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Applications of Continued Fraction Identities

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Abstract – In present paper we established four new expressions on q-product identities with the applications of continued fractions in recent results established by the author [7].

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I. Introduction

For $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) \tag{1.1}$$

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{(n-1)}) \tag{1.2}$$

$$(a_1,a_2,a_3,...,a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty...(a_k;q)_\infty \tag{1.3}$$

Ramanujan has defined general theta function, as

$$f(a,b) = \sum_{-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \;;\; |ab| < 1, \tag{1.4}$$

Jacobi's triple product identity [1, p.35] is given, as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{1.5}$$

Special cases of Jacobi's triple products identity are given, as

$$\Phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty \tag{1.6}$$

$$\Psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.7}$$

$$f(-q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.8}$$

Equation (1.8) is known as Euler's pentagonal number theorem. Euler's another well known identity is as

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Ref.

(1) B.C. Berndt : *Ramanujan's Notebooks, Part III*, Springer-Verlag, NewYork, 1991.

$$(q;q^2)_\infty^{-1} = (-q;q)_\infty \tag{1.9}$$

Throughout this paper we use the following representations

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty = (q^a,q^b,q^c\cdots q^t;q^n)_\infty \tag{1.13}$$

$$(q^a;q^n)_\infty(q^a;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty = (q^a,q^a,q^c\cdots q^t;q^n)_\infty \tag{1.14}$$

Notes

Computation of q-product identities:

we can have following q-products identities, as

$$(q^2;q^2)_\infty = \prod_{n=0}^{\infty}(1-q^{2n+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{2(4n)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+1)+2})\times$$

$$\times \prod_{n=0}^{\infty}(1-q^{2(4n+2)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+3)+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{8n+2}) \times \prod_{n=0}^{\infty}(1-q^{8n+4})\times$$

$$\times \prod_{n=0}^{\infty}(1-q^{8n+6}) \times \prod_{n=0}^{\infty}(1-q^{8n+8})$$

$$= (q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty$$

$$= (q^2,q^4,q^6,q^8;q^8)_\infty \tag{1.15}$$

$$(q^4;q^4)_\infty = \prod_{n=0}^{\infty}(1-q^{4n+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{4(3n)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+1)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+2)+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{12n+4}) \times \prod_{n=0}^{\infty}(1-q^{12n+8}) \times \prod_{n=0}^{\infty}(1-q^{12n+12})$$

$$= (q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty$$

$$= (q^4,q^8,q^{12};q^{12})_\infty \tag{1.16}$$

$$(q^4;q^{12})_\infty = \prod_{n=0}^{\infty}(1-q^{12n+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{12(5n)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+1)+4})\times$$

$$\times \prod_{n=0}^{\infty}(1-q^{12(5n+2)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+3)+4})\times$$

$$\times \prod_{n=0}^{\infty}(1-q^{12(5n+4)+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{60n+4}) \times \prod_{n=0}^{\infty}(1-q^{60n+16}) \times \prod_{n=0}^{\infty}(1-q^{60n+28})\times$$

$$\times \prod_{n=0}^{\infty}(1-q^{60n+40}) \times \prod_{n=0}^{\infty}(1-q^{60n+52})$$

$$= (q^4;q^{60})_\infty (q^{16};q^{60})_\infty (q^{28};q^{60})_\infty (q^{40};q^{60})_\infty (q^{52};q^{60})_\infty$$

$$= (q^4,q^{16},q^{28},q^{40},q^{52};q^{60})_\infty \tag{1.17}$$

Similarly we can compute following, as

$$(q^4;q^{12})_\infty = (q^4;q^{60})_\infty (q^{16};q^{60})_\infty (q^{28};q^{60})_\infty (q^{40};q^{60})_\infty (q^{52};q^{60})_\infty$$

$$= (q^4,q^{16},q^{28},q^{40},q^{52};q^{60})_\infty \tag{1.18}$$

$$(q^6;q^6)_\infty = (q^6;q^{24})_\infty (q^{12};q^{24})_\infty (q^{18};q^{24})_\infty (q^{24};q^{24})_\infty$$

