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Heat Conductance, a Boundary Value Problem Involving Certain Product of Special Functions

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Ref.

1. **Churchill, R.V.** Fourier series and boundary value problems, McGraw-Hill Book Co., New York (1942).

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Heat Conductance, a Boundary Value Problem Involving Certain Product of Special Functions

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Abstract - The object of this paper is to discuss an application to certain products containing the H-function of several complex variables in boundary value problems. The results established in this paper are general nature & hence encompass several cases of interest.

Keywords : The product of Fox's H-function, M-series, a general class of polynomials and the multivariable H-function.

I. Introduction

Boundary value problem with Fox's H-function, M-series & multivariable H-function were studied by many authors, Churchill, R.V.[1], Mohammed, T.[3], Shrivastava, H.M. [6], Sharma, M.[4] etc.

Further, an integral involving Fox's H-function & heat conduction and on simultaneous operational calculus involving a product of Fox's H-function and the multivariable were studied by Bajpai [7], Chourasia [9] respectively.

This paper deals the problem of determining a function θ(x,t), representing the temperature in a non-homogeneous bar with ends at x = ± in which the thermal conductivity is proportional to $(1 - x^2)$ and if the lateral surface of the bar is insulated, it satisfies the partial differential equation of heat conduction Churchill [1] ,

$$\frac{\partial\theta}{\partial t}=b\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial\theta}{\partial x}\right], \qquad (1)$$

where b is a constant, provided thermal coefficient is constant. The boundary conditions of the problem are that both ends of a bar at

$$x = \pm 1 \qquad (2)$$

are also insulated because the conductivity vanishes there and the initial conditions

$$\theta\ (x,0) = f(x);\ -1 \text{ ¡ } x \text{ ¡ } 1, \qquad (3)$$

II. Result Required

(i) *The finite integral*

$$\int_{-1}^{1}(1-x^2)^{\alpha-1}P_\nu^\mu(x)\,{}_PF_Q\left[\begin{matrix}A_P\\B_Q\end{matrix};\beta(1-x^2)^d\right]$$

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$$H_{p,q}^{m,n}\left[M(1-x^2)^k \left| \begin{matrix} (e_p,E_p) \\ (f_q,F_q) \end{matrix} \right.\right] {}_{P_1}M_{Q_1}^{\alpha'}[M_1(1-x^2)^{k_1}]$$

$$.S_{v'}^{u'}[M_2(1-x^2)]H\left[\prod_{i=1}^{r} z_i(1-x^2)^{\sigma_i}\right]dx$$

$$=\sum_{G=1}\sum_{s,s',s'',t=0}^{\infty}\frac{(A_1)_t\ldots(A_p)_t\,\beta^t(-1)^s M^{g_s}\phi(g_s)}{(B_1)_t\ldots(B_Q)_t\; t!\,s!\;F_G\;s'!}$$

$$.\frac{\pi 2^{\mu}\,(-v')_{u's'}\,A_{v's'}\,M_2^{s'}(a_1)_{s''}\ldots(a_{P_1})_{s''}\;M_1^{s''}}{\Gamma\left(1-\frac{\mu}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2}\right)(b_1)_{s''}\ldots(b_{Q_1})_{s''}\;\Gamma(\alpha's''+1)}$$

$$.H_{A+2,C+2:(B',D');\ldots;(B^{(r)},D^{(r)})}^{0,\lambda+2\;:(u',v');\ldots;(u^{(r)},v^{(r)})}\left[\begin{matrix}[----------],[1-\alpha-td-kg_s-k_1s''-s'\pm\frac{\mu}{2}:\sigma_1,\ldots,\sigma_r], \\ [(c):\psi',\ldots,\psi^{(r)}]:[-\alpha-td-kgs\;-k_1s''-s'-\frac{\nu}{2}:\sigma_1,\ldots,\sigma_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}):\;[(b');\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [1-\alpha-td-kg_s-k_1s''-s'+\frac{\nu}{2}:\sigma_1,\ldots,\sigma_r],[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\;z_1,\ldots,z_r\right], \qquad (4)$$

where $\mathrm{Re}\left(\alpha+k\frac{f_j'}{F_j'}+\sum_{i=1}^{r}\sigma_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>\frac{1}{2}|\mathrm{Re}(\mu)|, j'=1,\ldots,m, j=1,\ldots,u^{(i)},\sigma_i>0,k>0,$

$k_1>0,|\arg(z_i)|<\frac{1}{2}T_i\pi,|\arg M|<\frac{1}{2}T'\pi,T'>0,u'$ is an arbitrary positive integer, the coefficients $A_{v',s'}(v',s'>0)$ are arbitrary constants, real or complex.

