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On Coefficient Estimates and Neighbourhood Problem for Generalized Sakaguchi Type Functions

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ON COEFFICIENT ESTIMATES AND NEIGHBOURHOOD PROBLEM FOR GENERALIZED SAKAGUCHI TYPE FUNCTIONS

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On Coefficient Estimates and Neighbourhood Problem for Generalized Sakaguchi Type Functions

B. Srutha Keerthi [α] & S. Chinthamani [σ]

Abstract - In the present investigation, we introduce a new class $k\text{-}U^m(\rho, \beta, \lambda, \mu, \gamma, t)$ of analytic functions with negative coefficients. The various results obtained here for this function include coefficient estimate and inclusion relationships involving the neighbourhoods of the analytic function.

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I. Introduction

Let A denote the family of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

that are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Denote by S the subclass of A of functions that are univalent in $\mathcal{U}$.

For $f \in A$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (1.2)$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in \mathcal{U}) \qquad (1.3)$$

Note that $f * g \in A$.

A function $f \in A$ is said to be in $k\text{-}\mathcal{U}S(\gamma)$, the class of k-uniformly starlike functions of order γ, $0 \le \gamma < 1$, if satisfies the condition

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$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma \quad (k \geq 0) \tag{1.4}$$

and a function $f \in A$ is said to be in k-$\mathcal{UC}(\gamma)$, the class of k-uniformly convex functions of order γ, $0 \leq \gamma < 1$, if satisfies the condition

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| + \gamma \quad (k \geq 0) \tag{1.5}$$

Uniformly starlike and uniformly convex functions were first introduced by Goodman [8] and then studied by various authors. It is known that $f \in k$-$\mathcal{UC}(\gamma)$ or $f \in k$-$\mathcal{US}(\gamma)$ if and only if $1 + \frac{zf''(z)}{f'(z)}$ or $\frac{zf'(z)}{f(z)}$, respectively, takes all the values in the conic domain $\mathcal{R}_{k,\gamma}$ which is included in the right half plane given by

$$\mathcal{R}_{k,\gamma} = \{w = u + iv \in C : u > k\sqrt{(u-1)^2 + v^2} + \gamma, \beta \geq 0 \text{ and } \gamma \in [0,1)\}. \tag{1.6}$$

Denote by $\mathcal{P}(P_{k,\gamma})$, $(\beta \geq 0, 0 \leq \gamma < 1)$ the family of functions p, such that $p \in \mathcal{P}$, where $\mathcal{P}$ denotes well-known class of caratheodary functions. The function $P_{k,\gamma}$ maps the unit disk conformally onto the domain $\mathcal{R}_{k,\gamma}$ such that $1 \in \mathcal{R}_{k,\gamma}$ and $\partial\mathcal{R}_{k,\gamma}$ is a curve defined by the equality

$$\partial\mathcal{R}_{k,\gamma} = \{w = u+iv \in C : u^2 = (k\sqrt{(u-1)^2 + v^2}+\gamma)^2, \beta \geq 0 \text{ and } \gamma \in [0,1)\}. \tag{1.7}$$

where $0 \leq \alpha < 1$, $|t| \leq 1$, $t \neq 1$. Note that $S_S(0,-1) = S_s$ and $S_s(\alpha,-1) = S_s(\alpha)$ is called Sakaguchi function of order α.

Let us define the linear multiplier differential operator $D^m_{\lambda,\mu}f$ [11] which is shown as follows:

$$D^m_{\lambda,\mu}f(z) = z + \sum_{n=2}^{\infty} \phi^m(\lambda,\mu,n)a_n z^n \tag{1.8}$$

where

$$\phi^m(\lambda,\mu,n) = [1 + (\lambda\mu n + \lambda - \mu)(n-1)]^m, \tag{1.9}$$

$0 \leq \mu \leq 1$ and $m \in N_0 = N \cup \{0\}$.

It should be remarked that the operator $D^m_{\lambda,\mu}$ is a generalization of many other linear operators considered earlier. In particular, for $f \in A$ we have the following:

- $D^m_{1,0}f(z) \equiv D^m f(z)$ the operator investigated by Salagean (see [14]).
- $D^m_{\lambda,0}f(z) \equiv D^m_\lambda f(z)$ the operator studied by Al-Oboudi (see [1]).