$$= (q^6,q^{12},q^{18},q^{24};q^{24})_\infty \tag{1.19}$$

$$(q^6;q^{12})_\infty = (q^6;q^{60})_\infty (q^{18};q^{60})_\infty (q^{30};q^{60})_\infty (q^{42};q^{60})_\infty (q^{54};q^{60})_\infty$$

$$= (q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty \tag{1.20}$$

$$(q^8;q^8)_\infty = (q^8;q^{48})_\infty (q^{16};q^{48})_\infty (q^{24};q^{48})_\infty (q^{32};q^{48})_\infty (q^{40};q^{48})_\infty (q^{48};q^{48})_\infty$$

$$= (q^8,q^{16},q^{24},q^{32},q^{40},q^{48};q^{48})_\infty \tag{1.21}$$

$$(q^8;q^{12})_\infty = (q^8;q^{60})_\infty (q^{20};q^{60})_\infty (q^{32};q^{60})_\infty (q^{44};q^{60})_\infty (q^{56};q^{60})_\infty$$

$$= (q^8,q^{20},q^{32},q^{44},q^{56};q^{60})_\infty \tag{1.22}$$

$$(q^8;q^{16})_\infty = (q^8;q^{48})_\infty (q^{24};q^{48})_\infty (q^{40};q^{48})_\infty$$

$$= (q^8,q^{24},q^{40};q^{48})_\infty \tag{1.23}$$

$$(q^{10};q^{20})_\infty = (q^{10};q^{60})_\infty (q^{30};q^{60})_\infty (q^{50};q^{60})_\infty$$

$$= (q^{10},q^{30},q^{50};q^{60})_\infty \tag{1.24}$$

$$(q^{12};q^{12})_\infty = (q^{12};q^{60})_\infty (q^{24};q^{60})_\infty (q^{36};q^{60})_\infty (q^{48};q^{60})_\infty (q^{60};q^{60})_\infty$$

$$= (q^{12},q^{24},q^{36},q^{48},q^{60};q^{60})_\infty \tag{1.25}$$

Notes

$$(q^{16};q^{16})_\infty = (q^{16};q^{48})_\infty (q^{32};q^{48})_\infty (q^{48};q^{48})_\infty$$
$$= (q^{16},q^{32},q^{48};q^{48})_\infty \quad (1.26)$$

$$(q^{20};q^{20})_\infty = (q^{20};q^{60})_\infty (q^{40};q^{60})_\infty (q^{60};q^{60})_\infty$$
$$= (q^{20},q^{40},q^{60};q^{60})_\infty \quad (1.27)$$

The outline of this paper is as follows. In sections 2, we have recorded some well known results on continued fraction identities and recent results on q-products identities given by the author[7], those are useful to the rest of the paper. In section 3, we state and prove four new results related to q-product identities with theapplications of continued fraction identities.

II. Preliminaries

In 1983 Denis [5], has introduced following continued fraction identity

$$(q^2;q^2)_\infty(-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1+\vdots}}}}}}} \quad (2.1)$$

The famous Rogers-Ramanujan continued fraction identity [3, (1.6)], is

$$\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\vdots}}}}} \quad (2.2)$$

A well known continued fraction identity due to Ramanujan [4, (4.21)], is

$$\frac{(-q^3;q^4)_\infty}{(-q;q^4)_\infty} = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^3+q^2}{1+\cfrac{q^5}{1+\cfrac{q^7+q^4}{1+\cfrac{q^9}{1+\cfrac{q^{11}+q^6}{1+\vdots}}}}}}} \quad (2.3)$$

Ref.

(7) M.P. Chaudhary : *On q-product identities*, pre-print.

One of the most celebrated continued fractional identities associated with Ramanujan's academic career, given by Rogers-Ramanujan [6], is

$$C(q)=\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}=1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cfrac{q^5}{1+\vdots}}}}} \tag{2.4}$$

Ref.

