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Analytical Approach to Partial Differential Equations Arising in Mechanics

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Abstract - In this article, we implement relatively new analytical approach, Adomian decomposition method (ADM), for solving some selected partial differential equations in mechanics. This method in applied mathematics can be used as alternative method for obtaining exact solution for different types of partial differential equations. The method takes the form of a convergent series with easily computable components. The results obtained with minimum amount of computation indicate that the method is reliable and accurate.

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ANALYTICAL APPROACH TO PARTIAL DIFFERENTIAL EQUATIONS ARISING IN MECHANICS

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Analytical Approach to Partial Differential Equations Arising in Mechanics

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Abstract - In this article, we implement relatively new analytical approach, Adomian decomposition method (ADM), for solving some selected partial differential equations in mechanics. This method in applied mathematics can be used as alternative method for obtaining exact solution for different types of partial differential equations. The method takes the form of a convergent series with easily computable components. The results obtained with minimum amount of computation indicate that the method is reliable and accurate.

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I. Introduction

In recent years, much attention has been devoted to the search for reliable and more efficient solution methods for the equations modeling physical phenomena in various fields of sciences and engineering. One of the methods which have received the attention is the Adomian decomposition method. Unlike the other traditional numerical methods, which are usually valid for short period of time, the solution of the presented equation is analytic for $-\infty \le x \le \infty$. Also, the method needs no discretization, linearization, transformation or perturbation unlike the prior act.

In general, we shall consider equation of the form;

$$\frac{\partial^\alpha u}{\partial t^\alpha} + a_0(x)u + a_1(x)\frac{\partial u}{\partial x} + a_2(x)\frac{\partial^2 u}{\partial x^2} + \ldots + a_n(x)\frac{\partial^n u}{\partial x^n} = q(x,t), \;\; t>0, x \in \Re, \qquad (1)$$

Subject to the initial conditions

$$u(x,0) = f(x), \qquad \alpha = 1, \;\; u(x,t) \to 0 \;\text{ as }\; |x| \to \infty, \qquad t>0 \;\;\text{and}$$

$$u(x,0) = f(x), \qquad \frac{\partial u(x,0)}{\partial t} = g(x), \;\text{ for }\;\; \alpha = 2$$

We shall proceed to discuss the basic concepts and theory of the decomposition method.

The basic concepts of Adomian decomposition method

The decomposition approach requires that the partial differential equation (1) be expressed in an operator form as

$$C_t^\alpha u + a_0(x)u + a_1(x)L_{1x}u + a_2(x)L_{2x}u + \ldots + a_n(x)L_{nx}u = q(x,t) \qquad (2)$$

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Where $L_{1x} = \frac{\partial}{\partial x}, \quad L_{2x} = \frac{\partial^2}{\partial x^2}, \ . \ . \ . \ L_{nx} = \frac{\partial^n}{\partial x^n}$

And the differential operator C_t^α is defined as $C_t^\alpha = \frac{\partial}{\partial t}$ *for* $\alpha = 1$ $C_t^\alpha = \frac{\partial^2}{\partial t^2}$ *for* $\alpha = 2$

The method is based on applying the operator L^{-1}, the inverse of the operator C_t^α on both sides of equation (2) to obtain

$$u(x,t) = \sum_{k=0}^{\alpha-1} \frac{\partial^k u(x,t) t^k}{\partial t^k k!} - L^{-1}\left(a_0(x)u + a_1(x)L_{1x}u + a_2(x)L_{2x}u + . \ . \ . + a_n(x)L_{nx}u - q(x,t)\right) \quad (3)$$

Where L^{-1} is considered a one fold integral for $\alpha = 1$ and two fold integral for $\alpha = 2$.

