Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 13 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Homology Invariant Functions for Lane- Emden Equation of Finite Polytropic Index

By M. A. Sharaf & L.A.Alaqal
King AbdulAziz University, Saudi Arabia

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GJSFR-F Classification : *MSC 2010: 55N35, 55P20, 83F05*

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Homology Invariant Functions for Lane-Emden Equation of Finite Polytropic Index

M. A. Sharaf [α] & L.A.Alaqal [σ]

Abstract - The present paper is devoted to establish a general *Mathematica* module to determine if a function is homology invariant or not. The module is described through its basic points, propose, input, output and computational steps. Applications of the module are also given.

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I. Introduction

The reduction of the differential equations is probably the most challenging problem in dynamics and physics. A general interpretation of reducibility includes various transformations and changes the original problem not only along mathematical lines but also in a physical sense.

What concerns us in the present paper is the reduction of the second order Lane-Emden equation of stellar structure into a first order through what is known as homology theorem. The important consequence of the use of homology theorem.

is that, if we can find two independent homology invariant functions, say u and v, then the Lane-Emden equation transformed to u and v variables is of order one.

Due to the important role of homology invariant functions in the reduction of Lane-Emden equation, the present paper is devoted to establish a general *Mathematica* module to determine if a function is homology invariant or not. The module is described through its basic points, propose, input, output and computational steps. Applications of the module are also given.

II. The Homology Theorem and Homology Invariant Functions

The Lane-Emden equation of finite polytropic index $n \neq -1, \pm\infty$ is given as (Prialnik 2007)

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \ . \tag{1}$$

This equation is subject to the initial conditions

Author α : Department of Astronomy, Faculty of Science, King AbdulAziz University, Jeddah, Saudi Arabia.
Email : sharaf_adel@hotmail.com
Author σ : Department of Mathematics, Faculty of Science, King AbdulAziz University, Jeddah, Saudi Arabia.
Email : laq700 @hotmail.com

$$\text{at} \quad \xi = 0 \ ; \ \theta = 1, \quad \frac{d\theta}{d\xi} = 0 \ . \tag{2}$$

a) Theorem

If $\theta(\xi)$ *is a solution of the Lane-Emden equation of index n then* $A^{\frac{2}{n-1}}\theta(A\xi)$*, where* A *is an arbitrary real constant is also a solution of the equation* .The proof of this theorem is given by Horedt (2004)

Thus, if one solution $\theta = \theta(\xi)$ of the Lane-Emden equation is known, we can derive a whole homologous family $\{\theta(\xi)\}$ of solutions. In particular, if θ is just the Lane-Emden function defined by the initial conditions of Equation (2), then its homologous family $\{\theta(\xi)\}$ defines a whole set of solutions that are all finite at the origin $\xi = 0$. Solutions that are finite at the origin are called *E*-solutions and denoted by θ_E. The Lane-Emden function defined by the initial conditions from Equation (2) is just a particular member of the set $\{\theta(\xi)\}$ of *E*-solutions. All *E*-solutions can be found from the Lane-Emden function through the homology transformation

$$\theta(\xi) \to A^{2/(n-1)} \ \theta(A\xi) \qquad n \neq -1, \pm\infty \ . \tag{3}$$

It should also be noted that, any solution $\theta_E = \theta_E(\xi)$ that is finite at the origin $\xi = 0$ is an *E*-solution, and its derivative is zero $(d\theta_E / d\xi)_{\xi=0} = 0$.

The general solution of the second order Lane-Emden equation must characterized by two integration constants. According to the homology theorem one of the two constants must be "trivial" in the sense that it defines merely the scale factor A of the homology transformation, and we should be able to transform the second order Lane-Emden equation into a first order differential equation(Chandrasekhar (1957).

b) Homology invariant functions

1-A function Q (say) is said to *homology invariant* if it is invariant to the homologous transformation:

$$\theta^*(\xi) = A^{\omega}\,\theta(A\xi) \ \text{ or } \ \theta^*\left(\frac{\xi}{A}\right) = A^{\omega}\,\theta(\xi) \ ; \ \omega = 2/(n-1) \ ; n \neq -1, \pm\infty, \tag{4}$$

So, to prove that, Q is homology invariant function, we have to prove that

$$Q^*(\xi) = Q(A\xi) \ \text{ or } \ Q^*\left(\frac{\xi}{A}\right) = Q(\xi) \tag{5}$$

2-The homology transformation for the derivatives are:

$$\left|\frac{d^k\theta^*(\xi)}{d\xi^k}\right|_{\xi=\frac{\xi}{A}} = A^{\omega+k}\,\frac{d^k\theta(\xi)}{d\xi^k} \ ; \ n \neq -1, \pm\infty, \tag{6}$$

Notes

III. Mathematica Module "Homoinvfunpol"

In this section we shall develop a *Mathematica Module* called "**HomoInvFunPol**".