Now, by making use of he differential operator $D^m_{\lambda,\mu}$, we define a new subclass of functions belonging to the class A.

Ref.

[8] Goodman, A. W. On uniformly starlike functions, J. Math. Anal. Appl. 155, 364-370, 1991.

Definition 1.1. *A function $f(z) \in A$ is said to be in the class $k\text{-}\mathcal{U}^m(\rho,\beta,\lambda,\mu,\gamma,t)$ if for all $z \in \mathcal{U}$,*

$$Re\left\{\frac{(1-t)[(\rho\beta z^3(D^m_{\lambda,\mu}f(z))''' + (2\rho\beta+\rho-\beta)z^2(D^m_{\lambda,\mu}f(z))'' + z(D^m_{\lambda,\mu}f(z))'}{\begin{array}{c}\{\rho\beta z^2[(D^m_{\lambda,\mu}f(z))'' - t^2(D^m_{\lambda,\mu}f(tz))''] + (\rho-\beta)z[(D^m_{\lambda,\mu}f(z))' \\ -t(D^m_{\lambda,\mu}f(tz))'] + (1-\rho+\beta)[D^m_{\lambda,\mu}f(z) - D^m_{\lambda,\mu}f(tz)]\}\end{array}}\right\}$$

$$\geq k\left|\frac{(1-t)[(\rho\beta z^3(D^m_{\lambda,\mu}f(z))''' + (2\rho\beta+\rho-\beta)z^2(D^m_{\lambda,\mu}f(z))'' + z(D^m_{\lambda,\mu}f(z))'}{\begin{array}{c}\{\rho\beta z^2[(D^m_{\lambda,\mu}f(z))'' - t^2(D^m_{\lambda,\mu}f(tz))''] + (\rho-\beta)z[(D^m_{\lambda,\mu}f(z))' \\ -t(D^m_{\lambda,\mu}f(tz))'] + (1-\rho+\beta)[D^m_{\lambda,\mu}f(z) - D^m_{\lambda,\mu}f(tz)]\}\end{array}} - 1\right| + \gamma$$

for $\lambda \geq \mu \geq 0$, $m,k \geq 0$, $|t| \leq 1$, $t \neq 1$, $0 \leq \beta \leq \rho \leq 1$.

Furthermore, we say that a function $f(z) \in k\text{-}\mathcal{U}^m(\rho,\beta,\lambda,\mu,\gamma,t)$ is in the subclass $k\text{-}\tilde{\mathcal{U}}^m(\rho,\beta,\lambda,\mu,\gamma,t)$ if $f(z)$ is of the following form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0, n \in N) \tag{1.10}$$

The aim of this paper is to study the coefficient bounds and certain neighbourhood results of the class $k\text{-}\tilde{\mathcal{U}}^m(\rho,\beta,\lambda,\mu,\gamma,t)$.

This subclass was motivated by Murat Cagler and Halit Orhan See [17].

Definition 1.2. *A function $f(z) \in A$ is said to be in the class $k\text{-}\mathcal{U}^m(\rho,\lambda,\mu,\gamma,t)$ if for all $z \in \mathcal{U}$*

$$Re\left\{\frac{(1-t)[\rho z^2(D^m_{\lambda,\mu}f(z))'' + z(D^m_{\lambda,\mu}f(z))']}{(1-\rho)[D^m_{\lambda,\mu}f(z) - D^m_{\lambda,\mu}f(tz)] + \rho z[(D^m_{\lambda,\mu}f(z))' - t(D^m_{\lambda,\mu}f(tz))']}\right\}$$

$$\geq k\left|\frac{(1-t)[\rho z^2(D^m_{\lambda,\mu}f(z))'' + z(D^m_{\lambda,\mu}f(z))']}{(1-\rho)[D^m_{\lambda,\mu}f(z) - D^m_{\lambda,\mu}f(tz)] + \rho z[(D^m_{\lambda,\mu}f(z))' - t(D^m_{\lambda,\mu}f(tz))']} - 1\right| + \gamma$$

for $\lambda \geq \mu \geq 0$, $m,k \geq 0$, $|t| \leq 1$, $t \neq 1$, $0 \leq \rho \leq 1$.

