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Existence of Classical Solutions for a Class Nonlinear Wave Equations

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EXISTENCE OF CLASSICAL SOLUTIONS FOR A CLASS NONLINEAR WAVE EQUATIONS

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Existence of Classical Solutions for a Class Nonlinear Wave Equations

Svetlin Georgiev Georgiev

Abstract - In this article we investigate the Cauchy problem for the equation $u_{tt} - u_{xx} = |u|^l$, $t \in [0, \infty)$, $x \in \mathbb{R}$, $l \in [0,1)$. At this moment, the cases $l \geq 1, l = 0$ are well studied. Here we answer of the open problem $l \in (0,1)$ using approach.

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I. Introduction

In this article we investigate the Cauchy problem

$$u_{tt} - u_{xx} = |u|^l, \quad t > 0, x \in \mathbb{R}, \tag{1.1}$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $u_0 \in \mathcal{C}^2(\mathbb{R})$, $u_1 \in \mathcal{C}^1(\mathbb{R})$ are given functions for which $|u_0(x)| \leq D$, $|u_1(x)| \leq D$ for every $x \in \mathbb{R}$, D is given positive constant, $l \in [0, 1)$ is fixed, u is unknown function.

The problem (1.1), (1.2) was considered in the cases when $l \geq 1$, $l = 0$, for local existence, global existence, blow up and etc, see for instance [2] and references therein. The case $l \in (0, 1)$ was opened. Our aim in this article is to give an answer in this case. We give an answer for local existence of classical solutions. The problem for uniqueness of the classical solutions(twice continuous - differentiable in x and in t) is opened yet.

For $M, N \subseteq \mathbb{R}$ with $\mathcal{C}^2(M, \mathcal{C}^2(N))$ we will denote the space of the functions u which are twice continuous - differentiable in $t \in M$ and twice continuous - differentiable in $x \in N$.

Our main results are as follows.

Theorem 1.1. *Let D be fixed positive constant and $u_0 \in \mathcal{C}^2(\mathbb{R})$, $u_1 \in \mathcal{C}^1(\mathbb{R})$ be fixed so that $|u_0(x)| \leq D$, $|u_1(x)| \leq D$ for every $x \in \mathbb{R}$, let also $l \in [0, 1)$ be fixed. Then there exist positive constants A and B so that the Cauchy problem (1.1), (1.2) has a solution $u \in \mathcal{C}^2([0, A], \mathcal{C}^2([0, B]))$.*

Theorem 1.2. *Let D be fixed positive constant and $u_0 \in \mathcal{C}^2(\mathbb{R})$, $u_1 \in \mathcal{C}^1(\mathbb{R})$ be fixed so that $|u_0(x)| \leq D$, $|u_1(x)| \leq D$ for every $x \in \mathbb{R}$, let also $l \in [0, 1)$ be fixed. Then there exists positive constant A so that the Cauchy problem (1.1), (1.2) has a solution $u \in \mathcal{C}^2([0, A], \mathcal{C}^2([0, \infty)))$.*

[2] Shatah, J., M. Struwe. Geometric Wave Equations, vol. 2 of Courant Lecture Notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York, NY, USA, 1998.

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Theorem 1.3. *Let D be fixed positive constant and $u_0 \in \mathcal{C}^2(\mathbb{R})$, $u_1 \in \mathcal{C}^1(\mathbb{R})$ be fixed so that $|u_0(x)| \le D$, $|u_1(x)| \le D$ for every $x \in \mathbb{R}$, let also $l \in [0,1)$ be fixed. Then there exists positive constant A so that the Cauchy problem* (1.1), (1.2) *has a solution $u \in \mathcal{C}^2([0,A],\mathcal{C}^2(\mathbb{R}))$.*

We note that when u_0 or u_1 is not identically equal to zero the Cauchy problem (1.1), (1.2) has a nontrivial solution.

To prove our main results we will use a new approach which is used in the author's article [1] for another class of nonlinear wave equations.

The article is organized as follows: in the next section we will prove Theorem 1.1, in the section 3 we will prove Theorem 1.2, in the section 4 we will prove Theorem 1.3. In the appendix we will prove some facts which are used in the proof of basic results.

II. Proof of Theorem 1.1

Let $\epsilon \in (0,1)$ be fixed. We choose enough small positive constants A and B so that

$$\epsilon D + B^2D(2+A) + (2+B)A^2D + A^2B^2D^l \le D,$$

$$\epsilon D + BD(2+A) + (2+B)A^2D + A^2BD^l \le D, \tag{2.1}$$

$$\epsilon D + 2B^2D + (2+B)AD + AB^2D^l \le D.$$

For example $\epsilon = \frac{1}{2}$, $D = 100$, $A = B = \frac{1}{1000000}$.

In this section we will prove that the Cauchy problem

$$u_{tt} - u_{xx} = |u|^l, \quad t \in [0,A], x \in [0,B], \tag{2.2}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in [0,B], \tag{2.3}$$

has a solution $u^{+1} \in \mathcal{C}^2([0,A],\mathcal{C}^2([0,B]))$.

We define the sets

$$N_{+1} = \Big\{u \in \mathcal{C}^2([0,A],\mathcal{C}^2([0,B])) : |u(t,x)| \le D, |u_t(t,x)| \le D, |u_x(t,x)| \le D$$

$$\forall t \in [0,A], \quad \forall x \in [0,B]\Big\},$$

$$N^*_{+1} = \Big\{u \in \mathcal{C}^2([0,A],\mathcal{C}^2([0,B])) : |u(t,x)| \le (1+\epsilon)D, |u_t(t,x)| \le (1+\epsilon)D,$$

$$|u_x(t,x)| \le (1+\epsilon)D \quad \forall t \in [0,A], \quad \forall x \in [0,B]\Big\},$$

in these sets we define a norm as follows

$$||u||_1 = \max_{t\in[0,A],x\in[0,B]} |u(t,x)|.$$

Lemma 2.1. *The sets N_{+1} and N^*_{+1} are closed, compact and convex spaces in $\mathcal{C}([0,A]\times[0,B])$ in the sense of norm $||\cdot||_1$.*

Proof. We will prove our assertion for N_{+1}.

