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Applications of Laplace Homotopy Analysis Method for Solving Fractional Heat- And Wave-like Equations

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Abstract - In this paper, we apply Laplace homotopy analysis method for solving various fractional heat- and wave-like equations. This method is combined form of homotopy analysis method and Laplace transform. The proposed algorithm presents a procedure of construct the base function and gives a high order deformation equation in simple form. The purpose of using the Laplace transform is to overcome the deficiency that is mainly caused by unsatisfied conditions in the other analytical techniques. Numerical examples demonstrate the capability of LHAM for fractional partial differential equations.

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1. A. Al-rabtah, V.S. Ertürk. Momani, S. Solutions of a fractional oscillator by using differential transform method, Computers and Mathematics with Applications 59 (2010), no.3, 1356-1362.

Applications of Laplace Homotopy Analysis Method for Solving Fractional Heat- And Wave-like Equations

V.G.Gupta[α] & Pramod Kumar[σ]

Abstract - In this paper, we apply Laplace homotopy analysis method for solving various fractional heat- and wave-like equations. This method is combined form of homotopy analysis method and Laplace transform. The proposed algorithm presents a procedure of construct the base function and gives a high order deformation equation in simple form. The purpose of using the Laplace transform is to overcome the deficiency that is mainly caused by unsatisfied conditions in the other analytical techniques. Numerical examples demonstrate the capability of LHAM for fractional partial differential equations.

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I. Introduction

In recent years, fractional calculus has been given considerable popularity, due mainly to its various applications in fluid mechanics, viscoelasticity, biology, electrical network, optics and signal processing, electrochemistry and so on. A review of some applications in continuum and statistical mechanics is given by Mainardi [2].One of the most recent works on the subject of fractional calculus i.e. the theory of derivatives and integrals of fractional order, is the book of Podlubny [7], which deals principally with fractional differential equations and today there are many works on fractional calculus [11,12,20,32].The importance of obtaining the exact and approximate solutions of fractional linear or nonlinear differential equations is still significant problem that needs new methods to discover the exact and approximate solutions. But these nonlinear differential equations are difficult to get their exact solutions so numerical methods have been used to handle these equations, some numerical methods have been developed, such as Laplace transform method [7,12], differential transform method [1,4,10], Adomian decomposition method [5,6,25,26,28], variational iteration method [3,15,34], homotopy perturbation method [9,17,18,27,35], homotopy perturbation transform method [29,31]. Another analytical approach that can be applied to solve linear or nonlinear equations is homotopy analysis method [21-24]. A systematic and clear exposition on HAM is given in [24]. This method has been successfully applied to solve many types of nonlinear problems, such as nonlinear Riccati differential equation with fractional order [8], nonlinear Vakhnenko equation [33], the Gluert-jet problem [30], fractional KdV-Bergers-Kurumoto equation [13], and so on.

The objective of present paper is to apply the modified homotopy analysis method namely Laplace homotopy analysis method [16], to provide symbolic approximate

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solutions for heat-like and wave-like fractional differential equations with variable coefficients. The Laplace homotopy analysis method is a combination of HAM and Laplace transform. This method is characterized by choosing the identity auxiliary linear operator. The proposed method work efficiently and the result so far are very encouraging and reliable. We would like to emphasize that the LHAM may be considered as an important and significant refinement of the previously developed techniques and can be viewed as an alternative to the recently developed methods such as Adomian decomposition method , variational iteration method , homotopy perturbation method , homotopy perturbation transform method. In this paper we have considered the effectiveness of LHAM for solving various heat-like and wave-like fractional differential equation variable coefficients. The organization of this paper is as follows: Basic definitions of fractional calculus is given in next section, the LHAM is presented in section 3. In section 4, heat-like and wave-like equations are solved to illustrate the applicability of considered method.

II. Basic Definitions

For the concept of fractional derivatives, we will adopt Caputo's definition which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variable and their integral order which is the case in most physical processes. Some basic definitions and properties of fractional calculus theory which we have used in this paper are given in this section.

Definition 2.1 : A real function $f(x)$, $x > 0$ is said to be in the space C_μ, $\mu \in R$, if there exist a real number $p\ (>\mu)$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C\,[0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu, m \in N \cup \{0\}$.