Remark 1.1. *When $\beta = 0$ in the class $k\text{-}\mathcal{U}^m(\rho,\beta,\lambda,\mu,\gamma,t)$, we get the class $k\text{-}\mathcal{U}^m(\rho,\lambda,\mu,\gamma,t)$ as in Definition 1.2.*

Definition 1.3. *A function $f(z) \in A$ is said to be in the class $k\text{-}\mathcal{UC}^m(\lambda,\mu,\gamma,t)$ if for all $z \in \mathcal{U}$,*

$$Re\left\{\frac{(1-t)[z^2(D^m_{\lambda,\mu}f(z))'' + z(D^m_{\lambda,\mu}f(z))']}{z[(D^m_{\lambda,\mu}f(z))' - t(D^m_{\lambda,\mu}f(tz))']}\right\}$$

$$\geq k\left|\frac{(1-t)[z^2(D^m_{\lambda,\mu}f(z))'' + z(D^m_{\lambda,\mu}f(z))']}{z[(D^m_{\lambda,\mu}f(z))' - t(D^m_{\lambda,\mu}f(tz))']} - 1\right| + \gamma$$

Ref.

[17] Murat Caglar and Halit Orhan, On Coefficient Estimates and Neighborhood Problem for Generalized Sakaguchi type Functions, arxiv: 1204.4546v1, [math.cv] 20 Apr. 2012.

for $\lambda \geq \mu \geq 0,\ m,k \geq 0,\ |t| \leq 1,\ t \neq 1.$

Definition 1.4. *A function $f(z) \in A$ is said to be in the class $k\text{-}\mathcal{U}^m(\alpha,\lambda,\mu,\gamma,t)$ if for all $z \in \mathcal{U}$,*

$$Re\left\{\frac{(1-t)[\alpha z^3(D^m_{\lambda,\mu}f(z))''' + (1+2\alpha)z^2(D^m_{\lambda,\mu}f(z))'' + z(D^m_{\lambda,\mu}f(z))']}{\alpha z^2[(D^m_{\lambda,\mu}f(z))'' - t^2(D^m_{\lambda,\mu}f(tz))''] + z[(D^m_{\lambda,\mu}f(z))' - t(D^m_{\lambda,\mu}f(tz))']}\right\}$$

$$\geq k\left|\frac{(1-t)[\alpha z^3(D^m_{\lambda,\mu}f(z))''' + (1+2\alpha)z^2(D^m_{\lambda,\mu}f(z))'' + z(D^m_{\lambda,\mu}f(z))']}{\alpha z^2[(D^m_{\lambda,\mu}f(z))'' - t^2(D^m_{\lambda,\mu}f(tz))''] + z[(D^m_{\lambda,\mu}f(z))' - t(D^m_{\lambda,\mu}f(tz))']} - 1\right| + \gamma$$

for $\lambda \geq \mu \geq 0,\ m,k \geq 0,\ |t| \leq 1,\ t \neq 1,\ 0 \leq \alpha \leq 1.$

Remark 1.2. *When $\rho = 1$ in the class $k\text{-}\mathcal{U}^m(\rho,\lambda,\mu,\gamma,t)$ and when $\alpha = 0$ in the class $k\text{-}\mathcal{U}^m(\alpha,\lambda,\mu,\gamma,t)$, we get the class $k\text{-}\mathcal{UC}^m(\lambda,\mu,\gamma,t)$ as in Definition 1.3.*

II. Coefficient Bounds of the Function Class

$k\text{-}\tilde{\mathcal{U}}^m(\rho,\beta,\lambda,\mu,\gamma,t)$

Firstly, we, shall need the following lemmas.

