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Analytical Solutions for Some of the Nonlinear Hyperbolic-Like Equations with Variable Coefficients

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Abstract - In this work Homotopy Analysis Method(HAM) is used for analytic treatment of the nonlinear hyperbolic-like equations with variable coefficients. This method can provide analytical solutions to the problems by just utilizing the initial conditions. The proposed iterative scheme finds the solution without any discretization, linearization or restrictive assumptions. The proposed method solves nonlinear problems without using Adomain polynomials which is the advantage of this method over Adomain Decomposition method. The results reveal that the HAM is very effective, fast, simple, convenient, flexible and accurate. Outcomes prove that HAM is in very good agreement with ADM,VIM HPM.

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I. Introduction

Recently various iterative methods are applied for getting Numerical and analytical solutions of Nonlinear hyperbolic-like equations with variable coefficients. [1,2,3,4,5,6]. In this paper Homotopy Analysis Method is applied to solve the proposed equations. HAM introduced by Liao [7,8,9,10,11] has been used by many mathematicians and engineers to solve various equations based on homotopy, which is a basic concept in topology. In recent years this method has been successfully employed to solve many types of nonlinear homogeneous or nonhomogeneous equations and systems of equation as well as problems in Science and engineering [12,13,14,15,16]. The validity of the HAM is independent of whether or not there exists small parameters in the considered equation. HAM provides us with a simple way to adjust and control the convergence of solution series. It provides us with freedom to use different base functions to approximate a nonlinear problem. Especially, it provides us with freedom of replacing a nonlinear partial differential equation of first order n into an infinite number of linear Differential equations of order k, where the order k is even unnecessarily to be equal to order n. Thus as long as Auxillary parameter h, Auxillary function H(r,t), Initial approximation $U_0(r,t)$ and linear operator L are properly chosen the solution expression converges to exact solution in the convergence region. Homotopy analysis method provides us the great freedom to choose all of them.

II. Basic Idea of Homotopy Analysis Method (HAM)

In this section the basic ideas of the homotopy analysis method are introduced. Here a description of the method [9] is given to handle the general nonlinear problem.

Ref.

1. Mohammad Ghoreishi,Adomain Decomposition method(ADM) for Non linear wave-like equations with variable coefficient,Applied Mathematical sciences,vol4,2010,no 49,2431-2444.

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$$Nu_0(t)=0, t>0, \tag{1}$$

Where N is a nonlinear operator and $u_0(t)$ is unknown function of the independent variable t.

a) Zero- order deformation equation

Let $u_0(t)$ denote the initial guess of the exact solution of Eq. (1), h$\neq 0$ an auxiliary parameter, $H(t) \neq 0$ an auxiliary function. and L an auxiliary linear operator with the property.

$$L(f(t)) = 0 when f(t) = 0 \tag{2}$$

The auxiliary parameter h, the auxiliary function$H(t)$, and the auxiliary linear operator L play important roles within the HAM to adjust and control the convergence region of solution series. Liao[10] constructs, using $q \in [0,1]$ as an embedding parameter, the so-called zero-order deformation equation.

$$(1-q)L[(\emptyset(t;q) - u_0(t)] = qhH(t)N[(\emptyset(t;q)], \tag{3}$$

Where $\emptyset(t;q)$ is the solution which depends on h, $H(t), L,\ u_0(t)$ and q. when q=0,the zero-order deformation Eq.(3) becomes

$$\emptyset(t;0) = u_0(t), \tag{4}$$

And when q=1, since h$\neq 0$ and $H(t) \neq 0$, the zero-order deformation Eq.(3) reduces to,

$$N[\emptyset(t;1)] = 0, \tag{5}$$

So, $\emptyset(t;1)$ is exactly the solution of the nonlinear Eq.(1). Define the so-called mth order deformation derivatives.

$$1u_m(t) = \frac{1}{m!}\frac{\partial^m \emptyset(t;q)}{\partial q^m} \tag{6}$$

I the power series (6) of $\emptyset(t;q)$ converges at q=1, then we gets the following series solution:

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t). \tag{7}$$

Where the terms $u_m(t)$ can be determined by the so-called high order deformation described below.

b) High- order deformation equation

Define the vector,

$$\overrightarrow{u_n} = \{u_0(t), u_1(t), u_2(t) \dots \dots u_n(t) \tag{8}$$

Differentialing Eq(4)m times with respect to embedding parameter q, the setting q=0 and dividingthem by $m!$, we have the so-called mth- oder deformation equation.

$$L[u_m(t) - \aleph_m u_{m-1}(t)] = hH(t)R_m(\overrightarrow{u_m}, t), \tag{9}$$

Where $$\aleph_m = \begin{cases} o, & m \leq 1 \\ 1, & otherwise \end{cases} \tag{10}$$

Ref.