(ii) *Orthogonality property of the associated Legendre polynomials*

$$\int_{-1}^{1} P_n^m(t)\,P_k^m(t)\,dt=\frac{2(m+1)!}{(2n+1)(n-m)!}\delta_{nk} \qquad (5)$$

where δ_{nk} is the Kroneckar delta defined by

$$\delta_{nk}\begin{cases}0, \text{if } n\neq k \\ 1 \text{ if } n=k\end{cases} \qquad (6)$$

Solution of (1):-

Assuming the following

$$f(x)=(1-x^2)^{\alpha-1}\;{}_PF_Q[A_P;B_Q;\beta(1-x^2)^d]$$

$$H_{p,q}^{m,n}\left[M(1-x^2)^k \left| \begin{matrix} (e_p,E_p) \\ (f_q,F_q) \end{matrix} \right.\right] {}_{P_1}M_{Q_1}^{\alpha'}[M_2(1-x^2)^{k_1}]$$

$$S_{v'}^{u'}(M_1(1-x^2))H\left(\prod_{i=1}^{r} z_i(1-x^2)^{\sigma_i}\right), \qquad (7)$$

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The solution of the problem (4) can be written as

$$\theta(x,t)=\sum_{N=0}^{\infty} A_N P_N^{\mu}(x)\, e^{-bN(N+1)t'}, \tag{8}$$

If $t'=0$ in (8), then by virtue of (7)

$$f(x)=(1-x^2)^{\alpha-1}\,{}_PF_Q[A_P;B_Q;\beta(1-x^2)^d]$$

$$.H_{p,q}^{m,n}\left[M(1-x^2)^k\left|\begin{matrix}(e_p,E_p)\\(f_q,F_q)\end{matrix}\right.\right]{}_{P_1}\overset{\alpha'}{M}_{Q_1}[M_2(1-x^2)^{k''}]$$

$$.S_{v'}^{u'}(M_1(1-x^2))H\left(\prod_{i=1}^{r} z_i(1-x^2)^{\sigma_i}\right)$$

$$=\sum_{N=0}^{\infty} A_N^{\mu} P_N^{\mu}(x), \tag{9}$$

Equation (7) is valid since f(x) is continuous in the closed interval $-1\le x\le 1$ and has a piecewise continuous derivative there, the Legendre series (9) associated with f(x) converges uniformly to f(x) in $-1+\in\le x\le 1-\in$, $0\le\in\le 1$.

Now multiplying both sides of (9) by $P_\nu^{\mu}(x)$ and integrating from − 1 to +1 with respect to x, we find

$$\int_{-1}^{1}(1-x^2)^{\alpha-1}\,{}_PF_Q[A_P;B_Q;\beta(1-x^2)^d]$$

$$.H_{p,q}^{m,n}\left[M(1-x^2)^k\left|\begin{matrix}(e_p,E_p)\\(f_q,F_q)\end{matrix}\right.\right]{}_{P_1}\overset{\alpha'}{M}_{Q_1}[M_2(1-x^2)^{k''}]$$

$$.S_{v'}^{u'}(M_1(1-x^2))H\left[\prod_{i=1}^{r} z_i(1-x^2)^{\sigma_i}\right]P_\nu^{\mu}(x)\,dx$$

$$=\sum_{N=0}^{\infty} A_N\int_{-1}^{1} P_N^{\mu}(x)P_\nu^{\mu}(x)\,dx, \tag{10}$$

Now using (4) and the orthogonal property of Legendre polynomials, (5) and (6), we get

$$A_N=\frac{2^{\mu-1}\pi(2\nu+1)(\nu-\mu)!}{(\nu+\mu)!\,\Gamma\left(1-\frac{\mu}{2}\pm\frac{\nu}{2}\right)}$$

$$.\sum_{G=1}^{m}\sum_{s,s',s'',t=0}^{\infty}\frac{(A_1)_t\ldots(A_p)_t\,\beta^t(-1)^s M^{g_s}\phi(g_s)}{(B_1)_t\ldots(B_Q)_t\ t!\ s!\ F_G\ s!}$$

$$.\frac{(a_1)_{s''}\ldots(a_{P_1})_{s''}\,M_2^{s''}(-v')_{u's'}\,A_{v',s'}\,M_1^{s'}}{(b_1)_{s''}\ldots(b_{Q_1})_{s''}\,\Gamma(\alpha's''+1)}$$