Lemma 2.1. *Let $w = u + iv$. Then*

$$Re\ w \geq \alpha \text{ if and only if } |w-(1+\alpha)| \leq |w+(1-\alpha)|.$$

Lemma 2.2. *Let $w = u + iv$ and α,γ are real numbers. Then*

$$Re\ w > \alpha|w-1| + \gamma \text{ if and only if } Re\{w(1+\alpha e^{i\theta}) - \alpha e^{i\theta}\} > \gamma.$$

Theorem 2.1. *The function $f(z)$ defined by (1.10) is in the class $k\text{-}\tilde{\mathcal{U}}^m(\rho,\beta,\lambda,\mu,\gamma,t)$ if and only if*

$$\Sigma\phi^m(\lambda,\mu,n)|n(k+1) - u_n(k+\gamma)||n(n-1)\rho\beta + (\rho-\beta)(n-1)+1|a_n \leq 1-\gamma, \tag{2.1}$$

where $\lambda \geq \mu \geq 0,\ m,k \geq 0,\ |t| \leq 1,\ t \neq 1,\ 0 \leq \gamma < 1,\ 0 \leq \beta \leq \rho \leq 1,$ $u_n = 1 + t + \cdots + t^{n-1}$. The result is sharp for the function $f(z)$ given by

$$f(z) = z - \sum_{n=2}^{\infty}\frac{1-\gamma}{\phi^m(\lambda,\mu,n)|n(k+1) - u_n(k+\gamma)||n(n-1)\rho\beta + (\rho-\beta)(n-1)+1|}z^n$$

Proof. By Definition 1.1 and by Lemma 2.2, we have,

$$Re\left\{\frac{(1-t)[\rho\beta z^3(D^m_{\lambda,\mu}f(z))''' + (2\rho\beta+\rho-\beta)z^2(D^m_{\lambda,\mu}f(z))'' + z(D^m_{\lambda,\mu}f(z))'](1+ke^{i\theta})}{\begin{array}{c}\{\rho\beta z^2[(D^m_{\lambda,\mu}f(z))'' - t^2(D^m_{\lambda,\mu}f(tz))''] + (\rho-\beta)z[(D^m_{\lambda,\mu}f(z))' - t(D^m_{\lambda,\mu}f(tz))'] \\ +(1-\rho+\beta)[D^m_{\lambda,\mu}f(z) - D^m_{\lambda,\mu}f(tz)]\}\end{array}} - ke^{i\theta}\right\} \geq \gamma,$$

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where $-\pi < \theta < \pi$, or equivalently

$$Re\left\{\frac{F(z)}{E(z)}\right\} \geq \gamma \tag{2.2}$$

where

$$\begin{aligned}F(z) = (1-t)[\rho\beta z^3(D^m_{\lambda,\mu}f(z))''' + (2\rho\beta+\rho-\beta)z^2(D^m_{\lambda,\mu}f(z))'' \\ + z(D^m_{\lambda,\mu}f(z))'](1+ke^{i\theta}) - ke^{i\theta}\{\rho\beta z^2[(D^m_{\lambda,\mu}f(z))'' - t^2(D^m_{\lambda,\mu}f(tz))''] \\ + (\rho-\beta)z[(D^m_{\lambda,\mu}f(z))' - t(D^m_{\lambda,\mu}f(tz))'] + (1-\rho+\beta)[D^m_{\lambda,\mu}f(z) - D^m_{\lambda,\mu}f(tz)]\}\end{aligned}$$

and

$$\begin{aligned}E(z) = \rho\beta z^2[(D^m_{\lambda,\mu}f(z))'' - t^2(D^m_{\lambda,\mu}f(tz))''] + (\rho-\beta)z[(D^m_{\lambda,\mu}f(z))' \\ - t(D^m_{\lambda,\mu}f(tz))'] + (1-\rho+\beta)[D^m_{\lambda,\mu}f(z) - D^m_{\lambda,\mu}f(tz)]\end{aligned} \tag{2.3}$$

By Lemma 2.1, (2.2) is equivalent to

$$|F(z) + (1-\gamma)E(z)| \geq |F(z) - (1+\gamma)E(z)| \quad \text{for} \quad 0 \leq \gamma < 1$$

But

$$\begin{aligned}&|F(z) + (1-\gamma)E(z)| \\ &\geq |1-t|\left\{\begin{array}{l}(2-\gamma)|z| \\ -\Sigma\phi^m(\lambda,\mu,n)|n(n-1)\rho\beta + (\rho-\beta)(n-1)+1||n + u_n(1-\gamma)|a_n|z|^n \\ -k\Sigma\phi^m|n(n-1)\rho\beta + (\rho-\beta)(n-1)+1||n - u_n|a_n|z|^n\end{array}\right\}\end{aligned}$$

