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Certain Indefinite Integrals Involving Lucas Polynomials and Harmonic Number

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Abstract - In this paper we have established certain indefinite integrals involving Harmonic number and Lucas Polynomials. The results represent here are assume to be new.

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I. Introduction and Preliminaries

a) Harmonic Number

The n^{th} harmonic number is the sum of the reciprocals of the first n natural numbers:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \tag{1.1}$$

Harmonic numbers were studied in antiquity and are important in various branches of number theory. They are sometimes loosely termed harmonic series, are closely related to the Riemann zeta function, and appear in various expressions for various special functions.
An integral representation is given by Euler

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx \tag{1.2}$$

The equality above is obvious by the simple algebraic identity below

$$\frac{1-x^n}{1-x} = 1 + x + ... + x^n \tag{1.3}$$

An elegant combinatorial expression can be obtained for H_n using the simple integral transform $x = 1 - u$:

$$H_n = \int_0^1 \frac{1-x^n}{1-x} = -\int_1^0 \frac{1-(1-u)^n}{u} du = \int_0^1 \frac{1-(1-u)^n}{u} du$$

$$= \int_0^1 \left[\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} u^{k-1} \right] du$$

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$$= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \int_0^1 u^{k-1} du$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} \binom{n}{k} \tag{1.4}$$

b) Lucas polynomials

The sequence of Lucas polynomials is a sequence of polynomials defined by the recurrence relation

$$L_n(x) = \begin{cases} 2x^0 = 2 & , \text{ if } n = 0 \\ 1x^1 = x & , \text{ if } n = 1 \\ x^1 L_{n-1}(x) + x^0 L_{n-2}(x) & , \text{ if } n \geq 2 \end{cases} \tag{1.5}$$

The first few Lucas polynomials are:

$$L_0(x) = 2$$

$$L_1(x) = x$$

$$L_2(x) = x^2 + 2$$

$$L_3(x) = x^3 + 3x$$

$$L_4(x) = x^4 + 4x^2 + 2$$

The ordinary generating function of the Lucas polynomials is

$$G_{\{L_n(x)\}}(t) = \sum_{n=0}^{\infty} L_n(x) t^n = \frac{2 - xt}{1 - t(x + t)}. \tag{1.6}$$

c) Polylogarithm

The polylogarithm (also known as Jonquire's function) is a special function $Li_s(z)$ that is defined by the infinite sum, or power series:

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \tag{1.7}$$

It is in general not an elementary function, unlike the related logarithm function. The above definition is valid for all complex values of the order s and the argument z where$|\ z\ |< 1$.The polylogarithm is defined over a larger range of z than the above definition allows by the process of analytic continuation.

The special case s = 1 involves the ordinary natural logarithm ($Li_1(z) = -ln(1 - z)$) while the special cases s = 2 and s = 3 are called the dilogarithm (also referred to as Spence's function) and trilogarithm respectively. The name of the function comes from the fact that it may alternatively be defined as the repeated integral of itself, namely that

$$Li_{s+1}(z) = \int_0^z \frac{Li_s(t)}{t} dt \tag{1.8}$$

Notes

Thus the dilogarithm is an integral of the logarithm, and so on. For nonpositive integer orders s, the polylogarithm is a rational function.
The polylogarithm also arises in the closed form of the integral of the FermiDirac distribution and the Bose-Einstein distribution and is sometimes known as the Fermi- Dirac integral or the Bose-Einstein integral. Polylogarithms should not be confused with polylogarithmic functions nor with the offset logarithmic integral which has a similar notation.

Notes

d) Generalized Gaussian Hypergeometric Function

Generalized ordinary hypergeometric function of one variable is defined by

$$
{}_AF_B\left[\begin{array}{lcc} a_1, a_2, \cdots, a_A & ; & \\ & & z \\ b_1, b_2, \cdots, b_B & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}
$$

or

$$
{}_AF_B\left[\begin{array}{lcc} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{lcc} (a_j)_{j=1}^{A} & ; & \\ & & z \\ (b_j)_{j=1}^{B} & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1.9}
$$

where denominator parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers.

II. Main Indefinite Integrals

$$
\int \frac{\sinh x\; H_1^{(x)} L_1(x)}{\sqrt{1-\cos x}}\,\mathrm{dx} = -\frac{1}{\sqrt{1-\cos x}}\left(\frac{8}{25}-\frac{6\iota}{25}\right)e^{(-1-\frac{\iota}{2})x}\sin\frac{x}{2}\times
$$
$$
\times\Big[2e^{2x}{}_3F_2\Big(-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota,\frac{1}{2}-\iota;e^{\iota x}\Big)-2e^{\iota x}{}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^{\iota x}\Big)-
$$
$$
-(2-\iota)xe^{2x}{}_2F_1\Big(-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota;e^{\iota x}\Big)-(2-\iota)xe^{\iota x}{}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^{\iota x}\Big)+
$$
$$
+(2-\iota)xe^{2x}-2e^{2x}\Big]+Constant \tag{2.1}
$$

