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Generalized I-convergent Difference Sequence Spaces defined by a Moduli Sequence

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Abstract - In this article we introduce the sequence space $c_0^I(F,\Delta^n)$ and $\ell_\infty^I(F,\Delta^n$ for the sequence of $\mathrm{F} = (\mathrm{fk})$ and given some inclusion relations.

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GENERALIZED I-CONVERGENT DIFFERENCE SEQUENCE SPACES DEFINED BY A MODULI SEQUENCE

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10. H.Kizmaz, on certain sequence spaces, Canadian Math.Bull.,24(1981), 169-176.

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Abstract - In this article we introduce the sequence space $c_0^I(F,\Delta^n)$ and $\ell_\infty^I(F,\Delta^n$ for the sequence of F = (fk) and given some inclusion relations.

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I. Introduction and Preliminaries

Let ω, ℓ_∞, c_0 be the set of all sequences of complex numbers, the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|, \text{where} K \in \mathbb{N} = 1, 2, 3......$$

The idea of difference sequence spaces was introduced by H. Kizmaz [10]. In 1981 , Kizmaz defined the sequence spaces as follow;

$$\ell_\infty(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in \ell_\infty\},$$

$$c(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in c\},$$

$$c_0(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in c_0\},$$

where

$$\Delta x = (x_k - x_{k+1}) \text{ and } \Delta^0 x = (x_k),$$

These are Banach space with the norm

$$\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty.$$

Later Colak and Et [2] defined the sequence spaces:

$$\ell_\infty(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in \ell_\infty\},$$

$$c(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in c\},$$

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$$c_0(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in c_0\},$$

where $n \in \mathbf{N}, \Delta^0 x = (x_k), \Delta x = (x_k - x_{k+1}),\ \ \Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1})$ and so that

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \begin{bmatrix} n \\ v \end{bmatrix} x_{k+v},$$

and so that these are Banach space with the norm

$$\|x\|_\Delta = \sum_{i=1}^{n} |x_i| + \|\Delta^n x\|_\infty.$$

The idea of modulus was defined by Nakano [15] in 1953. A function $f : [0, \infty) \to [0, \infty)$ is called a modulus if

(*i*) $f(t) = 0$ if and only if $t = 0$,

(*ii*) $f(t + u) \leq f(t) + f(u)$, for all $t, u \geq 0$,

(*iii*) f is increasing and

(*iv*) f is continuous from the right at 0.

Let X be a sequence spaces.Then the sequence spaces $X(f)$ is defined as

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\}$$

for a modulus f. Maddox and Ruckle[14,16]

Kolak[11,12] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$,that is

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.$$

After then Gaur and Mursaleen[9] defined the following sequence spaces

$$\ell_\infty(F, \Delta) = \{x = (x_k) : (\Delta x_k) \in \ell_\infty(F)\},$$

$$c_0(F, \Delta) = \{x = (x_k) : (\Delta x_k) \in c_0(F)\},$$

for a sequence of moduli $F = (f_k)$.

we defined the following sequence spaces:

$$\ell_\infty(F, \Delta^n) = \{x = (x_k) : (\Delta^n x_k) \in \ell_\infty(F)\},$$

$$c_0(F, \Delta^n) = \{x = (x_k) : (\Delta^n x_k) \in c_0(F)\},$$

for a sequence of moduli $F = (f_k)$. We will give the necessary and sufficient conditions for the inclusion relations between $X(\Delta^n)$ and $Y(F, \Delta^n)$, where $X, Y = \ell_\infty$ or c_0. Sequence of moduli have been studied by C.A.Bektas and R. Colak[1] and many other authours.

The notion of statical convergence was introduced by H.Fast[6]. Later on it was studied by J.A.Fridy [7,8] from the sequence space point view and linked with the summability theory.

The notion of I-convergence is a generalization of the statical convergence. It was studied at initial stage by Kostyrko, Salat and Wilezynski [13]. Later on it was studied by Salat [19],Salat, Tripathy and Ziman [20], Demric[3]

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15. H.Nakano, Concave modulars, J.Math Soc. Japan., 5(1953),29-49.

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Let $\mathbb{N}$ be a non empty set. Then a family of sets $I \subseteq 2^N$ (power set of $\mathbb{N}$) is said to be an ideal if I is additive i.e $(A, B) \in I \Rightarrow (A \cup B) \in I$ and i.e $A \in I, B \subseteq A \Rightarrow B \in I$. A non empty family of sets $\pounds(I) \subseteq 2^N$ is said to be filter on N if and only if $\Phi \notin \pounds(I)$ for $A, B \in \pounds(I)$ we have $(A \cap B) \in \pounds(I)$ and for each $A \in \pounds(I)$ and $A \subseteq B$ implies $B \in \pounds(I)$.

