Note on Certain Field of Fractions

By S. Usaini & S. M. Tudunkaya

Kano University of Science and Technology, Nigeria

Abstract - The set of some real rhotrices of the same dimension $D^*$ was defined in [2] to be an integral domain. An example of a finite field $M[R_3]$ was given in [4] based on this definition also and on the construction of finite fields presented in [3]. It was discovered that the finite subcollection of the elements of $M[R_3]$ as contained in $D^*$ is not closed under rhotrix addition and hence not an integral domain. More generally, $D^*$ is not an integral domain as it is not closed under rhotrix addition. This problem affects the field of fractions constructed in [8]. A solution to this problem is provided in this article and the construction method of such fields is reviewed. This reviewed version gives the generalization of such construction as the n-dimensional rhotrices are used.

Keywords : n-dimensional rhotrix; Quotient rhotrix; Integral domain; Field of fraction.

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Note On Certain Field of Fractions

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1. Introduction

The idea of classifying the set of all rhotrices of dimension 3 as abstract structures was presented in [1] and [2]. In [2] the set of some 3-dimensional real rhotrices

$$D^* = \langle (R - ZD), +, \circ \rangle \quad (1.1)$$

was defined to be an integral domain under rhotrix addition and multiplication, where $R$ is the set of all real rhotrices of dimension 3 as defined in [6] by

$$R = \left\{ \begin{bmatrix} a & b & c & d \\ e & 0 & 0 & 0 \end{bmatrix} : a, b, c, d, e \in \mathbb{R} \right\},$$

$$ZD = \left\{ \begin{bmatrix} a & b & d & e \\ -b & 0 & 0 & 0 \end{bmatrix} : a, b, d, e, 0 \in \mathbb{R} \text{ and at least one of } a, b, d, e \neq 0 \right\}.$$

Recall that an integral domain is a commutative ring with out zero divisors. However, $D^*$ is not even a ring because the additive closure is not there. This can be seen as follows: Let $R, Q \in D^*$ such that $h(R) = c$ and $h(Q) = -c$. If $R + Q = S$ then $h(S) = 0$ and at least one of $a, b, d, e \neq 0$ which implies that $S \notin D^*$. 

Author α α : Department of Mathematics, Kano University of Science and Technology, Wudil, P.M.B. 3244 Kano, Nigeria.
E-mails: kunyasco@yahoo.com, tudunkayaunique@yahoo.com

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A method of constructing finite fields through the use of rhotrices was presented in [3]. The cardinalities of such fields were also provided. In [4] an example of a particular finite field $M[R_3]$ whose cardinality is $|M[R_3]| = 3^5 = 243$ was given. This field gave the clear picture of the problem in $D^*$.

In [1] a hearty rhotrix of dimension 3 was defined to be a rhotrix $S$ having all its entries (except the heart) as zero. Thus, an integer hearty rhotrix of dimension 3 is of the form $\begin{pmatrix} 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $c \neq 0 \in \mathbb{Z}$. Therefore we may define the set of all integer hearty rhotrices of the same dimension as

$$H = \left\{ \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix} : c \neq 0 \in \mathbb{Z} \right\}$$  \hspace{1cm} (1.2)

For any two base rhotrices $R, S$ the quotient $\frac{R}{S}$ is defined in [5] to be the quotient rhotrix as follows:

$$\frac{R}{S} = R \circ S^{-1}, \text{ provided } h(S) \neq 0.$$

(1.3)

The concept of n-dimensional heart-oriented rhotrix multiplication was introduced in [7]. A rhotrix $R_n$ of dimension $n$ have $|R_n|$ entries where $|R_n| = \frac{1}{2}(n^2 + 1)$ as indicated in [6]. Thus any given rhotrix $R_n$ with entries $a_1, a_2, \ldots, a_{\frac{1}{2}(n^2 + 1)}$, is generally represented as

$$R_n = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{\frac{1}{2}(n^2 + 1)} & a_{\frac{1}{2}(n^2 + 3)} & a_{\frac{1}{2}(n^2 + 5)} & \ldots & a_{\frac{1}{2}(n^2 + 1)} \end{pmatrix}$$

The method of constructing field of fractions through the use of base rhotrices based on definitions (1.1) and (1.3) was presented in [8]. To take care of the problem that affects this construction as pointed out earlier; this article aims at reviewing and generalizing the construction method using definitions (1.2) and (1.3).
II. A Particular Field of Fractions of an Integral Domain

**Theorem 2.1**

Let $H_n$ be the set of all $n$-dimensional integer hearty rhotrices. If $H_n^* = H_n \cup \{O_n\}$ then $H_n^*$ is an integral domain, where $O_n$ is an $n$-dimensional zero rhotrix.

