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A Unified Study of Fourier Series Involving Generalized Hypergeometric Function

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Abstract – In this paper, we make an application of an integral involving sine function, exponential function, the product of Kamp´e de F´eriet functions and the I-function to evaluate three fourier series. We also evaluate a multiple integral involving the Ifunction to make its application to derive a multiple exponential Fourier series. Some known and interesting particular cases are also given at the end.

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A Unified Study of Fourier Series Involving Generalized Hypergeometric Function

Yashwant Singh [α] & Naseem A.Khan [σ]

Abstract - In this paper, we make an application of an integral involving sine function, exponential function, the product of Kamp´e de F´eriet functions and the I-function to evaluate three fourier series. We also evaluate a multiple integral involving the Ifunction to make its application to derive a multiple exponential Fourier series. Some known and interesting particular cases are also given at the end.

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I. Introduction

$$I^{m,n}_{p_i,q_i:r}[z] = I^{m,n}_{p_i,q_i:r}\left[z \;\middle|\; \begin{matrix} (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right]$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \phi(\xi) z^{\xi}\, d\xi \tag{1.1}$$

where

$$\phi(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j\xi)\}}{\Sigma\{\prod_{j=m+1}^{q_i}\{\Gamma(1-b_{ji}+\beta_{ji}\xi)\}\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\}} \tag{1.2}$$

For the convergence and other details of the I-function, we refer the original paper of Saxena[5]. Saxena [5] has proved that the integral on the right hand side of (1.1) is absolutely convergent when $\Omega > 0$ and $|\arg z| < \frac{1}{2}\pi\Omega$, where
Kampé de Fériet hypergeometric function will be represented as follows.

$$F\left(\begin{matrix} p \\ \mu \\ q \\ \sigma \end{matrix} \;\middle|\; \begin{matrix} a_1,\cdots,a_p \\ b_1,b'_1,\cdots,b_\mu,b'_\mu \\ c_1,\ldots,c_q \\ d_1,d'_1,\cdots,d_\sigma,d'_\sigma \end{matrix} \;\middle|\; xy \right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{m+n}\prod_{j=1}^{\mu}\{(b_j)_m(b'_j)_n\}}{\prod_{j=M+1}^{q}(c_j)_{m+n}\prod_{j=1}^{\sigma}\{(d_j)_m(d'_j)_n\}}\frac{x^m y^n}{m!n!} \tag{1.3}$$

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Ref.
[5] Saxena, V.P.: Formal solution of certain new pair of dual integral equations involving H-function, Proc. Nat. Acad. Sci. India, A52, (1982), 366-275.

$$(p+\nu < q+\sigma+1 \text{ or } p+\nu = q+\sigma+1 \text{ and } |x|+|y| < min(1, 2^{q-p+1}));$$

$$= -\frac{1}{4\pi^2 K}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\psi(s,t)\Gamma(-s)\Gamma(-t)(-x)^s(-y)^t ds\ dt$$

where

$$K = \frac{\prod_{j=1}^{p}\Gamma(a_j)\prod_{j=1}^{\mu}\{\Gamma(b_j)\Gamma(b'_j)\}}{\prod_{j=M+1}^{q}\Gamma(c_j)\prod_{j=1}^{\sigma}\{\Gamma(d_j)\Gamma(d'_j)\}} \tag{1.4}$$

and

$$\psi(s,t) = \frac{\prod_{j=1}^{p}(a_j+s+t)\prod_{j=1}^{\mu}\{\Gamma(b_j+s)\Gamma(b'_j+t)\}}{\prod_{j=M+1}^{q}\Gamma(c_j+s+t)\prod_{j=1}^{\sigma}\{\Gamma(d_j+s)\Gamma(d'_j+t)\}} \tag{1.5}$$

if we put $\nu = 0 = \sigma$,then it changes in the following form;

$$F\left(\begin{array}{c|c|c} p & a_1,\cdots,a_p & \\ \mu & ----- & xy \\ q & c_1,\ldots,c_q & \\ \sigma & ---- & \end{array}\right) = {}_pF_q\left(\begin{array}{cc} a_1,\cdots,a_p & \\ & ;\ x+y \\ c_1,\ldots,c_q & \end{array}\right) \tag{1.7}$$

For further detail one can refer the monography by Appell and Kampé de Fériet[1]. Mishra[4] has evaluated

$$\int_0^\pi (\sin x)^{w-1}e^{imx}{}_pF_q\left[\begin{array}{l}\alpha_p;\\ \beta_q;\end{array}C(sinx)^{2h}\right]dx \;=\; \frac{\pi e^{im\pi/2}}{2^{w-1}}\sum_{r=0}^{\infty}\frac{(\alpha_p)_r C^r\Gamma(w+2hr)}{(\beta_q)_r r! 4^{hr}\Gamma(\frac{\omega+2hr\pm M+1}{2})} \tag{1.7}$$