An ideal $I \subseteq 2^N$ is called non trivial if $I \neq 2^N$. A non trivial ideal $I \subseteq 2^N$ is called admissible if $\{(x) : x \in \mathbb{N}\} \subseteq I$. A non trivial ideal is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. For each ideal I, there exist a filter $\pounds(I)$ corresponding to I, i.e $\pounds(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$,where $K^c = N - K$.

Definition 1.1. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I$.In this case we write $I - \lim x_k = L$.

Definition 1.2. A sequence $(x_k) \in \omega$ is said to be I-null if L=0. In this case we write $I - \lim x_k = 0$.

Definition 1.3. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$, there exist a number $m = m(\epsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in I$.

Definition 1.4. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exist $M > 0$ such that $\{K \in \mathbb{N} : |x_k| \geq M\}$.

We need the following Lemmas.

Lemma 1.5. The condition $\sup_k f_k(t) < \infty, t > 0$ hold if and only if there is a point $t_0 > 0$ such that $\sup_k f_k(t_0) < \infty$ (see [1,9]).

Lemma 1.6. The condition $\inf_k f_k(t) > 0$ hold if and only if there exist is a point $t_0 > 0$ such that $\inf_k f_k(t_0) > 0$ (see [1,9]).

Lemma 1.7. Let $K \in \pounds(I)$ and $M \subseteq N$. If $M \neq I$ then $M \cap K \neq I$ (see [20]).

Lemma 1.8. If $I \subseteq 2^N$ and $M \subseteq N$. If $M \neq I$ then $M \cap K \neq I$ (see [13]).

II. Main Results

In this article we introduce the following classes of sequence spaces.

$$c_0^I(F, \Delta^n) = \{(x_k) \in \omega : I - \lim f_k(|\Delta^n x_k|) = 0\} \in I,$$

$$\ell_\infty^I(F, \Delta^n) = \{(x_k) \in \omega : I - \sup_k f_k(|\Delta^n x_k|) < \infty\} \in I$$

Theorem 2.1. For a sequence $F = f_k$ of moduli, the following statements are equivalent:

(a) $\ell_\infty^I(\Delta^n) \subseteq \ell_\infty^I(F, \Delta^n)$,

(b) $c_0^I(\Delta^n) \subseteq c_0^I(F, \Delta^n)$,

(c) $\sup_k f_k(t) < \infty, (t > 0)$.

Proof. (a) implies (b) is obvious .

(b) implies (c). Let $c_0^I(\Delta^n) \subseteq c_0^I(F, \Delta^n)$. Suppose that (c) is not true. Then by Lemma (1.5)

$$\sup_k f_k(t) = \infty, \text{ for all } t > 0,$$

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1. C.A.Bektas, R.Colak, Generalized difference sequence spaces defined by a sequence of moduli, Soochow. J.Math., 29(2)(2003), 215-220 164.

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and therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}(\frac{1}{i}) > i, \quad \text{for each} \quad i = 1, 2, 3...... \tag{1}$$

Define $x = (x_k)$ as follow

$$x_k = \begin{cases} \frac{1}{i} \text{ if } \ k = k_i, i = 1, 2, 3.....; \\ 0 \quad \text{otherwise.} \end{cases}$$

Then $x \in c_0^I(\Delta^n)$ but by (1), $x \notin \ell_\infty^I(F, \Delta^n)$ which contradicts (b). Hence (c) must hold.
(c) implies (a). Let (c) be satisfied and $x \in \ell_\infty^I(F, \Delta^n)$. If we suppose that $x \notin \ell_\infty^I(F, \Delta^n)$ then

$$\sup_k f_k(|\Delta^n x_k|) = \infty \text{ for } \ \Delta^n x \in \ell_\infty^I$$

If we take $t = |\Delta^n x|$ then $\sup_k f_k(t) = \infty$ which contradicts (c).
Hence $\ell_\infty^I(\Delta^n) \subseteq \ell_\infty^I(F, \Delta^n)$.

Theorem 2.2. For a sequence $F = f_k$ is a sequence of moduli, the following statements are equivalent:

(a) $c_0^I(F, \Delta^n) \subseteq c_0^I(\Delta^n)$,

(b) $c_0^I(F, \Delta^n) \subseteq \ell_\infty^I(\Delta^n)$,

(c) $\inf_k f_k(t) > 0, (t > 0)$.

Proof. (a) implies (b) is obvious.