In what follows, the module will be described through its basic points, propose, input, output and computational steps. Applications of the module are also given

- *Propose*

To determine if a function is homology invariant or not.

- *Input*

F: A function of ξ, which we required to known if it is homology invariant or not

n: The polytropic index, such that $n \neq -1, \pm\infty$

k1, k2, k3, k4 : Nonnegative integers which represent the order of the derivatives that may exist in the function $F(\xi)$

- *Output*

1 - Message informing that, the function is homology invariant or not homology invariant

2 - Full proof of the result

- *Module List*

Notes

Appendix A : Mathematica Module HomoInvFunPol

```
HomoInvFunPol[F_, n_, k1_, k2_, k3_, k4_] := Module[{},

   P1 = s;  2/(n - 1);
   P2 = k1  s^k1  k2  s^k2
        k3  s^k3,  k4  s^k4;
   W1 = s  A  A  ;
   W2 = k1  s - A^k1  A^k1  k1 ,
                A
        k2  s - A^k2  A^k2  k2 ,
                A
        k3  s - A^k3  A^k3  k3 ,
                A
        k4  s - A^k4  A^k4  k4 ;
                A
   T = (z_/r_)^m_  z^m r^m,  h_ g_^m_  h^m g^m,
        h^u_^m_  h^m u ;
   Fs  F  P1  P2;
   H  Fs  A; G  H  W1  W2
     FullSimplify; S  G  T  FullSimplify;
   TF  F  S ;
   Print["The Given function F", F ]; Which[TF  True,
     Print[" is Homology invariant function "],
     Print["  The proof  "],
     Print["  F", Fs],
     Print["  F  A", H],
     Print["  Appling the general rules for homology
invariant functions we get  "],
     Print["  F  A", S],
     Print["  So we find that F  A equal to F ,
this is the required to be proved.  "],
    TF  False , Print["The function F  is not
 Homology invariant function"], Print["  The proof  "],
     Print["  F", Fs],
     Print["  F  A", H],
     Print["  Appling the general rules for homology
invariant functions we get  "],
     Print["  F  A", S  FullSimplify],
     Print["  So we find that F  A not equal to F ,
this is the required to be proved.  "]]];
```

Notes

a) Applications

1-Is the function $F_1(\xi) = -\xi\psi^n(\xi)/\psi'(\xi)$ is homology invariant or not?.

Appling the module with $k1 = 1, k2 = k3, = k4 = 0$, we get the following message, and the proof of the result

```
The Given function F □ □□ □ □ □^n / □□ □
is Homology invariant function
□ The proof
□   F □ □ □ □□ □ □s □ □^n / □s □ □ □
□   F □ □ A □ □□ □ □s □ □/A □^n / A □s □ □/A □
□ Appling the general rules for homology
invariant functions we get
□   F □ □ A □ □□ □ □ □ □^n / □ □ □ □
□ So we find that F □ □ A □ equal to F □ □ ,
this is the required to be proved.
```

The above function $F_1(\xi) = -\xi\psi'(\xi/\psi(\xi)$ was introduced by Milne(1930) ,it plays an important role in fitting up solutions at the surface of the composite stellar models (Menzel et al 1963).

$$F_1(\xi) = 3 \times \frac{\text{local densiy } \rho(r)}{\text{mean densiy } \bar{\rho}(r) \text{ within radius r}}$$

2-Is the function $F_2(\xi) = -\xi\psi'(\xi)/\psi(\xi)$ is homology invariant or not.