Where $(\alpha)_p$ denotes $\alpha_1,\cdots,\alpha_p;\Gamma(a\pm b)$ represents$\Gamma(a+b),\Gamma(a-b)$; h is a positive integer;$p<q$ and $Re(w)>0$.Recall the following elementary integrals:

$$\int_0^\pi e^{i(m-n)x}dx = \begin{cases}\pi & , \quad m=n\ ;\\ 0 & , \quad m\neq n\ ;\end{cases} \tag{1.8}$$

$$\int_0^\pi e^{imx}\cos nx\, dx = \begin{cases}\frac{\pi}{2} & , \quad m=n\neq 0\ ;\\ \pi & , \quad m=n=0\ ;\\ 0 & , \quad m=n\ ;\end{cases} \tag{1.9}$$

$$\int_0^\pi e^{imx}\sin nx\, dx = \begin{cases}i\frac{\pi}{2} & , \quad m=n\ ;\\ 0 & , \quad m\neq n\ ;\end{cases} \tag{1.10}$$

Ref.

[4] Mishra,S.: *Integrals involving Legendre functions,generalized hypergeometric series and Fox's H-function,and Fourier-Legendre series for products of generalized hypergeometric functions*, Indian J. Pure Appl.Math., **21**(1990), 805-812.

Notes

Provided either both m and n are odd or both m and n are even integers.

For brevity, we shall use the following notations.

$$\frac{\prod_{k=1}^{E}(e_k)_{r+t}\prod_{k=1}^{F}(f_k)_r\prod_{k=1}^{F'}(f'_k)_t}{\prod_{k=1}^{G}(g_k)_{r+t}\prod_{k=1}^{H}(h_k)_r\prod_{k=1}^{H'}(h'_k)_t}=\epsilon$$

$$\frac{\prod_{k_1=1}^{E_1}(e_{1k_1})_{r_1+t_1}\prod_{k_1=1}^{F_1}(f_{1k_1})_{r_1}\prod_{k_1=1}^{F'_1}(f'_{1j_1})_{t_1}}{\prod_{k_1=1}^{G_1}(g_{1k_1})_{r_1+t_1}\prod_{k_1=1}^{H_1}(h_{1k_1})_{r_1}\prod_{k_1=1}^{H'_1}(h_{1k_1})_{t_1}}=\epsilon_1$$

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$$\frac{\prod_{k_n=1}^{E_n}(e_{nk_n})_{r_n+t_n}\prod_{k_n=1}^{F_n}(f_{nk_n})_{r_n}\prod_{k_n=1}^{F'_n}(f'_{nj_n})_{t_n}}{\prod_{k_n=1}^{G_n}(g_{nk_n})_{r_n+t_n}\prod_{k_n=1}^{H_n}(h_{nk_n})_{r_n}\prod_{k_n=1}^{H_n}(h'_{nk_n})_{t_n}}=\epsilon_n$$

II. Integral

The integrals to be evaluated are:

$$\int_0^{\pi}(\sin\ x)^{w-1}e^{imx}F^{E;F;F'}_{G;H;H'}\left[\begin{array}{ll}(e);(f);(f'); & \alpha(sin\ x)^{2\rho}\\ (g);(h);(h'); & \beta(sin\ x)^{2\gamma}\end{array}\right]$$

$$\times I^{p_i,q_i:r}_{m,n}\left[\ z(\sin\ x)^{2\sigma}\ \left|\begin{array}{l}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,P_i}\\ \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{array}\right.\right]dx\ =\ \frac{\sqrt{(\pi)}e^{im\pi/2}}{2^{\omega-1}}\sum_{r,t=0}^{\infty}\epsilon\frac{(\alpha/4^{\rho})^r(\beta/4^{\gamma})^t}{r!\ t!}$$

$$\times I^{m,n+1}_{p_i+1,qi+2:r}\left[\ \tfrac{z}{4^{\sigma}}\ \left|\begin{array}{l}(1-\omega-2\rho r-2\gamma t,2\sigma;1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,pi}\\ \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\left(\frac{1-\omega-2\rho r-2\gamma t\pm m}{2},\sigma;1\right)\end{array}\right.\right] \quad (2.1)$$

provided that $|\arg z|<\frac{1}{2}\pi\Omega$,and $\mathbf{Re}(\mathbf{w})>\mathbf{0};\alpha,\beta,\rho,\gamma,\sigma,\mathbf{z}$ are positive integers, where

$$\Omega\ \equiv\ \sum_{j=1}^{m}\beta_j+\sum_{j=1}^{n}\alpha_j\ -\ \sum_{j=m+1}^{q_i}\beta_{ji}\ -\ \sum_{j=n+1}^{p}\alpha_{ji}\ >\ 0.$$