Definition 2.2 : The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-t)^{\alpha-1} f(t)\,dt\ ,\quad \alpha > 0,\ x > 0 \qquad \ldots(1)$$

$$J^0 f(x) = f(x) \qquad \ldots(2)$$

Properties of the operator J^α can be found in [14, 19], we mention only the following:

(i) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$

(ii) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$

(iii) $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$.

Definition 2.3 : The fractional derivative of $f(x)$ in the Caputo sense is defined as [14]

$$D_*^\alpha f(x) = J^{m-\alpha} D_*^m f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt \qquad \ldots(3)$$

Ref.

19. R. Gorenflow, F. Mainardi. Fractional Calculus: integral and differential equations of fractional order, in: A Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York (1997), 223-276.

For $m-1<\alpha\le m,\ m\in N,\ x>0,\ f\in C_{-1}^{m}$.

Also, we need here three basic properties

(i) $D_*^{\alpha} J^{\alpha} f(x)=f(x)$

(ii) $J^{\alpha} D_*^{\alpha} f(x)=f(x)-\sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!},\ x>0$

(iii) $D_*^{\alpha} x^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}x^{\gamma-\alpha}\ ;x>0,\gamma>0.$

For $m-1<\alpha\le m, m\in N, \mu\ge -1$ and $f\in C_{\mu}^{m}$.

Lemma 2.1. If $m-1<\alpha\le m, m\in N$, then the Laplace transform of the fractional derivative $D_*^{\alpha} f(t)$ is

$$L(D_*^{\alpha} f(t))=s^{\alpha}\bar{f}(s)-\sum_{k=0}^{m-1} f^{(k)}(0^+)s^{\alpha-k-1},\ t>0 \qquad \ldots(4)$$

Where $\bar{f}(s)$ is the Laplace transform of f (t).

III. Laplace Homotopy Analysis Method

The homotopy analysis method which provides an analytical approximate solution has been applied to various linear or nonlinear problems by many workers. In this section, we apply the modified homotopy analysis method [16].This modification is based on the Laplace transform of the fractional differential equations. To illustrate the basic idea of this method, let us consider the following fractional differential equation

$$D_t^{\alpha}u(t)=f(u,u_x,u_{xx})\ ,1<\alpha\le 2\ ,t\ge 0 \qquad \ldots(5)$$

Subject to the initial conditions

$$u(x,0)=g_1(x)\quad ,\quad u_t(x,0)=g_2(x) \qquad \ldots(6)$$

Where f a linear or nonlinear function and D_t^{α} is a fractional differential operator.

The operator form of nonlinear fPDE (5) can be written as follows:

$$D_t^{\alpha}u(x,t)=A(u,u_x,u_{xx})+B(u,u_x,u_{xx})+C(x,t)\quad ,1<\alpha\le 2\ ,t>0 \qquad \ldots(7)$$

Where A and B are linear and nonlinear operators respectively which might include other fractional derivatives of order less than ∝ and C is the known analytic function.

Now Taking the Laplace transform of both sides of eq. (5) and using (6), we have

$$L\left(D_t^{\alpha}u(x,t)\right)=L\left(A(u,u_x,u_{xx})+B(u,u_x,u_{xx})+C(x,t)\right) \qquad \ldots(8)$$

$$s^{\alpha}\bar{u}(x,s)-s^{\alpha-1}g_1(x)-s^{\alpha-2}g_2(x)=L\left(A(u,u_x,u_{xx})+B(u,u_x,u_{xx})+C(x,t)\right)$$

$$\bar{u}(x,s)=\frac{g_1(x)}{s}+\frac{g_2(x)}{s^2}+\frac{1}{s^{\alpha}}L\left(A(u,u_x,u_{xx})+B(u,u_x,u_{xx})+C(x,t)\right) \qquad \ldots(9)$$

Where $L\left(u(x,t)\right)=\bar{u}(x,s)$

The so-called zero-order deformation equation of the Laplace equation (9) has the form

Ref.