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$$.H^{0,\lambda+2\ :(u',v');...;(u^{(r)},v^{(r)})}_{A+2,C+2:(B',D');...;(B^{(r)},D^{(r)})}\left[\begin{matrix}[---------],[1-\alpha-td-kg_s\pm\frac{\mu}{2}-k''s''-s':\sigma_1,...,\sigma_r],\\ [(c):\psi',...,\psi^{(r)}]:[-\alpha-td-kg_s-\frac{\nu}{2}-k''s''-s':\sigma_1,...,\sigma_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',...,\theta^{(r)}):[(b');\phi'];...;[(b^{(r)}):\phi^{(r)}];\\ [1-\alpha-td-kg_s+\frac{\nu}{2}-k''s''-s':\sigma_1,...,\sigma_r],[(d'):\delta'];...;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,...,z_r\right], \quad (11)$$

Notes

With the help of (8) and (9) the solution of the problem (1) is obtained in the form

$$\theta(x,t)=\pi 2^{\mu-1}\frac{(2\nu+1)(\nu-\mu)!}{(\nu+\mu)!\Gamma\left(1-\frac{\mu}{2}\pm\frac{\nu}{2}\right)}$$

$$.\sum_{G=1}^{m}\sum_{s,s',s'',t=0}^{\infty}e^{-bN(N+1)t'}\frac{(A_1)_t...(A_p)_t\,\beta^t}{(B_1)_t...(B_Q)_t\ \ t!\,s!}\frac{P_N^{\mu}(x)(-1)^s M^{g_s}\phi(g_s)}{f_G\ \ s!\,s'!}$$

$$.\frac{(a_1)_{s''}...(a_{P_1})_{s''}\ M_2^{s''}(-v')_{u's'}\ A_{v',s'}\ M_1^{s'}}{(b_1)_{s''}...(b_{Q_1})_{s''}\ \Gamma(\alpha's''+1)}$$

$$.H^{0,\lambda+2\ :(u',v');...;(u^{(r)},v^{(r)})}_{A+2,C+2:(B',D');...;(B^{(r)},D^{(r)})}\left[\begin{matrix}[---------],[1-\alpha-td-kg_s-k''s'-s'\pm\frac{\mu}{2}:\sigma_1,...,\sigma_r],\\ [(c):\psi',...,\psi^{(r)}]:[-\alpha-td-kg_s-k''s''-s'-\frac{\nu}{2}:\sigma_1,...,\sigma_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',...,\theta^{(r)}):[(b');\phi'];...;[(b^{(r)}):\phi^{(r)}];\\ [1-\alpha-td-kg_s+-k''s''-s'+\frac{\nu}{2}:\sigma_1,...,\sigma_r],[(d'):\delta'];...;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,...,z_r\right] \quad (12)$$

$$\text{where}\quad \mathrm{Re}\left(\alpha+k\frac{f_{j'}}{F_{j'}}+\sum_{i=1}^{r}\sigma_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>\frac{1}{2}|\mathrm{Re}(w)|, j=1,...,m;\sigma_i,k,k'',T_i>0,$$

$$|\arg z_i|<\frac{1}{2}T_i\pi, i=1,...,r,\alpha>0,P\le Q,|M_2|<1,P_1\le Q,|\beta|<1,\arg M|<\frac{1}{2}T'\pi,T'>0.$$

Special Cases :-

(1) Putting $\lambda=A, u^{(i)}=1, v^{(i)}=B^{(i)}, D^{(i)}=D^{(i)}+1, \forall i=1,...,r$ in (12), we obtain

$$\theta(x,t)=\frac{\pi 2^{\mu-1}(2\nu+1)(\nu-\mu)}{(\nu+\mu)!\Gamma\left(1-\frac{\mu}{2}\pm\frac{\nu}{2}\right)}$$

$$=\sum_{G=1}^{m}\sum_{s,s',s'',N,t=0}^{\infty}\frac{e^{-b\,N(N+1)t'}(A_1)_t...(A_p)_t\,\beta t}{(B_1)_t...(B_q)_t}$$

$$.\frac{P_N^{\mu}(x)(-1)^s M^{g_s}\phi(g_s)\,(a_1)_{s''}...(a_{P_1})_{s''}\ M_2^{s''}(-v')_{u's'}\ A_{v',s'}\ M_1^{s'}}{t!\,s!\,s'!\,(b_1)_{s''}...(b_{Q_1})_{s''}\ \ s'!}$$

$$.F^{A+2:B';...;B^{(r)}}_{C+2:D';...;D^{(r)}}\left[\begin{matrix}[---------],[1-\alpha-td-kg_s-M_2s''-s'\pm\frac{\mu}{2}:\sigma_1,...,\sigma_r],\\ [1-(c):\psi',...,\psi^{(r)}]:[-\alpha-td-kg_s-M_2s''-s'-\frac{\nu}{2}:\sigma_1,...,\sigma_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',...,\theta^{(r)}):\ [1-(b');\phi'];...;[(1-b^{(r)}):\phi^{(r)}];\\ [1-\alpha-td-kg_s-M_2s''-s'+\frac{\nu}{2}:\sigma_1,...,\sigma_r],[(1-d'):\delta'];...;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ -z_1,...,-z_r\right], \qquad (13)$$

valid under the same conditions as derivable from (12).