Also,

$$\begin{aligned}&|F(z) - (1+\gamma)E(z)| \\ &\leq |1-t|\left\{\begin{array}{l}\gamma|z| \\ +\Sigma\phi^m(\lambda,\mu,n)|n(n-1)\rho\beta + (\rho-\beta)(n-1)+1||n - u_n(1+\gamma)|a_n|z|^n \\ +k\Sigma\phi^m(\lambda,\mu,n)|n(n-1)\rho\beta + (\rho-\beta)(n-1)+1||n - u_n|a_n|z|^n\end{array}\right\}\end{aligned}$$

and so

$$\begin{aligned}&|F(z) + (1-\gamma)E(z)| - |F(z) - (1+\gamma)E(z)| \\ &\geq \left\{\begin{array}{l}2(1-\gamma)|z| \\ -\displaystyle\sum_{n=2}^{\infty} 2\phi^m(\lambda,\mu,n)|n(k+1) - u_n(k+\gamma)||n(n-1)\rho\beta + (\rho-\beta)(n-1)+1|a_n|z|^n\end{array}\right\} \\ &\geq 0\end{aligned}$$

or

$$\Sigma\phi^m(\lambda,\mu,n)|n(k+1) - u_n(k+\gamma)||n(n-1)\rho\beta + (\rho-\beta)(n-1)+1|a_n \leq (1-\gamma)$$

Conversely, suppose that (2.1) holds, then we must show that (2.2) is true upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, the above inequality reduces to

$$Re\left\{\frac{(1-\gamma) - \Sigma\phi^m(\lambda,\mu,n)[n(n-1)\rho\beta + (\rho-\beta)(n-1)+1][n(k+1) - u_n(k+\gamma)]a_n z^{n-1}}{1 - \Sigma\phi^m(\lambda,\mu,n)[n(n-1)\rho\beta + (\rho-\beta)(n-1)+1]u_n a_n z^{n-1}}\right\} \geq 0$$

Since $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to

$$Re\left\{\frac{(1-\gamma) - \sum_{n=2}^{\infty}\phi^m(\lambda,\mu,n)[n(n-1)\rho\beta + (\rho-\beta)(n-1)+1][n(k+1) - u_n(k+\gamma)]a_n r^{n-1}}{1 - \Sigma\phi^m(\lambda,\mu,n)[n(n-1)\rho\beta + (\rho-\beta)(n-1)+1]u_n a_n r^{n-1}}\right\} \geq 0$$

Letting $r \to 1^-$, we have desired concluison.

Corollary 2.1. *Let $\beta = 0$ in (2.1) then we have the result for the class defined in Definition 1.2 as*

$$\sum \phi^m(\lambda,\mu,n)|n(k+1)-u_n(k+\gamma)||\rho(n-1)+1|a_n \leq (1-\gamma)$$

where $\lambda \geq \mu \geq 0$, $m,k \geq 0$, $|t| \leq 1$, $t \neq 1$, $0 \leq \gamma < 1$, $0 \leq \rho \leq 1$, $u_n = 1+t+\cdots+t^{n-1}$.

Corollary 2.2. *Let $\rho = 1$, $\beta = 0$ in (2.1) then we have the result for the class defined in Definition 1.3 as*

$$\sum \phi^m(\lambda,\mu,n)|n(k+1)-u_n(k+\gamma)|na_n \leq (1-\gamma)$$

where $\lambda \geq \mu \geq 0$, $m,k \geq 0$, $|t| \leq 1$, $t \neq 1$, $0 \leq \gamma < 1$, $u_n = 1+t+\cdots+t^{n-1}$.

Theorem 2.2. *The funciton $f(z)$ defined by (1.10) is in the class $k\text{-}\tilde{\mathcal{U}}^m(\alpha,\lambda,\mu,\gamma,t)$ if and only if*

$$\sum \phi^m(\lambda,\mu,n)|n(k+1)-u_n(k+\gamma)||\alpha(n-1)+1|a_n \leq 1-\gamma$$

where $\lambda \geq \mu \geq 0$, $m,k \geq 0$, $|t| \leq 1$, $t \neq 1$, $0 \leq \gamma < 1$, $0 \leq \alpha \leq 1$, $u_n = 1+t+\cdots+t^{n-1}$.
The result is sharp for the function $f(z)$ given by

$$f(z) = z - \sum_{n=2}^{\infty} \frac{1-\gamma}{\phi^m(\lambda,\mu,n)|n(k+1)-u_n(k+\gamma)||\alpha(n-1)+1|} z^n$$

Proof. The same procedure is followed as in Theorem 2.1 to prove this result.