Appling the module with $k1 = 1, k2 = k3, = k4 = 0$, we get the following message, and the proof of the result

```
The Given function F □ □□ □ □□ □ / □ □ □
is Homology invariant function
□ The proof
□   F □ □ □ □□ □ □s □ □ □ / □s □ □
□   F □ □ A □ □□ □ □s □ □/A □ / A □s □ □/A □
□ Appling the general rules for homology
invariant functions we get
□   F □ □ A □ □□ □ □ □ □ □ / □ □ □
□ So we find that F □ □ A □ equal to F □ □ ,
this is the required to be proved.
```

Also, the function $F_2(\xi) = -\xi\psi'(\xi)/\psi(\xi)$ was introduced by Milne (1932), and plays the same important role as the function $F_1(\xi)$ in fitting up solutions at the surface of the composite stellar models (Menzel et al 1963).

$$F_2(\xi) = \frac{1}{n+1}\,\frac{\left(\frac{1}{\gamma-1}\right)\text{grav.itational energy}}{\text{internal energy}}\ \text{per unit mass at r}$$

where $\gamma = C_p / C_v$, C_p and C_v are the specific heats at constant pressure and constant volume respectively.

Notes

Appendix B : Applications of the *Mathematica* Module HomoInvFunPol

F3 □ □ □ 2/(n□1) □ □ □ ;

F4 □ □ □ (1□n)/(□1□n) □ □ ' □ ;

F5 □ □ □k □ □ ' □ □ □ □ □1□ 1/2 □1□k □1□n ;

F6 □ □ □k □ □ □ k n□1 n□1 2 □ ' □ ;

F7 □ □ □k□1 □ □ □ □1□ 1/2 k □1□n □ □ ' □ ;

F8 □ □ □k□1 □ □ ' □ □1□k□ 2 k/(1□n) □ □ ' ' □ ;

F9 □ □ □k □ □ □ k □ ;

F10 □ □ □k □ □ ' □ k n□1 n□1

We apply our module with appropriate values of to the above new functions and we get homology

F3: k1=k2=k3=k4=0

The Given function F□□□□ 2/(□1□n) □□□

is Homology invariant function

□ The proof

□ F□□□□□ 2/(□1□n) □s□□

□ F□□□A□(□/A)^(2/(□1□n)) □s □/A

□ Appling the general rules for homology invariant functions we get

□ F□□□A□□ 2/(□1□n) □□□

□ So we find that F□□□A□ equal to F□□□ , this is the required to be proved.

F4: k1=1,k2=k3=k4=0

The Given function F□□□□ (1□n)/(□1□n) □□□□

is Homology invariant function

□ The proof

□ F□□□□□ (1□n)/(□1□n) □s□□□

□ F□□□A□(□/A)^((1□n)/(□1□n)) □s□ □/A

□ Appling the general rules for homology invariant functions we get

□ F□□□A□□ (1□n)/(□1□n) □□□

□ So we find that F□□□A□ equal to F□□□ , this is the required to be proved.

Notes

F5: k1=1,k2=k3=k4=0

The Given function $F\square\square\square\square^{k}\ \square\square\square^{1\square\frac{1}{2}\square 1\square k\square\square 1\square n\square}\ \square\square\square$

is Homology invariant function

□ The proof

□ $F^{\square}\square\square\square\square^{k}\ \square_s\square\square^{1\square\frac{1}{2}\square 1\square k\square\square 1\square n\square}\ \square_s\square\square$

□ $F^{\square}\square\square A\square\left(\frac{\square}{A}\right)^{k}\ \square_s\frac{\square}{A}^{1\square\frac{1}{2}\square 1\square k\square\square 1\square n\square}\ \square_s\frac{\square}{A}$

□ Appling the general rules for homology invariant functions we get

□ $F^{\square}\square\square A\square\square\square^{k}\ \square\square\square^{1\square\frac{1}{2}\square 1\square k\square\square 1\square n\square}\ \square\square\square$

□ So we find that $F^{\square}\square\square A\square$ equal to $F\square\square$, this is the required to be proved.

F6 : k1=1,k2=k3=k4=0

The Given function $F\square\square\square\frac{\square^{k}\ \square\square\square^{\frac{1}{2}\square 1\square k\square 1\square n\square n\square}}{\square^{\square}\square\square}$

is Homology invariant function

□ The proof

□ $F^{\square}\square\square\square\frac{\square^{k}\ \square_s\square\square^{\frac{1}{2}\square 1\square k\square 1\square n\square n\square}}{\square_s{}^{\square}\square\square}$

□ $F^{\square}\square\square A\square\frac{\frac{\square}{A}^{k}\ \square_s\frac{\square}{A}^{\frac{1}{2}\square 1\square k\square 1\square n\square n\square}}{\square_s{}^{\square}\frac{\square}{A}}$

□ Appling the general rules for homology invariant functions we get

□ $F^{\square}\square\square A\square\frac{\square^{k}\ \square\square\square^{\frac{1}{2}\square 1\square k\square 1\square n\square n\square}}{\square^{\square}\square\square}$

□ So we find that $F^{\square}\square\square A\square$ equal to $F\square\square$, this is the required to be proved.