(2) Letting r = 2 in (13), we have

$$\theta(x,t)=\pi 2^{\mu-1}\frac{(2\nu+1)(\nu-\mu)!}{(\nu+\mu)!\Gamma\left(1-\frac{\mu}{2}\pm\frac{\nu}{2}\right)}$$

$$.\sum_{G=1}^{m}\sum_{s,s',s'',t=0}^{\infty}e^{-bN(N+1)t'}\frac{(A_1)_t...(A_p)_t\,\beta^t}{(B_1)_t...(B_Q)_t\ t!}\frac{P_N^{\mu}(x)(-1)^s M^{g_s}\phi(g_s)(-v')_{u's'}A_{v',s'}M_1^{s'}}{f_G\ s!\ s'!}$$

$$.\frac{(a_1)_{s''}...(a_{P_1})_{s''}M_2^{s''}}{(b_1)_{s''}...(b_{Q_1})_{s''}\Gamma(\alpha's''+1)}$$

$$.S^{A+2:B';...;B''}_{C+2:D';...;D''}\left[\begin{matrix}[---------],[1-\alpha-td-kg_s-M_2s''-s'\pm\frac{\mu}{2}:\sigma_1,\sigma_2],\\ [1-(c):\psi',...,\psi^{(r)}]:[-\alpha-td-kg_s-M_2s''-s'-\frac{\nu}{2}:\sigma_1,\sigma_2],\end{matrix}\ \begin{matrix}[(a):\theta',\theta''):\ [1-(b');\phi'];[(b''):\phi''];\\ [1-\alpha-td-kg_s-s''-s'+\frac{\nu}{2}:\sigma_1.\sigma_2],[1-(d'):\delta'].;[(d''):\delta''];\end{matrix}\ -z_1,-z_2\right],(14)$$

valid under the same conditions as derivable from (15).

(3) Taking λ = A = C = 0 the results in (12) reduces to the following result

$$\theta(x,t)=\pi 2^{\mu-1}\frac{(2\nu+1)(\nu-\mu)!}{(\nu+\mu)!\Gamma\left(1-\frac{\mu}{2}\pm\frac{\nu}{2}\right)}$$

$$.\sum_{G=1}^{m}\sum_{s,s',s'',N,t'=0}^{\infty}e^{-bN(N+1)t'}\frac{(A_1)_t...(A_p)_t\,\beta^t}{(B_1)_t...(B_Q)_t\ t!\ s!}.\frac{P_N^{\mu}(x)(-1)^s M^{g_s}\phi(g_s)}{f_G}$$

$$.\frac{(a_1)_{s''}...(a_{P_1})_{s''}M_2^{s''}(-v')_{u's'}A_{v',s'}M_1^{s'}}{(b_1)_{s''}...(b_{Q_1})_{s''}\Gamma(\alpha's''+1)s'!}$$

$$.H^{0,2:(u',v');...;(u^{(r)},v^{(r)})}_{2,2:(B',D');...;(B^{(r)},D^{(r)})}\left[\begin{matrix}[---------],[1-\alpha-td-kg_s-M_2s''-s'\pm\frac{\mu}{2}:\sigma_1,...,\sigma_r],\\ [(c):\psi',...,\psi^{(r)}]:[-\alpha-td-kg_s-M_2s''-s'-\frac{\nu}{2}:\sigma_1,...,\sigma_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',...,\theta^{(r)}):\ [(b');\phi'];...;[(b^{(r)}):\phi^{(r)}];\\ [1-\alpha-td-kg_s+-M_2s''-s'+\frac{\nu}{2}:\sigma_1,...,\sigma_r],[(d'):\delta'];...;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,...,z_r\right], \qquad (15)$$

valid under the same conditions as derivable from (12).

(4) Letting $k, \alpha', v' \to 0$ in (4), we have a known result given in ([8], eq.(1.3), p.227).

(5) Also taking $k, \alpha', v' \to 0$ in (12), we get a result given in ([8], eq. (2.1), p.228).

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Notes

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