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Coincidence and Common Fixed Point Theorem in Cone Metric Spaces

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5. K. Jha; A common Fixed Point Theorem in a cone metric space; Kathmandu University Journal of Science, Engg. Tech. Vol. 5 No. 1 Jan. 2009 1 - 5.

Coincidence and Common Fixed Point Theorem in Cone Metric Spaces

Piyush Kumar Tripathi[α] & A. Kumar[σ]

Abstract - In this paper we have proved some coincidence and common fixed point theorem in cone metric space by using Jungck type contractive condition and generalized the results of Huang Long Guang and Zhang Xian.

I. Introduction

Common fixed point theorems have been proved by many authors by using contractive conditions for single valued mapping [5], [6], [7]. The concept of weakly commuting mapping defined by S. Sessa [9] was generalized by Singh and Pant [10]. They proved some common fixed point theorem by using weak commutative condition of mappings. G. Jungck [3] has generalized this condition by defining compatible mapping and proved some common fixed point theorem.

Recently Huang Long [4] replaced the set of real numbers by an ordered Banach space and obtained some fixed point theorems for self mapping satisfying different contractive condition. Dejan Ilic and Rakocevic [1] and K. Jha [5] have proved some fixed point theorems in cone metric space by using commuting mapping. Abbas and Jungck [6] has proved some common fixed point theorem for non commuting mappings in the set of cone metric spaces. P. Raja [8] has proved new results by extending Banach contraction principle in complete cone metric spaces. Jungck type contraction [2] leads to a new development regarding the weaker forms of commuting mappings which is stated as:

Let Y be an arbitrary nonempty set and (X, d) is a metric space. Let f and I be maps from Y with the values in X. Consider the following properties,

a. $d(fx, fy) \leq kd(Ix, Iy), \quad xy \in X, \quad 0 \leq k < 1$

b. $d(fx, fy) < kd(Ix, Iy), \quad xy \in X,$

c. $d(fx, fy) \leq d(Ix, Iy), \quad xy \in X$

In the scenario of existing literature in metric fixed point theory the pair of mapping (f, I) satisfying properties a, b and c are called Jungck contraction, Jungck strictly contraction and Jungck non expensive contraction. Piyush tripathi [10] proved some common fixed point theorems and extended some well known results by using Jungck type contraction.

The papers of Jungck [2] and piyush tripathi [9] motivated us to establish coincidence and common fixed point theorem for two self mapping in a complete cone

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metric spaces by using Jungck type contractive condition which is the extended result of Huang Long [4].

The following definitions regarding cone metric space is defined in the paper of Huang Long [4].

Definition-1.1 Suppose E as the real Banach space and P is a subset of E. Then P is cone if and only if

(i) P is closed, non-empty and $P \neq \{0\}$

(ii) If $\alpha, \beta \in R$ and $\alpha, \beta \geq , 0$ and if $x, y \in P$ then $\alpha x + \beta y \in P$

(iii) If $x \in P$ and $- x \in P \Rightarrow x = 0$

For a cone P which is subset of E Huang Long [4] defined a partial ordering $<<$ with respect to P as $x << y$ if and only if $y - x \in$ int P, where int P is interior of P

Definition-1.2 : P is called normal cone if there exist a number $K > 0$ such that $o \leq x \leq y \Rightarrow \|x\| \leq K \|y\|$

Definition-1.3: The cone P is said to be regular if every increasing sequence which is bounded above is convergent.

Definition-1.4 : Let X is a non - empty set. Let d is a mapping from $X \times X \rightarrow E$ and if d satisfies the following conditions.

(i) $d(x, y) > 0 \quad \forall x, y \in X$

(ii) $d(x, y) = 0$ iff $x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, y) \leq d(x, z) + d(y, z) \qquad \forall\ x y z \in X$

Then d is called the cone metric on X and the pair (X, d) is called the cone metric space.

Definition-1.5: A sequence $\{x_n\}$ in a cone metric space (X, d) is said to be convergent and converges to $x \in X$ if for every $c \in E$ with o << c, $\exists N$ such that $n > N$, $d(x_n, x) << c$, and x is called the limit of $\{x_n\}$.

i.e. $\lim x_n = x$ or $x_n \rightarrow x$ for $n \rightarrow \infty$

Definition-1.6: A sequence $\{x_n\}$ in a cone metric space (X, d) is said to be convergent and converges to $x \in X$ if for every $c \in E$ with $o << c \exists N$ such that $n > N$, $d(x_n, x) << c$.

Definition-1.7 : A cone metric space (X, d) is said to be complete cone metric space if every Cauchy sequence is convergent in (X, d).

Definition-1.8 : A cone metric space (X, d) is said to be sequentially compact if for any sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ is convergent in X. To prove our main result, the following Lemma [2] are very important.

Lemma – 1.1 [4] : Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x then $\{x_n\}$ is a Cauchy sequence.

Lemma – 1.2 [4] : Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if

Ref.

4. H.L. Guang, Z. Xian : Cone metric spaces and fixed point theorems of contractive mapping J. Math. Anal. Appl. 332 (2007) 1468 - 1476.

$$d\ (x_m, x_n)\ \to 0\ \ (m, n\ \to\ \infty)$$

Lemma – 1.3 [4] : Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \to x,\ \ y_n \to y\ \ (n \to \infty)$

Then $$d\ (x_n, y_n) \to d\ (x, y) \text{ when } n \to \infty$$

II. Main Results

Notes

Theorem-2.1: Let (X, d) be a cone metric space, P be a normal cone with constant K. Suppose $T, I : X \to X$ are self mappings satisfies the condition

(i) $d(Tx\ Ty) \leq kd(Ix\ Iy)$ for all $x, y \in Y$, where $k \in [0, 1)$.