Ref.

[1] Georgiev, S. Global existence for nonlinear wave equation. Dyn. PDE, Vol. 7, No. 3, pp. 207-216.

Firstly we will prove that N_{+1} is a closed space with respect $||\cdot||_1$. For this we will propose two ways, the first one is to be proved that N_{+1} is a completely normed space with respect the norm $||\cdot||_1$, using Weierstrass - Stone theorem, the second one is based on the definition - in other words we will prove that it contains its limit points.

First proof. Let $\{u_n\}$ is a sequence of elements of the space N_{+1} which is a fundamental sequence and it is well known that there exists $U \in \mathcal{C}([0,A]\times[0,B])$ so that $\lim_{n\longrightarrow\infty} u_n = U$ with respect the norm $||\cdot||_1$. Then for every $\epsilon > 0$ there exists $N_1 = N_1(\epsilon) > 0$ so that for every $n > N_1$ we have

$$||u_n - U||_1 < \epsilon.$$

From Weierstrass - Stone theorem there exists a sequence $\{p_l\}$ of trigonometric polynomials so that $||p_l - U||_1 \longrightarrow 0$ when $l \longrightarrow \infty$. We have $p_l \in \mathcal{C}^2([0,A]\times[0,B])$ and there exists $L_1 = L_1(\epsilon) > 0$ so that for every $L > L_1$ we have

$$||p_L - U||_1 < \epsilon.$$

We fix $L > L_1$ and put $u = p_L$. From here, for every $n > N_1$,

$$||u_n - u||_1 \leq ||u_n - U||_1 + ||U - u||_1 < 2\epsilon.$$

Consequently the fundamental sequence u_n of elements of N_{+1} is convergent to the element $u \in \mathcal{C}^2([0,A]\times[0,B])$ with respect the norm $||\cdot||_1$. Now we will prove that $u \in N_{+1}$. Evidently $|u(t,x)| \leq D$ for every $(t,x) \in [0,A]\times[0,B]$. Now we suppose that there exists $(\tilde{t},\tilde{x}) \in [0,A]\times[0,B]$ so that

$$|u_t(\tilde{t},\tilde{x})| > D.$$

Then there exists $\epsilon_2 > 0$ so that

$$|u_t(\tilde{t},\tilde{x})| \geq D + \epsilon_2.$$

From here there exists $\delta_5 = \delta_5(\epsilon_2) > 0$ such that from $|h| < \delta_5$, $h \neq 0$, $(\tilde{t},\tilde{x}) \in [0,A]\times[0,B]$ we have

$$\Big|\frac{u(\tilde{t},\tilde{x}+h) - u(\tilde{t},\tilde{x})}{h}\Big| \geq D + \epsilon_2.$$

On the other hand, since $u_n(\tilde{t},\tilde{x}) \longrightarrow u(\tilde{t},\tilde{x})$ in sense of $||\cdot||_1$, as $n \longrightarrow \infty$, follows that there exists $\delta_6 = \delta_6(\epsilon_2) > 0$ so that we have from $|h| < \delta_6$, $h \neq 0$, $(\tilde{t}+h,\tilde{x}) \in [0,A]\times[0,B]$

$$\Big|\tfrac{u_n(\tilde{t}+h,\tilde{x}) - u_n(\tilde{t},\tilde{x})}{h} - \tfrac{u(\tilde{t}+h,\tilde{x}) - u(\tilde{t},\tilde{x})}{h}\Big| < \epsilon_2$$

and since $|(u_n)_t| \leq D$ in $[0,A]\times[0,B]$

$$\Big|\frac{u_n(\tilde{t}+h,\tilde{x}) - u_n(\tilde{t},\tilde{x})}{h}\Big| \leq D$$

for enough large n. From here, for enough large n and for $|h| < \min\{\delta_5,\delta_6\}$, $h \neq 0$, $(\tilde{t}+h,\tilde{x}) \in [0,A]\times[0,B]$ we have

$$\epsilon_2 = D + \epsilon_2 - D$$

Notes

$$\leq \left|\frac{u(\tilde{t}+h,\tilde{x})-u(\tilde{t},\tilde{x})}{h}\right| - \left|\frac{u_n(\tilde{t}+h,\tilde{x})-u_n(\tilde{t},\tilde{x})}{h}\right|$$

$$\leq \left|\frac{u(\tilde{t}+h,\tilde{x})-u(\tilde{t},\tilde{x})}{h} - \frac{u_n(\tilde{t}+h,\tilde{x})-u_n(\tilde{t},\tilde{x})}{h}\right| < \epsilon_2,$$

which is a contradiction. Therefore $|u_t| \leq D$ in $[0, A] \times [0, B]$. As in above we can prove that $|u_x(t, x)| \leq D$ in $[0, A] \times [0, B]$. Consequently $u \in N_{+1}$ and N_{+1} is closed in $\mathcal{C}([0, A] \times [0, B])$ in sense of $|| \cdot ||_1$.

Second proof. Let u is a limit point of N_{+1}.

Then there exists a sequence $\{u_n\}$ of elements of N_{+1} and $u_n \longrightarrow_{n\longrightarrow\infty} u$ in the sense of the norm $|| \cdot ||_1$.

Evidently $|u(t, x)| \leq D$ for every $(t, x) \in [0, A] \times [0, B]$.

