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A New Class of Harmonic Univalent Functions Defined by an Integral Operator

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Ref.

[2]J. Clunie, T. Scheil- Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math., **9**(1984), 3-25.

A New Class of Harmonic Univalent Functions Defined by an Integral Operator

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Abstract - We define and investigate a new class of harmonic univalent functions defined by Salagean integral operator. We obtain coefficient inequalities and distortion bounds for the functions in this class.

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I. Introduction

A continuous complex valued function $f = u + iv$ defined in a complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|, z \in D$.(See Clunie and Sheil-Small[2]).

Denote by $\mathcal{H}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense preserving in the unit disc $U = \{z : |z| < 1\}$ so that $f = h + \overline{g}$ is normalized by $f(0) = h(0) = f_z'(0) - 1 = 0$.

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U. We let:

$$A_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1}z^{n+1} + ..., z \in U\}, \quad with \quad A_1 = A.$$

We let $\mathcal{H}[a, n]$ denote the class of analytic functions in $U$of the form

$$f(z) = a + a_n z^n + a_{n+1}z^{n+1} + ..., z \in U.$$

The integral operator I^n is defined in [4] by:

$$(i) \quad I^0 f(z) = f(z);$$

$$(ii) \quad I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt;$$

$$(iii) \quad I^n f(z) = I(I^{n-1}f(z)), \quad n \in \mathbb{N} - \{0\}, \quad f \in A.$$

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Ahuja and Jahangiri [1] defined the class $H(n)$, $n \in \mathbb{N}$, consisting of all univalent harmonic functions $f = h + \overline{g}$ that are sense preserving in U and h and g are of the form:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{1.1}$$

For $f = h + \overline{g}$ given by (1.1) the integral operator I^n is defined as:

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)}, z \in U, \tag{1.2}$$

where

$$I^n h(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k$$

and

$$I^n g(z) = \sum_{k=1}^{\infty} k^{-n} b_k z^k.$$

For fixed positive integers n and for $0 \leq \alpha < 1, \beta \geq 0$ we let $H(n, \alpha, \beta)$ denote the class of univalent harmonic functions of the form (1.1) that satisfy the condition:

$$Re\left\{\frac{I^n f(z)}{I^{n+1} f(z)}\right\} > \beta\left|\frac{I^n f(z)}{I^{n+1} f(z)} - 1\right| + \alpha. \tag{1.3}$$

The subclass $H^-(n, \alpha, \beta)$ consists of functions $f_n = h + \overline{g_n}$ in $H(n, \alpha, \beta)$ so that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.4}$$

II. The Main Results

In the first theorem, we introduce a sufficient coefficient bound for harmonic functions in $H(n, \alpha, \beta)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1.1). If*

$$\sum_{k=1}^{\infty}\{\ (n, \alpha, \beta)|a_k| + \theta(n, \alpha, \beta)|b_k|\} \leq 2, \tag{2.1}$$

where

$$(n, \alpha, \beta) = \frac{k^{-n}(1+\beta) - (\beta + \alpha)k^{-(n+1)}}{1 - \alpha},$$

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[1]O.P. Ahuja, J.M. Jahangiri, *Multivalent harmonic starlike functions*, Ann. Univ. Marie Curie-Sklodowska Sect. A, LV 1(2001), 1-13.

and $$\theta(n,\alpha,\beta)=\frac{k^{-n}(1+\beta)+(\beta+\alpha)k^{-(n+1)}}{1-\alpha},$$

$a_1=1,\quad 0\leq\alpha<1,\quad \beta\geq 0,\quad n\in\mathbb{N},$ *then* $f\in H(n,\alpha,\beta)$.

Proof. According to (1.2) and(1.3) we only need to show that

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$$Re\Big(\frac{I^nf(z)-\alpha I^{n+1}f(z)-\beta e^{i\theta}|I^nf(z)-I^{n+1}f(z)|}{I^{n+1}f(z)}\Big)\geq 0.$$

