On Coefficient Estimates and Neighbourhood Problem for Generalized Sakaguchi Type Functions

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Abstract - In the present investigation, we introduce a new class $k - U^m (\rho, \beta, \lambda, \mu, \gamma, t)$ of analytic functions with negative coefficients. The various results obtained here for this function include coefficient estimate and inclusion relationships involving the neighbourhoods of the analytic function.

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GJSFR-F Classification : MSC 2010: 11B65, 05A10
On Coefficient Estimates and Neighbourhood Problem for Generalized Sakaguchi Type Functions

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Abstract - In the present investigation, we introduce a new class \( k \cdot U^m (\rho, \beta, \lambda, \mu, \gamma, t) \) of analytic functions with negative coefficients. The various results obtained here for this function include coefficient estimate and inclusion relationships involving the neighbourhoods of the analytic function.

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\section{I. Introduction}

Let \( A \) denote the family of functions of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1) \]

that are analytic in the open unit disk \( U = \{ z : |z| < 1 \} \). Denote by \( S \) the subclass of \( A \) of functions that are univalent in \( U \).

For \( f \in A \) given by (1.1) and \( g(z) \) given by

\[ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (1.2) \]

their convolution (or Hadamard product), denoted by \( (f * g) \), is defined as

\[ (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in U) \quad (1.3) \]

Note that \( f * g \in A \).

A function \( f \in A \) is said to be in \( k \cdot U S(\gamma) \), the class of \( k \)-uniformly starlike functions of order \( \gamma \), \( 0 \leq \gamma < 1 \), if satisfies the condition

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\[ \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > k \left| \frac{zf'(z)}{f(z)} - 1 \right| + \gamma \quad (k \geq 0) \]  

(1.4)

and a function \( f \in A \) is said to be in \( k-\mathcal{UC}(\gamma) \), the class of \( k \)-uniformly convex functions of order \( \gamma \), \( 0 \leq \gamma < 1 \), if satisfies the condition

\[ \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > k \left| \frac{zf''(z)}{f'(z)} \right| + \gamma \quad (k \geq 0) \]  

(1.5)

Uniformly starlike and uniformly convex functions were first introduced by Goodman [8] and then studied by various authors. It is known that \( f \in k-\mathcal{UC}(\gamma) \) or \( f \in k-\mathcal{US}(\gamma) \) if and only if \( 1 + \frac{zf''(z)}{f'(z)} \) or \( \frac{zf'(z)}{f(z)} \), respectively, takes all the values in the conic domain \( \mathcal{R}_{k,\gamma} \) which is included in the right half plane given by

\[ \mathcal{R}_{k,\gamma} = \{ w = u + iv \in C : u > k\sqrt{(u-1)^2 + v^2 + \gamma}, \beta \geq 0 \text{ and } \gamma \in [0,1) \}. \]

(1.6)

Denote by \( \mathcal{P}(P_{k,\gamma}) \), \( (\beta \geq 0, 0 \leq \gamma < 1) \) the family of functions \( p \), such that \( p \in \mathcal{P} \), where \( \mathcal{P} \) denotes well-known class of caratheodary functions. The function \( P_{k,\gamma} \) maps the unit disk conformally onto the domain \( \mathcal{R}_{k,\gamma} \) such that \( 1 \in \mathcal{R}_{k,\gamma} \) and \( \partial \mathcal{R}_{k,\gamma} \) is a curve defined by the equality

\[ \partial \mathcal{R}_{k,\gamma} = \{ w = u + iv \in C : u^2 = (k\sqrt{(u-1)^2 + v^2 + \gamma})^2, \beta \geq 0 \text{ and } \gamma \in [0,1) \}. \]

(1.7)

where \( 0 \leq \alpha < 1 \), \( |t| \leq 1 \), \( t \neq 1 \). Note that \( S_{S}(0,-1) = S_{s} \) and \( S_{s}(\alpha,-1) = S_{s}(\alpha) \) is called Sakaguchi function of order \( \alpha \).

Let us define the linear multiplier differential operator \( D_{\lambda,\mu}^m f \) [11] which is shown as follows:

\[ D_{\lambda,\mu}^m f(z) = z + \sum_{n=2}^{\infty} \phi^m(\lambda,\mu,n)a_n z^n \]

(1.8)

where

\[ \phi^m(\lambda,\mu,n) = [1 + (\lambda m + \lambda - \mu)(n-1)]^m, \]

(1.9)

\( 0 \leq \mu \leq 1 \) and \( m \in N_0 = N \cup \{0\} \).