$$
\int \frac{\sin x\; H_1^{(x)} L_1(x)}{\sqrt{1-\cosh x}}\,\mathrm{dx} = \frac{1}{25\sqrt{1-\cosh x}}e^{-\iota x}(e^x-1)\Big[-(8+6\iota){}_3F_2\Big(\frac{1}{2}-\iota,\frac{1}{2}-\iota,1;\frac{3}{2}-\iota,\frac{3}{2}-\iota;e^x\Big)-
$$
$$
-(8-6\iota)e^{2\iota x}{}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;\cosh x+\sinh x\Big)+5x\Big\{(2-\iota){}_2F_1\Big(\frac{1}{2}-\iota,1;\frac{3}{2}-\iota;e^x\Big)+
$$
$$
+(2+\iota)e^{2\iota x}{}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;\cosh x+\sinh x\Big)\Big\}\Big]+Constant \tag{2.2}
$$

$$
\int \frac{\cos x\; H_1^{(x)} L_1(x)}{\sqrt{1-\cosh x}}\,\mathrm{dx} = -\frac{1}{25\sqrt{1-\cosh x}}e^{-\iota x}(e^x-1)\Big[(6-8\iota){}_3F_2\Big(\frac{1}{2}-\iota,\frac{1}{2}-\iota,1;\frac{3}{2}-\iota,\frac{3}{2}-\iota;e^x\Big)+
$$
$$
+(6+8\iota)e^{2\iota x}{}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;\cosh x+\sinh x\Big)+5x\Big\{(1+2\iota){}_2F_1\Big(\frac{1}{2}-\iota,1;\frac{3}{2}-\iota;e^x\Big)+
$$
$$
+(1-2\iota)e^{2\iota x}{}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;\cosh x+\sinh x\Big)\Big\}\Big]+Constant \tag{2.3}
$$

$$\int \frac{\sin x\ H_1^{(x)} L_1(x)}{\sqrt{1-\sin x}}\,\mathrm{dx} = \frac{2}{\sqrt{1-\sin x}}\Big(\cos\frac{x}{2}-\sin\frac{x}{2}\Big)\Bigg[\frac{1}{\sqrt{2}}\Bigg\{\pi\tanh^{-1}\left(\frac{\tan\frac{x}{4}+1}{\sqrt{2}}\right)+$$
$$+\frac{1}{2}\Big(4\iota Li_2\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-4\iota Li_2\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-(\pi-2x)\Big(\log\Big(1-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-$$
$$-\log\Big(1+(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)\Big)\Big)\Bigg\}-(x-2)\sin\frac{x}{2}-(x+2)\cos\frac{x}{2}\Bigg]+Constant \tag{2.4}$$

$$\int \frac{\cot x\ H_1^{(x)} L_1(x)}{\sqrt{1-\sin x}}\,\mathrm{dx} = \frac{1}{2\sqrt{1-\sin x}}\Big(\cos\frac{x}{2}-\sin\frac{x}{2}\Big)\Big[4\iota Li_2(-e^{\frac{\iota x}{2}})+4\iota Li_2(-\iota e^{\frac{\iota x}{2}})-$$
$$-4\iota Li_2(\iota e^{\frac{\iota x}{2}})-4\iota Li_2(e^{\frac{\iota x}{2}})-\iota\pi x+2x\log(1-e^{\frac{\iota x}{2}})+2x\log(1-\iota e^{\frac{\iota x}{2}})-2x\log(1+e^{\frac{\iota x}{2}})-$$
$$-2x\log(1+\iota e^{\frac{\iota x}{2}})+2\pi\log(1-\iota e^{\frac{\iota x}{2}})+2\pi\log(1+\iota e^{\frac{\iota x}{2}})-2\pi\log\Big(\sin\frac{x+\pi}{4}\Big)-$$
$$-2\pi\log\Big(-\cos\frac{x+\pi}{4}\Big)\Big]+Constant \tag{2.5}$$

$$\int \frac{\tan x\ H_1^{(x)} L_1(x)}{\sqrt{1-\sec x}}\,\mathrm{dx} = \frac{1}{8\sqrt{\frac{(-1+e^{\iota x})^2}{1+e^{2\iota x}}}\sqrt{1+e^{2\iota x}}}\Big[-\Big(\iota(-1+e^{\iota x})\Big(4Li_2\Big(\frac{1}{2}-\frac{1}{2}\sqrt{1+e^{2\iota x}}\Big)-$$
$$-4Li_2\Big(e^{-2\sinh^{-1}(e^{\iota x})}\Big)-\log^2(-e^{2\iota x})-2\log^2\Big(\frac{1}{2}\Big(1+\sqrt{1+e^{2\iota x}}\Big)\Big)+$$
$$+4\log(-e^{2\iota x})\log\Big(\frac{1}{2}\Big(1+\sqrt{1+e^{2\iota x}}\Big)\Big)-8\iota x\log\Big(\sqrt{1+e^{2\iota x}}+e^{\iota x}\Big)+4\sinh^{-1}(e^{\iota x})^2+$$
$$+8\iota x\tanh^{-1}\Big(\sqrt{1+e^{2\iota x}}\Big)+8\sinh^{-1}(e^{\iota x})\log\Big(1-e^{-2\sinh^{-1}(e^{\iota x})}\Big)-$$
$$-4\log(-e^{2\iota x})\tanh^{-1}\Big(\sqrt{1+e^{2\iota x}}\Big)\Big)\Big)\Big]+Constant \tag{2.6}$$