We suppose that $u \notin \mathcal{C}^1([0, A] \times [0, B])$. Without loss of generality we suppose that u_t does not exist in a point $(t, x) \in [0, A] \times [0, B]$. Then there exists $\epsilon > 0$ so that for every $\delta_1 = \delta_1(\epsilon) > 0$ and $|h| < \delta_1$, $h \neq 0$, $(t + h, x) \in [0, A] \times [0, B]$, we have

$$\left|\frac{u(t + h, x) - u(t, x)}{h}\right| > \epsilon. \tag{2.4}$$

On the other hand since $u_n \in \mathcal{C}^2([0, A], \mathcal{C}^2([0, B]))$ we have that there exists $\delta_2 = \delta_2(\epsilon) > 0$ so that from $|h| < \delta_2$, $h \neq 0$, $(t + h, x) \in [0, A] \times [0, B]$, we have

$$\left|\frac{u_n(t + h, x) - u_n(t, x)}{h}\right| < \frac{\epsilon}{3}. \tag{2.5}$$

Also, from $u_n \longrightarrow_{n\longrightarrow\infty} u$ in the sense of the norm $|| \cdot ||_1$ we have for enough large n and $|h| < \min\{\delta_1, \delta_2\}$, $h \neq 0$, $(t + h, x) \in [0, A] \times [0, B]$ that

$$\left|\frac{u_n(t+h,x)-u_n(t,x)}{h} - \frac{u(t+h,x)-u(t,x)}{h}\right| < \frac{\epsilon}{3}. \tag{2.6}$$

Then from (2.6), (2.5), (2.4) we obtain for $|h| < \min\{\delta_1, \delta_2\}$, $h \neq 0$, $(t+h, x) \in [0, A]\times[0, B]$, for enough large n,

$$\epsilon < \left|\frac{u(t+h,x)-u(t,x)}{h}\right| \leq \left|\frac{u_n(t+h,x)-u_n(t,x)}{h}\right| + \left|\frac{u_n(t+h,x)-u_n(t,x)}{h} - \frac{u(t+h,x)-u(t,x)}{h}\right| < 2\frac{\epsilon}{3},$$

which is a contradiction with our assumption that $u_t(t, x)$ does not exist. If we suppose that $u_x(t, x)$ does not exist in a point $(t, x) \in [0, A] \times [0, B]$, as in above we will go to a contradiction. Therefore $u \in \mathcal{C}^1([0, A], \mathcal{C}^1([0, B]))$.

We note that from $u_n \longrightarrow u$ as $n \longrightarrow \infty$ and $u_n, u \in \mathcal{C}^1([0, A], \mathcal{C}^1([0, B]))$ follows that for every $\tilde{\epsilon}$ there exists $\tilde{\delta}(\tilde{\epsilon}) > 0$ so that from $|h| < \tilde{\delta}$, $h \neq 0$, $(t + h, x) \in [0, A] \times [0, B]$, we have

$$\left|\frac{u_n(t + h, x) - u_n(t, x)}{h} - \frac{u(t + h, x) - u(t, x)}{h}\right| < \tilde{\epsilon},$$

from where we conclude that $u_{nt} \longrightarrow u_t$ in sense of $|| \cdot ||_1$ as $n \longrightarrow \infty$. In the same way we have $u_{nx} \longrightarrow u_x$ when $n \longrightarrow \infty$ in sense of $|| \cdot ||_1$.

We suppose that $u \notin \mathcal{C}^2([0, A], \mathcal{C}^2([0, B])$. Without loss of generality we suppose that u_{tt} does not exist in a point $(t, x) \in [0, A] \times [0, B]$. Then there exists $\epsilon_1 > 0$ so that for every $\delta_3 = \delta_3(\epsilon_1) > 0$ and $|h| < \delta_3$, $h \neq 0$, $(t + h, x) \in [0, A] \times [0, B]$ we have

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$$\left|\frac{u_t(t+h,x)-u_t(t,x)}{h}\right| > \epsilon_1. \tag{2.7}$$

On the other hand since $u_n \in \mathcal{C}^2([0,1],\mathcal{C}^2(B_1))$ we have that there exists $\delta_4=\delta_4(\epsilon_1)>0$ so that from $|h|<\delta_4$, $h\neq 0$, $(t+h,x)\in[0,A]\times[0,B]$ we have

$$\left|\frac{(u_n)_t(t+h,x)-(u_n)_t(t,x)}{h}\right| < \frac{\epsilon_1}{3}. \tag{2.8}$$

Notes

Also, from $u_{nt} \longrightarrow_{n\longrightarrow\infty} u_t$ in the sense of the norm $||\cdot||_1$ we have for enough large n and $|h|<\min\{\delta_3,\delta_4\}$, $h\neq 0$, $(t+h,x)\in[0,A]\times[0,B]$ that

$$\left|\frac{(u_n)_t(t+h,x)-(u_n)_t(t,x)}{h}-\frac{u_t(t+h,x)-u_t(t,x)}{h}\right| < \frac{\epsilon_1}{3}. \tag{2.9}$$

Then from (2.9), (2.8), (2.7) we obtain for $|h|<\min\{\delta_3,\delta_4\}$, $h\neq 0$, $(t+h,x)\in[0,A]\times[0,B]$, for enough large n,

$$\begin{aligned}\epsilon_1 &< \left|\frac{u_t(t+h,x)-u_t(t,x)}{h}\right|\\ &\leq \left|\frac{(u_n)_t(t+h,x)-(u_n)_t(t,x)}{h}\right| + \left|\frac{(u_n)_t(t+h,x)-(u_n)_t(t,x)}{h}-\frac{u_t(t,x)-u_t(t,x)}{h}\right| < 2\frac{\epsilon_1}{3},\end{aligned}$$

which is a contradiction with our assumption that u_{tt} does not exists in a point $(t,x)\in[0,A]\times[0,B]$. If we suppose that u_{xx} does not exist in a point $(t,x)\in[0,A]\times[0,B]$ as in above we can go to a contradiction. Therefore $u\in\mathcal{C}^2([0,A],\mathcal{C}^2([0,B]))$. As in the first proof(above) we have that $|u(t,x)|\leq D$, $|u_t(t,x)|\leq D$, $|u_x(t,x)|\leq D$ for every $(t,x)\in[0,A]\times[0,B]$, i.e. $u\in N_{+1}$. Consequently N_{+1} contains its limit points.