Recently Chaudhary [7], has introduce following q-product identities

$$(q^2,q^4,q^6;q^8)_\infty[(-q;q^2)^2_\infty+(q;q^2)^2_\infty]=2(-q^4;q^8)^2_\infty \tag{2.5}$$

$$(q^2,q^4,q^6,q^8;q^8)_\infty[(-q;q^2)^2_\infty-(q;q^2)^2_\infty]=4q\frac{(q^{16},q^{32},q^{48};q^{48})_\infty}{(q^8,q^{24},q^{40};q^{48})_\infty} \tag{2.6}$$

$$\frac{(-q;q^2)^2_\infty+(q;q^2)^2_\infty}{(-q;q^2)^2_\infty-(q;q^2)^2_\infty}=\frac{(-q^4;q^8)^2_\infty(q^8,q^8,q^{24},q^{24},q^{40},q^{40};q^{48})_\infty}{2q} \tag{2.7}$$

$$(-q;q^2)^2_\infty(q;q^2)^2_\infty(q^2;q^2)^2_\infty=(q^2,q^2,q^4;q^4)_\infty \tag{2.8}$$

$$\frac{(-q;q^2)_\infty(-q^3;q^6)_\infty-(q;q^2)_\infty(q^3;q^6)_\infty}{(-q;q^2)_\infty\times(-q^3;q^6)_\infty\times(q;q^2)_\infty\times(q^3;q^6)_\infty}$$

$$=\frac{2q(-q^2;q^4)^2_\infty(q^4,q^8,q^{16},q^{20},q^{24};q^{24})_\infty}{(q^2,q^4,q^6,q^8;q^8)_\infty(q^6,q^{12},q^{18};q^{24})_\infty} \tag{2.9}$$

$$\frac{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty-(q^3;q^6)_\infty(q^5;q^{10})_\infty}{(-q^3;q^6)_\infty\times(-q^5;q^{10})_\infty\times(q^3;q^6)_\infty\times(q^5;q^{10})_\infty}$$

$$=\frac{(q^4,q^8,q^{12};q^{12})_\infty}{(q^6,q^{12},q^{18},q^{24};q^{24})_\infty}\times$$

$$\times\frac{2q^3}{(q^2,q^6,q^{10};q^{12})_\infty(q^{10},q^{20},q^{30},q^{30},q^{40},q^{50};q^{60})_\infty} \tag{2.10}$$

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(7) M.P. Chaudhary : *On q-product identities*, pre-print.

And,

$$\frac{[(q;q^2)_\infty(q^{15};q^{30})_\infty]+[(-q;q^2)_\infty(-q^{15};q^{30})_\infty]}{[(q;q^2)_\infty(q^{15};q^{30})_\infty][(-q;q^2)_\infty(-q^{15};q^{30})_\infty]}$$

$$=\frac{(q^{12},q^{20},q^{24},q^{36},q^{40},q^{48},q^{60},q^{60};q^{60})_\infty}{(q^{10},q^{30},q^{30},q^{50},q^{60};q^{60})_\infty}\times$$

$$\times\frac{2}{(q^2,q^4,q^6,q^8,q^8;q^8)_\infty(q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty} \tag{2.11}$$

III. Main Results

In this section, we established and proved following identities with the applications of continued fraction identities in the q-product identities, recently given by Chaudhary [7], as

$$(q^2,q^4,q^6,q^8;q^8)_\infty[(-q;q^2)_\infty^2-(q;q^2)_\infty^2]$$

$$=\frac{4q(q^{16},q^{32};q^{48})_\infty}{(q^8,q^{40};q^{48})_\infty}\times\cfrac{1}{1-\cfrac{q^{24}}{1+\cfrac{q^{24}(1-q^{24})}{1-\cfrac{q^{72}}{1+\cfrac{q^{48}(1-q^{48})}{1-\cfrac{q^{120}}{1+\cfrac{q^{72}(1-q^{72})}{1+\vdots}}}}}}} \tag{3.1}$$

