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Modified H-Tran sform and Pathway Fractional Integral Operator

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MODIFIED H-TRAN SFORM AND PATHWAY FRACTIONAL INTEGRAL OPERATOR

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Ref.

7. M. Saigo, R. K. Saxena and J.Ram, On the two dimensional generalized Weyl fractional calculus associated with two dimensional H-transform, J.Fractional Calculus, 8(1995), 63-73.

Modified H-Transform and Pathway Fractional Integral Operator

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Keywords : Pathway fractional integral operator, modified H-function transform, H- function of one variable, Whittaker function, Wright's generalized Bessel function.

I. Introduction

The modified H-function transform was introduced by Saigo, Saxena and Ram [7] and is defined in the following manner:

$$h(s) \quad = \quad h^{M,\,N}_{P,\,Q}\left[F(x)\,;\rho\,,s\right]$$

$$= \int_{d}^{\infty}(sx)^{\rho-1}\; H^{M,N}_{P,Q}\left[(sx)^{k}\left|\begin{matrix}(c_j,\gamma_j)_{1,\,P}\\(d_j,\delta_j)_{1,\,Q}\end{matrix}\right.\right]F(x)\,dx\,,\quad \text{for } k>0, \tag{1}$$

$$F(x) \;=\; f\,(\,a'\sqrt{x^2-d^2}\,)\,U(x-d),\quad x>d>0\,, \tag{2}$$

where U(x-d) is the well- known Heaviside unit function.

Further we assume that h(s) exists and belongs to U. where U is the class of functions f(x) on $R_+ = (0,\infty)$, which is infinitely differentiable with partial derivatives of any order such that

$$f(x) \;=\; \begin{bmatrix} 0\;\;(|x|^{w_1})\;\;\text{as}\;\;x\to 0\\ 0\;\;(|x|^{-w_2})\;\text{as}\;\;x\to\infty\end{bmatrix}. \tag{3}$$

The transform defined by (1) exists provided that following (sufficient) conditions are satisfied:

(i) $\left|\arg s\right| < \frac{1}{2}\pi\,\Omega/k\,,$

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where $\Omega = \sum_{j=1}^{N} \gamma_j - \sum_{j=N+1}^{P} \gamma_j + \sum_{j=1}^{M} d_j - \sum_{j=M+1}^{Q} d_j$

(ii) $\mathrm{Re}(w_1) + 1 > 0$,

(iii) $\mathrm{Re}(\rho - w_2) + k \max_{1 \le j \le N} \left[\mathrm{Re}\left(\frac{c_j - 1}{\gamma_j} \right) \right] < 0$.

The Fox's H-function or simply H-function was introduced by Charles Fox [5]. This function is defined and represented by means of the following Mellin-Barnes type of contour integral:

$$H_{P,Q}^{M,N}[z] = H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (a_j,\alpha_j)_{1,P} \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right] = \frac{1}{2\pi i} \int_L \theta(s_1) z^{s_1} ds_1 , \tag{4}$$

where $i = (-1)^{1/2}$, $z \neq 0$ and

$$\theta(s_1) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j s_1) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j s_1)}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + \beta_j s_1) \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j s_1)} . \tag{5}$$

The nature of the contour L in (4), the conditions of convergence of the integral (4), the asymptotic expansion of the H-function and some of its special cases can be referred to the work of Srivastava, Gupta and Goyal [6] and Mathai and Saxena [4].

The Pathway fractional integral operator introduced by Nair [8] and is defined in the following manner:

$$\left(P_{0_+}^{(\eta,\alpha)} f \right)(x) = x^{\eta} \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1 - \frac{a(1-\alpha)t}{x} \right]^{\frac{\eta}{(1-\alpha)}} f(t)\,dt , \tag{6}$$

where $f(x) \in L(a,b)$, $\eta \in C$, $R(\eta) > 0$, $a > 0$ and 'pathway parameter' $\alpha < 1$.

The pathway model is introduced by Mathai [1] and studied further by Mathai and Haubold ([2], [3]). For real scalar α, the pathway model for scalar random variables is represented by the following probability density function (p.d.f.):

$$f(x) = c|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}} , \tag{7}$$

$-\infty < x < \infty$, $\delta > 0$, $\beta \ge 0$, $(1 - a(1-\alpha)|x|^{\delta}) > 0$, $\gamma > 0$, where c is the normalizing constant and α is called the pathway parameter. For real α, the normalizing constant is as follows:

$$c = \frac{1}{2} \frac{\delta\,[a(1-\alpha)]^{\frac{\gamma}{\delta}}\, \Gamma\left(\frac{\gamma}{\delta} + \frac{\beta}{1-\alpha} + 1 \right)}{\Gamma\left(\frac{\gamma}{\delta} \right) \Gamma\left(\frac{\beta}{1-\alpha} + 1 \right)}, \quad \text{for } \alpha < 1, \tag{8}$$

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5. Fox, The G and H-functions as symmetrical Fourier kernels,Trans.Amer.Math. Soc., 98(1961), 395- 429.