**Proof**

It suffices to show that $H_n^*$ is isomorphic to an integral domain $\mathbb{Z}$. That is $\cong H_n^* \mathbb{Z}$.

Define a mapping $\tau : \mathbb{Z} \rightarrow H_n^*$ by $\tau(c) = C_n$.

For homomorphism, let $c, d \in \mathbb{Z}$, then

(i) $\tau(c + d) = C_n + D_n = \tau(c) + \tau(d)$  
(ii) $\tau(cd) = C_n \circ D_n = \tau(c)\tau(d)$

Therefore $\tau$ is a homomorphism.

Since $\forall \tau(c) \in H_n^*$ there exists $c \in \mathbb{Z}$ such that $\tau(c) = C_n$ then $\tau$ is onto.

Now let $\tau(c), \tau(d) \in H_n^*$ such that $\tau(c) = \tau(d)$.

$\tau(c) = \tau(d) \Rightarrow C_n = D_n \Rightarrow c = d$.

Thus $\tau$ is one to one.

Hence $H_n^* \cong \mathbb{Z}$.

**Definition 2.1**

Let $H_n^*$ and $H_n$ be as in Theorem 2.1 above. Then a relation $\sim$ on $H_n^* \times H_n$ defined by cross multiplication as $(C_{n_1}, D_{n_1}) \sim (C_{n_2}, D_{n_2})$ if $C_{n_1} \circ D_{n_2} = C_{n_2} \circ D_{n_1}$, $\exists C_{n_1}, C_{n_2} \in H_n^*$; $D_{n_1}, D_{n_2} \in H_n$.

**Proposition 2.1**

The relation $\sim$ as defined in Definition (2.1) is an equivalence relation.

**Proof**

Reflexivity and Symmetry of the relation are obvious.

For transitivity, let $C_{n_1} \circ D_{n_2} = C_{n_2} \circ D_{n_1}$ and $C_{n_2} \circ D_{n_3} = C_{n_3} \circ D_{n_2}$. To show that $C_{n_1} \circ D_{n_3} = C_{n_3} \circ D_{n_1}$ we have

$$
(C_{n_1} \circ D_{n_2}) \circ D_{n_2} = (C_{n_1} \circ D_{n_2}) \circ D_{n_3}
$$

$$
= (C_{n_2} \circ D_{n_1}) \circ D_{n_3}
$$

$$
= (C_{n_2} \circ D_{n_3}) \circ D_{n_1}
$$

$$
= (C_{n_3} \circ D_{n_2}) \circ D_{n_1}
$$

$$
= (C_{n_3} \circ D_{n_1}) \circ D_{n_2}
$$

$\therefore (C_{n_1} \circ D_{n_3}) = (C_{n_3} \circ D_{n_1})$ by cancellation law.

We denote by $\frac{C_n}{D_n}$ the equivalence class of $(C_n, D_n)$ in $H_n^* \times H_n$ and define $H_n^*[H_n^{-1}]$ to be the set of all the equivalence classes $\frac{C_n}{D_n}$, where $C_n \in H_n^*$ and $D_n \in H_n^*$. 

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For all \( \frac{C_{n1}}{D_{n1}}, \frac{C_{n2}}{D_{n2}} \in H_n^*[H_n^{-1}] \) we define addition and multiplication on \( H_n^*[H_n^{-1}] \) as follows:

\[
\frac{C_{n1}}{D_{n1}} + \frac{C_{n2}}{D_{n2}} = \frac{C_{n1} \circ D_{n2} + C_{n2} \circ D_{n1}}{D_{n1} \circ D_{n2}} \quad \text{and} \quad \frac{C_{n1}}{D_{n1}} \cdot \frac{C_{n2}}{D_{n2}} = \frac{C_{n1} \circ C_{n2}}{D_{n1} \circ D_{n2}}.
\]

**Proposition 2.2**

The operations \((+')\), \((\cdot')\) as defined above are well-defined.