It should be remarked that the operator \( D_{\lambda,\mu}^m f \) is a generalization of many other linear operators considered earlier. In particular, for \( f \in A \) we have the following:

- \( D_{1,0}^m f(z) \equiv D^m f(z) \) the operator investigated by Salagean (see [14]).
- \( D_{\lambda,0}^m f(z) \equiv D_{\lambda}^m f(z) \) the operator studied by Al-Oboudi (see [1]).

Now, by making use of he differential operator \( D_{\lambda,\mu}^m \), we define a new subclass of functions belonging to the class \( A \).
**Definition 1.1.** A function \( f(z) \in A \) is said to be in the class \( k-U^m(\rho, \beta, \lambda, \mu, \gamma, t) \) if for all \( z \in U \),

\[
\text{Re} \left\{ \frac{(1 - t)[(\rho z^2(D_{\lambda,\mu}^m f(z))'' + z(D_{\lambda,\mu}^m f(z))']}{(1 - \rho)[D_{\lambda,\mu}^m f(z) - D_{\lambda,\mu}^m f(tz)] + \rho z[(D_{\lambda,\mu}^m f(z))' - t(D_{\lambda,\mu}^m f(tz))']} \right\} \geq k \left[ \frac{(1 - t)[\rho z^2(D_{\lambda,\mu}^m f(z))'' + z(D_{\lambda,\mu}^m f(z))']}{(1 - \rho)[D_{\lambda,\mu}^m f(z) - D_{\lambda,\mu}^m f(tz)] + \rho z[(D_{\lambda,\mu}^m f(z))' - t(D_{\lambda,\mu}^m f(tz))']} - 1 \right] + \gamma
\]

for \( \lambda \geq \mu \geq 0 \), \( m, k \geq 0 \), \(|t| \leq 1 \), \( t \neq 1 \), \( 0 \leq \beta \leq \rho \leq 1 \).

Furthermore, we say that a function \( f(z) \in k-U^m(\rho, \beta, \lambda, \mu, \gamma, t) \) is in the subclass \( k-U^m(\rho, \beta, \lambda, \mu, \gamma, t) \) if \( f(z) \) is of the following form:

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0, n \in N) \quad (1.10)
\]

The aim of this paper is to study the coefficient bounds and certain neighbourhood results of the class \( k-U^m(\rho, \beta, \lambda, \mu, \gamma, t) \).

This subclass was motivated by Murat Cagler and Halit Orhan See [17].

**Definition 1.2.** A function \( f(z) \in A \) is said to be in the class \( k-U^m(\rho, \lambda, \mu, \gamma, t) \) if for all \( z \in U \)

\[
\text{Re} \left\{ \frac{(1 - t)[z^2(D_{\lambda,\mu}^m f(z))'' + z(D_{\lambda,\mu}^m f(z))']}{z[(D_{\lambda,\mu}^m f(z))' - t(D_{\lambda,\mu}^m f(tz))']} \right\} \geq k \left[ \frac{(1 - t)[z^2(D_{\lambda,\mu}^m f(z))'' + z(D_{\lambda,\mu}^m f(z))']}{z[(D_{\lambda,\mu}^m f(z))' - t(D_{\lambda,\mu}^m f(tz))']} - 1 \right] + \gamma
\]

for \( \lambda \geq \mu \geq 0 \), \( m, k \geq 0 \), \(|t| \leq 1 \), \( t \neq 1 \), \( 0 \leq \beta \leq \rho \leq 1 \).

**Remark 1.1.** When \( \beta = 0 \) in the class \( k-U^m(\rho, \beta, \lambda, \mu, \gamma, t) \), we get the class \( k-U^m(\rho, \lambda, \mu, \gamma, t) \) as in Definition 1.2.

**Definition 1.3.** A function \( f(z) \in A \) is said to be in the class \( k-U^m(\lambda, \mu, \gamma, t) \) if for all \( z \in U \),

\[
\text{Re} \left\{ \frac{(1 - t)[z^2(D_{\lambda,\mu}^m f(z))'' + z(D_{\lambda,\mu}^m f(z))']}{z[(D_{\lambda,\mu}^m f(z))' - t(D_{\lambda,\mu}^m f(tz))']} \right\} \geq k \left[ \frac{(1 - t)[z^2(D_{\lambda,\mu}^m f(z))'' + z(D_{\lambda,\mu}^m f(z))']}{z[(D_{\lambda,\mu}^m f(z))' - t(D_{\lambda,\mu}^m f(tz))']} - 1 \right] + \gamma
\]
for \( \lambda \geq \mu \geq 0, m, k \geq 0, |t| \leq 1, t \neq 1 \).