The case $r=0$ is obvious. For $0<r<1$ it follows that

$$Re\Big(\frac{I^nf(z)-\alpha I^{n+1}f(z)-\beta e^{i\theta}|I^nf(z)-I^{n+1}f(z)|}{I^{n+1}f(z)}\Big)=$$

$$=Re\Big\{\frac{(1-\alpha)z+\sum_{k=2}^{\infty}a_kz^k[\gamma^n-\alpha\gamma^{n+1}]}{z+\sum_{k=2}^{\infty}\gamma^{n+1}a_kz^k+(-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}\overline{z^k}}+$$

$$+\frac{(-1)^n\sum_{k=1}^{\infty}\overline{b_k}\overline{z^k}[\gamma^n+\alpha\gamma^{n+1}]}{z+\sum_{k=2}^{\infty}\gamma^{n+1}a_kz^k+(-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}\overline{z^k}}-$$

$$-\frac{\beta e^{i\theta}|\sum_{k=2}^{\infty}a_kz^k[\gamma^n-\gamma^{n+1}]+(-1)^n\sum_{k=1}^{\infty}\overline{b_k}\overline{z^k}[\gamma^n+\gamma^{n+1}]|}{z+\sum_{k=2}^{\infty}\gamma^{n+1}a_kz^k+(-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}\overline{z^k}}\Big\}=$$

$$=Re\Big\{\frac{1-\alpha+\sum_{k=2}^{\infty}a_kz^{k-1}[\gamma^n-\alpha\gamma^{n+1}]}{1+\sum_{k=2}^{\infty}\gamma^{n+1}a_kz^{k-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}\overline{z^k}z^{-1}}+$$

$$\frac{(-1)^n\sum_{k=1}^{\infty}\overline{b_k}\overline{z^k}z^{-1}[\gamma^n+\alpha\gamma^{n+1}]}{1+\sum_{k=2}^{\infty}\gamma^{n+1}a_kz^{k-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}\overline{z^k}z^{-1}}-$$

$$\frac{\beta e^{i\theta}z^{-1}|\sum_{k=2}^{\infty}[\gamma^n-\gamma^{n+1}]a_kz^k+(-1)^n\sum_{k=1}^{\infty}[\gamma^n+\gamma^{n+1}]\overline{b_k}\overline{z^k}|}{1+\sum_{k=2}^{\infty}\gamma^{n+1}a_kz^{k-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}\overline{z^k}z^{-1}}\Big\}=$$

$$= Re\frac{(1-\alpha)+A(z)}{1+B(z)}, \quad where \quad \gamma = \frac{1}{k}.$$

For $z = re^{i\theta}$ we have

$$A(re^{i\theta}) = \sum_{k=2}^{\infty}(\gamma^n - \alpha\gamma^{n+1})a_k r^{k-1}e^{(k-1)\theta i} +$$

$$+(-1)^n\sum_{k=1}^{\infty}(\gamma^n + \gamma^{n+1}\alpha)\overline{b_k}r^{k-1}e^{-(k+1)\theta i} - \beta\mathcal{D}(n+1, n, \alpha),$$

where

$$\mathcal{D}(n+1, n, \alpha) = |\sum_{k=2}^{\infty}(\gamma^n - \gamma^{n+1})a_k r^{k-1}e^{-ki\theta} + (-1)^n\sum_{k=1}^{\infty}(\gamma^n + \gamma^{n+1})\overline{b_k}r^{k-1}e^{-ki\theta}|,$$

and

$$B(re^{i\theta}) = \sum_{k=2}^{\infty}\gamma^{n+1}a_k r^{k-1}e^{(k-1)\theta i} + (-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}r^{k-1}e^{-(k+1)\theta i}.$$

Setting $\frac{1-\alpha+A(z)}{1+B(z)} = (1-\alpha)\frac{1+w(z)}{1-w(z)}$.

The proof will be complete if we can show that $|w(z)| \leq r < 1$. This is the case since, by the condition (2.1), we can write:

$$|w(z)| = |\frac{A(z) - (1-\alpha)B(z)}{A(z) + (1-\alpha)B(z) + 2(1-\alpha)}| \leq$$

$$\leq \frac{\sum_{k=1}^{\infty}[(1+\beta)(\gamma^n - \gamma^{n+1})|a_k| + (1+\beta)(\gamma^n + \gamma^{n+1})|b_k|]r^{k-1}}{4(1-\alpha) - \sum_{k=1}^{\infty}\{[\gamma^n(1+\beta) - \delta\gamma^{n+1}]|a_k| + [\gamma^n(1+\beta) + \delta\gamma^{n+1}]|b_k|\}r^{k-1}}$$

$$< \frac{\sum_{k=1}^{\infty}(1+\beta)(\gamma^n - \gamma^{n+1})|a_k| + (\gamma^n + \gamma^{n+1})(1+\beta)|b_k|}{4(1-\alpha) - \sum_{k=1}^{\infty}\{[\gamma^n(1+\beta) - \delta\gamma^{n+1}]|a_k| + [\gamma^n(1+\beta) + \delta\gamma^{n+1}]|b_k|\}} \leq 1,$$

where $\delta = \beta + 2\alpha - 1$.