$\sum_{k=0}^{\alpha-1} \frac{\partial^k u(x,t) t^k}{\partial t^k k!}$ is the term(s) that arise from the initial condition(s)

The method assumes a series solution for $u(x,t)$ given by;

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (4)$$

The components $u_n(x,t)$ will be determined recursively. By substituting (4) into both sides of (3) we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = \sum_{k=0}^{\alpha=1} \frac{\partial^k u(x,t) t^k}{\partial t^k} - L^{-1}\left(a_0(x)\sum_{n=0}^{\infty} u_n(x,t)\right) + L^{-1}\left(a_1(x)L_{1x}\sum_{n=0}^{\infty} u_n(x,t)\right)$$

$$+L^{-1}\left(a_2(x)L_{2x}\sum_{n=0}^{\infty} u_n(x,t)\right) + \ . \ . \ . + L^{-1}\left(a_n(x)L_{nx}\sum_{n=0}^{\infty} u_n(x,t)\right) \quad (5)$$

We next determine the components $u_n(x,t)$ for which $n \geq 0$. We first identify the zeroth component $u_0(x,t)$ by all terms that arise from the initial conditions. The remaining components are determined by using the preceding component. Each term of series (4) is given by the recursive relations

$$u_0(x,t) = \sum_{n=0}^{\alpha-1} \frac{\partial^k u(x,t) t^k}{\partial t^k k!} - L^{-1}\left(q(x,t)\right)$$

$$u_{n+1}(x,t) = -L^{-1}\left(a_0(x)u_n(x,t) + a_1(x)L_{1x}u_n(x,t) + . \ . \ . + a_n(x)L_n u_{xn}(x,t)\right) \quad (6)$$

In practice, all terms of the series (4) cannot be computed. Thus, the solution if equation (1) is approximated by the truncated by the truncated series of the form

$$\phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t)$$

And

$$\lim_{N \to \infty} \phi_N(x,t) = u(x,t) \quad (7)$$

The decomposition series solution by (7), in general converges very rapidly in real physical problems (G. Adomian 1993). The theoretical treatment of convergence of the decomposition method has been investigated by (Y. Cherruault and G. Adomian 1993).

Notes

The method provides the solution of (1) in the form of a rapidly convergent series that may lead to the exact solution in the case of linear differential equations. The accuracy of the method can be improved by accommodating more terms in our decomposition series.

II. Numerical Results

Problem 1

Consider the equation;

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$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, \quad t>0,\ x\in\Re \tag{8}$$

Subject to the initial condition $(x,0)=\sin x$

We apply ADM operator to equation (8)

$$Lu(x,t)=L_{2x}u(x,t) \tag{9}$$

Operating L^{-1} on both sides of (9) and by imposing the initial condition

$$u(x,t)=u(x,0)+L^{-1}L_{2x}u(x,t) \tag{10}$$

L^{-1} is considered as one fold integral

From (10), we obtain the recurrence relations

$$\begin{aligned} u_0(x,t)&=u(x,0)=\sin x \\ u_{n+1}(x,t)&=L^{-1}(L_{2x}u_n(x,t) \end{aligned} \tag{11}$$

From (11), we can proceed to compute the first few terms of the series

$$u_1(x,t)=L^{-1}(L_{2x}u_0(x,t))=\int_0^t\left(\frac{\partial^2 \sin x}{\partial x^2}\right)dt=-t\sin x$$

$$u_2(x,t)=L^{-1}(L_{2x}u_1(x,t))=-\int_0^t\left(\frac{\partial^2 t\sin x}{\partial x^2}\right)dt=\frac{t^2\sin x}{2!}$$

$$u_3(x,t)=L^{-1}(L_{2x}u_2(x,t))=-\int_0^t\left(\frac{\partial^2 t^2\sin x}{2!\partial x^2}\right)dt=-\frac{t^3\sin x}{3!}$$

$$u_4(x,t)=L^{-1}(L_{2x}u_3(x,t))=-\int_0^t\left(\frac{\partial^2 t^3\sin x}{3!\partial x^2}\right)dt=\frac{t^4\sin x}{4!}$$

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$$u_n(x,t)=\frac{(-1)t^n\sin x}{n!}$$

The solution in series form is given by

$$u(x,t)=\sin x-t\sin x+\frac{t^2\sin x}{2!}-\frac{t^3\sin x}{3!}+\frac{t^4\sin x}{4!}+\ldots$$

$$u(x,t) = \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} \ . \ . \ . \right)\sin x = e^{-t}\sin x \tag{12}$$

For application purpose, only the first ten terms of the series will be computed. Table (1) compares the approximate solution of equation (8) with the theoretical solution $u(x,t) = e^{-t}\sin x$. It is obvious that the results are in agreement with the theoretical solution. Higher accuracy of the approach can be obtained by computing more components of the series.