**Definition 1.4.** A function \( f(z) \in A \) is said to be in the class \( k-U^m(\alpha, \lambda, \mu, \gamma, t) \) if for all \( z \in U \),

\[
Re \left\{ \frac{(1-t)[\alpha z^2(D^m_{\lambda,\mu}f(z))'' + (1+2\alpha)z^2(D^m_{\lambda,\mu}f(z))'' + z(D^m_{\lambda,\mu}f(z))']}{\alpha z^2[(D^m_{\lambda,\mu}f(z))'' - t^2(D^m_{\lambda,\mu}f(tz))'' + z[(D^m_{\lambda,\mu}f(z))' - t(D^m_{\lambda,\mu}f(tz))')] - 1} + \gamma \right| \geq k
\]

for \( \lambda \geq \mu \geq 0, m, k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \alpha \leq 1 \).

**Remark 1.2.** When \( \rho = 1 \) in the class \( k-U^m(\rho, \lambda, \mu, \gamma, t) \) and when \( \alpha = 0 \) in the class \( k-U^m(\alpha, \lambda, \mu, \gamma, t) \), we get the class \( k-UC^m(\lambda, \mu, \gamma, t) \) as in Definition 1.3.

### II. Coefficient Bounds of the Function Class

**k-U^m(\rho, \lambda, \mu, \gamma, t)**

Firstly, we, shall need the following lemmas.

**Lemma 2.1.** Let \( w = u + iv \). Then

\[
Re \, w \geq \alpha \text{ if and only if } |w - (1 + \alpha)| \leq |w + (1 - \alpha)|.
\]

**Lemma 2.2.** Let \( w = u + iv \) and \( \alpha, \gamma \) are real numbers. Then

\[
Re \, w > \alpha|w - 1| + \gamma \text{ if and only if } Re\{w(1 + e^{i\theta}) - \alpha e^{i\theta}\} > \gamma.
\]

**Theorem 2.1.** The function \( f(z) \) defined by (1.10) is in the class \( k-U^m(\rho, \beta, \lambda, \mu, \gamma, t) \) if and only if

\[
\Sigma \phi^m(\lambda, \mu, n)|n(k+1) - u_n(k+\gamma)||n(n-1)\rho \beta + (\rho - \beta)(n-1) + 1|a_n \leq 1 - \gamma,
\]

(2.1)

where \( \lambda \geq \mu \geq 0, m, k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1, 0 \leq \beta \leq \rho \leq 1, u_n = 1 + t + \cdots + t^{n-1} \). The result is sharp for the function \( f(z) \) given by

\[
f(z) = z - \sum_{n=2}^{\infty} \phi^m(\lambda, \mu, n)|n(k+1) - u_n(k+\gamma)||n(n-1)\rho \beta + (\rho - \beta)(n-1) + 1|z^n
\]

**Proof.** By Definition 1.1 and by Lemma 2.2, we have,

\[
\left\{ \frac{(1-t)[\rho \beta z^3(D^m_{\lambda,\mu}f(z))'' + (2\rho \beta + \rho - \beta)z^2(D^m_{\lambda,\mu}f(z))'' + z(D^m_{\lambda,\mu}f(z))']}{\rho \beta z^2[(D^m_{\lambda,\mu}f(z))'' - t^2(D^m_{\lambda,\mu}f(tz))'' + z[(D^m_{\lambda,\mu}f(z))' - t(D^m_{\lambda,\mu}f(tz))'] - ke^{i\theta}}\right\} \geq \gamma,
\]
where $-\pi < \theta < \pi$, or equivalently
\[
\Re \left\{ \frac{F(z)}{E(z)} \right\} \geq \gamma
\]  
(2.2)