Using Arzela - Ascoli Theorem the set N_{+1} is a compact set in $\mathcal{C}([0,A]\times[0,B])$ in sense of $||\cdot||_1$.

Let now $\lambda\in[0,1]$ is arbitrary chosen and fixed and $u_1,u_2\in N_{+1}$. Then for $(t,x)\in[0,A]\times[0,B]$ we have $\lambda u_1(t,x)+(1-\lambda)u_2(t,x)\in\mathcal{C}^2([0,A],\mathcal{C}^2([0,B]))$ and

$$|u_i(t,x)|\leq D, |u_{it}(t,x)|\leq D \quad |u_{ix}(t,x)|\leq D \quad \text{for} \quad i=1,2,$$

$$|\lambda u_1(t,x)+(1-\lambda)u_2(t,x)|\leq\lambda|u_1(t,x)|+(1-\lambda)|u_2(t,x)|\leq\lambda D+(1-\lambda)D=D,$$

$$|\lambda u_{1t}(t,x)+(1-\lambda)u_{2t}(t,x)|\leq\lambda|u_{1t}(t,x)|+(1-\lambda)|u_{2t}(t,x)|\leq\lambda D+(1-\lambda)D=D,$$

$$|\lambda u_{1x}(t,x)+(1-\lambda)u_{2x}(t,x)|\leq\lambda|u_{1x}(t,x)|+(1-\lambda)|u_{2x}(t,x)|\leq\lambda D+(1-\lambda)D=D.$$

Therefore N_{+1} is convex.

As in above we can prove that N^*_{+1} is closed, compact and convex in $\mathcal{C}([0,A]\times[0,B])$ in sense of $||\cdot||_1$.

For $u\in N^*_{+1}$ we define the operators

$$K_{+1}(u)(t,x)=(1+\epsilon)u(t,x),$$

$$L_{+1}(u)(t,x)=-\epsilon u(t,x)+\int_0^x\int_0^\sigma u(t,y)dyd\sigma-\int_0^x\int_0^\sigma\Big(u_0(y)+tu_1(y)\Big)dyd\sigma$$

$$-\int_0^t \int_0^\tau u(s,x)dsd\tau - \int_0^t \int_0^\tau \int_0^x \int_0^\sigma |u|^l(s,y)dyd\sigma dsd\tau,$$

$$P_{+1}(u)(t,x) = K_{+1}(u)(t,x) + L_{+1}(u)(t,x).$$

Our first observation is as follows.

Lemma 2.2. *Let $u \in N^*_{+1}$ be a fixed point of the operator P_{+1}. Then u is a solution to the Cauchy problem (2.2), (2.3).*

Proof. Since $u \in N^*_{+1}$ is a fixed point of the operator P_{+1} we have for every $t \in [0, A]$ and $x \in [0, B]$

$$u(t,x) = P_{+1}(u)(t,x) = K_{+1}(u)(t,x) + L_{+1}(u)(t,x)$$

$$= (1+\epsilon)u(t,x) - \epsilon u(t,x) + \int_0^x \int_0^\sigma u(t,y)dyd\sigma - \int_0^x \int_0^\sigma \Big(u_0(y) + tu_1(y)\Big)dyd\sigma$$

$$-\int_0^t \int_0^\tau u(s,x)dsd\tau - \int_0^t \int_0^\tau \int_0^x \int_0^\sigma |u|^l(s,y)dyd\sigma dsd\tau$$

$$= u(t,x) + \int_0^x \int_0^\sigma u(t,y)dyd\sigma - \int_0^x \int_0^\sigma \Big(u_0(y) + tu_1(y)\Big)dyd\sigma$$

$$-\int_0^t \int_0^\tau u(s,x)dsd\tau - \int_0^t \int_0^\tau \int_0^x \int_0^\sigma |u|^l(s,y)dyd\sigma dsd\tau,$$

whereupon for every $t \in [0, A]$ and every $x \in [0, B]$ we have

$$0 = \int_0^x \int_0^\sigma u(t,y)dyd\sigma - \int_0^x \int_0^\sigma \Big(u_0(y) + tu_1(y)\Big)dyd\sigma$$
$$-\int_0^t \int_0^\tau u(s,x)dsd\tau - \int_0^t \int_0^\tau \int_0^x \int_0^\sigma |u|^l(s,y)dyd\sigma dsd\tau. \tag{2.10}$$

Now we differentiate the last equality with respect t and we get, for $t \in [0, A]$, $x \in [0, B]$,

$$0 = \int_0^x \int_0^\sigma \Big(u_t(t,y) - u_1(y)\Big)dyd\sigma - \int_0^t u(s,x)ds - \int_0^t \int_0^x \int_0^\sigma |u|^l(s,y)dyd\sigma ds. \tag{2.11}$$

We differentiate the last equality with respect the time variable t and we obtain

$$0 = \int_0^x \int_0^\sigma u_{tt}(t,y)dyd\sigma - u(t,x) - \int_0^x \int_0^\sigma |u|^l(t,y)dyd\sigma, \quad t \in [0, A], x \in [0, B].$$

Now we differentiate the last equality with respect x we find

$$0 = \int_0^x u_{tt}(t,y)dy - u_x(t,x) - \int_0^x |u|^l(t,y)dy, \quad t \in [0, A], x \in [0, B].$$

After we differentiate the last equality in x we obtain

$$0 = u_{tt} - u_{xx} - |u|^l, \quad t \in [0, A], x \in [0, B],$$

in other words u satisfies the equation (2.2).