16. M. Zurigat. Solving fractional oscillators using Laplace homotopy analysis method. Annals of the University of Craiova, Mathematics and Computer Science Series, 38(4) (2011), 1-11.

$$(1-q)\left[\bar{\phi}(x,s;q)-\bar{u}_0(x,s)\right]=q\,\hbar\left[\begin{array}{l}\bar{\phi}(x,s;q)-\frac{g_1(x)}{s}-\frac{g_2(x)}{s^2}-\\ \frac{1}{s^{\alpha}}L\left(A(\phi(x,t),\phi_x(x,t),\phi_{xx}(x,t))+B(\phi(x,t),\phi_x(x,t),\phi_{xx}(x,t))+C(x,t)\right)\end{array}\right] \quad (10)$$

Where $q\in[0,1]$ is an embedding parameter, when q=0 and q=1, we have $\bar{\phi}(x,s;0)=\bar{u}_0(x,s)$ and $\bar{\phi}(x,s;1)=\bar{u}(x,s)$ respectively. Thus, as q increasing from 0 to 1, $\bar{\phi}(x,s;q)$ varies from $\bar{u}_0(x,s)$ to $\bar{u}(x,s)$. Expanding $\bar{\phi}(x,s;q)$ in Taylor series with respect to q, one has

$$\bar{\phi}(x,s,q)=\bar{u}_0(x,s)+\sum_{m=1}^{\infty}\bar{u}_m(x,s)\,q^m \quad ...(11)$$

Where

$$\overline{u_m}(x,s)=\frac{1}{m!}\frac{\partial^m}{\partial x^m}\bar{\phi}(x,s,q)\ \Big|_{q=0} \quad ...(12)$$

Define the vectors

$$\vec{\bar{u}}_0(x,s)=\left\{\bar{u}_0(x,s),\bar{u}_1(x,s),\bar{u}_2(x,s),..\,.\bar{u}_m(x,s)\right\} \quad ...(13)$$

Differentiating Equation (10) m times with respect to the embedding parameter q, and then
Setting q=0, h= -1 and finally dividing them by m!, we have the so-called mth-order deformation

$$\bar{u}_m(x,s)=\chi_m\bar{u}_{m-1}(x,s)-\Re_m(\vec{\bar{u}}_{m-1}(x,s) \quad ...(14)$$

Where

$$\Re_m(\vec{\bar{u}}_{(m-1)}(x,s))=\bar{u}_{(m-1)}(x,s)-\frac{1}{s^{\alpha}}\left(\frac{1}{m-1!}\frac{\partial^{m-1}}{\partial q^{m-1}}[L(A(\phi,\phi_x(x,t,q),\phi_{xx}(x,t,q))+B(\phi,\phi_x(x,t,q),\phi_{xx}(x,t,q)))\right)_{q=0}$$

$$-\left(\frac{g_1(x)}{s}+\frac{g_2(x)}{s^2}+\frac{1}{s^{\alpha}}L(C(x,t))\right)(1-\chi_m) \quad ...(15)$$

And $$\chi_m=\begin{cases}0 & ,\ m\le 1\\ 1 & ,\ m>1\end{cases} \quad ...(16)$$

Applying the inverse Laplace transform of both sides of (14), then we have a power series solution $u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)$ o f(5)

IV. Numerical Results

In this section we shall illustrate the Laplace homotopy analysis method (LHAM) to fractional heat- and wave-like equations.

Example 4.1 : Consider the one dimensional fractional Heat-like equation with variable coefficient.

$$D_t^{\alpha}u(x,t)=\frac{1}{2}x^2u_{xx}(x,t) \qquad ;0<x<1\,,0<\alpha\le 1\,,t>0 \quad ...(17)$$

Subject to the initial conditions $u(x,0)=x^2$ ...(18)

Taking the Laplace transform of both sides of eq. (17) and using (18) , we have

$$\bar{u}(x,s)-\frac{1}{s}x^2-\frac{1}{2}\frac{1}{s^\alpha}L\left(x^2u_{xx}\right)=0 \qquad ...(19)$$