**Proof**

Suppose \( \frac{C_{n1}'}{D_{n1}'} = \frac{C_{n1}}{D_{n1}} \) and \( \frac{C_{n2}'}{D_{n2}'} = \frac{C_{n2}}{D_{n2}} \); then \( C_{n1}' \circ D_{n1} = D_{n1}' \circ C_{n1} \) and \( C_{n2}' \circ D_{n2} = D_{n2}' \circ C_{n2} \), so that\( (C_{n1}' \circ D_{n2}' + C_{n2}' \circ D_{n1}')D_{n1}D_{n2} = C_{n1}' \circ D_{n1} \circ D_{n2}' \circ D_{n2} + C_{n2}' \circ D_{n2} \circ D_{n1}' \circ D_{n1} \)

\[= C_{n1} \circ D_{n1}' \circ D_{n2}' \circ D_{n2} + C_{n2} \circ D_{n2}' \circ D_{n1}' \circ D_{n1} \]

\[= (C_{n1} \circ D_{n2} + C_{n2} \circ D_{n1})D_{n1}' \circ D_{n2}' \]

implying that \( \frac{C_{n1}'}{D_{n1}'} + \frac{C_{n2}'}{D_{n2}'} = \frac{C_{n1}}{D_{n1}} + \frac{C_{n2}}{D_{n2}} \).

Similarly \((C_{n1}' \circ C_{n2}')D_{n1} \circ D_{n2} = (C_{n1} \circ C_{n2})D_{n1}' \circ D_{n2}' \) implies that \( \frac{C_{n1}'}{D_{n1}'} \circ \frac{C_{n2}'}{D_{n2}'} = \frac{C_{n1} \circ C_{n2}}{D_{n1} \circ D_{n2}} \).

By definition (1.3) the equivalence class \( \frac{C_n}{D_n} = C_n \circ D_n^{-1} \) since \( D_n \neq 0_n \in H_n \). Therefore, for all \( D_n \in H_n \), \( 0_n \in H_n^* \), \( \frac{0_n}{D_n} = 0_n \circ D_n^{-1} = 0_n = 0_n \circ I_n = 0_n \frac{I_n}{D_n} \). Thus \( \frac{0_n}{I_n} = \frac{D_n}{I_n} \) is the additive identity and \( -\frac{C_n}{D_n} = \frac{-C_n}{D_n} \) is the additive inverse. Similarly, \( \frac{I_n}{I_n} = \frac{D_n}{I_n} \) is the multiplicative identity.

**Theorem 2.2**

With the above definitions and the definitions of the operations \((+')\) and \((\cdot')\), the set of the equivalence classes \( H_n^*[H_n^{-1}] \) is a commutative ring.

**Proof**

One should check that the properties of a ring are fulfilled. But the proof follows from the fact that addition and multiplication are the regular addition and multiplication of fractions.

**Proposition 2.3**

The function \( \psi : H_n^* \to H_n^*[H_n^{-1}] \) defined by \( \psi(C_n) = \frac{C_n}{I_n} \) is a ring homomorphism whose kernel is \( \{ C_n \in H_n^* : C_n \circ' D_n = 0 \text{ for some } D_n \in H_n^*[H_n^{-1}] \} \).
Proof

Let \( C_{n1}, C_{n2} \in H_n^* \), then

\[
\psi(C_{n1} + C_{n2}) = \frac{C_{n1} + C_{n2}}{I_n} = (C_{n1} + C_{n2}) \circ I_n = C_{n1} \circ I_n + C_{n2} \circ I_n = \frac{C_{n1}}{I_n} + \frac{C_{n2}}{I_n} = \psi(C_{n1}) + \psi(C_{n2})
\]

\[
\psi(C_{n1} \circ C_{n2}) = \frac{C_{n1} \circ C_{n2}}{I_n} = (C_{n1} \circ C_{n2}) \circ I_n = C_{n1} \circ I_n \circ C_{n2} \circ I_n = \frac{C_{n1}}{I_n} \circ \frac{C_{n2}}{I_n} = \psi(C_{n1}) \circ \psi(C_{n2})
\]

\[
\psi(I_n) = \frac{I_n}{I_n}.
\]

Now \( C_n \in \ker \psi \) if and only if \( \frac{C_n}{I_n} = \frac{0}{I_n} \), if and only if \( C_n \circ I_n = 0 \circ I_n = 0_n \), which imply that \( \ker \psi = \{0_n\} \).

Recall from [1] that, the set \( M = \{ nI : n \in \mathbb{Z} \} \) where \( I \) is the unity element of the commutative ring of 3-dimensional rhotrices \( R \) is a subring and submonoid of \( R \) under multiplication (\( \circ \)). Thus the set \( M_n = \{ nI_n : n \in \mathbb{Z} \} \) is a subring and submonoid of the commutative ring \( R_n^* \) of n-dimensional rhotrices. Therefore any submonoid, \( H_n \) of \( R_n^* \) with property that for all \( Q_n \neq 0 \in R_n^* \) and \( S_n \in H_n \), \( Q_n \circ S_n \neq 0 \) can serve in the above construction for the generalization of proposition 2.3 as stated in the following proposition.