Corollary 2.3. *Take $\alpha = 0$, then we get the result as in Corollary 2.2.*

III. Neighbourhood of the Function Class $k\text{-}\mathcal{U}^m(\rho,\beta,\lambda,\mu,\gamma,t)$

Following the earlier investigations (based upon the familiar concept of neighbourhoods of analytic functions) by Goodman [7], Ruscheweyh [12], Altintas et al. ([2, 3]) and others including Srivastava et al. ([15, 16]), Orhan ([9]), Deniz et al. [6], Catas [4].

Definition 3.1. *Let $\lambda \geq \mu \geq 0$, $m,k \geq 0$, $|t| \leq 1$, $t \neq 1$, $0 \leq \gamma < 1$, $\alpha \geq 0$, $u_n = 1+t+\cdots+t^{n-1}$ we define the α-neighbourhood of a function $f \in A$ and denote by $N_\alpha(f)$ consisting of all functions $g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in S$ $(b_n \geq 0, n \in N)$ satisfying*

$$\sum \frac{\phi^m(\lambda,\mu,n)|n(k+1)-u_n(k+\gamma)||n(n-1)\rho\beta+(\rho-\beta)(n-1)+1|}{1-\gamma}|a_n - b_n| \leq \alpha$$

Ref.

[6] Deniz, E. and Orhan, H. Some properties of certain subclasses of analytic functions with negative coefficients by using generalized Ruscheweyh derivative operator, Czechoslovak Math. J., 60 (135), 699-713, 2010.

Theorem 3.1. *Let $f \in k\text{-}\tilde{\mathcal{U}}^m(\rho,\beta,\lambda,\mu,\gamma,t)$ and for all real θ, we have $\gamma(e^{i\theta}-1)-2e^{i\theta}\neq 0$. For any complex number ϵ with $|\epsilon|<\alpha$ $(\alpha\geq 0)$, if f satisfies the following condition:*

$$\frac{f(z)+\epsilon z}{1+\epsilon}\in k-\tilde{\mathcal{U}}^m(\rho,\beta,\lambda,\mu,\gamma,t),$$

then $N_\alpha(f)\subset k\text{-}\tilde{\mathcal{U}}^m(\rho,\beta,\lambda,\mu,\gamma,t)$.

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Proof. It is obvious that $f\in k\text{-}\tilde{\mathcal{U}}^m(\rho,\beta,\lambda,\mu,\gamma,t)$ if and only if

$$\left|\frac{u(z)(1+ke^{i\theta})-(ke^{i\theta}+1+\gamma)v(z)}{u(z)(1+ke^{i\theta})+(1-ke^{i\theta}-\gamma)v(z)}\right|<1\quad(-\pi<\theta<\pi)$$

where

$$u(z)=(1-t)[\rho\beta z^3(D^m_{\lambda,\mu}f(z))'''+(2\rho\beta+\rho-\beta)z^2(D^m_{\lambda,\mu}f(z))''+z(D^m_{\lambda,\mu}f(z))']$$
$$v(z)=\rho\beta z^2[(D^m_{\lambda,\mu}f(z))''-t^2(D^m_{\lambda,\mu}f(tz))'']+(\rho-\beta)z[(D^m_{\lambda,\mu}f(z))'-t(D^m_{\lambda,\mu}f(tz))'] +(1-\rho+\beta)[D^m_{\lambda,\mu}f(z)-D^m_{\lambda,\mu}f(tz)]$$

for any complex number S with $|S|=1$, we have

$$\frac{u(z)(1+ke^{i\theta})-(ke^{i\theta}+1+\gamma)v(z)}{u(z)(1+ke^{i\theta})+(1-ke^{i\theta}-\gamma)v(z)}\neq S$$

