transform

The Present value of a series of payments given by,

$$(PV)_t = \sum_{t=1}^{T} \frac{C(t)}{(1+r)^t} \tag{1}$$

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Where, $(PV)_t$ = Present discounted value at time t

$C(t)$ = Cash flow

r = rate of discount

t = Period

Here we assume the Present value with continuous compounding .It is the current value of a stream of cash flows. In other words, it is the amount that we would be willing to pay today in order to receive a cash flow or a series of them in the future.

Now by using an exponential series we can write equation (1) as,

$$(PV)_t = \sum_{t=1}^{T} e^{-rt}\, C(t) \tag{2}$$

In the limiting case replacing summation to an integral, equation (2) can be written as

$$(PV)_r = \int_0^T e^{-rt}\, C(t)\, dt \tag{3}$$

Again here T is some finite quantity. So if we consider as $T \to \infty$, equation (3) will becomes

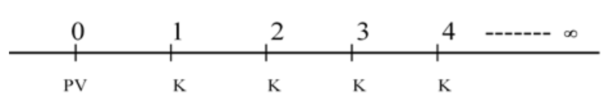
$$(PV)_r = \int_0^\infty e^{-rt}\, C(t)\, dt \tag{4}$$

This equation is the exact replica of Laplace transform in mathematics.

Therefore,
$$(PV)_r = L\left[C(t)\right] \tag{5}$$

III. Laplace Transforms and Present Value Rules for Some Cash Flows

Using Present value equation: Consider the case of constant cash payment K made at the end of each year at interest rate r as shown in the following time line,



Here the cash flow is continuous forever Therefore the Present value is given by an infinite geometric series:

$$PV = \frac{K}{1+r} + \frac{K}{(1+r)^2} + \frac{K}{(1+r)^3} + ------ \tag{6}$$

Dividing both sides by (1+r) we get,

$$\frac{PV}{1+r} = \frac{K}{(1+r)^2} + \frac{K}{(1+r)^3} + \frac{K}{(1+r)^4} ------ \tag{7}$$

Notes

Notes

Subtracting equation (7) from (6) we get,

$$PV - \frac{PV}{1+r} = \frac{K}{1+r}$$

On solving we get the Present value of perpetuity.

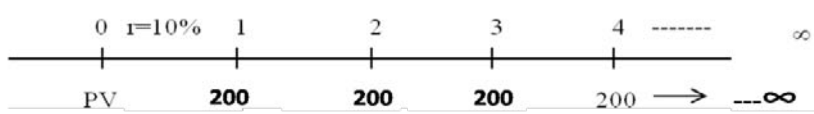
Using Laplace transform equation: If the cash flow is constant say K then the Present discounted value of a stream at interest rate r is given by,

$$L[K] = K\int_0^{\infty} e^{-rt}\, dt = \frac{K}{r}$$

This is the same formula as above.

For example: An insurance company has just launched a security that will pay Rs.200 indefinitely, starting the first payment next year. How much should this security be worth today if the appropriate return is 10% ?

We solve this example by using the time line,



$$PV = \frac{K}{r} = \frac{200}{0.10} = Rs.2000$$

Using Present value equation: Consider the payments in perpetuity increases at a certain growth rate g as shown on the time line:

The Present value of a growing perpetuity can be written as the following infinite series-

$$PV = \frac{K}{1+r} + \frac{K(1+g)}{(1+r)^2} + \frac{K(1+g)^2}{(1+r)^3} + ------ \tag{8}$$

Multiplying both sides by $\frac{(1+g)}{(1+r)}$. Hence we get,

$$PV\frac{(1+g)}{(1+r)} = \frac{K(1+g)}{(1+r)^2} + \frac{K(1+g)^2}{(1+r)^3} + \frac{K(1+g)^3}{(1+r)^4} ------ \tag{9}$$

Subtracting equation (9) from (8) we get,

$$PV - \frac{PV(1+g)}{1+r} = \frac{K}{1+r}$$

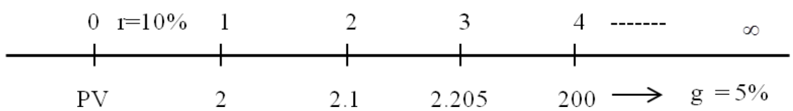
On solving we get the Present value of a growing perpetuity.

Using Laplace transform equation: For an exponential or geometric cash flow the Present discounted value of a stream growing at rate g, is given by:

$$L[C(t)] = K\, L\left[e^{g\,t}\right] = \frac{K}{r-g} \quad if \;\; r > g \tag{10}$$

This is the geometric growth stream or Present value of growing perpetuity having cash flow after the first period divided by the difference between the discount rate and the growth rate and the growth rate must be less than the interest rate.