(ii) $T(Y) \subseteq I(Y)$.

(iii) Either $T(Y)$ or $I(Y)$ is complete.

Then T and I have a coincidence point in X.

Proof: Since $T(Y) \subseteq I(Y)$ Choose $x_0, x_1 \in Y$, such that $Ix_1 = T(x_0)$ and $x_1, x_2 \in Y$, $Ix_2 = Tx_1$, hence we can construct a sequence $\{x_n\}$ such that $Ix_{n+1} = Tx_n$

$$\begin{aligned} d\left(Ix_{n+1}\ \ Ix_n\right) &= d\ \left(Tx_n,\ Tx_{n-1}\right) \\ &\leq kd\ \left(Ix_n,\ Ix_{n-1}\right) \end{aligned}$$

$$\begin{aligned} d\left(Ix_n\ \ Ix_{n-1}\right) &= d\ \left(Tx_{n-1},\ Tx_{n-2}\right) \\ &\leq kd\ \left(Ix_{n-1},\ Ix_{n-2}\right) \end{aligned}$$

$$d\left(I.x_{n+1}, Ix_n\right) \leq k^2 d\ \left(Ix_{n-1},\ Ix_{n-2}\right)$$

continuing this process we get

$$d\left(Ix_{n+1}, Ix_n\right) \leq k^n d\ \left(Ix_1,\ Ix_0\right)$$

So for n > m

$$\begin{aligned} d\left(Ix_n, Ix_m\right) &\leq d\ \left(Ix_n,\ Ix_{n-1}\right) + d\ \left(Ix_{n+1},\ Ix_{n-2}\right) + \ldots..\ d\left(Ix_{m+1},\ Ix_m\right) \\ &\leq \left(k^{n-1} + k^{n-2} + \ldots..\ k^m\right)\ d\ \left(Ix_1,\ Ix_0\right) \\ &\leq \frac{k^m}{1-k}\ d\ \left(Ix_1,\ Ix_0\right) \end{aligned}$$

We get

$$\| d(Ix_n, Ix_m) \| \leq \frac{k^m}{1-k}\ K\ \| d\left(Ix_1,\ Ix_0\right) \|$$

$$\Rightarrow d\left(Ix_n,\ Ix_m\right) \to 0 \qquad as \qquad n,\ m \to \infty$$

Hence $\{Ix_n\} = \{Tx_{n-1}\}$ is a Cauchy sequence in $T(Y) \subseteq I(Y)$. Suppose $I(Y)$ is complete in X so there is $p \in I(Y)$, $z \in Y$ such that $Ix_n, Tx_n \to p$ as $n \to \infty$ and $Iz = p$.

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Putting $x = x_n$ and $y = z$ in (i) .

$$d(Tx_n, Tz) \le kd(Ix_n, Iz)$$

$$d(Tx_n, Tz) \le kd(Ix_n, p)$$

as $n \to \infty$,

$$d(p, Tz) \le kd(p, p) \Rightarrow d(p, Tz) = 0$$

So we get $Tz = Iz = p$. Therefore z is a coincidence point of T and I.

Again if $T(Y)$ is complete then $Tx_n \to p \in T(Y) \subseteq I(Y)$ hence as above z is coincidence point of T and I.

Theorem-2.2: Let (X, d) be a cone metric space, P be a normal cone with constant K. Suppose $T, I : X \to X$ are self mappings satisfies the condition

(i) $d(Tx\ Ty) \le kd\ (Ix\ Iy)$ for all x, y ∈ Y, where k ∈ [0, 1).

(ii) $T(Y) \subseteq I(Y)$.

(iii) Either $T(X)$ or $I(X)$ is complete.

(iv) T and I are commuting at their coincidence point.

Then T and I have a unique common fixed point in X.

Proof: If we take $Y = X$ in Theorem 2.1 then we get sequence $\{Tx_n\}$ which is a Cauchy sequence. Suppoose g(X) is complete then $Tx_n \to p \in g(X)$, $z \in X$ such that $Iz = p$ and $Iz = Tz = p$. Since T and I are commuting at their coincidence point so $TIz = ITz$ and $Tp = Ip$.

Putting $x = z,\ y = Tz$, in (i),

$$d(Tz, TTz) \le kd(Iz, ITz),$$

$$d(p, Tp) \le kd(Iz, TIz)$$

$$d(p, Tp) \le k^2 d(p, Tp)$$

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$$d(p, Tp) \le k^n d(p, Tp)$$

As $n \to \infty$,

$$d(p, Tp) \to 0$$

$$Tp = Ip = p$$

So p is a common fixed point of T and I. For uniqueness suppose p' and q' are two common fixed points of T and I,

Notes

From (i) $d(Tp',Tq') \le k(Ip',Iq') = k(Tp',Tq') \le k^2(Ip',Iq'). \le k^n(Ip',Iq')$, as $n \to \infty$ $d(Tp',Tq') = d(p',q') = 0 \Rightarrow p' = q'$. Therefore T and I have unique common fixed point. The following theorem was proved by Huang Long-Guang and Zhang Xian [4].

Theorem - 2.2.[4] : Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition d $(Tx, Ty) \le kd(x, y)$, for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X and for any x ∈ X iterative sequence $(T^n x)$ converges to the fixed point.

Notes

Proof: In the our Theorem 2.2, mapping I is considered as identity mapping then the theorem stated by Huang Long [4] becomes special case of our proved Theorem 2.2

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