10. S.J.Liao,Beyond perturbation:Introduction to Homotopy Analysis Method, Chapman &Hall/CRC press.Boca Raton 2003.

And

$$R_m(\overrightarrow{u_m}, t) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\emptyset(t;q)]}{\partial q^{m-1}} \tag{11}$$

For any given nonlinear operator N, the term $R_m(\overrightarrow{u_m}, t)$ can be easily expressed by (11). Thus, we can gain $u_1(t), u_2(t) \ldots\ldots$ by means of solving the linear high-order deformation Eq. (9) one after the other order in order. The mth − order approximation of u(t) is given by

$$u(t) = \sum_{k=0}^{m} u_k(t). \tag{12}$$

III. Illustrative Examples

Example1:
Let us consider the nonlinear hyperbolic-like equation

$$u_{tt}=u_{xx}+u_x u-e^x t-e^{2x}t^2 \tag{13}$$

With initial conditions

$$U(x,y,0)=0,\ u_t(x,y,0)=e^x \tag{14}$$

We apply homotopy analysis method to Eq. (13) and (14), as follows:
since m≥1, $\chi_m=1$ set h= -1 and H(r,t)=1,L==$\partial^2 u/\partial t^2$ in(9)then (9) becomes

$$u_m(x,t) = u_{m-1}(x,t) - L^{-1}(R_m(u_{m-1}, x, t)$$

$$R_m(u_{m-1}, x, t) = \frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{\partial^2 u_{m-1}}{\partial x^2} - \frac{\partial u_{m-1}}{\partial x} + e^x\, t + e^{2x} t^2 \tag{15}$$

Where

$$R_m(u_{m-1},x,t) =\partial^2 u_{m-1}/\partial t^2-\partial^2 u_{m-1}/\partial x^2 -\partial u_{m-1}/\partial x u_{m-1}+e^x t+e^{2x}t^2$$

And solution for u_0:
Now we can select

$$u_0(x,t)= e^x t \tag{16}$$

Applying (16) in (15) we obtain the following successive approximations:

$$U_1(x,t)= e^x t \tag{17}$$

$$U_2(x,t)= e^x t \tag{18}$$

$$U_3(x,t)= e^x t \tag{19}$$

The Final solution is

$$U(x,t)= e^x t \tag{20}$$

Example2:

Let us consider the following nonlinear two dimensional Hyperbolic-like equation with variable coefficients

$$u_{tt} = \frac{\partial^2}{\partial x \partial y}\left(u_{xx} u_{yy}\right) - \frac{\partial^2}{\partial x \partial y}\left(xyu_x u_y\right) - u \tag{21}$$

With initial conditions

$$U(x,y,0)=0 \quad u_t=(x,y,0)=e^{xy} \tag{22}$$

Ref.

12. A.Molabahrami,F.Khani,The homotopy analysis method to solve Burgers-Huxley equation,Non linear analysis:Real world Applications10(2009)589-600.

We apply Homotopy Analysis method to (21) and (22) as follows
since $m\geq1$, $\chi_m=1$ set $h=-1$ and $H(r,t)=1, L==\partial^2u/\partial t^2$ in (9)then (9) becomes

$$U_m(x,y,t)=u_{m-1}(x,t)-L^{-1}(R_m(u_{m-1},x,y,t)) \tag{23}$$

Where

$$R(u_{m-1},x,y,t)=\partial^2u_{m-1}/\partial t^2-\partial^2/\partial x\partial y(\partial^2u_{m-1}/\partial x^2\partial^2u_{m-1}/\partial y^2)+\partial^2/\partial x\partial y(xy\partial u_{m-1}/\partial x\partial u_{m-1}/\partial y)+u_{m-1}$$

And solution for u_0:
Now we can select

$$u_0(x,y,t)=e^{xy}t \tag{24}$$

Applying (24) in (23) we obtain the following successive approximations:

$$u_1(x,y,t)=e^{xy}(\frac{-t^3}{3!}),$$
$$u_2(x,y,t)=e^{xy}(\frac{t^5}{5!}) \tag{25}$$