$$\int \frac{\cot x\ H_1^{(x)} L_1(x)}{\sqrt{1-\operatorname{cosec} x}}\,\mathrm{dx} = \frac{1}{8(e^{\iota x}-\iota)}\sqrt{\frac{(e^{\iota x}-\iota)^2}{-1+e^{2\iota x}}}\Bigg[\sqrt{-1+e^{2\iota x}}\Bigg\{\frac{1}{\sqrt{-1+e^{2\iota x}}}\sqrt{1-e^{2\iota x}}\times$$
$$\times\Big(-4Li_2\Big(\frac{1}{2}-\frac{1}{2}\sqrt{1-e^{2\iota x}}\Big)+\log^2(e^{2\iota x})+2\log^2\Big(\frac{1}{2}\Big(1+\sqrt{1-e^{2\iota x}}\Big)\Big)-$$
$$-4\log\Big(\frac{1}{2}\Big(1+\sqrt{1-e^{2\iota x}}\Big)\Big)\log(e^{2\iota x})\Big)+4(2\iota x-\log(e^{2\iota x}))\tan^{-1}\sqrt{-1+e^{2\iota x}}\Bigg\}-$$
$$-4\sqrt{1-e^{2\iota x}}\Big(Li_2\Big(e^{-2\iota\sin^{-1}(e^{\iota x})}\Big)+2\iota x\log(\sqrt{1-e^{2\iota x}}+\iota e^{\iota x})+\sin^{-1}(e^{\iota x})^2-$$
$$-2\iota\sin^{-1}(e^{\iota x})\log\Big(1-e^{-2\iota\sin^{-1}(e^{\iota x})}\Big)\Big)\Bigg]+Constant \tag{2.7}$$

$$\int \frac{\tan x\ H_1^{(x)} L_1(x)}{\sqrt{1-\cos x}}\,\mathrm{dx} = \frac{1}{2\sqrt{2-2\cos x}}\sin\frac{x}{2}\Big[8\iota Li_2\Big(-\frac{(1+\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)+8\iota Li_2\Big(\frac{(1-\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)+$$
$$+8\iota Li_2\Big(-\frac{(1+\iota)(\cos\frac{x}{2}+\iota\sin\frac{x}{2})}{\sqrt{2}}\Big)+8\iota Li_2\Big(\frac{(1+\iota)(\sin\frac{x}{2}-\iota\cos\frac{x}{2})}{\sqrt{2}}\Big)+2\iota x^2-2\iota\pi x+$$
$$+4x\log\Big(1-\frac{(1-\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)+4x\log\Big(1+\frac{(1+\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)-4\pi\log\Big(1-\frac{(1-\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)-$$

Notes

$$-4\pi\log\Big(1+\frac{(1+\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)+16\sin^{-1}\Big(\frac{\sqrt{2+\sqrt{2}}}{2}\Big)\log\Big(1-\frac{(1-\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)-$$
$$-16\sin^{-1}\Big(\frac{\sqrt{2+\sqrt{2}}}{2}\Big)\log\Big(1+\frac{(1+\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)+4\pi\log\Big(2\sin\frac{x}{2}+\sqrt{2}\Big)-$$
$$-4x\log\Big(-\frac{(1+\iota)\sin\frac{x}{2}}{\sqrt{2}}-\frac{(1-\iota)\cos\frac{x}{2}}{\sqrt{2}}+1\Big)-4x\log\Big(-\frac{(1-\iota)\sin\frac{x}{2}}{\sqrt{2}}+\frac{(1+\iota)\cos\frac{x}{2}}{\sqrt{2}}+1\Big)-$$
$$-32\sin^{-1}\Big(\frac{\sqrt{2+\sqrt{2}}}{2}\Big)\tanh^{-1}\Big(\frac{(\sqrt{2}-2)\cot\frac{x+\pi}{4}}{\sqrt{2}}\Big)+\iota\pi^2\Big]+Constant \qquad (2.8)$$

Notes

III. Derivation of the Integrals

Involving the same parallel method of ref[8] , one can derive the integrals.

IV. Conclusion

In our work we have established certain indefinite integrals involving Lucas Polynomials , Harmonic Number , and Hypergeometric function . However, one can establish such type of integrals which are very useful for different field of engineering and sciences by involving these integrals.Thus we can only hope that the development presented in this work will stimulate further interest and research in this important area of classical special functions.

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