$$\int_0^{\pi}\cdots\int_0^{\pi}(\sin x)^{w_1-1}\cdots(sinx)^{w_n-1}e^{i(m_1x_1+\cdots+m_nx_n)}$$

$$\times F^{E_1;F_1;F_1'}_{G_1;H_1;H_1'}\left[\begin{matrix}(e_1);(f_1);(f_1); & \alpha_1(sinx_1)^{2\rho_1}\\ (g_1);(h_1);(h_1'); & \beta_1(sinx_1)^{2\gamma_1}\end{matrix}\right]\cdots F^{E_n;F_n;F_n'}_{G_n;H_n;H_n'}\left[\begin{matrix}(e_n);(f_n);(f_n'); & \alpha_n(sinx_n)^{2\rho_n}\\ (g_n);(h_n);(h_n'); & \beta_n(sinx_n)^{2\gamma_n}\end{matrix}\right]$$

$$\times I^{p_i,q_i:r}_{m,n}\left[\ z(\sin x_1)^{2\sigma_1}\cdots(\sin x_n)^{2\sigma_n}\ \right]dx_1\cdots dx_n$$

$$=\frac{(\pi)^n e^{i(m_1+\cdots+m_n)\pi/2}}{2^{(\omega_1+\cdots+\omega_n)-n}}\sum_{r_1,t_1=0}^{\infty}\cdots\sum_{r_n,t_n}^{\infty}(\epsilon_1\cdots\epsilon_n)\frac{(\alpha_1/4^{\rho_1})^{r_1}(\beta_1/4^{\gamma_1})^{t_1}}{r_1!\ t_1!}\cdots\frac{(\alpha_n/4^{\rho_n})^{r_n}(\beta_n/4^{\gamma_n})^{t_n}}{r_n!\ t_n!}$$

$$\times I^{m,n+n}_{p_i+n,q_i+2n:r}\left[\frac{z}{4^{(\sigma_1+\cdots+\sigma_n)}}\left|\begin{matrix}(1-\omega_1-2\rho_1r_1-2\gamma_1t_1,2\sigma_1;1)\cdots(1-\omega_n-2\rho_nr_n-2\gamma_nt_n,2\sigma_n;1),\ (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},\ (b_{ji},\beta_{ji})_{m+1,q_i}\ \left(\frac{1-\omega_1-2\rho_1r_1-2\gamma_1t_1\pm m_1}{2},\sigma_1;1\right)\cdots\left(\frac{1-\omega_n-2\rho_nr_n-2\gamma_nt_n\pm m_n}{2},\sigma_n;1\right)\end{matrix}\right.\right] \tag{2.2}$$

provided that all the conditions of (2.1) are satisfied and $\mathbf{Re(w_i)>0};\sigma_i,\alpha_i,\beta_i,\rho_i,\gamma_i,\mathbf{z_i}$ are positive integers $(i=1,\cdots,n)$

Proof: To prove (2.1), expand the I-Function into the mellin-Barnes type integral. Now, on changing the order of integration,which is permissible under the conditions stated with the integral, the integral readily follows from (1.7)

On applying the same procedure as above the integral (2.2) can be derived easily.

III. Exponential Fourier Series

Let

$$f(x)\ =\ (\sin x)^{w-1}F^{E;F;F'}_{G;H;H'}\left[\begin{matrix}(e);(f);(f'); & \alpha(sinx)^{2\rho}\\ (g);(h);(h'); & \beta(sinx)^{2\gamma}\end{matrix}\right]$$

$$\times I^{m,n}_{p_i,q_i:r}\left[z(\sin x)^{2\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dx\ =\ \sum_{p=-\infty}^{\infty}A_pe^{-ipx} \tag{3.1}$$

which is valid due to $f(x)$ is continuous and of bounded variation with interval $(0,\pi)$. Now,multiplying by e^{imx} both sides in (3.1) and integrating it with respect to x from 0 to π, and then making an appcal to (1.8)and (2.1),we get

$$A_p\ =\ \frac{e^{im\pi/2}}{2^{\omega-1}}\sum_{r,t=0}^{\infty}\epsilon\ \frac{(\alpha/4^{\rho})^r}{r!}\ \frac{(\beta/4^{\gamma})^t}{t!}$$

Notes

$$\times I^{m,n+1}_{p_i+1,q_i+1:r}\left[\frac{z}{4^\sigma}\left|\begin{array}{l}(1-\omega-2\rho r-2\gamma t,2\sigma;1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\left(\frac{1-\omega-2\rho r-2\gamma t\pm m}{2},\sigma;1\right)\end{array}\right.\right] \quad (3.2)$$