F7: k1=1,k2=k3=k4=0

The Given function $F\square\square\square\square^{1\square k}\ \square\square\square^{1\square\frac{1}{2}k\square 1\square n\square}\ \square^{\square}\square\square$

is Homology invariant function

□ The proof

□ $F^{\square}\square\square\square\square^{1\square k}\ \square_s\square\square^{1\square\frac{1}{2}k\square 1\square n\square}\ \square_s{}^{\square}\square\square$

□ $F^{\square}\square\square A\square\left(\frac{\square}{A}\right)^{1\square k}\ \square_s\frac{\square}{A}^{1\square\frac{1}{2}k\square 1\square n\square}\ \square_s{}^{\square}\frac{\square}{A}$

□ Appling the general rules for homology invariant functions we get

□ $F^{\square}\square\square A\square\square\square^{1\square k}\ \square\square\square^{1\square\frac{1}{2}k\square 1\square n\square}\ \square^{\square}\square\square$

□ So we find that $F^{\square}\square\square A\square$ equal to $F\square\square$, this is the required to be proved.

F8: k1=1,k2=2,k3=k4=0

The Given function $F\square\square\square\square^{1\square k}\ \square^{\square}\square\square^{1\square k\square\frac{2k}{1\square n}}\ \square^{\square\square}\square\square$

is Homology invariant function

□ The proof

□ $F^{\square}\square\square\square\square^{1\square k}\ \square_s{}^{\square}\square\square^{1\square k\square\frac{2k}{1\square n}}\ \square_s{}^{\square\square}\square\square$

□ $F^{\square}\square\square A\square\left(\frac{\square}{A}\right)^{1\square k}\ \square_s{}^{\square}\frac{\square}{A}^{1\square k\square\frac{2k}{1\square n}}\ \square_s{}^{\square\square}\frac{\square}{A}$

□ Appling the general rules for homology invariant functions we get

□ $F^{\square}\square\square A\square\square\square^{1\square k}\ \square^{\square}\square\square^{1\square k\square\frac{2k}{1\square n}}\ \square^{\square\square}\square\square$

□ So we find that $F^{\square}\square\square A\square$ equal to $F\square\square$, this is the required to be proved.

F9: k1=k2=k3=k4=0

The Given function $F\square\square\square\square^{k}\ \square\square\square^{\frac{1}{2}k\square 1\square n\square}$

is Homology invariant function

□ The proof

□ $F^{\square}\square\square\square\square^{k}\ \square_s\square\square^{\frac{1}{2}k\square 1\square n\square}$

□ $F^{\square}\square\square A\square\left(\frac{\square}{A}\right)^{k}\ \square_s\frac{\square}{A}^{\frac{1}{2}k\square 1\square n\square}$

□ Appling the general rules for homology invariant functions we get

□ $F^{\square}\square\square A\square\square\square^{k}\ \square\square\square^{\frac{1}{2}k\square 1\square n\square}$

□ So we find that $F^{\square}\square\square A\square$ equal to $F\square\square$, this is the required to be proved.

F10: k1=1,k2=k3=k4=0

The Given function $F\square\square\square\square^{k}\ \square^{\square}\square\square^{\frac{k\square 1\square n\square}{1\square n}}$

is Homology invariant function

□ The proof

□ $F^{\square}\square\square\square\square^{k}\ \square_s{}^{\square}\square\square^{\frac{k\square 1\square n\square}{1\square n}}$

□ $F^{\square}\square\square A\square\left(\frac{\square}{A}\right)^{k}\ \square_s{}^{\square}\frac{\square}{A}^{\frac{k\square 1\square n\square}{1\square n}}$

□ Appling the general rules for homology invariant functions we get

□ $F^{\square}\square\square A\square\square\square^{k}\ \square^{\square}\square\square^{\frac{k\square 1\square n\square}{1\square n}}$

□ So we find that $F^{\square}\square\square A\square$ equal to $F\square\square$, this is the required to be proved.

In concluded the present paper, a general *Mathematica* module was established to determine if a function is homology invariant or not. The module is described through its basic points, propose, input, output and computational steps. Applications of the module are also given

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