The harmonic univalent functions

$$f(z) = z + \sum_{k=2}^{\infty}\frac{1}{(n, \alpha, \beta)}x_k z^k + \sum_{k=1}^{\infty}\frac{1}{\theta(n, \alpha, \beta)}\overline{y_k z^k},$$

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where $n \in \mathbb{N}, 0 \leq \alpha < 1, \beta \geq 0$ and $\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 1$, show that the coefficient bound given by (2.1) is sharp.

In the following theorem it is show that the condition (2.1)is also necessary for the function $f_n = h + \overline{g_n}$, where h and g_n are of the form (1.4).

Notes

Theorem 2.2. *Let $f_n = h + \overline{g_n}$ be given by (1.4). Then $f_n \in H^-(n, \alpha, \beta)$ if and only if*

$$\sum_{k=1}^{\infty}[\ (n,\alpha,\beta)a_k + \theta(n,\alpha,\beta)b_k] \leq 2, \tag{2.2}$$

$$a_1 = 1, 0 \leq \alpha < 1, n \in \mathbb{N}.$$

Proof. Since $H^-(n,\alpha,\beta) \subset H(n,\alpha,\beta)$, we only need to prove the "only if" part of the theorem. For functions f_n of the form (1.4), we note that the condition

$$Re\{\frac{I^n f(z)}{I^{n+1}f(z)}\} > \beta|\frac{I^n f(z)}{I^{n+1}f(z)} - 1| + \alpha$$

is equivalent to

$$Re\{\frac{(1-\alpha)z - \sum_{k=2}^{\infty}(\gamma^n - \alpha\gamma^{n+1})a_k z^k}{z - \sum_{k=2}^{\infty}\gamma^{n+1}a_k z^k + (-1)^{2n}\sum_{k=1}^{\infty}\gamma^{n+1}b_k\overline{z^k}} +$$

$$+\frac{(-1)^{2n-1}\sum_{k=1}^{\infty}(\gamma^n + \gamma^{n+1}\alpha)b_k\overline{z^k}}{z - \sum_{k=2}^{\infty}\gamma^{n+1}a_k z^k + (-1)^{2n}\sum_{k=1}^{\infty}\gamma^{n+1}b_k\overline{z^k}} -$$

$$-\frac{\beta e^{i\theta}|-\sum_{k=2}^{\infty}(\gamma^n + \gamma^{n+1})a_k z^k + (-1)^{2n-1}\sum_{k=1}^{\infty}(\gamma^n - \gamma^{n+1})\overline{b_k}\overline{z^k}|}{z - \sum_{k=2}^{\infty}\gamma^{n+1}a_k z^k + (-1)^{2n+1}\sum_{k=1}^{\infty}\gamma^{n+1}b_k\overline{z^k}}\} \geq 0, \tag{2.3}$$

where $\gamma = \frac{1}{k}$.

The above required condition (2.3) must hold for all values of $z \in U$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$ and using $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$ we must have

$$\frac{(1-\alpha)-\sum_{k=2}^{\infty}[\gamma^n(1+\beta)-(\alpha+\beta)\gamma^{n+1}]a_k r^{k-1}}{1-\sum_{k=2}^{\infty}\gamma^{n+1}a_k r^{k-1}+\sum_{k=1}^{\infty}\gamma^{n+1}b_k r^{k-1}}- \tag{2.4}$$

$$-\frac{\sum_{k=1}^{\infty}[\gamma^n(1+\beta)+\gamma^{n+1}(\beta+\alpha)]b_k r^{k-1}}{1-\sum_{k=2}^{\infty}\gamma^{n+1}a_k r^{k-1}+\sum_{k=1}^{\infty}\gamma^{n+1}b_k r^{k-1}}\geq 0.$$

If the condition (2.3) does not hold, then the expression in (2.4) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which this quotient in (2.4) is negative. This contradicts the required condition for $f_n \in H^-(n,\alpha,\beta)$ and so the proof is complete.

The following theorem gives the distortion bounds for functions in $H^-(n,\alpha,\beta)$ which yields a covering results for this class.