where
\[
F(z) = (1-t)[\rho \beta z^3(D^m_{\lambda,\mu} f(z))'' + (2\rho \beta + \rho - \beta)z^2(D^m_{\lambda,\mu} f(z))'')
+ z(D^m_{\lambda,\mu} f(z))''' + t(D^m_{\lambda,\mu} f(tz))'']
+ (\rho - \beta) z(D^m_{\lambda,\mu} f(z))' - t(D^m_{\lambda,\mu} f(tz))' + (1 - \rho + \beta)(D^m_{\lambda,\mu} f(z) - D^m_{\lambda,\mu} f(tz))
\]

and
\[
E(z) = \rho \beta z^2[(D^m_{\lambda,\mu} f(z))'' - t^2(D^m_{\lambda,\mu} f(tz))''] + (\rho - \beta) z[(D^m_{\lambda,\mu} f(z))' - t(D^m_{\lambda,\mu} f(tz))'] + (1 - \rho + \beta)(D^m_{\lambda,\mu} f(z) - D^m_{\lambda,\mu} f(tz))
\]  
(2.3)

By Lemma 2.1, (2.2) is equivalent to
\[
|F(z) + (1 - \gamma)E(z)| \geq |F(z) - (1 + \gamma)E(z)| \quad \text{for } 0 \leq \gamma < 1
\]

But
\[
|F(z) + (1 - \gamma)E(z)|
\geq |1 - t|
\left\{ \begin{array}{l}
(2 - \gamma)|z| \\
-\Sigma \phi^n(\lambda, \mu, n)|n(n-1)\rho \beta + (\rho - \beta)(n-1) + 1||n + u_n(1 - \gamma)|a_n|z|^n
\end{array} \right\}
\]

Also,
\[
|F(z) - (1 + \gamma)E(z)|
\leq |1 - t|
\left\{ \begin{array}{l}
\gamma|z| \\
+\Sigma \phi^n(\lambda, \mu, n)|n(n-1)\rho \beta + (\rho - \beta)(n-1) + 1||n - u_n(1 + \gamma)|a_n|z|^n
\end{array} \right\}
\]

and so
\[
|F(z) + (1 - \gamma)E(z)| - |F(z) - (1 + \gamma)E(z)|
\geq \left\{ \begin{array}{l}
2(1 - \gamma)|z|
\geq 
-\sum_{n=2}^{\infty} 2\phi^n(\lambda, \mu, n)|n(k+1) - u_n(k+\gamma)||n(n-1)\rho \beta + (\rho - \beta)(n-1) + 1||a_n|z|^n
\geq 0
\end{array} \right\}
\]

or
\[
\Sigma \phi^n(\lambda, \mu, n)|n(k+1) - u_n(k+\gamma)||n(n-1)\rho \beta + (\rho - \beta)(n-1) + 1||a_n|z|^n \leq (1 - \gamma)
\]

Conversely, suppose that (2.1) holds, then we must show that (2.2) is true upon choosing the values of $z$ on the positive real axis where $0 \leq z = r < 1$, the above inequality reduces to
\[
\Re \left\{ \frac{(1 - \gamma) - \Sigma \phi^n(\lambda, \mu, n)|n(n-1)\rho \beta + (\rho - \beta)(n-1) + 1||n(k+1) - u_n(k+\gamma)|a_n|z^{n-1}}{1 - \Sigma \phi^n(\lambda, \mu, n)|n(n-1)\rho \beta + (\rho - \beta)(n-1) + 1||u_n|a_n|z^{n-1}} \right\} \geq 0
\]

Since $\Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to
\[
\Re \left\{ \frac{(1 - \gamma) - \sum_{n=2}^{\infty} \phi^n(\lambda, \mu, n)|n(n-1)\rho \beta + (\rho - \beta)(n-1) + 1||n(k+1) - u_n(k+\gamma)|a_n|z^{n-1}}{1 - \sum_{n=2}^{\infty} \phi^n(\lambda, \mu, n)|n(n-1)\rho \beta + (\rho - \beta)(n-1) + 1||u_n|a_n|z^{n-1}} \right\} \geq 0
\]
Letting \( r \to 1^- \), we have desired conclusion.

**Corollary 2.1.** Let \( \beta = 0 \) in (2.1) then we have the result for the class defined in Definition 1.2 as

\[
\sum \phi^m(\lambda, \mu, n)|n(k+1) - u_n(k+\gamma)||\rho(n-1)+1|a_n \leq (1-\gamma)
\]

where \( \lambda \geq \mu \geq 0, m, k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1, 0 \leq \rho \leq 1, u_n = 1 + t + \cdots + t^{n-1}.