(b) implies (c). Let $c_0^I(F, \Delta^n) \subseteq \ell_\infty^I(\Delta^n)$. Suppose that (c) is not true. Then by Lemma (1.6)

$$\inf_k f_k(t) = 0, \quad (t > 0)$$

and therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}(i^2) < \frac{1}{i} \quad \text{for each} \quad i = 1, 2, 3...... \tag{2}$$

Define $x = (x_k)$ as follow

$$x_k = \begin{cases} i^2, \text{ if } \ k = k_i \quad i = 1, 2, 3.....; \\ 0 \quad \text{otherwise.} \end{cases}$$

By (2) $x \in c_0^I(F, \Delta^n)$ but $x \notin \ell_\infty^I(\Delta^n)$ which contradicts (b). Hence (c) must hold.
(c) implies (a). Let (c) be satisfied and $x \in c_0^I(F, \Delta^n)$ that is

$$I - \lim_k f_k(|\Delta^n x_k|) = 0.$$

Suppose that $x \notin c_0^I(\Delta^n)$. Then for some number $\epsilon_0 > 0$ and positive integer k_0 we have $|\Delta^n x_k| \leq \epsilon_0$ for $k > k_0$. Therefore $f_k(\epsilon_0) \geq f_k(|\Delta^n x_k|)$ for $k > k_0$ and hence $\lim_k f_k(\epsilon_0) > 0$, which contradicts our assumption that $x \notin c_0^I(\Delta^n)$.

Thus $c_0^I(F, \Delta^n) \subseteq c_0^I(\Delta^n)$.

Theorem 2.3. The inclusion $\ell_\infty^I(F, \Delta^n) \subseteq c_0^I(\Delta^n)$ holds if and only if

$$\lim_k f_k(t) = \infty \quad \text{for} \quad t > 0. \tag{3}$$

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Proof. Let $\ell_\infty^I(F,\Delta^n) \subseteq c_0^I(\Delta^n)$ such that $\lim_k f_k(t) = \infty$ for $t > 0$ doesn't hold. Then there is a number $t_0 > 0$ and a sequence (k_i) of positive integer such that

$$f_{k_i}(t_0) \leq M < \infty. \qquad (4)$$

define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} t_0, & \text{if } \ k = k_i \ \ i = 1,2,3.....; \\ 0 & \text{otherwise.} \end{cases}$$

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Thus $x \in \ell_\infty^I(F,\Delta^n)$ by (4). But $x \notin c_0^I(\Delta^n)$, so that (3) must hold.If $\ell_\infty^I(F,\Delta^n) \subseteq c_0^I(\Delta^n)$. Conversely, let (3) hold. If $x \in \ell_\infty^I(F,\Delta^n)$,then $f_k(|\Delta^n x_k|) \leq M < \infty$, for $k = 1,2,3$.......Suppose that $x \notin c_0^I(\Delta^n)$. Then for some number $\epsilon_0 > 0$ and positive integer k_0 we have $|\Delta^n x_k| < \epsilon_0$ for $k \geq k_0$. Therefore $f_k(\epsilon_0) \geq f_k(|\Delta^n x_k|) \leq M$ for $k \geq k_0$, which contradicts(3). Hence $x \in c_0^I(\Delta^n)$.

Theorem 2.4. The inclusion $\ell_\infty^I(\Delta^n) \subseteq c_0^I(F,\Delta^n)$ holds if and only if

$$\lim_k f_k(t) = 0, \quad \text{for } t > 0. \qquad (5)$$

Proof. Suppose that $\ell_\infty^I(\Delta^n) \subseteq c_0^I(F,\Delta^n)$ but (5) doesn't hold,

Then

$$\lim_k f_k(t_0) = l \neq 0, \quad \text{for some } t_0 > 0 \qquad (6).$$

Define the sequence $x = (x_k)$ by

$$x_k = t_0 \sum_{v=0}^{k-n} (-1)^n \begin{bmatrix} n+k-v-1 \\ k-v \end{bmatrix}$$

for $k = 1,2,3,....$ Then $x \notin c_0^I(F,\Delta^n)$ by (6). Hence (5) must hold.
conversely, let $x \in \ell_\infty^I(\Delta^n)$ and suppose that (5) holds.
Then$|\Delta^n x_k| \leq M < \infty$ for $K = 1,2,3....$ There for $f_k(|\Delta^n x_k|) \leq f_k(M)$ for $k = 1,2,3.....$ and $\lim_k f_k(|\Delta^n x_k|) \leq \lim_k f_k(M) = 0$ by (5). Hence $x \in c_0^I(F,\Delta^n)$

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