Theorem 2.3. *Let $f_n \in H^-(n,\alpha,\beta)$. Then for $|z| = r < 1$ we have*

$$|f_n(z)| \leq (1+b_1)r + [\theta(n,\alpha,\beta) - \omega(n,\alpha,\beta)b_1]r^{n+2}$$

and

$$|f_n(z)| \geq (1-b_1)r - \{\phi(n,\alpha,\beta) - \omega(n,\alpha,\beta)b_1\}r^{n+2},$$

where

$$\phi(n,\alpha,\beta) = \frac{1-\alpha}{(1/2)^n(1+\beta)-(1/2)^{n+1}(\alpha+\beta)},$$

$$\omega(n,\alpha,\beta) = \frac{(1+\beta)+(\alpha+\beta)}{(1/2)^n(1+\beta)-(1/2)^{n+1}(\alpha+\beta)}.$$

Proof. We prove the right side inequality for $|f_n|$. The proof for the left hand inequality can be done using similar arguments. Let $f_n \in H^-(n,\alpha,\beta)$. Taking the absolute value of f_n then by Theorem 2.2, we can obtain :

$$|f_n(z)| = |z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{n-1}\sum_{k=1}^{\infty} b_k \overline{z^k}| \leq$$

$$\leq r + \sum_{k=2}^{\infty} a_k r^k + \sum_{k=1}^{\infty} b_k r^k = r + b_1 r + \sum_{k=2}^{\infty}(a_k+b_k)r^k \leq$$

$$\leq r + b_1 r + \sum_{k=2}^{\infty}(a_k+b_k)r^2 =$$

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$$= (1+b_1)r + \phi(n,\alpha,\beta)\sum_{k=2}^{\infty}\frac{1}{\phi(n,\alpha,\beta)}(a_k+b_k)r^2 \leq$$

$$\leq (1+b_1)r + \phi(n,\alpha,\beta)r^{n+2}\sum_{k=2}^{\infty}[\ \ (n,\alpha,\beta)a_k + \theta(n,\alpha,\beta)b_k] \leq$$

$$\leq (1+b_1)r + [\phi(n,\alpha,\beta) - \omega(n,\alpha,\beta)b_1]r^{n+2}.$$

Notes

The following covering result follows from the left hand inequality in Theorem 2.3.

Corollary 2.4. *Let* $f_n \in H^-(n,\alpha,\beta)$. *Then for* $|z| = r < 1$ *we have* $\{w : |w| < 1 - b_1 - [\phi(n,\alpha,\beta) - \omega(n,\alpha,\eta)b_1] \subset f_n(U)\}$.

Next we determine the extreme points of closed convex hulls of $H^-(n,\alpha,\beta)$, denoted by clco$H^-(n,\alpha,\beta)$.

Theorem 2.5. *Let* f_n *be given by (1.4). Then* $f_n \in H^-(n,\alpha,\beta)$ *if and only if*

$$f_n(z) = \sum_{k=1}^{\infty}[x_k h_k(z) + y_k g_{n_k}(z)],$$

where $h(z) = z$,

$$h_k(z) = z - \frac{1-\alpha}{k^{-n}(1+\beta) - (\beta+\alpha)k^{-(n+1)}}z^k, k = 2, 3, ...$$

and

$$g_{n_k}(z) = z + (-1)^{n-1}\frac{1-\alpha}{k^{-n}(1+\beta) + (\beta+\alpha)k^{-(n+1)}}\overline{z}^k, k = 1, 2, 3, ...$$

$$x_k \geq 0, y_k \geq 0, \sum_{k=1}^{\infty}(x_k + y_k) = 1.$$

In particular, the extreme points of $H^-(n,\alpha,\beta)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. For functions f_n of the form (2.1) we have:

$$f_n(z) = \sum_{k=2}^{\infty}[x_k h_k(z) + y_k g_{n_k}(z)] =$$

$$= \sum_{k=1}^{\infty}(x_k + y_k)z - \sum_{k=2}^{\infty}\frac{1-\alpha}{k^{-n}(1+\beta) - (\beta+\alpha)k^{-(n+1)}}x_k z^k +$$

$$+(-1)^{n-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{k^{-n}(1+\beta) + (\beta+\alpha)k^{-(n+1)}}y_k \overline{z}^k.$$

Then

$$\sum_{k=2}^{\infty} x_k \frac{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+1)}}{1-\alpha} \cdot \frac{(1-\alpha)}{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+1)}} +$$

$$\sum_{k=1}^{\infty} y_k \frac{k^{-n}(1+\beta)+(\beta+\alpha)k^{-(n+1)}}{1-\alpha} \frac{1-\alpha}{k^{-n}(1+\beta)+(\beta+\alpha)k^{-(n+1)}}$$

$$= \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k = 1 - x_1 \leq 1$$

and so $f_n(z) \in H^-(n, \alpha, \beta)$.