Now in view of eq. (14) and (15), we have

Notes

$$\bar{u}_m(x,s)=\chi_m u_{m-1}(x,s)-\left[\bar{u}_{m-1}(x,s)-\frac{1}{2s^\alpha}L\left(x^2u_{(m-1)xx}(x,t)\right)-\frac{1}{s}x^2(1-\chi_m)\right] \qquad ...(20)$$

$$\bar{u}_0(x,s)=\frac{x^2}{s}$$

$$\bar{u}_1(x,s)=\frac{x^2}{s^{\alpha+1}}$$

$$\bar{u}_2(x,s)=\frac{x^2}{s^{2\alpha+1}}$$

Now

$$\bar{u}(x,s)=\bar{u}_0(x,s)+\bar{u}_1(x,s)+\bar{u}_2(x,s)+.. \qquad ...(21)$$

$$\bar{u}(x,s)=x^2\left(\frac{1}{s}+\frac{1}{s^{\alpha+1}}+\frac{1}{s^{2\alpha+1}}+...\right) \qquad ...(22)$$

Taking the inverse Laplace transform of both sides of (22), we have

$$u(x,t)=x^2\left(1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+...\right) \qquad ...(23)$$

For the special case $\alpha=1$

$$u(x,t)=x^2e^t$$

Example 4.2 *:* Consider the two dimensional fractional Heat-like equation:

$$D_t^\alpha u(x,t)=u_{xx}(x,t)+u_{yy}(x,t) \quad ;0<x,y<2\pi\ ,0<\alpha\le 1\ ,t>0 \qquad ...(24)$$

Subject to the initial conditions

$$u(x,y,0)=\sin x\sin y \qquad ...(25)$$

Taking the Laplace transform of both sides of eq. (24) and using (25), we have

$$\bar{u}(x,s)-\frac{1}{s}\sin x\sin y-\frac{1}{s^\alpha}L\left(u_{xx}+u_{yy}\right)=0 \qquad ...(26)$$

In view of equation (14) and (15), we have

$$\bar{u}_m(x,y,s)=\chi_m\bar{u}_{m-1}(x,y,s)-\left[\bar{u}_{m-1}(x,y,s)-\frac{1}{s^\alpha}L\left(u_{(m-1)xx}+u_{(m-1)yy}\right)-\frac{1}{s}\sin x\sin y(1-\chi_m)\right] ...(27)$$

$$\bar{u}_0(x,y,s)=\frac{1}{s}\sin x\sin y$$

$$\bar{u}_1(x,y,s)=-2\sin x\sin y\frac{1}{s^{\alpha+1}}$$

$$\bar{u}_2(x,y,s) = 4\sin x \sin y \frac{1}{s^{2\alpha+1}}$$

$$\bar{u}(x,s) = \bar{u}_0(x,s) + \bar{u}_1(x,s) + \bar{u}_2(x,s) + \ldots \qquad \ldots(28)$$

Taking the inverse Laplace transform of both side of eq. (28), we have

$$u(x,t) = \sin x \sin y \left(1 - 2\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots\right) \qquad \ldots(29)$$

Notes

If we take $\alpha=1$

$$u(x,t) = e^{-2t} \sin x \sin y$$

Example 4.3 *:* Consider the three dimensional inhomogeneous fractional Heat-like equation

$$D_t^{\alpha} u(x,y,z,t) = x^4 y^4 z^4 + \frac{1}{36}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right) \qquad \ldots(30)$$

$$0 < x,y,z \le 1 \;, 0 < \alpha \le 1 \,, t > 0$$

Subject to the initial conditions

$$u(x,y,z,0) = 0 \qquad \ldots(31)$$

Taking the Laplace transform of both sides of eq. (30) and using (31), we have

$$\bar{u}(x,y,z,s) = \frac{1}{s^{\alpha}} L\left(x^4 y^4 z^4 + \frac{1}{36}(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz} \right) \qquad \ldots(32)$$

Now in view of eq. (14) and (15), we have

$$\bar{u}_m(x,y,z,s) = \chi_m u_{m-1}(x,y,z,s) - \begin{bmatrix} u_{m-1}(x,y,z,s) - \dfrac{1}{s^{\alpha}} L\left(\dfrac{1}{36}(x^2 u_{(m-1)xx} + y^2 u_{(m-1)yy} + z^2 u_{(m-1)zz})\right) \\ -\dfrac{1}{s^{\alpha}} x^4 y^4 z^4 (1-\chi_m) \end{bmatrix} \qquad \ldots(33)$$

$$\bar{u}_0(x,y,z,s) = 0$$

$$\bar{u}_1(x,y,z,s) = -\frac{1}{s^{\alpha+1}} x^4 y^4 z^4$$

$$\bar{u}_2(x,y,z,s) = \frac{1}{s^{2\alpha+1}} x^4 y^4 z^4$$

$$\bar{u}_3(x,y,z,s) = \frac{1}{s^{3\alpha+1}} x^4 y^4 z^4$$

Now

$$\bar{u}(x,y,z,s) = \bar{u}_0(x,y,z,s) + \bar{u}_1(x,y,z,s) + \bar{u}_2(x,y,z,s) + \ldots \qquad \ldots(34)$$