$$\frac{(-q;q^2)_\infty(-q^3;q^6)_\infty-(q;q^2)_\infty(q^3;q^6)_\infty}{(-q;q^2)_\infty\times(-q^3;q^6)_\infty\times(q;q^2)_\infty\times(q^3;q^6)_\infty}$$

$$=\frac{2q(-q^2;q^4)_\infty^2(q^4,q^8,q^{16},q^{20};q^{24})_\infty}{(q^2,q^4,q^6,q^8;q^8)_\infty(q^6,q^{18};q^{24})_\infty}\times$$

$$\times\cfrac{1}{1-\cfrac{q^{12}}{1+\cfrac{q^{12}(1-q^{12})}{1-\cfrac{q^{36}}{1+\cfrac{q^{24}(1-q^{24})}{1-\cfrac{q^{60}}{1+\cfrac{q^{36}(1-q^{36})}{1+\vdots}}}}}}} \tag{3.2}$$

$$\frac{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty-(q^3;q^6)_\infty(q^5;q^{10})_\infty}{(-q^3;q^6)_\infty\times(-q^5;q^{10})_\infty\times(q^3;q^6)_\infty\times(q^5;q^{10})_\infty}$$

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$$=\frac{2q^3(q^8;q^{12})_\infty(q^4,q^{16},q^{28},q^{40},q^{54};q^{60})_\infty}{(q^6,q^{12},q^{18},q^{24};q^{24})_\infty(q^2,q^{10};q^{12})_\infty(q^{10},q^{20},q^{30},q^{30},q^{40},q^{50};q^{60})_\infty}\times$$

$$\times\cfrac{1}{1-\cfrac{q^{6}}{1+\cfrac{q^{6}(1-q^{6})}{1-\cfrac{q^{18}}{1+\cfrac{q^{12}(1-q^{12})}{1-\cfrac{q^{30}}{1+\cfrac{q^{18}(1-q^{18})}{1+\vdots}}}}}}} \tag{3.3}$$

Ref.

(7) M.P. Chaudhary : *On q-product identities*, pre-print.

Notes

And,

$$\frac{[(q;q^2)_\infty(q^{15};q^{30})_\infty]+[(-q;q^2)_\infty(-q^{15};q^{30})_\infty]}{[(q;q^2)_\infty(q^{15};q^{30})_\infty][(-q;q^2)_\infty(-q^{15};q^{30})_\infty]}$$

$$=\frac{2(q^{12},q^{20},q^{24},q^{36},q^{40},q^{48};q^{60})_\infty}{(q^2,q^4,q^6,q^8,q^8;q^8)_\infty(q^6,q^{10},q^{18},q^{30},q^{42},q^{50},q^{54},q^{60};q^{60})_\infty}\times$$

$$\times\left[\cfrac{1}{1-\cfrac{q^{30}}{1+\cfrac{q^{30}(1-q^{30})}{1-\cfrac{q^{90}}{1+\cfrac{q^{60}(1-q^{60})}{1-\cfrac{q^{150}}{1+\cfrac{q^{90}(1-q^{90})}{1+\vdots}}}}}}}\right]^2 \quad (3.4)$$

Proof of (3.1): Making suitable arrangements in the q-products identities given in the right hand side of (2.6), and further apply (2.1) for $q=q^{24}$, we get (3.1).

Proof of (3.2): Making suitable arrangements in the q-products identities given in the right hand side of (2.9), and further apply (2.1) for $q=q^{12}$, we get (3.2).

Proof of (3.3): Making suitable arrangements in the q-products identities given in the right hand side of (2.10), and further apply (1.17), and (2.1) for $q=q^{6}$, we get (3.3).

Proof of (3.4): Making suitable arrangements in the q-products identities given in the right hand side of (2.11), and further apply (2.1) for $q=q^{30}$, we get (3.4).

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Notes

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