An application to (3.1) and (3.2) gives the required exponential Fourier series

Notes

$$(2\sin x)^{w-1}F^{E;F;F'}_{G;H;H'}\left[\begin{array}{ll}(e);(f);(f'); & \alpha(sinx)^{2\rho}\\ (g);(h);(h'); & \beta(sinx)^{2\gamma}\end{array}\right]$$

$$\times \bar{H}^{P,Q}_{M,N}\left[z(\sin x)^{2\sigma}\left|\begin{array}{l}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{array}\right.\right]$$

$$=\sum_{p=-\infty}^{\infty}\sum_{r,t=0}^{\infty} e^{ip(\pi/2-x)}\ \epsilon\frac{(\alpha/4^\rho)^r}{r!}\ \frac{(\beta/4^\gamma)^t}{t!}$$

$$\times I^{m,n+1}_{p_i+1,q_i+1:r}\left[\frac{z}{4^\sigma}\left|\begin{array}{l}(1-\omega-2\rho r-2\gamma t,2\sigma;1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\left(\frac{1-\omega-2\rho r-2\gamma t\pm m}{2},\sigma;1\right)\end{array}\right.\right]. \quad (3.3)$$

IV. Cosine Fourier Series

Let

$$f(x) = (\sin x)^{w-1}F^{E;F;F'}_{G;H;H'}\left[\begin{array}{ll}(e);(f);(f'); & \alpha(sinx)^{2\rho}\\ (g);(h);(h'); & \beta(sinx)^{2\gamma}\end{array}\right]$$

$$\times I^{m,n}_{pi,q_i:r}\left[z(\sin x)^{2\sigma}\left|\begin{array}{l}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{array}\right.\right] = \frac{B_0}{2}+\sum_{p=1}^{\infty}B_p\ cospx \quad (4.1)$$

Integrating both sides with respect to x from 0 to π, we get

$$\frac{B_0}{2} = \frac{1}{\sqrt{(\pi)}}\sum_{r,t=0}^{\infty}\ \epsilon\ \frac{(\alpha)^r}{r!}\ \frac{(\beta)^t}{t!}$$

$$\times I^{m,n+1}_{p_i+1,q_i+1:r}\left[z\left|\begin{array}{l}(\frac{2-\omega}{2}-\rho r-\gamma t,2\sigma;1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\left(\frac{1-\omega}{2}-2\rho r-2\gamma t,\sigma;1\right)\end{array}\right.\right] \quad (4.2)$$

Now, multiplying by e^{imx} both sides in (4.1) and integrating it with respect to x frome 0 to π ,and finally ,making an application to (1.8),(1.9) and (2.1), we derive

$$B_p = \frac{e^{ip\pi/2}}{2^{\omega-1}}\sum_{r,t=0}^{\infty}\ \epsilon\ \frac{(\alpha/4^\rho)^r}{r!}\ \frac{(\beta/4^\gamma)^t}{t!}$$

$$\times I^{m,n+1}_{p_i+1,qi+2:r}\left[\frac{z}{4^{\sigma}}\left|\begin{array}{l}(1-\omega-2\rho r-2\gamma t,2\sigma;1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\left(\frac{1-\omega-2\rho r-2\gamma t\pm m}{2},\sigma;1\right)\end{array}\right.\right] \quad (4.3)$$

using (4.2), (4.3), from(4.1) we get required cosine Fourier Series.

$$(\sin x)^{w-1}F^{E;F;F'}_{G;H;H'}\left[\begin{array}{ll}(e);(f);(f'); & \alpha(sinx)^{2\rho}\\ (g);(h);(h'); & \beta(sinx)^{2\gamma}\end{array}\right]$$

$$\times I^{m,n}_{p_i,q_i:r}\left[z(\sin x)^{2\sigma}\left|\begin{array}{l}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ \\ (b_j,\beta_j)_{1,m},(b_j,\beta_j)_{m+1,q_i}\end{array}\right.\right]=\frac{1}{\sqrt{(\pi)}}\sum_{r,t=0}^{\infty}\epsilon\frac{(\alpha)^r}{r!}\frac{(\beta)^t}{t!}$$

$$\times I^{m,n+1}_{p_i+1,q_i+1:r}\left[z\left|\begin{array}{l}(\frac{2-\omega}{2}-\rho r-\gamma t,2\sigma;1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\left(\frac{1-\omega}{2}-2\rho r-2\gamma t),\sigma;1\right)\end{array}\right.\right]$$

$$+\sum_{p=-\infty}^{\infty}\sum_{r,t=0}^{\infty}\epsilon\ e^{ip\pi/2}\ cospx\ \frac{(\alpha/4^{\rho})^r}{r!}\ \frac{(\beta/4^{\gamma})^t}{t!}.\frac{1}{2^{\omega-2}}$$

$$\times I^{m,n+1}_{p_i+1,q_i+2:r}\left[\frac{z}{4^{\sigma}}\left|\begin{array}{l}(1-\omega-2\rho r-2\gamma t,2\sigma;1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ \\ (b_j,\beta_j)_{1,n},(b_{ji},\beta_{ji})_{m+1,q_i}\left(\frac{1-\omega-2\rho r-2\gamma t\pm m}{2},\sigma;1\right)\end{array}\right.\right]. \quad (4.4)$$