**Corollary 2.2.** Let \( \rho = 1, \beta = 0 \) in (2.1) then we have the result for the class defined in Definition 1.3 as

\[
\sum \phi^m(\lambda, \mu, n)|n(k+1) - u_n(k+\gamma)|na_n \leq (1-\gamma)
\]

where \( \lambda \geq \mu \geq 0, m, k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1, 0 \leq \alpha \leq 1, u_n = 1 + t + \cdots + t^{n-1}.

The result is sharp for the function \( f(z) \) given by

\[
f(z) = z - \sum_{n=2}^{\infty} \frac{1-\gamma}{\phi^m(\lambda, \mu, n)|n(k+1) - u_n(k+\gamma)||\alpha(n-1)+1|} z^n
\]

**Proof.** The same procedure is followed as in Theorem 2.1 to prove this result.

**Corollary 2.3.** Take \( \alpha = 0 \), then we get the result as in Corollary 2.2.

### III. Neighbourhood of the Function Class

\( k-U^m(\rho, \beta, \lambda, \mu, \gamma, t) \)

Following the earlier investigations (based upon the familiar concept of neighbourhoods of analytic functions) by Goodman [7], Ruscheweyh [12], Altintas et al. (2, 3) and others including Srivastava et al. ([15, 16]), Orhan ([9]), Deniz et al. [6], Catas [4].

**Definition 3.1.** Let \( \lambda \geq \mu \geq 0, m, k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1, \alpha \geq 0, u_n = 1 + t + \cdots + t^{n-1} \) we define the \( \alpha \)-neighbourhood of a function \( f \in A \) and denote by \( N_\alpha(f) \) consisting of all functions \( g(z) = z - \sum_{n=2}^{\infty} b_nz^n \in S \) \((b_n \geq 0, n \in N)\) satisfying

\[
\sum \frac{\phi^m(\lambda, \mu, n)|n(k+1) - u_n(k+\gamma)||n(n-1)||\rho(\rho-\beta)(n-1)+1|}{1-\gamma} |a_n - b_n| \leq \alpha
\]
Theorem 3.1. Let \( f \in k-\tilde{U}^m(\rho, \beta, \lambda, \mu, \gamma, t) \) and for all real \( \theta \), we have \( \gamma(e^{i\theta} - 1) - 2e^{i\theta} \neq 0 \). For any complex number \( \epsilon \) with \( |\epsilon| < \alpha \) \((\alpha \geq 0)\), if \( f \) satisfies the following condition:

\[
\frac{f(z) + \epsilon z}{1 + \epsilon} \in k - \tilde{U}^m(\rho, \beta, \lambda, \mu, \gamma, t),
\]

then \( N_\alpha(f) \subset k-\tilde{U}^m(\rho, \beta, \lambda, \mu, \gamma, t) \).

Proof. It is obvious that \( f \in k-\tilde{U}^m(\rho, \beta, \lambda, \mu, \gamma, t) \) if and only if

\[
\frac{|u(z)(1 + ke^{i\theta}) - (ke^{i\theta} + 1 + \gamma)v(z)|}{|u(z)(1 + ke^{i\theta}) + (1 - ke^{i\theta} - \gamma)v(z)|} < 1 \quad (-\pi < \theta < \pi)
\]

where

\[
u(z) = (1 - t)[\rho \beta z^2(D_{\lambda,\mu}^m f(z))^m + (2\rho \beta + \rho - \beta)z^2(D_{\lambda,\mu}^m f(z))'' + z(D_{\lambda,\mu}^m f(z))']
\]

\[
v(z) = \rho \beta z^2[(D_{\lambda,\mu}^m f(z))'' - t^2(D_{\lambda,\mu}^m f(tz))''] + (\rho - \beta)z[(D_{\lambda,\mu}^m f(z))' - t(D_{\lambda,\mu}^m f(tz))']
\]

\[
+ (1 - \rho + \beta)[D_{\lambda,\mu}^m f(z) - D_{\lambda,\mu}^m f(tz)]
\]

for any complex number \( S \) with \( |S| = 1 \), we have

\[
\frac{u(z)(1 + ke^{i\theta}) - (ke^{i\theta} + 1 + \gamma)v(z)}{u(z)(1 + ke^{i\theta}) + (1 - ke^{i\theta} - \gamma)v(z)} \neq S
\]