Now taking the inverse Laplace transform of both sides of eq. (34), we have

$$u(x,y,z,t) = x^4 y^4 z^4 \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right) \qquad \ldots(35)$$

If we take $\alpha=1$

$$u(x,y,z,t) = x^4 y^4 z^4 (e^t - 1)$$

Example 4.4 *:* Consider the one dimensional wave-like equation:

$$D_t^{\alpha} u(x,t) = \frac{1}{2} x^2 u_{xx}(x,t) \quad , 1 < \alpha \leq 2 \ , t > 0 \qquad \ldots(36)$$

Subject to the initial conditions

$$u(x,0) = x \ , \ u_t(x,0) = x^2 \qquad \ldots(37)$$

Taking the Laplace transform of both sides of eq. (36) and using (37), we have

$$\bar{u}(x,s) = \frac{x}{s} + \frac{x^2}{s^2} + \frac{1}{2s^{\alpha}} L\left(x^2 u_{xx}\right) \qquad \ldots(38)$$

Notes

Now in view of equation (14) and (15), we have

$$\bar{u}_m(x,s) = \chi_m \bar{u}_{m-1}(x,s) - \left[\bar{u}_{m-1}(x,s) - \frac{1}{2s^{\alpha}} L\left(x^2 \bar{u}_{(m-1)}(x,t)\right) - \left(\frac{x}{s} + \frac{x^2}{s^2}\right)(1-\chi_m) \right] \qquad \ldots(39)$$

$$\bar{u}_0(x,s) = \frac{x}{s} + \frac{x^2}{s^2}$$

$$\bar{u}_1(x,s) = \frac{x^2}{s^{\alpha+2}}$$

$$\bar{u}_2(x,s) = \frac{x^2}{s^{2\alpha+2}}$$

Now

$$\bar{u}(x,s) = \bar{u}_0(x,s) + \bar{u}_1(x,s) + \bar{u}_2(x,s) + \ldots \qquad \ldots(40)$$

Now taking the inverse Laplace transform of both sides of eq. (40), we have

$$u(x,t) = x + x^2\left(t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots\right) \qquad \ldots(41)$$

If we take $\propto=1$

$$u(x,t) = x + x^2(e^t - 1)$$

Example 4.5 *:* Consider the two dimensional fractional Wave-like equation:

$$D_t^{\alpha} u(x,y,t) = \frac{1}{12}\left(x^2 u_{xx}(x,y,t) + y^2 u_{yy}(x,y,t)\right) , \ 0 < x, y < 1, 1 < \alpha \leq 2 \ , t > 0 \qquad \ldots(42)$$

Subject to the initial conditions

$$u(x,y,0) = x^4 \ , \ u_t(x,y,0) = y^4 \qquad \ldots(43)$$

Taking the inverse Laplace transform of both sides of eq. (42) and using (43), we have

$$\bar{u}(x,y,s) = \left(\frac{1}{s}x^4 + \frac{1}{s^2}y^4\right) + \frac{1}{12s^{\alpha}} L\left(x^2 u_{xx} + y^2 u_{yy}\right) \qquad \ldots(44)$$

Now in view of eq. (14) and (15), we have

$$\bar{u}_m(x,y,s) = \chi_m \bar{u}_{m-1}(x,y,s) - \left[\bar{u}_{m-1}(x,y,s) - \frac{1}{12s^{\alpha}} L\left(x^2 \bar{u}_{(m-1)xx} + y^2 \bar{u}_{(m-1)yy}\right) - \left(\frac{x^4}{s} + \frac{y^4}{s^2}\right)(1-\chi_m) \right]$$

$$\ldots(45)$$

$$\bar{u}_0(x,y,s)=\frac{1}{s}x^4+\frac{1}{s^2}y^4$$

$$\bar{u}_1(x,y,s)=\frac{1}{s^{\alpha+1}}x^4+\frac{1}{s^{\alpha+2}}y^4$$

$$\bar{u}_2(x,y,s)=\frac{1}{s^{2\alpha+1}}x^4+\frac{1}{s^{2\alpha+2}}y^4$$

$$\bar{u}(x,y,s)=\bar{u}_0(x,y,s)+\bar{u}_1(x,y,s)+\bar{u}_2(x,y,s)+\ldots \qquad \ldots(46)$$