Now we put $t = 0$ in (2.10) and we find

$$0 = \int_0^x \int_0^\sigma \Big(u(0,y) - u_0(y)\Big)dyd\sigma, \quad x \in [0, B],$$

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after which we differentiate it twice in x and we get

$$u(0,x) = u_0(x), \quad x \in [0,B].$$

We put $t=0$ in (2.11) and we have

$$0 = \int_0^x \int_0^\sigma \Big(u_t(0,y) - u_1(y)\Big) dy d\sigma, \quad x \in [0,B],$$

we differentiate twice the last equality with respect x and we find

$$u_t(0,x) = u_1(x), \quad x \in [0,B].$$

Consequently u satisfies the initial conditions (2.3).

The above lemma motivate us to search fixed points of the operator P_{+1}. For this purpose we will use the following fixed point theorem.

Theorem 2.3. *(see [3], Corollary 2.4, pp. 3231) Let X be a nonempty closed convex subset of a Banach space Y. Suppose that T and S map X into Y such that*
(i) *S is continuous, $S(X)$ resides in a compact subset of Y;*
(ii) *$T : X \longrightarrow Y$ is expansive and onto.*
Then there exists a point $x^\star \in X$ with $Sx^\star + Tx^\star = x^\star$.

Here we will use the following definition for expansive operator.

Definition. (see [3], pp. 3230) Let (X,d) be a metric space and M be a subset of X. The mapping $T : M \longrightarrow X$ is said to be expansive, if there exists a constant $h > 1$ such that

$$d(Tx,Ty) \geq hd(x,y) \quad \forall x,y \in M.$$

Lemma 2.4. *The operator $K_{+1} : N_{+1} \longrightarrow N^*_{+1}$ is an expansive operator and onto.*

Proof. Firstly we will see that $K_{+1} : N_{+1} \longrightarrow N^*_{+1}$. Let $u \in N_{+1}$. Then $u \in \mathcal{C}^2([0,A],\mathcal{C}^2([0,B]))$ and $|u(t,x)| \leq D$, $|u_t(t,x)| \leq D$, $|u_x(t,x)| \leq D$ for every $t \in [0,A]$ and $x \in [0,B]$. From here $K_{+1}(u) = (1+\epsilon)u \in \mathcal{C}^2([0,A],\mathcal{C}^2([0,B]))$ and $|K_{+1}(u)(t,x)| = (1+\epsilon)|u(t,x)| \leq (1+\epsilon)D$, $\left|\frac{\partial}{\partial t}K_{+1}(u)(t,x)\right| = (1+\epsilon)|u_t(t,x)| \leq (1+\epsilon)D$, $\left|\frac{\partial}{\partial x}K_{+1}(u)(t,x)\right| = (1+\epsilon)|u_x(t,x)| \leq (1+\epsilon)D$ for every $t \in [0,A]$ and $x \in [0,B]$. Consequently $K_{+1} : N_{+1} \longrightarrow N^*_{+1}$.

Let now $u, v \in N_{+1}$. Then

$$||K_{+1}(u) - K_{+1}(v)|| = (1+\epsilon)||u-v||,$$

i.e. the operator $K_{+1} : N_{+1} \longrightarrow N^*_{+1}$ is an expansive operator with a constant $h = 1+\epsilon$.

Now we will see that the operator $K_{+1} : N_{+1} \longrightarrow N^*_{+1}$ is onto. Indeed, let $v \in N^*_{+1}$. Then $u = \frac{v}{1+\epsilon} \in N_{+1}$ and $K_{+1}(u)(t,x) = v(t,x)$ for every $t \in [0,A]$ and $x \in [0,B]$. Therefore $K_{+1} : N_{+1} \longrightarrow N^*_{+1}$ is onto.

Lemma 2.5. *The operator $L_{+1} : N_{+1} \longrightarrow N_{+1}$ is a continuous operator.*

Proof. Let $u \in N_{+1}$, from where $|u(t,x)| \leq D$, $|u_t(t,x)| \leq D$, $|u_x(t,x)| \leq D$ for every $t \in [0,A]$ and $x \in [0,B]$, also $|u_0(x)| \leq D$, $|u_1(x)| \leq D$ for every $x \in [0,B]$. From the definition of the operator L_{+1}, for $t \in [0,A]$, $x \in [0,B]$, we have

$$|L_{+1}(u)(t,x)| \leq \epsilon|u(t,x)| + \int_0^x \int_0^\sigma |u(t,y)| dy d\sigma + \int_0^x \int_0^\sigma \Big(|u_0(y)| + t|u_1(y)|\Big) dy d\sigma$$

$$+\int_0^t\int_0^\tau |u(s,x)|dsd\tau + \int_0^t\int_0^\tau\int_0^x\int_0^\sigma |u|^l(s,y)dyd\sigma dsd\tau$$

$$\leq \epsilon D + B^2D(2+A) + A^2D + A^2B^2D^l \leq D,$$

in the last inequality we use the first inequality of (2.1).

For $t\in[0,A]$, $x\in[0,B]$, we have

$$\frac{\partial}{\partial x}L_{+1}(u)(t,x) = -\epsilon u_x(t,x) + \int_0^x u(t,y)dy - \int_0^x \Big(u_0(y)+tu_1(y)\Big)dy$$

$$-\int_0^t\int_0^\tau u_x(s,x)dsd\tau - \int_0^t\int_0^\tau\int_0^x |u|^l(s,y)dydsd\tau$$

and from here, for $t\in[0,A]$ and $x\in[0,B]$, we get

$$\left|\frac{\partial}{\partial x}L_{+1}(u)(t,x)\right| \leq \epsilon|u_x(t,x)| + \int_0^x |u(t,y)|dy + \int_0^x\Big(|u_0(y)|+t|u_1(y)|\Big)dy$$

$$+\int_0^t\int_0^\tau |u_x(s,x)|dsd\tau + \int_0^t\int_0^\tau\int_0^x |u|^l(s,y)dydsd\tau$$

$$\leq \epsilon D + BD(2+A) + A^2D + A^2BD^l \leq D,$$

in the last inequality we use the second inequality of (2.1).