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$$= \frac{1}{2} \frac{\delta\, [a(\alpha-1)]^{\frac{\gamma}{\delta}}\, \Gamma\left(\frac{\beta}{\alpha-1}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right) \Gamma\left(\frac{\beta}{\alpha-1}-\frac{\gamma}{\delta}\right)}, \text{ for } \frac{1}{\alpha-1}-\frac{\gamma}{\delta}>0, \alpha>1, \tag{9}$$

$$= \frac{1}{2} \frac{\delta\, (a\,\beta)^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)} \quad \text{for} \quad \alpha \to 1. \tag{10}$$

For $\alpha < 1$, it is a finite range density with $1 - a(1-\alpha)|x|^{\delta} > 0$ and (7) remains in the extended generalized type-1 beta family. The pathway density in (7), for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.

For $\alpha > 1$, we have

$$f(x) = c\, |x|^{\gamma-1} \left[1 + a\,(\alpha-1)|x|^{\delta}\right]^{-\frac{\beta}{\alpha-1}}, \tag{11}$$

$-\infty < x < \infty$, $\delta > 0$, $\beta \geq 0$, $\alpha > 1$, which is the extended generalized type-2 beta model for real x. It includes the type-2 beta density, the F density, the Student-t density, the Cauchy density and many more.

Here it is considered only the case of pathway parameter $\alpha < 1$. For $\alpha \to 1$ (7) and (11) take the exponential form, since

$$\lim_{\alpha\to 1} c|x|^{\gamma-1}\left[1-a\,(1-\alpha)|x|^{\delta}\right]^{\frac{\eta}{1-\alpha}} = \lim_{\alpha\to 1} c|x|^{\gamma-1}\left[1+a\,(\alpha-1)|x|^{\delta}\right]^{-\frac{\eta}{\alpha-1}}$$

$$= c\,|x|^{\gamma-1} e^{-a\,\eta|x|^{\delta}} \tag{12}$$

This includes the generalized Gamma-, the Weibull-, the Chi-square, the Laplace-, the Maxwell- Boltzmann and other related densities.

When $\alpha \to 1_{-}$, $\left[1-\frac{a\,(1-\alpha)\,t}{x}\right]^{\frac{\eta}{1-\alpha}} \to e^{-\frac{a\,\eta}{x}t}$, then operator (6) reduces to the Laplace integral transform of f with parameter $\frac{a\eta}{x}$:

$$(P_{0_+}^{(\eta,1)} f)(x) = x^{\eta} \int_0^{\infty} e^{-\frac{a\,\eta}{x}t} f(t)\, dt = x^{\eta} L_f\left(\frac{a\eta}{x}\right). \tag{13}$$

When $\alpha = 0, a = 1$, then replacing η by $\eta - 1$ in (6) the integral operator reduces to the Riemann-Liouville fractional integral operator.

II. Main Theorem

If

$$h(s) = \int_d^\infty (sx)^{\rho-1} H^{M,N}_{P,Q}\left[(sx)^k \middle| \begin{matrix}(c_j,\gamma_j)_{1,P}\\(d_j,\delta_j)_{1,Q}\end{matrix}\right] F(x)\,dx\,, \tag{14}$$

and

$$(P_{0_+}^{(\eta,\alpha)} f)(x) = x^\eta \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)\,dt\,, \tag{15}$$

then

$$P_{0_+}^{(\eta,\alpha)}[h(s)] = \frac{s^{\eta+\rho}}{[a(1-\alpha)]^\rho}\Gamma\left(\frac{\eta}{1-\alpha}+1\right)\int_d^\infty x^{\rho-1} H^{M,N+1}_{P+1,Q+1}\left[\left(\frac{sx}{a(1-\alpha)}\right)^k \middle| \begin{matrix}(c_j,\gamma_j)_{1,P},(1-\rho,k)\\(d_j,\delta_j)_{1,Q},(\frac{\eta}{\alpha-1}-\rho,k)\end{matrix}\right] F(x)\,dx\,, \tag{16}$$

where $F(x) = f(a'\sqrt{x^2-d^2})\,U(x-d),\ x>d>0$ as defined in (2) provided that

(i) $\Omega>0,\ |\arg s| < \frac{1}{2}\pi\Omega/k$,

(ii) $\mathrm{Re}(w_1)+1>0,\ \mathrm{Re}(\rho-w_2)+k\max_{1\le j\le N}\left[\mathrm{Re}\left(\frac{c_j-1}{\gamma_j}\right)\right]<0$,

(iii) $f(x)\in L(a,b),\ \eta\in C,\ R(\eta)>0,\ a>0,\ \alpha<1$,

(iv) $R(1+\frac{\eta}{1-\alpha})>0$.