V. Sine Fourier Series

Let

$$f(x)=(\sin x)^{w-1}F^{E;F;F'}_{G;H;H'}\left[\begin{array}{ll}(e);(f);(f'); & \alpha(sinx)^{2\rho}\\ (g);(h);(h'); & \beta(sinx)^{2\gamma}\end{array}\right]$$

$$\times I^{m,n}_{p_i,q_i:r}\left[z(\sin x)^{2\sigma}\left|\begin{array}{l}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{array}\right.\right]=\sum_{p=-\infty}^{\infty}C_p\ sin\ px. \quad (5.1)$$

Multiplying by e^{imx} both sides in (5.1) and the integrating it with respect to x frome 0 to π ,and making to (1.10) and (2.1), we obtain

$$C_p=\frac{e^{ip\pi/2}}{2^{\omega-1}}\sum_{r,t=0}^{\infty}\epsilon\frac{(\alpha/4^{\rho})^r}{r!}\frac{(\beta/4^{\gamma})^t}{t!}.$$

Notes

$$\times I^{m,n+1}_{p_i+1,q_i+2:r}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(1-\omega-2\rho r-2\gamma t,2\sigma;1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1.p_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\left(\frac{1-\omega-2\rho r-2\gamma t\pm m}{2},\sigma;1\right)\end{matrix}\right.\right]. \quad (5.2)$$

Now making an application of (5.1) and (5.2), we get required Sine Fourier Series.

Notes

$$(2\sin x)^{w-1}F^{E;F;F'}_{G;H;H'}\left[\begin{matrix}(e);(f);(f'); & \alpha(sinx)^{2\rho}\\ (g);(h);(h'); & \beta(sinx)^{2\gamma}\end{matrix}\right]$$

$$\times I^{m,n}_{p_i,q_i:r}\left[z(\sin x)^{2\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]$$

$$=\sum_{p=-\infty}^{\infty}\sum_{r,t=0}^{\infty}\frac{2\ \epsilon\ e^{ip\pi/2}}{i}sin\ px\ \ \epsilon\ \frac{(\alpha/4^{\rho})^r}{r!}\ \ \frac{(\beta/4^{\gamma})^t}{t!}$$

$$\times I^{m,n+1}_{p_i+1,q_i+2:r}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(1-\omega-2\rho r-2\gamma t,2\sigma;1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\left(\frac{1-\omega-2\rho r-2\gamma t\pm m}{2},\sigma;1\right)\end{matrix}\right.\right]. \quad (5.3)$$

VI. Multiple Exponential Fourier Series

Let

$$f(x_1,\cdots,x_n) = (\sin x)^{w_1-1}\cdots(\sin x)^{w_n-1}F^{E_1;F_1;F_1'}_{G_1;H_1;H_1'}\left[\begin{matrix}(e_1);(f_1);(f_1'); & \alpha_1(sinx_1)^{2\rho_1}\\ (g_1);(h_1);(h_1'); & \beta_1(sinx_1)^{2\gamma_1}\end{matrix}\right]$$

$$\cdots F^{E_n;F_n;F_n'}_{G_n;H_n;H_n'}\left[\begin{matrix}(e_n);(f_n);(f_n'); & \alpha_n(sinx_n)^{2\rho_n}\\ (g_n);(h_n);(h_n'); & \beta_n(sinx_n)^{2\gamma_n}\end{matrix}\right]$$

$$\times I^{m,n}_{p_i,q_i:r}\left[z(\sin x_1)^{2\sigma_1}\cdots(\sin x_n)^{2\sigma_n}\left|\begin{matrix}(a_j,\alpha_j)_{1,},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]$$

$$=\sum_{p_1=-\infty}^{\infty}\cdots\sum_{p_n=-\infty}^{\infty}A_{p_1\cdots p_n}e^{-i(p_1x_1+\cdots+p_nx_n)}. \quad (6.1)$$

Equation (6.1) is valid, since $f(x_1,\cdots,x_n)$ is continuous and of bounded variation in the open interval $(0,\pi)$. In the series (6.1),to calculate $A_{p_1\cdots p_n}$we fix $x_1,\cdots,x_{n-1}$,so that

$$\sum_{p_1=-\infty}^{\infty}\cdots\sum_{p_{n-1}=-\infty}^{\infty}A_{p_1\cdots p_{n-1}}e^{-i(p_1x_1+\cdots+p_{n-1}x_{n-1})}$$

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depends only on p_n.