Also, for $t\in[0,A]$, $x\in[0,B]$, we have

$$\frac{\partial}{\partial t}L_{+1}(u)(t,x) = -\epsilon u_t(t,x) + \int_0^x\int_0^\sigma u_t(t,y)dyd\sigma - \int_0^x\int_0^\sigma u_1(y)dyd\sigma$$

$$-\int_0^t u(s,x)ds - \int_0^t\int_0^x\int_0^\sigma |u|^l(s,y)dyd\sigma ds$$

and

$$\left|\frac{\partial}{\partial t}L_{+1}(u)(t,x)\right| \leq \epsilon|u_t(t,x)| + \int_0^x\int_0^\sigma |u_t(t,y)|dyd\sigma + \int_0^x\int_0^\sigma |u_1(y)|dyd\sigma$$

$$+\int_0^t |u(s,x)|ds + \int_0^t\int_0^x\int_0^\sigma |u|^l(s,y)dyd\sigma ds$$

$$\leq \epsilon D + 2B^2D + AD + AB^2D^l \leq D,$$

in the last inequality we use the third inequality of (2.1).

From the above estimates follows that $L_{+1}: N_{+1} \longrightarrow N_{+1}$.

Let now $\{u_n\}$ is a sequence of elements of N_{+1} and $u\in N_{+1}$ and $u_n\longrightarrow u$ when $n\longrightarrow\infty$ in the sense of the topology of the set N_{+1}, i.e. for every $\epsilon_1>0$ there exists $N_1=N_1(\epsilon_1)>0$ so that for every $n>N_1$ and $t\in[0,A]$, $x\in[0,B]$, we have

$$|u_n(t,x)-u(t,x)|<\epsilon_1, |(u_n)_x(t,x)-u_x(t,x)|<\epsilon_1, |(u_n)_t(t,x)-u_t(t,x)|<\epsilon_1.$$

From here, for every $\epsilon_2>0$ there exists $N_2=N_2(\epsilon_2)>0$ so that for every $n>N_2$ and for every $t\in[0,A]$, $x\in[0,B]$, we have $\Big||u_n|^l(t,x)-|u|^l(t,x)\Big|<\epsilon_2$ and

$$|u_n(t,x)-u(t,x)|<\epsilon_2, |(u_n)_x(t,x)-u_x(t,x)|<\epsilon_2, |(u_n)_t(t,x)-u_t(t,x)|<\epsilon_2,$$

$$|L_{+1}(u_n)(t,x)-L_{+1}(u)(t,x)| \leq \epsilon|u_n(t,x)-u(t,x)| + \int_0^x\int_0^\sigma |u_n(t,y)-u(t,y)|dyd\sigma$$

Notes

Notes

$$+\int_0^t\int_0^\tau|u_n(s,x)-u(s,x)|dsd\tau+\int_0^t\int_0^\tau\int_0^x\int_0^\sigma\Big||u_n|^l(s,y)-|u|^l(s,y)\Big|dyd\sigma dsd\tau$$

$$<\epsilon_2\Big(\epsilon+B^2+A^2+A^2B^2\Big),$$

$$\Big|\frac{\partial}{\partial x}L_{+1}(u_n)(t,x)-\frac{\partial}{\partial x}L_{+1}(u)(t,x)\Big|\leq\epsilon|(u_n)_x(t,x)-u_x(t,x)|+\int_0^x|u_n(t,y)-u(t,y)|dy$$

$$+\int_0^t\int_0^\tau|(u_n)_x(s,x)-u_x(s,x)|dsd\tau+\int_0^t\int_0^\tau\int_0^x\Big||u_n|^l(s,y)-|u|^l(s,y)\Big|dydsd\tau$$

$$<\epsilon_2\Big(\epsilon+B+A^2+A^2B\Big),$$

$$\Big|\frac{\partial}{\partial t}L_{+1}(u_n)(t,x)-\frac{\partial}{\partial t}L_{+1}(u)(t,x)\Big|\leq\epsilon|(u_n)_t(t,x)-u_t(t,x)|+\int_0^x\int_0^\sigma|(u_n)_t(t,y)-u_t(t,y)|dyd\sigma$$

$$+\int_0^t|u_n(s,x)-u(s,x)|ds+\int_0^t\int_0^x\int_0^\sigma\Big||u_n|^l(s,y)-|u|^l(s,y)\Big|dyd\sigma ds$$

$$<\epsilon_2\Big(\epsilon+B^2+A+AB^2\Big),$$

Therefore $L_{+1}(u_n)\longrightarrow L_{+1}(u)$ when $n\longrightarrow\infty$ in the sense of the topology of the space N_{+1}, i.e. the operator $L_{+1}:N_{+1}\longrightarrow N_{+1}$ is a continuous operator.

Using Lemma 2.1, Lemma 2.4, Lemma 2.5 we apply Theorem 2.3 as the operator T in Theorem 2.3 corresponds of the operator K_{+1}, the operator S in Theorem 2.3 corresponds of L_{+1}, the set X in Theorem 2.3 corresponds of N_{+1}, Y in Theorem 2.3 corresponds of N^*_{+1} and follows that the operator P_{+1} has a fixed point $u^{+1}\in N_{+1}$. From here and from Lemma 2.2 follows that u^{+1} is a solution to the Cauchy problem (2.2), (2.3).

III. Proof of Theorem 1.2

In the previous section we prove that if the positive constants A and B satisfy the conditions (2.1) then the Cauchy problem

$$u_{tt}-u_{xx}=|u|^l,\quad t\in[0,A],x\in[0,B],$$

$$u(0,x)=u_0(x),u_t(0,x)=u_1(x),\quad x\in[0,B],$$

has a solution $u^{+1}\in\mathcal{C}^2([0,A],\mathcal{C}^2([0,B]))$.