In other words, we must have

$$(1-S)u(z)(1+ke^{i\theta})-(ke^{i\theta}+1+\gamma-S(ke^{i\theta}-1+\gamma))v(z)\neq 0$$

which is equivalent to

$$z-\frac{\begin{array}{c}\{\Sigma\phi^m(\lambda,\mu,n)(\rho\beta(n(n-1))+(\rho-\beta)(n-1)+1)\\ \times((n-u_n)(1+ke^{i\theta}-Ske^{i\theta})-S(n+u_n)-u_n\gamma(1-S))\}\end{array}}{\gamma(S-1)-2S}a_nz^n\neq 0$$

However, $f\in k\text{-}\tilde{\mathcal{U}}^m(\rho,\beta,\lambda,\mu,\gamma,t)$ if and only if $\dfrac{(f*h)(z)}{z}\neq 0,\ z\in\mathcal{U}-\{\ 0\}$ where $h(z)=z-\sum_{n=2}^{\infty}c_nz^n$ and

$$c_n=\frac{\begin{array}{c}\{\sum\phi^m(\lambda,\mu,n)(\rho\beta(n(n-1))+(\rho-\beta)(n-1)+1)\\ \times((n-u_n)(1+ke^{i\theta}-Ske^{i\theta})-S(n+u_n)-u_n\gamma(1-S))\}\end{array}}{\gamma(S-1)-2S}$$

we note that

$$|c_n|\leq\frac{\sum\phi^m(\lambda,\mu,n)|\rho\beta(n(n-1))+(\rho-\beta)(n-1)+1||n(1+k)-u_n(k+\gamma)|}{1-\gamma}$$

Since $\dfrac{f(z)+\epsilon z}{1+\epsilon} \in k\text{-}\tilde{\mathcal{U}}^m(\rho,\beta,\lambda,\mu,\gamma,t)$, therefore

$z^{-1}\left(\dfrac{f(z)+\epsilon z}{1+\epsilon} * h(z)\right) \neq 0$ which is equivalent to

$$\frac{(f*h)(z)}{(1+\epsilon)z} + \frac{\epsilon}{1+\epsilon} \neq 0 \tag{3.1}$$

Now suppose that $\left|\dfrac{(f*h)(z)}{z}\right| < \alpha$. Then by (3.1), we must have

$$\left|\frac{(f*h)(z)}{(1+\epsilon)z} + \frac{\epsilon}{1+\epsilon}\right| \geq \frac{|\epsilon|}{|1+\epsilon|} - \frac{1}{|1+\epsilon|}\left|\frac{(f*h)(z)}{z}\right| > \frac{|\epsilon|-\alpha}{|1+\epsilon|} \geq 0$$

this is a contradiction by $|\epsilon| < \alpha$ and however, we have $\left|\dfrac{(f*h)(z)}{z}\right| \geq \alpha$. If $g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in N_\alpha(f)$, then

$$\alpha - \left|\frac{(g*h)(z)}{z}\right| \leq \left|\frac{((f-g)*h)(z)}{z}\right| \leq \sum_{n=2}^{\infty} |a_n - b_n| c_n |z^n|$$

$$< \sum_{n=2}^{\infty} \frac{\phi^m(\lambda,\mu,n)|\rho\beta(n(n-1)) + (\rho-\beta)(n-1)+1||n(1+k) - u_n(k+\gamma)|}{1-\gamma}|a_n - b_n| \leq \alpha$$

Corollary 3.1. *When $\beta = 0$ in Theorem 3.1, we get the result for the class $k\text{-}\tilde{\mathcal{U}}^m(\rho,\lambda,\mu,\gamma,t)$.*

Corollary 3.2. *When $\rho = 1$, $\beta = 0$ in Theorem 3.1, we get the result for the class $k\text{-}\tilde{\mathcal{U}}\mathcal{C}^m(\lambda,\mu,\gamma,t)$.*

Remark 3.1. *Using the similar procedure, we can prove the result as in Theorem 3.1 for the class $k\text{-}\tilde{\mathcal{U}}^m(\alpha,\lambda,\mu,\gamma,t)$ in which $\alpha = 0$ implies the result for the class $k\text{-}\tilde{\mathcal{U}}\mathcal{C}^m(\lambda,\mu,\gamma,t)$.*

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