For example: A company is expected to pay Rs.2 of dividend per share that wills increases 5% forever. If investors require 10% return on the company's stocks, how much should investors pay for the stocks? The cash flows are as follows:



Notes

$$PVG = \frac{K}{r-g} = \frac{2}{0.10-0.05} = Rs.40 \tag{11}$$

For an arithmetic cash flow the Present discounted value of a stream at rate r, is given by:

$$L\left[C(t)\right] = \mathrm{K}\, L[t] = \frac{K}{r^2} \tag{12}$$

This shows that an arithmetic growth stream is equivalent to receiving one consol per period in perpetuity. This rule is widely used in finance for solving Preset value.

The above rules are commonly used transforms but more useful are the general properties of the Laplace transforms in an algebraic fashion .Let us look at some of the main properties.

Property 1: Linearity: The Laplace Transform is a linear operator. Hence if the Laplace Transforms of the cash flows f(t) and g(t) both exists then we have for any arbitrary constants (a,b) that:

$$L\left\{a\ f(t) + b\ g(t)\right\} = a\ L\left\{f(t)\right\} + b\ \ L\left\{g(t)\right\} \tag{13}$$

This property allows us to deduce more complex transforms to simple transforms.

Property 2: Geometric scaling:

$$L\left\{e^{\alpha t} c(t)\right\} = V(r-\alpha) \quad for\ \alpha < r \tag{14}$$

This property shows that the scaling a cash flow by geometric term is equivalent to corresponding reduction in the rate of discount.

Property 3: Multiplication by t:

$$\mathrm{L}\left\{\mathrm{t\ C(t)}\right\} = -\mathrm{V}'\ (\mathrm{r}) \tag{15}$$

We can confirm this property by using the derivative of exponential function.

Property 4: Time shifting:

$$\left.\begin{array}{ll} \mathrm{L}\left\{\mathrm{C(a+bt)}\right\} & \mathrm{for\ \ t} \geq a/b \\ 0 & \mathrm{for\ \ t} < a/b \end{array}\right\} = \mathrm{e}^{r\,a/b} \left(\frac{1}{b}\right) \mathrm{P\ (r/b)} \tag{16}$$

This property applies the change of variable theorem of integral calculus and helpful for finding cash flows with altered time schedules.

Property 5: Time derivative:

$$L\{C'(t)\} = r\,L\{C(t)\} - C(0) \tag{17}$$

This property identifies a fundamental linear relationship between Laplace transform for cash flows and their time derivatives. For the confirmation we use integration by parts:

$$\int u\,dv = uv - \int v\,du$$

When we evaluate the integral over relevant range 0 to ∞ for the Laplace transform and impose a standard assumption in present value problems that the marginal present value of the cash flow vanishes as t gets large.

All present value rules of section second can be derived from this time derivative property of Laplace transform and hence having particular significance in finance.

For the confirmation we rewrite the time-derivative property by using notation

$(PV)_r = L\left[C(t)\right]$, as:

$$L[C(t)] = \frac{C(0)}{r} + \frac{L\left[C'(t)\right]}{r} \tag{18}$$

Apply the property, for some cash flows.

Ex: For S (t) =K$\Rightarrow$ S (0) =K & $S'(t)$ =0, we get the consol rule by using property 5.

Alternatively, each asset is valued as if its cash flow were projected at a constant level equal to the current rate plus the present value of the time derivative of the cash flow.

For geometric cash flow (2.I)

$S(t) = e^{\alpha t} \Rightarrow S(0) = 1$ & $S'(t) = \alpha\, e^{\alpha t}$, we get

$$L[e^{\alpha t}] = \frac{1}{\theta} + \frac{\alpha L\left[e^{\alpha t}\right]}{\theta} \tag{19}$$

Alternatively, we could combine the consol rule with property 2 to establish the rule for geometric growth.

Similarly we can derive the rule for arithmetic growth by using equation (19) or combining the consol rule and property 3.

IV. Conclusion

In this article we have presented the close relationship between present discounted value in finance and Laplace transform. We can solve the present discounted value examples within a few minutes by using Laplace equation method. The result seems to be new & to have a potential to increase the practical utility of Laplace transform especially in finance. However it is important to notice that frequency domain is possible appreciate also in the real world & applied in the areas like economics or finance. But the Laplace transform is the big source for present discounted value function to illustrate the enhanced problems.

Notes

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