Taking the inverse Laplace transform of both sides of eq. (46), we have

$$u(x,y,t)=x^4\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots\right)+y^4\left(t+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right) \qquad \ldots(47)$$

If we take ∝=1

$$u(x,y,t)=x^4e^t+y^4(e^t-1)$$

Example 4.6 : Consider the three dimensional fractional Wave-like equation of the form:

$$D_t^{\alpha}u(x,y,z,t)=(x^2+y^2+z^2)+\frac{1}{2}(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}) \qquad \ldots(48)$$

$$0<x,y,z<1 \quad , \ 1<\alpha\le 2 \ , \ t>0$$

Subject to the initial conditions

$$u(x,y,z,0)=0 \quad , \ u_t(x,y,z,0)=(x^2+y^2-z^2) \qquad \ldots(49)$$

Taking the Laplace transform of both sides of the eq. (48) and using (49), we have

$$\bar{u}(x,s)=\frac{1}{s^2}(x^2+y^2-z^2)+\frac{1}{s^{\alpha}}L\left((x^4+y^4+z^4)+\frac{1}{2}(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right) \qquad \ldots(50)$$

Now in view of eq. (14) and (15), we have

$$\bar{u}_m(x,y,z,s)=\chi_m\bar{u}_{m-1}-\left[\begin{array}{l}\bar{u}_{m-1}-\frac{1}{2s^{\alpha}}L\left(x^2u_{(m-1)xx}+y^2u_{(m-1)yy}+z^2u_{(m-1)zz}\right)-\\ \left(\frac{1}{s^2}(x^2+y^2-z^2)+\frac{1}{s^{\alpha+1}}(x^2+y^2+z^2)\right)(1-\chi_m)\end{array}\right] \qquad \ldots(51)$$

$$\bar{u}_0(x,y,z,s)=\frac{1}{s^2}(x^2+y^2-z^2)+\frac{1}{s^{\alpha+1}}(x^2+y^2+z^2)$$

$$\bar{u}_1(x,y,z,s)=\frac{1}{s^{\alpha+2}}(x^2+y^2-z^2)+\frac{1}{s^{2\alpha+1}}(x \ \ +y \ \ +z \ \ )$$

$$\bar{u}_2(x,y,z,s)=\frac{1}{s^{2\alpha+2}}(x^2+y^2-z^2)+\frac{1}{s^{3\alpha+1}}(x \ \ +y \ \ +z \ \ )$$

$$\bar{u}(x,y,z,s)=\bar{u}_0(x,y,z,s)+\bar{u}_1(x,y,z,s)+\bar{u}_2(x,y,z,s)+\ldots \qquad \ldots(52)$$

Now taking the inverse Laplace transform of both sides of eq. (52), we have

$$u(x,y,z,t)=(x^2+y^2)\left(t+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+1)}+\ldots\right)$$

Notes

$$+z^2(-t+\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{t^{2\alpha+1}}{\Gamma(2\alpha+1)}+...) \qquad ...(53)$$

If we take $\propto=2$

$$u(x,y,z,t)=(x^2+y^2)e^t+z^2e^{-t}-(x^2+y^2+z^2)$$

V. Conclusion

Notes

The main concern of this article is to apply the Laplace homotopy analysis method to construct an analytical solution for heat- and wave-like partial differential equations of fractional order with variable coefficients. The method was used in a direct way without using linearization, perturbation, or restrictive assumptions. Also its small size of computation in comparison with the computational size required in other numerical methods and its rapid convergence shows that the LHAM is reliable and introduces a significant improvement in solving partial differential equations over existing methods. Finally, the appearance of fractional differential equations as models in some field of applied mathematics makes it necessary to investigate the method (analytical or numerical) for such equations and we hope the LHAM can be applied for some other engineering system with less computational work.

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