In other words, we must have

\[
(1 - S)u(z)(1 + ke^{i\theta}) - (ke^{i\theta} + 1 + \gamma - S(ke^{i\theta} - 1 + \gamma))v(z) \neq 0
\]

which is equivalent to

\[
\left\{ \sum \phi^m(\lambda, \mu, n)(\rho \beta (n(n - 1)) + (\rho - \beta)(n - 1) + 1)
\right\}
\]

\[
z = \frac{((n - u_n)(1 + ke^{i\theta} - S ke^{i\theta}) - S(n + u_n) - u_n \gamma(1 - S))}{\gamma(S - 1) - 2S} a_n z^n \neq 0
\]

However, \( f \in k-\tilde{U}^m(\rho, \beta, \lambda, \mu, \gamma, t) \) if and only if \( \frac{(f \ast h)(z)}{z} \neq 0 \), \( z \in U - \{0\} \)

where \( h(z) = z - \sum_{n=2}^{\infty} c_n z^n \) and

\[
\left\{ \sum \phi^m(\lambda, \mu, n)(\rho \beta (n(n - 1)) + (\rho - \beta)(n - 1) + 1)
\right\}
\]

\[
c_n = \frac{((n - u_n)(1 + ke^{i\theta} - S ke^{i\theta}) - S(n + u_n) - u_n \gamma(1 - S))}{\gamma(S - 1) - 2S}
\]

we note that

\[
|c_n| \leq \frac{\sum \phi^m(\lambda, \mu, n)(\rho \beta (n(n - 1)) + (\rho - \beta)(n - 1) + 1|n(1 + k) - u_n(k + \gamma)|}{1 - \gamma}
\]
Since \( \frac{f(z) + \varepsilon z}{1 + \varepsilon} \in k-\tilde{U}^m(\rho, \beta, \lambda, \mu, \gamma, t) \), therefore
\[
z^{-1} \left( \frac{f(z) + \varepsilon z}{1 + \varepsilon} \ast h(z) \right) \neq 0 \]
which is equivalent to
\[
\frac{(f \ast h)(z)}{(1 + \varepsilon)z} + \frac{\varepsilon}{1 + \varepsilon} \neq 0
\]
(3.1)

Now suppose that \( \left| \frac{(f \ast h)(z)}{z} \right| < \alpha \). Then by (3.1), we must have
\[
\left| \frac{(f \ast h)(z)}{(1 + \varepsilon)z} + \frac{\varepsilon}{1 + \varepsilon} \right| \geq \frac{|\varepsilon|}{1 + |\varepsilon|} - \frac{1}{1 + |\varepsilon|} \left| \frac{(f \ast h)(z)}{z} \right| > |\varepsilon| - \alpha \geq 0
\]
this is a contradiction by \( |\varepsilon| < \alpha \) and however, we have \( \left| \frac{(f \ast h)(z)}{z} \right| \geq \alpha \). If
\[
g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in N_\alpha(f), \text{ then}
\]
\[
\alpha - \left| \frac{(g \ast h)(z)}{z} \right| \leq \left| \frac{((f - g) \ast h)(z)}{z} \right| \leq \sum_{n=2}^{\infty} |a_n - b_n| |c_n| |z^n|
\]
\[
< \sum_{n=2}^{\infty} \phi^m(\lambda, \mu, n) |\rho \beta (n(n-1)) + (\rho - \beta)(n-1) + 1||n(1+k) - u_n(k+\gamma)| |a_n - b_n| \leq \alpha
\]

**Corollary 3.1.** When \( \beta = 0 \) in Theorem 3.1, we get the result for the class \( k-\tilde{U}^m(\rho, \lambda, \mu, \gamma, t) \).

**Corollary 3.2.** When \( \rho = 1, \beta = 0 \) in Theorem 3.1, we get the result for the class \( k-\tilde{U}^C^m(\lambda, \mu, \gamma, t) \).

**Remark 3.1.** Using the similar procedure, we can prove the result as in Theorem 3.1 for the class \( k-\tilde{U}^m(\alpha, \lambda, \mu, \gamma, t) \) in which \( \alpha = 0 \) implies the result for the class \( k-\tilde{U}^C^m(\lambda, \mu, \gamma, t) \).

**IV. Acknowledgement**

The first author thanks for the support given by Science and Engineering Research Board, New Delhi - 110 016, Project No. SR/S4/MS 716/10 with titled “On certain analytic univalent functions and sakaguchi type functions”.

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