The Final solution is

$$U(x,y,t)=e^{xy}(t-t^3/3!+t^5/5!-t^7/7!-\ldots\ldots\ldots\ldots)=e^{xy}\sin t \tag{26}$$

Example3:
Let us consider the following Nonlinear Hyperbolic-like equation

$$U_{tt}=u^2\partial^2/\partial x^2(u_xu_{xx}u_{xxx})+u^2{}_x\partial^2/\partial x^2(u^3{}_{xx})-18u^5+u,\quad 0<x<1,t>0 \tag{27}$$

With initial conditions

$$U(x,0)=e^x\quad u_t(x,0)=e^x \tag{28}$$

We apply Homotopy Analysis method to (27) and (28) as follows
since $m\geq1$, $\chi_m=1$ set $h=-1$ and $H(r,t)=1, L==\partial^2u/\partial t^2$ in(9)then (9) becomes

$$U_m(x,t)=u_{m-1}(x,t)-L^{-1}(R_m(u_{m-1},x,t)) \tag{29}$$

Where

$$R(u_{m-1},x,y,t)=\partial^2u_{m-1}/\partial t^2-(u_{m-1})^2\partial^2/\partial x^2(\partial u_{m-1}/\partial x\partial^2u_{m-1}/\partial x^2\partial^3u_{m-1}/\partial x^3)-(\partial u_{m-1}/\partial x)^2\partial^2/\partial x^2(\partial^2u_{m-1}/\partial x^2)^3+18(u_{m-1})^5-u_{m-1}$$

And solution for u_0:
Now we can select

$$u_0(x,t)=e^x(1+t) \tag{30}$$

Applying (30) in (29) we obtain the following successive approximations:

$$U_1(x,t)=e^x(t^2/2!+t^3/3!) \tag{31}$$

$$U_2(x,t)=e^x(t^4/4!+t^5/5!) \tag{32}$$

$$U_3(x,t)=e^x(t^6/6!+t^7/7!) \tag{33}$$

Notes

Notes

The Final solution is

$$U(x,t)= e^{x} (1+t^2/2!+t^3/3!+t^4/4!+.........) = e^{x+t} \quad (34)$$

Example4:

Finally we solve another nonlinear Hyperbolic-like equation

$$U_{tt}=x^2\partial/\partial x(u_x u_{xx})- x^2(u_{XX})^2+u, \quad 0<x<1,t>0 \quad (35)$$

With initial conditions

$$u(x,t)=x^2, \frac{\partial u(x,t)}{\partial t}=x^2 t \quad (36)$$

We apply Homotopy Analysis method to (35) and (36) as follows since m≥1, χ_m=1 set h= -1 and H(r,t)=1,L==∂^2u/∂t^2 in(9)then (9) becomes

$$U_m(x,t)=u_{m-1}(x,t)-L^{-1}(R_m(u_{m-1},x,t)) \quad (37)$$

Where

$$R_m(u_{m-1},x,t) =\partial^2 u_{m-1}/\partial t^2-x^2\partial/\partial x(\partial u_{m-1}/\partial x\partial^2 u_{m-1}/\partial x^2)+x^2(\partial^2 u_{m-1}/\partial x^2)^2-u_{m-1}$$

And solution for u_0:

Now we can select

$$u_0(x,t)= x^2 (1+t) \quad (38)$$

Applying (38) in (37) we obtain the following successive approximations:

$$U_1(x,t)= x^2 (1+t+t^2/2!+t^3/3!) \quad (39)$$

$$U_2(x,t)= x^2 (1+t+t^2/2!+t^3/3!+t^4/4!+t^5/5!) \quad (40)$$

The Final solution is

$$U(x,t)=x^2 (1+t+t^2/2!+t^3/3!+t^4/4!+t^5/5!+..........) = x^2e^t \quad (41)$$

IV. Conclusion

In this paper exact solutions for some of the hyperbolic-like equations have been established. The Homotopy Analysis method(HAM) is successfully used to develop these solutions. This work shows that HAM has significant advantages over the existing techniques. It avoids the need for calculating the Adomain polynomials which can be difficult in some cases. The reliability of the method and reduction in the size of computational domain give this method wider applicability. The results shows that HAM is a powerful mathematical tool for finding the exact and approximate solutions of the nonlinear equations.

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