**Proposition 2.4**

\( R_n^*[H_n^{-1}] \) as constructed above is a ring, and there is a homomorphism \( \psi : R_n^* \to R_n^*[H_n^{-1}] \) given by \( \psi(Q_n) = \frac{Q_n}{I_n} \).

**Proof**

The proof follows from propositions 2.2 and 2.3.

As defined in [1], a diagonal rhotrix of dimension 3 is a rhotrix whose two non-diagonal entries are all zero. Let \( D \) be the set of all n-dimensional diagonal rhotrices then it is easy for someone to verify that \( W = D \cup \{ O_n, I_n \} \), where \( O_n \) is the n-dimensional additive identity; \( I_n \) is the n-dimensional multiplicative identity is a group and is normal in \( R_n^* \) under multiplication.

**Proposition 2.5**

Let \( R_n^* \) be a commutative ring of n-dimensional rhotrices, and let \( H_n \) be a submonoid of \( R_n^* \) such that \( Q_n \circ S_n \neq 0_n \) for every \( Q_n \neq 0_n \in R_n^* \) and \( S_n \in H_n \). Then every ideal of \( R_n^*[H_n^{-1}] \) has the form \( W[H_n^{-1}] \), for suitable \( W \) normal in \( R_n^* \).

**Proof**

Since all the elements of \( W[H_n^{-1}] \) are also elements of \( R_n^*[H_n^{-1}] \) and \( I_n \in H_n \) then obviously \( W[H_n^{-1}] \) is an additive subgroup of \( R_n^*[H_n^{-1}] \).
For all \( \frac{C_n}{D_n} \in R_n^*[H_n^{-1}] \); \( \frac{W_n}{D_n'} \in W[H_n^{-1}] \); \( \frac{C_n \circ W_n}{D_n \circ D'_n} \in W[H_n^{-1}] \) since \( C_n \circ W_n \in W, D_n \circ D'_n \in H_n \).

\( W \) is normal in \( R_n^* \) implies that \( \frac{W_n}{D_n'} \circ \frac{C_n}{D_n} \in W[H_n^{-1}] \).

**Proposition 2.6**

\( H_n^*[H_n^{-1}] \) is an integral domain.

**Proof**

Suppose \( \frac{C_{n_1}}{D_{n_1}} \circ \frac{C_{n_2}}{D_{n_2}} = 0_n \in H_n^*[H_n^{-1}] \), that is \( \frac{C_{n_1} \circ C_{n_2}}{D_{n_1} \circ D_{n_2}} = 0_n \)

\( \Rightarrow (C_{n_1} \circ C_{n_2}, D_{n_1} \circ D_{n_2}) - (0_n, I_n) \) and \( C_{n_1} \circ C_{n_2} \circ D_n = 0 \) for some \( D_n \in H_n \).

\( C_{n_1} \circ C_{n_2} \circ D_n = 0_n \in H_n^* \), which is an integral domain, and \( D_n \neq 0_n \), thus \( C_{n_1} \circ C_{n_2} = 0_n \).

So either \( C_{n_1} \) or \( C_{n_2} \) is \( 0_n \) and consequently either \( \frac{C_{n_1}}{D_{n_1}} \) or \( \frac{C_{n_2}}{D_{n_2}} \) is \( 0_n \).

**Theorem 2.3**

The set \( H_n^*[H_n^{-1}] \) of all equivalence classes \( \frac{C_n}{D_n} \) is a field.

**Proof**

From Theorem 2.2, \( H_n^*[H_n^{-1}] \) is a commutative ring with unity \( \frac{I_n}{I_n} \). So we just need to show that every non zero element of \( H_n^*[H_n^{-1}] \) has multiplicative inverse.

Suppose \( \frac{C_n}{D_n} \neq 0_n \), then \( C_n \neq 0_n \), so \( C_n \in H_n \) which implies that \( \frac{C_n}{D_n} \in H_n^*[H_n^{-1}] \).

Clearly, \( \frac{C_n}{D_n} \circ \frac{D_n}{C_n} = \frac{C_n \circ D_n}{D_n \circ C_n} = \frac{I_n}{I_n} \). Thus \( \frac{D_n}{C_n} \) is the multiplicative inverse of \( \frac{C_n}{D_n} \).

### III. Conclusion

In this short note, amendment concerning some definitions in [2] and [8] with their generalizations were provided. The steps observed in the construction of field of fractions illustrated in [8] were also amended respectively.

**References Références Referencias**