Furthermore ,it must be the coefficient of Fourier exponential series in x_n of $f(x_1,\cdots,x_n)$ over $0<x_n<\pi$.

Now multiplying by $e^{im_nx_n}$ both sides in (6.1) and integrating with respect to x_n from 0 to π,we get

$$(\sin x_1)^{w_1-1}\cdots(\sin x_n)^{w_n-1}F^{E_1;F_1;F_1'}_{G_1;H_1;H_1'}\left[\begin{array}{ll}(e_1);(f_1);(f_1'); & \alpha_1(sinx_1)^{2\rho_1}\\(g_1);(h_1);(h_1'); & \beta_1(sinx_1)^{2\gamma_1}\end{array}\right]$$

$$\cdots F^{E_{n-1};F_{n-1};F_{n-1}'}_{G_{n-1};H_{n-1};H_{n-1}'}\left[\begin{array}{ll}(e_{n-1});(f_{n-1});(f_{n-1}'); & \alpha_{n-1}(sinx_{n-1})^{2\rho_{n-1}}\\(g_{n-1});(h_{n-1});(h_{n-1}'); & \beta_{n-1}(sinx_{n-1})^{2\gamma_{n-1}}\end{array}\right]$$

$$\times\int_0^\pi(sinx_n)^{w_n-1}e^{im_nx_n}F^{E_n;F_n;F_n'}_{G_n;H_n;H_n'}\left[\begin{array}{ll}(e_n);(f_n);(f_n'); & \alpha_n(sinx_n)^{2\rho_n}\\(g_n);(h_n);(h_n'); & \beta_n(sinx_n)^{2\gamma_n}\end{array}\right]$$

$$\times I^{m,n}_{p_i,q_i:r}\left[z(\sin x_1)^{2\sigma_1}\cdots(\sin x_n)^{2\sigma_n}\left|\begin{array}{l}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ \\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{array}\right.\right]dx_n$$

$$=\sum_{p_1=-\infty}^{\infty}\cdots\sum_{p_{n-1}=-\infty}^{\infty}A_{p_1\cdots p_{n-1}}e^{-i(p_1x_1+\cdots+p_nx_n)}+\sum_{p_n=-\infty}^{\infty}\int_0^\pi(e^{i(m_n-p_n)x_n}dx \tag{6.2}$$

using(1.8) and (2.1),from (6.2), respectively,we get

$$A_{p_1\cdots p_n}=\sum_{r_1,t_1=0}^{\infty}\cdots\sum_{r_n,t_n=0}^{\infty}\frac{e^{i(p_1+\cdots+p_n)\pi/2}}{2^{(\omega_1+\cdots+\omega_n)-n}}\quad(\epsilon_1\cdots\epsilon_n)$$

$$\times\frac{(\alpha_1/4^{\rho_1})^{r_1}}{r_1!}\frac{(\beta_1/4^{\gamma_1})^{t_1}}{t_1!},\cdots\cdots\cdots\cdots,\frac{(\alpha_n/4^{\rho_n})^{r_n}}{r_n!}\frac{(\beta_n/4^{\gamma_n})^{t_n}}{t_n!}$$

$$\times I^{m,n+1}_{p_i+1,q_i+1:r}\left[\frac{z}{4^{(\sigma_1+\cdots+\sigma_n)}}\left|\begin{array}{l}(1-\omega_1-2\rho_1r_1-2\gamma_1t_1,2\sigma_1;1)\cdots(1-\omega_n-2\rho_nr_n-2\gamma_nt_n,2\sigma_n;1\\ \\(b_j,\beta_j)_{1,m},\quad(b_{ji},\beta_{ji})_{m+1,q_i}\quad\left(\frac{1-\omega_1-2\rho_1r_1-2\gamma_1t_1\pm m_1}{2},\sigma_1;1\right)\end{array}\right.\right.$$

$$\left.\begin{array}{l}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ \\ \cdots\left(\frac{1-\omega_n-2\rho_nr_n-2\gamma_nt_n\pm m_n}{2},\sigma_n;1\right)\end{array}\right]. \tag{6.3}$$

Using (6.3) in (6.1), we get required multiple exponential Fourier series.