Let A and B be the same constants as in the Section 2. We consider the Cauchy problem

$$\begin{aligned}&u_{tt}-u_{xx}=|u|^l,\quad t\in[0,A],\quad x\in[B,2B],\\&u(0,x)=u_0(x),u_t(0,x)=u_1(x),\quad x\in[B,2B].\end{aligned}\tag{3.1}$$

We define the sets

$$N_{+2}=\Big\{u\in\mathcal{C}^2([0,A],\mathcal{C}^2([B,2B])):|u(t,x)|\leq D,|u_t(t,x)|\leq D,|u_x(t,x)|\leq D$$

$$\forall t\in[0,A],\quad\forall x\in[B,2B]\Big\},$$

$$N^*_{+2} = \Big\{u \in \mathcal{C}^2([0,A], \mathcal{C}^2([B,2B])) : |u(t,x)| \leq (1+\epsilon)D, |u_t(t,x)| \leq (1+\epsilon)D,$$

$$|u_x(t,x)| \leq (1+\epsilon)D \quad \forall t \in [0,A], \quad \forall x \in [B,2B]\Big\},$$

in these sets we define a norm as follows

$$||u||_1 = \max_{t\in[0,A], x\in[B,2B]} |u(t,x)|,$$

in this way the sets N_{+2} and N^*_{+2} are closed, convex and compact sets in $\mathcal{C}([0,A]\times[B,2B])$. For $u \in N^*_{+2}$ we define the operators

$$K_{+2}(u)(t,x) = (1+\epsilon)u(t,x),$$

$$L_{+2}(u)(t,x) = -\epsilon u(t,x) + \int_B^x \int_B^\sigma u(t,y)dyd\sigma - \int_B^x \int_B^\sigma \Big(u_0(y) + tu_1(y)\Big)dyd\sigma$$

$$-\int_0^t \int_0^\tau \Big(u(s,x) - u^{+1}(s,B) - (x-B)u_x^{+1}(s,B)\Big)dsd\tau - \int_0^t \int_0^\tau \int_B^x \int_B^\sigma |u|^l(s,y)dyd\sigma dsd\tau,$$

$$P_{+2}(u)(t,x) = K_{+2}(u)(t,x) + L_{+2}(u)(t,x).$$

As in the Section 2 we prove that the Cauchy problem (3.1) has a solution $u^{+2} \in \mathcal{C}^2([0,A], \mathcal{C}^2([B,2B]))$ for which we have, for $t \in [0,A]$, $x \in [B,2B]$,

$$0 = \int_B^x \int_B^\sigma u^{+2}(t,y)dyd\sigma - \int_B^x \int_B^\sigma \Big(u_0(y) + tu_1(y)\Big)dyd\sigma$$

$$-\int_0^t \int_0^\tau \Big(u^{+2}(s,x) - u^{+1}(s,B) - (x-B)u_x^{+1}(s,B)\Big)dsd\tau - \int_0^t \int_0^\tau \int_B^x \int_B^\sigma \Big|u^{+2}\Big|^l(s,y)dyd\sigma dsd\tau \tag{3.2}$$

Now we put $x = B$ in (3.2) and we obtain

$$0 = \int_0^t \int_0^\tau \Big(u^{+2}(s,B) - u^{+1}(s,B)\Big)dsd\tau, \quad t \in [0,A],$$

after we differentiate twice in t the last equality we get

$$u^{+2}(t,B) = u^{+1}(t,B), \quad t \in [0,A]. \tag{3.3}$$

Now we differentiate in x the equality (3.2), after which we put $x = B$ and we find

$$0 = \int_0^t \int_0^\tau \Big(u_x^{+2}(s,B) - u_x^{+1}(s,B)\Big)dsd\tau, \quad t \in [0,A],$$

after we differentiate the last equality twice in t we obtain

$$u_x^{+2}(t,B) = u_x^{+1}(t,B), \quad t \in [0,A].$$

From (3.3) we have

$$u_t^{+1}(t,B) = u_t^{+2}(t,B), u_{tt}^{+1}(t,B) = u_{tt}^{+2}(t,B), \quad t \in [0,A].$$

From here, from (3.3) and from

$$u_{tt}^{+2}(t,B) - u_{xx}^{+2}(t,B) = \Big|u^{+2}\Big|^l(t,B), \quad t \in [0,A],$$

Notes

$$u_{tt}^{+1}(t,B)-u_{xx}^{+1}(t,B)=\Big|u^{+1}\Big|^{l}(t,B),\quad t\in[0,A],$$

we conclude that

$$u_{xx}^{+2}(t,B)=u_{xx}^{+1}(t,B),\quad t\in[0,A].$$

Notes

Consequently the function

$$\tilde{u}=\begin{cases}u^{+1} & t\in[0,A],x\in[0,B],\\ u^{+2} & t\in[0,A],x\in[B,2B],\end{cases}$$

is a solution to the Cauchy problem

$$u_{tt}-u_{xx}=|u|^{l},\quad t\in[0,A],x\in[0,2B],$$

$$u(0,x)=u_0(x),u_t(0,x)=u_1(x),\quad x\in[0,2B],$$

which belongs in the space $\mathcal{C}^2([0,A],\mathcal{C}^2([0,2B]))$.
Now consider the Cauchy problem

$$u_{tt}-u_{xx}=|u|^{l},\quad t\in[0,A],\quad x\in[2B,3B],$$

$$u(0,x)=u_0(x),u_t(0,x)=u_1(x),\quad x\in[2B,3B].$$

We define the sets

$$N_{+3}=\Big\{u\in\mathcal{C}^2([0,A],\mathcal{C}^2([2B,3B])):|u(t,x)|\le D,|u_t(t,x)|\le D,|u_x(t,x)|\le D$$