Proof: In order to prove the main theorem, substituting the value of h(s) from (14) in the left hand side of (15), we find that

$$P_{0_+}^{(\eta,\alpha)}[h(s)] = s^\eta \int_{t=0}^{\left[\frac{s}{a(1-\alpha)}\right]} \left[1-\frac{a(1-\alpha)t}{s}\right]^{\frac{\eta}{(1-\alpha)}} \left\{\int_{x=d}^\infty (tx)^{\rho-1} H^{M,N}_{P,Q}\left[(tx)^k \middle| \begin{matrix}(c_j,\gamma_j)_{1,P}\\(d_j,\delta_j)_{1,Q}\end{matrix}\right] F(x)\,dx\right\} dt \tag{17}$$

Now interchanging the orders of x and t integrals which is permissible under given conditions, we get

Ref.

2. A. M. Mathai, H. J. Haubold, Pathway model, Super statistics, Tsalle's statistics and a generalized measure of entropy, Physica A, 375(2007), 110-122.

$$P_{0_+}^{(\eta,\alpha)}[h(s)] = s^{\eta}\int_{x=d}^{\infty} F(x)\ x^{\rho-1}\left\{\int_{t=0}^{\left[\frac{s}{a(1-\alpha)}\right]}\left[1-\frac{a(1-\alpha)\,t}{s}\right]^{\frac{\eta}{(1-\alpha)}} t^{\rho-1}\ H_{P,Q}^{M,N}\left[(tx)^k \middle| \begin{matrix}(c_j,\gamma_j)_{1,P}\\ (d_j,\delta_j)_{1,Q}\end{matrix}\right]dt\right\}dx \qquad (18)$$

To evaluate the t-integral, we express the H- function in terms of Mellin-Barnes contour integrals with the help of (4) and change the order of contour integrations and t-integral. After evaluating the t- integral and reinterpreting the result thus obtained in terms of H-function, we easily arrive at the right hand side of (16) after a little simplification.

When $\alpha \to 1_-$, then (16) tends to

$$\lim_{\alpha\to 1_-} P_{0_+}^{(\eta,\alpha)}[h(s)] = \frac{s^{\eta+\rho}}{(a\,\eta)^{\rho}}\int_d^{\infty} x^{\rho-1}\ H_{P+1,Q}^{M,N+1}\left[\left(\frac{sx}{a\,\eta}\right)^k \middle| \begin{matrix}(c_j,\gamma_j)_{1,P},\ (1-\rho,k)\\ (d_j,\delta_j)_{1,Q}\end{matrix}\right]F(x)\,dx \qquad (19)$$

III. Special Cases

(I) If we reduce the Fox's H-function involved in (14) to Whittaker function by using a known result [4, p.155], a little simplification will give the following Corollary 1. If

$$h(s) = \int_d^{\infty}(sx)^{\rho-1}\ e^{-\frac{1}{2}sx}\ W_{b,c}(sx)\,F(x)\,dx\,, \qquad (20)$$

and

$$(P_{0_+}^{(\eta,\alpha)}f)(x) = x^{\eta}\int_0^{\left[\frac{x}{a(1-\alpha)}\right]}\left[1-\frac{a(1-\alpha)\,t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)\,dt\,, \qquad (21)$$

then

$$P_{0_+}^{(\eta,\alpha)}[h(s)] = \frac{s^{\eta+\rho}}{[a(1-\alpha)]^{\rho}}\Gamma\left(\frac{\eta}{1-\alpha}+1\right)\int_d^{\infty} x^{\rho-1}\ H_{2,3}^{2,1}\left[\frac{sx}{a(1-\alpha)} \middle| \begin{matrix}(1-\rho,1),\ (1-b,1)\\ (\frac{\eta}{\alpha-1}-\rho,1), (\frac{1}{2}-c,1),\ (\frac{1}{2}+c,1)\end{matrix}\right]F(x)\,dx\,. \qquad (22)$$

The conditions of validity of the aforementioned corollary can be easily derived from our main theorem.

When $\alpha \to 1_-$ then (22) tends to

$$\lim_{\alpha\to 1_-} P_{0_+}^{(\eta,\alpha)}[h(s)] = \frac{s^{\eta+\rho}}{[a\,\eta]^{\rho}}\int_d^{\infty} x^{\rho-1}\ H_{2,2}^{2,1}\left[\frac{sx}{a\,\eta} \middle| \begin{matrix}(1-\rho,1),\ (1-b,1)\\ (\frac{1}{2}-c,1),\ (\frac{1}{2}+c,1)\end{matrix}\right]F(x)\,dx\,. \qquad (23)$$

Ref.