Let

$$(\sin x_1)^{w_1-1}\cdots(\sin x_n)^{w_n-1}F^{E;F;F'}_{G;H;H'}\left[\begin{array}{ll}(e_1);(f_1);(f_1'); & \alpha_1(sinx_1)^{2\rho_1}\\(g_1);(h_1);(h_1'); & \beta_1(sinx_1)^{2\gamma_1}\end{array}\right]$$

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$$\times F^{E_n;F_n;F'_n}_{G_n;H_n;H'_n}\left[\begin{array}{ll}(e_n);(f_n);(f'_n); & \alpha_n(sinx_n)^{2\rho_n}\\(g_n);(h_n);(h'_n); & \beta_n(sinx_n)^{2\gamma_n}\end{array}\right]$$

$$\times I^{m,n}_{p_i,q_i:r}\left[z(\sin x_1)^{2\sigma_1}\cdots(\sin x_n)^{2\sigma_n}\left|\begin{array}{l}(a_j,\alpha_j)_{1,n},(a_j,\alpha_j)_{n+1,p_i}\\ \\(b_j,\beta_j)_{1},m,(b_{ji},\beta_{ji})_{m+1,q_i}\end{array}\right.\right]$$

$$=\sum_{p_1\cdots p_n=-\infty}^{\infty}\cdots\sum_{r_1\cdots r_n,t_1\cdots t_n=0}^{\infty}\frac{(\epsilon_1\cdots\epsilon_n)}{2^{(\omega_1+\cdots+\omega_n)-n}}\times e^{-i(p_1n_1+\cdots+p_nn_n)}\ .\ e^{(p_1+\cdots+p_n)\ \ 2}$$

$$\frac{(\alpha_1/4^{\rho_1})^{r_1}}{r_1!}\ \frac{(\beta_1/4^{\gamma_1})^{t_1}}{t_1!},\cdots\cdots\cdots,\frac{(\alpha_n/4^{\rho_n})^{r_n}}{r_n!}\ \frac{(\beta_n/4^{\gamma_n})^{t_n}}{t_n!}$$

$$\times I^{m,n+n}_{p_i+n,q_i+n}\left[\frac{z}{4^{(\sigma_1+\cdots+\sigma_n)}}\left|\begin{array}{l}(1-\omega_1-2\rho_1r_1-2\gamma_1t_1,2\sigma_1;1)\cdots(1-\omega_n-2\rho_nr_n-2\gamma_nt_n,2\sigma_n;1),\\ \\(b_j,\beta_j)_{1,m},\quad(b_{ji},\beta_{ji})_{m+1,q_i}\quad\left(\frac{1-\omega_1-2\rho_1r_1-2\gamma_1t_1\pm m_1}{2},\sigma_1;1\right)\end{array}\right.\right.$$

$$\left.\begin{array}{l}(a_{ji},\alpha_{ji})_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ \\ \cdots\left(\frac{1-\omega_n-2\rho_nr_n-2\gamma_nt_n\pm m_n}{2},\sigma_n;1\right)\end{array}\right]. \qquad (6.4)$$

VII. Particular Cases

Setting $\beta_1,\cdots,\beta_n=0$ in (2.2),we get

$$\int_0^{\pi}\cdots\int_0^{\pi}(\sin x_1)^{w_1-1}\cdots(sinx_n)^{w_n-1}e^{i(m_1x_1\cdots m_nx_n)}$$

$$\times{}_{E_1+F_1}F_{G_1+H_1}\left[\begin{array}{ll}(e_1);(f_1); & \\ & \alpha_1(sinx_1)^{2\rho_1}\\(g_1);(h_1); & \end{array}\right]\cdots{}_{E_n+F_n}F_{G_n+H_n}\left[\begin{array}{ll}(e_n);(f_n); & \\ & \alpha_n(sinx_n)^{2\rho_n}\\(g_n);(h_n); & \end{array}\right]$$

$$\times I^{m,n}_{p_i,q_i:r}\left[z(\sin x_1)^{2\sigma_1}\cdots(\sin x_n)^{2\sigma_n}\left|\begin{array}{l}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ \\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{array}\right.\right]dx_1\cdots dx_n$$

$$=\frac{(\pi)^ne^{i(m_1+\cdots+m_n)\pi/2}}{2^{(\omega_1+\cdots+\omega_n)-n}}\sum_{r_1\cdots r_n=0}^{\infty}\frac{\prod_{k_1=1}^{E_1}(e_{1k_1})_{r_1}\prod_{k_1=1}^{F_1}(f_{1k_1})_{r_1}}{\prod_{k_1=1}^{G_1}(g_{1k_1})_{r_1}\prod_{k_1=1}^{H_1}(h_{1k_1})_{r_1}}\cdots$$

$$\cdots \frac{\prod_{k_n=1}^{E_n}(e_{nk_n})_{r_n}\prod_{k_n=1}^{F_n}(f_{nk_n})_{r_n}}{\prod_{k_n=1}^{G_n}(g_{nk_n})_{r_n}\prod_{k_n=1}^{H_n}(h_{nk_n})_{r_n}}\frac{(\alpha_1/4^{\rho_1})^{r_1}}{r_1!}\cdots\frac{(\alpha_n/4^{\rho_n})^{r_n}}{r_n!}$$

$$\times I^{m,n+n}_{p_i+2,q_i+2:r}\left[\frac{z}{4^{(\sigma_1+\cdots+\sigma_n)}}\left|\begin{matrix}(1-\omega_1-2\rho_1r_1,2\sigma_1;1)\cdots(1-\omega_n-2\rho_nr_n,2\sigma_n;1),\\ (b_j,\beta_j)_{1,m},\quad (b_{ji},\beta_{ji})_{m+1,q_i}\quad \left(\frac{1-\omega_1-2\rho_1r_1\pm m_1}{2},\sigma_1;1\right)\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ \cdots\left(\frac{1-\omega_n-2\rho_nr_n\pm m_n}{2},\sigma_n;1\right)\end{matrix}\right]. \quad (7.1)$$