Conversely, suppose $f_n(z) \in H^-(n, \alpha, \beta)$. Letting

$$x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$$

$$x_k = \frac{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+1)}}{1-\alpha} \cdot a_k, k = 2, 3, ...$$

and

$$y_k = \frac{k^{-n}(1+\beta)+(\beta+\alpha)k^{-(n+1)}}{1-\alpha} \cdot b_k, k = 1, 2, 3, ...$$

we obtain the required representation, since

$$f_n(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{n-1} \sum_{k=1}^{\infty} b_k \overline{z}^k =$$

$$= z - \sum_{k=2}^{\infty} \frac{1-\alpha}{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+1)}} x_k z^k +$$

$$+(-1)^{n-1} \sum_{k=1}^{\infty} \frac{1-\alpha}{k^{-n}(1+\beta)+(\beta+\alpha)k^{-(n+1)}} y_k \overline{z}^k =$$

$$= z - \sum_{k=2}^{\infty} [z - h_k(z)] x_k - \sum_{k=1}^{\infty} [z - g_{n_k}(z)] y_k =$$

$$= [1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k] z + \sum_{k=2}^{\infty} x_k h_k(z) + \sum_{k=1}^{\infty} y_k g_{n_k}(z) =$$

$$= \sum_{k=1}^{\infty} [x_k h_k(z) + y_k g_{n_k}(z)].$$

Notes

Now we show that $H^{-}(n,\alpha,\beta)$ is closed under convex combination of its members.

Theorem 2.6. *The family $H^{-}(n,\alpha,\beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, ...$ suppose that $f_n^i \in H^{-}(n,\alpha,\beta)$, where

$$f_n^i(z) = z + \sum_{k=2}^{\infty} a_k^i z^k + (-1)^{n-1} \sum_{k=1}^{\infty} b_k^i \overline{z}^k,$$

then by Theorem 2.2,

$$\sum_{k=1}^{\infty} \frac{k^{-n}(1+\beta) - (\beta+\alpha)k^{-(n+1)}}{1-\alpha} a_k^i + \sum_{k=1}^{\infty} \frac{k^{-n}(1+\beta) + (\beta+\alpha)k^{-(n+1)}}{1-\alpha} b_k^i \le 2, \tag{2.5}$$

for $\sum_{i=1}^{\infty} t_i = 1, 0 \le t_i \le 1$, the convex combination of f_n^i may be written as

$$\sum_{i=1}^{\infty} t_i f_n^i(z) = z - \sum_{k=2}^{\infty}(\sum_{i=1}^{\infty} t_i a_k^i) z^k + (-1)^{n-1} \sum_{k=1}^{\infty}(\sum_{i=1}^{\infty} t_i b_k^i) \overline{z}^k.$$

Then by (2.4)

$$\sum_{k=1}^{\infty} \frac{k^{-n}(1+\beta) - (\beta+\alpha)k^{-(n+1)}}{1-\alpha} (\sum_{i=1}^{\infty} t_i a_k^i) +$$

$$+ \sum_{k=1}^{\infty} \frac{k^{-n}(1+\beta) + (\beta+\alpha)k^{-(n+1)}}{1-\alpha} (\sum_{i=1}^{\infty} t_i b_k^i) =$$

$$= \sum_{i=1}^{\infty} t_i [\sum_{k=1}^{\infty} \frac{k^{-n}(1+\beta) - (\beta+\alpha)k^{-(n+1)}}{1-\alpha} a_k^i +$$

$$+ \sum_{k=1}^{\infty} \frac{k^{-n}(1+\beta) + (\beta+\alpha)k^{-(n+1)}}{1-\alpha} b_k^i] \le 2 \sum_{i=1}^{\infty} t_i = 2$$

and therefore $\sum_{i=1}^{\infty} t_i f_n^i(z) \in H^{-}(n,\alpha,\beta)$.

The beautiful results for harmonic functions, was obtained by P. T. Mocanu in [3].

Ref.

[3]P. T. Mocanu, *Three-cornered hat harmonic functions*, Complex Variables and Elliptic Equation, **12**(2009), 1079-1084.

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Notes