4. A.M. Mathai and R. K. Saxena, The H-functions with Applications in Statistics and other Disciplines, John Wiley and Sons (1974).

(II) Reducing Fox's H-function in (14) to exponential function using known result [6, p.18] then we have the following

If

$$h(s) = \int_d^\infty (sx)^{\rho-1} e^{-sx} F(x)\,dx \,, \tag{24}$$

and

$$(P_{0_+}^{(\eta,\alpha)} f)(x) = x^\eta \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)\,dt \,, \tag{25}$$

then

$$P_{0_+}^{(\eta,\alpha)}[h(s)] = \frac{s^{\eta+\rho}}{[a(1-\alpha)]^\rho} B\left(\rho, \frac{\eta}{1-\alpha}+1\right) \int_d^\infty x^{\rho-1} {}_1F_1\left[(\rho); (\rho+\frac{\eta}{1-\alpha}+1); -\frac{sx}{a(1-\alpha)}\right] F(x)\,dx \,.. \tag{26}$$

The conditions of validity of the aforementioned corollary can be easily derived from our main theorem.

When $\alpha \to 1_-$ then (26) tends to

$$\lim_{\alpha\to 1_-} P_{0_+}^{(\eta,\alpha)}[h(s)] = \frac{s^{\eta+\rho}\Gamma(\rho)}{[a\eta]^\rho} \int_d^\infty x^{\rho-1} \left(1+\frac{sx}{a\eta}\right)^{-\rho} F(x)\,dx \,. \tag{27}$$

(III) If we reduce the Fox's H-function involved in (14) to Wright's generalized Bessel function by using a known result [6, p.19], after a little simplification we have

If

$$h(s) = \int_d^\infty (sx)^{\rho-1} J_\lambda^\nu(x) F(x)\,dx \,, \tag{28}$$

and

$$(P_{0_+}^{(\eta,\alpha)} f)(x) = x^\eta \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)\,dt \,, \tag{29}$$

then

$$P_{0_+}^{(\eta,\alpha)}[h(s)] = \frac{s^{\eta+\rho}}{[a(1-\alpha)]^\rho} \Gamma\left(\frac{\eta}{1-\alpha}+1\right) \int_d^\infty x^{\rho-1} {}_1\Psi_2\left[\begin{matrix}(\rho,1) & ; \\ (1+\lambda,\nu),(\rho+\frac{\eta}{1-\alpha}+1,1); & -\frac{sx}{a(1-\alpha)}\end{matrix}\right] F(x)\,dx \,, \tag{30}$$

Ref.

6. H. M. Srivastava, K. C. Gupta, S. P. Goyal, The H- function of one and two variables with applications, South Asian publisher, New Delhi (1982).

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where ${}_1\psi_2$ is wright's hypergeometric function and the conditions of validity of the aforementioned corollary can be easily derived from our main theorem.

When $\alpha \to 1_-$ then (30) tends to

$$\lim_{\alpha\to 1_-} P_{0_+}^{(\eta,\alpha)}[h(s)] = \frac{s^{\eta+\rho}}{[a\eta]^{\rho}} \int_d^{\infty} x^{\rho-1} \; {}_1\psi_1\left[\begin{matrix}(\rho,1)\,; \\ (1+\lambda,\nu);\end{matrix} \; -\frac{sx}{a\eta}\right] F(x)\,dx \quad . \tag{31}$$

References Références Referencias

1. M. Mathai, A pathway to matrix – variate gamma and normal densities, Linear Algebra and its Applications, 396(2005), 317-328.
2. A. M. Mathai, H. J. Haubold, Pathway model, Super statistics, Tsalle's statistics and a generalized measure of entropy, Physica A, 375(2007), 110-122.
3. A. M. Mathai, H. J. Haubold, On generalized distributions and pathways, Physics Letters, 372(2008), 2109-2113.
4. A.M. Mathai and R. K. Saxena, The H-functions with Applications in Statistics and other Disciplines, John Wiley and Sons (1974).
5. Fox, The G and H-functions as symmetrical Fourier kernels,Trans.Amer.Math. Soc., 98(1961), 395- 429.
6. H. M. Srivastava, K. C. Gupta, S. P. Goyal, The H- function of one and two variables with applications, South Asian publisher, New Delhi (1982).
7. M. Saigo, R. K. Saxena and J.Ram, On the two dimensional generalized Weyl fractional calculus associated with two dimensional H-transform, J.Fractional Calculus, 8(1995), 63-73.
8. Seema S. Nair, Pathway fractional integration operator, Fractional calculus & Applied analysis, 12(3)(2009), 237-252.

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