Further setting $\alpha_1,\cdots,\alpha_n=0$ in (7.1),we obtain

$$\int_0^{\pi}\cdots\int_0^{\pi}(\sin x_1)^{w_1-1}\cdots(sinx_n)^{w_n-1}e^{i(m_1x_1+\cdots+m_nx_n)}$$

$$\times I^{n,N}_{p_i,q_i:r}\left[z(\sin x_1)^{2\sigma_1}\cdots(\sin x_n)^{2\sigma_n}\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dx_1\cdots dx_n$$

$$=\frac{(\pi)^ne^{i(m_1+\cdots+m_n)\pi/2}}{2^{(\omega_1+\cdots+\omega_2)-n}}$$

$$\times I^{m,n+1}_{p_i+n,q_i+n:r}\left[\frac{z}{4^{(\sigma_1+\cdots+\sigma_n)}}\left|\begin{matrix}(1-\omega_1,2\sigma_1;1)\cdots(1-\omega_n,2\sigma_n;1),\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\left(\frac{1-\omega_1\pm m_1}{2},\sigma_1;1\right)\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(a_j,\alpha_j)_{1,},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ \cdots\left(\frac{1-\omega_n\pm m_n}{2},\sigma_n;1\right)\end{matrix}\right]. \quad (7.2)$$

Now setting $\alpha=\beta=0$ in (3.3) we establish

$$(\sin x)^{w-1}I^{m,n}_{p_i,q_i}\left[z(\sin x)^{2\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]$$

$$=\sum_{p=-\infty}^{\infty}\frac{e^{ip(\frac{\pi}{2}-x)}}{2^{w-1}}I^{m,n+1}_{P=p_i+1,q_i+2:r}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(1-\omega,2\sigma;1),(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\left(\frac{1-\omega_1\pm p}{2},\sigma;1\right)\end{matrix}\right.\right]. \quad (7.3)$$

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Letting $p = 2l$ as l is a integer,from (7.3), we establish

$$L.H.S.\ of\ (7.3) = \frac{1}{\sqrt{\pi}} I^{m,n+1}_{p_i+1,q_i+1:r}\left[z \left| \begin{matrix} (\frac{2-\omega}{2}, \sigma; 1), (a_j, \alpha_j)_{1,n}, (a_{ji}, \alpha_{ji})_{n+1,p_i} \\ (b_j, \beta_j)_{1,m}, (b_{ji}, \beta_{ji})_{m+1,q_i} \left(\frac{1-\omega}{2}, \sigma; 1\right) \end{matrix} \right. \right]$$

$$+\frac{1}{2^{w-2}} \sum_{p_n=1}^{\infty} cosl\pi \quad cos2lx\ I^{m,n+1}_{p_i+1,q_i+2:r}\left[\frac{z}{4^{\sigma}} \left| \begin{matrix} (1-\omega, \sigma; 1), (a_j, \alpha_j)_{1,n}, (a_{ji}, \alpha_{ji})_{n+1,p_i} \\ (b_j, \beta_j)_{1,m}, (b_{ji}, \beta_{ji})_{m+1,q_i} \left(\frac{1-\omega\pm 2l}{2}, \sigma; 1\right) \end{matrix} \right. \right] \quad (7.4)$$

Further letting $p = (2l + 1)$ as l is an integer,from (7.3) we obtain

$$= \frac{1}{2^{w-2}} \sum_{p=1}^{\infty} sin(2l+1)\pi/2 \quad .sin(2l+1)x$$

$$\times I^{m,n+1}_{p_i+1,q_i+2:r}\left[\frac{z}{4^{\sigma}} \left| \begin{matrix} (1-\omega, 2\sigma; 1), (a_j, \alpha_j)_{1,n}, (a_{ji}, \alpha_{ji})_{n+1,p_i} \\ (b_j, \beta_j)_{1,m}, (b_{ji}, \beta_{ji})_{m+1,q_i} \left(\frac{1-\omega\pm(2l+1)}{2}, \sigma; 1\right) \end{matrix} \right. \right] \quad (7.5)$$

Similarly, remaining particular cases can be evaluated by (4.4) and (5.3) applying the same techniques.

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