Table 1 : Numerical result when $t = 0.5$

x	Exact solution	Adomian result	Error
0.0	0.00000000	0.00000000	0.00000000
0.2	0.12049904	0.12049904	0.00000000
0.4	0.23619416	0.23619416	0.00000000
0.6	0.34247297	0.34247297	0.00000000
0.8	0.43509847	0.43509847	0.00000000
1.0	0.51037794	0.51037788	0.00000006
1.2	0.56531030	0.56531030	0.00000000
1.4	0.59770548	0.59770548	0.00000000
1.6	0.60627204	0.60627210	0.00000006
1.8	0.59066844	0.59066850	0.00000006
2.0	0.55151671	0.55151671	0.00000000

Problem 2

Let us consider the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{x^2 \partial^2 u}{2\partial x^2}, \quad t > 0 \tag{13}$$

Subject to the initial conditions; $u(x,o) = 0, \ \dfrac{\partial u(x,0)}{\partial t} = x^2$

With the theoretical solution $u(x,t) = x + x^2 \sinh t$

We apply ADM operator to equation (13)

$$Lu(x,t) = \frac{x^2 L_{2x} u(x,t)}{2} \tag{14}$$

Operating L^{-1} on both sides of (14) and impose the initial conditions, we obtain

$$u(x,t) = \sum_{k=0}^{1} \frac{\partial^k u(x,t) t^k}{\partial t^k k!} + \frac{L^{-1}\left(x^2 L_{2x} u(x,t)\right)}{2} \tag{15}$$

Where $\sum_{k=0}^{1} \dfrac{\partial^k u(x,t) t^k}{\partial t^k k!}$ is the term that arises from the initial conditions and $L^{-1} = \int_0^t \int_0^t dt dt$

From equation (15),

$$u_0(x,t) = x + x^2 t$$

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$$u_{n+1}(x,t)=\frac{L^{-1}\left(x^2L_{2x}u_n(x,t)\right)}{2} \tag{16}$$

From equation (16), we compute the first few terms of the series as follows;

$$u_1(x,t)=\frac{L^{-1}\left(x^2L_{2x}(x+x^2)\right)}{2}=\int_0^t\int_0^t x^2tdtdt=\frac{x^2t^3}{3!}$$

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$$u_2(x,t)=\frac{L^{-1}\left(x^2L_{2x}(x^2t^3)\right)}{2(3!)}=\frac{1}{2(3!)}\int_0^t\int_0^t x^2t^3dtdt=\frac{x^2t^5}{5!}$$

$$u_3(x,t)=\frac{L^{-1}\left(x^2L_{2x}(x^2t^5)\right)}{2(5!)}=\frac{1}{2(5!)}\int_0^t\int_0^t x^2t^5dtdt=\frac{x^2t^7}{7!}$$

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$$u_n(x,t)=\frac{x^2t^{2n+1}}{(2n+1)!},\quad n\geq 1 \tag{17}$$

Thus,

$$u(x,t)=x+x^2\left(\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\right),\quad n\geq 0 \tag{18}$$

Table (2) compares the result obtained using the ADM with the theoretical solution

Table 2 : Numerical result when $t=0.5$

x	Exact solution	Adomian result	Error
0.0	0.00000000	0.00000000	0.00000000
0.2	0.22084381	0.22084381	0.00000000
0.4	0.48337525	0.48337525	0.00000000
0.6	0.78759451	0.78759443	0.00000008
0.8	1.13350096	1.13350099	0.00000003
1.0	1.52109531	1.52109531	0.00000000
1.2	1.95037724	1.95037725	0.00000001
1.4	2.42134676	2.42134680	0.00000004
1.6	2.93400396	2.93400399	0.00000003
1.8	3.48834859	3.48834880	0.00000021
2.0	4.08438122	4.08438124	0.00000002

III. Conclusions

In this work, we have applied decomposition method to obtain an approximate solution to heat and wave equations. A comparison of the results with the exact solutions suggests that Adomian decomposition method is accurate, reliable and easy to use. It is interesting to note that higher accuracy of the method can be obtained by accommodating more terms of the series.

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