$$\forall t\in[0,A],\quad \forall x\in[2B,3B]\Big\},$$

$$N_{+3}^{*}=\Big\{u\in\mathcal{C}^2([0,A],\mathcal{C}^2([2B,3B])):|u(t,x)|\le(1+\epsilon)D,|u_t(t,x)|\le(1+\epsilon)D,$$

$$|u_x(t,x)|\le(1+\epsilon)D\quad\forall t\in[0,A],\quad\forall x\in[2B,3B]\Big\},$$

in these sets we define a norm as follows

$$||u||_1=\max_{t\in[0,A],x\in[2B,3B]}|u(t,x)|,$$

in this way the sets N_{+3} and N_{+3}^{*} are are closed, convex and compact sets in $\mathcal{C}([0,A]\times[2B,3B])$.

For $u\in N_{+3}^{*}$ we define the operators

$$K_{+3}(u)(t,x)=(1+\epsilon)u(t,x),$$

$$L_{+3}(u)(t,x)=-\epsilon u(t,x)+\int_{2B}^{x}\int_{2B}^{\sigma}u(t,y)dyd\sigma-\int_{2B}^{x}\int_{2B}^{\sigma}\Big(u_0(y)+tu_1(y)\Big)dyd\sigma$$

$$-\int_0^t\int_0^\tau\Big(u(s,x)-u^{+2}(s,2B)-(x-2B)u^{+2}(s,2B)\Big)dsd\tau-\int_0^t\int_0^\tau\int_{2B}^{x}\int_{2B}^{\sigma}|u|^{l}(s,y)dyd\sigma dsd\tau,$$

$$P_{+3}(u)(t,x)=K_{+3}(u)(t,x)+L_{+3}(u)(t,x).$$

And etc.

The function

$$u^+ = \begin{cases} u^{+1} & t\in[0,A], x\in[0,B],\\ u^{+2} & t\in[0,A], x\in[B,2B],\\ u^{+3} & t\in[0,A], x\in[2B,3B],\\ \dots \end{cases}$$

is a solution to the Cauchy problem

$$u_{tt}-u_{xx}=|u|^l,\quad t\in[0,A], x\in[0,\infty),$$
$$u(0,x)=u_0(x), u_t(0,x)=u_1(x),\quad x\in[0,\infty),$$

which belongs to the space $\mathcal{C}^2([0,A],\mathcal{C}^2([0,\infty)))$.

IV. Proof of Theorem 1.3

Let A and B are the same constants as in the Section 2. Now consider the Cauchy problem

$$\begin{aligned} &u_{tt}-u_{xx}=|u|^l,\quad t\in[0,A],\quad x\in[-B,0],\\ &u(0,x)=u_0(x), u_t(0,x)=u_1(x),\quad x\in[-B,0]. \end{aligned}\tag{4.1}$$

We define the sets

$$N_{-1}=\Big\{u\in\mathcal{C}^2([0,A],\mathcal{C}^2([-B,0])):|u(t,x)|\le D, |u_t(t,x)|\le D, |u_x(t,x)|\le D$$
$$\forall t\in[0,A],\quad \forall x\in[-B,0]\Big\},$$
$$N^*_{-1}=\Big\{u\in\mathcal{C}^2([0,A],\mathcal{C}^2([-B,0])):|u(t,x)|\le(1+\epsilon)D, |u_t(t,x)|\le(1+\epsilon)D,$$
$$|u_x(t,x)|\le(1+\epsilon)D\quad \forall t\in[0,A],\quad \forall x\in[-B,0]\Big\},$$

in these sets we define a norm as follows

$$||u||=\max_{t\in[0,A],x\in[-B,0]}|u(t,x)|,$$

in this way the sets N_{-1} and N^*_{-1} are are closed, convex and compact sets in $\mathcal{C}([0,A]\times[-B,0])$.

For $u\in N^*_{-1}$ we define the operators

$$K_{-1}(u)(t,x)=(1+\epsilon)u(t,x),$$

$$L_{-1}(u)(t,x)=-\epsilon u(t,x)+\int_x^0\int_\sigma^0 u(t,y)dyd\sigma-\int_x^0\int_\sigma^0\Big(u_0(y)+tu_1(y)\Big)dyd\sigma$$

$$-\int_0^t\int_0^\tau\Big(u(s,x)-u^+(s,0)-xu^+_x(s,0)\Big)dsd\tau-\int_0^t\int_0^\tau\int_x^0\int_\sigma^0|u|^l(s,y)dyd\sigma dsd\tau,$$

$$P_{-1}(u)(t,x) = K_{-1}(u)(t,x) + L_{-1}(u)(t,x).$$

As in the Section 2 and in the Section 3 we prove that the Cauchy problem (4.1) has a solution $u^{-1} \in \mathcal{C}^2([0,A], \mathcal{C}^2([-B,0]))$. And etc.
The function

$$u^- = \begin{cases} u^{-1} & t \in [0,A], x \in [-B,0], \\ u^{-2} & t \in [0,A], x \in [-2B,-B], \\ \dots \end{cases}$$

is a solution to the Cauchy problem

$$u_{tt} - u_{xx} = |u|^l, \quad t \in [0,A], x \in (-\infty,0],$$

$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), \quad x \in (-\infty,0],$$

which belongs to the space $\mathcal{C}^2([0,A], \mathcal{C}^2(-\infty,0])$, and the function

$$u = \begin{cases} u^+ & t \in [0,A], x \in [0,\infty), \\ u^- & t \in [0,A], x \in (-\infty,0], \end{cases}$$

is a solution to the Cauchy problem (1.1), (1.2) which belongs to the space $\mathcal{C}^2([0,A], \mathcal{C}^2(\mathbb{R}))$.

Notes

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