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Note Oncertain Field of Fractions

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Abstract - The set of some real rhotrices of the same dimension D^* was defined in [2] to be an integral domain. An example of a finite field $M[R_3]$ was given in [4] based on this definition also and on the construction of finite fields presented in [3]. It was discovered that the finite sub collection of the elements of $M[R_3]$ as contained in D^* is not closed under rhotrix addition and hence not an integral domain. More generally, D^* is not an integral domain as it is not closed under rhotrix addition. This problem affects the field of fractions constructed in [8]. A solution to this problem is provided in this article and the construction method of such fields is reviewed. This reviewed version gives the generalization of such construction as the n-dimensional rhotrices are used.

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NOTE ONCERTAIN FIELD OF FRACTIONS

Strictly as per the compliance and regulations of :

Note Oncertain Field of Fractions

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I. Introduction

The idea of classifying the set of all rhotrices of dimension 3 as abstract structures was presented in [1] and [2].In [2] the set of some 3-dimensional real rhotrices

$$D^* = \langle (R - ZD), +, \circ \rangle \qquad (1.1)$$

was defined to be an integral domain under rhotrix addition and multiplication, where R is the set of all real rhotrices of dimension 3 as defined in [6] by

$$R = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \in \Re \right\},$$

$$ZD = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & 0 & d \\ & e & \end{array} \right\rangle : a, b, d, e, 0 \in \Re \text{ and at least one of } a, b, d, e \neq 0 \right\}.$$

Recall that an integral domain is a commutative ring with out zero divisors. However, D^* is not even a ring because the additive closure is not there. This can be seen as follows: Let $R, Q \in D^*$ such that $h(R) = c$ and $h(Q) = -c$. If $R + Q = S$ then $h(S) = 0$ and at least one of $a, b, d, e \neq 0$ which implies that $S \notin D^*$.

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Ref.

1. Mohammed (2007a). Enrichment exercises through extension to rhotrices. Int. J. Math. Educ. Sci. Technol.38(2007b), 131-136.

A method of constructing finite fields through the use of rhotrices was presented in [3]. The cardinalities of such fields were also provided. In [4] an example of a particular finite field $M[R_3]$ whose cardinality is $|M[R_3]| = 3^5 = 243$ was given. This field gave the clear picture of the problem in D^*.

In [1] a hearty rhotrix of dimension 3 was defined to be a rhotrix S having all its entries(except the heart) as zero. Thus, an integer hearty rhotrix of dimension 3 is of the form $\left\langle \begin{array}{ccc} & 0 & \\ 0 & c & 0 \\ & 0 & \end{array} \right\rangle$, $c \neq 0 \in \mathbb{Z}$. Therefore we may define the set of all integer hearty rhotrices of the same dimension as

$$H = \left\{ \left\langle \begin{array}{ccc} & 0 & \\ 0 & c & 0 \\ & 0 & \end{array} \right\rangle : c \neq 0 \in \mathbb{Z} \right\} \tag{1.2}$$

For any two base rhotrices R, S the quotient $\frac{R}{S}$ is defined in [5] to be the quotient rhotrix as follows:

$$\frac{R}{S} = R \circ S^{-1} \text{ , provided } h(S) \neq 0 . \tag{1.3}$$

The concept of n-dimensional heart-oriented rhotrix multiplicationwas introduced in [7]. A rhotrix $R_n$of dimension n have $|R_n|$entries where $|R_n| = \frac{1}{2}(n^2+1)$as indicated in [6]. Thus any given rhotrix R_n with entries $a_1, a_2, \ldots, a_{\frac{1}{2}(n^2+1)}$, is generally represented as

$$R_n = \left\langle \begin{array}{ccccccccc}
 & & & & a_1 & & & & \\
 & & & a_2 & a_3 & a_4 & & & \\
 & & a_5 & a_6 & a_7 & a_8 & a_9 & & \\
 & \cdots\ \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & \\
a_{\frac{n^2-2n+5}{4}} & \cdots\ \cdots & & \cdots & a_{\frac{1}{4}(n^2+3)} & \cdots & \cdots & \cdots & a_{\frac{n^2+2n+1}{4}} \\
 & \cdots\ \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & a_{\frac{1}{2}(n^2+1)-8} & a_{\frac{1}{2}(n^2+1)-7} & a_{\frac{1}{2}(n^2+1)-6} & a_{\frac{1}{2}(n^2+1)-5} & a_{\frac{1}{2}(n^2+1)-4} & & \\
 & & & a_{\frac{1}{2}(n^2+1)-3} & a_{\frac{1}{2}(n^2+1)-2} & a_{\frac{1}{2}(n^2+1)-1} & & & \\
 & & & & a_{\frac{1}{2}(n^2+1)} & & & &
\end{array} \right\rangle$$

The method of constructing field of fractions through the use of base rhotrices based on definitions (1.1) and (1.3) was presented in [8]. To take care of the problem that affects this construction as pointed out earlier; this article aims at reviewing and generalizing the construction method using definitions (1.2) and (1.3).

Ref.

3. S.M. Tudunkaya and S.O. Makanjuola (2010). Rhotrices and the construction of finite fields. Bulletin of Pure and Applied Sciences, Vol. 29 E Issue 2, pp225-229

II. A Particular Field of Fraction of an Integral Domain

Theorem 2.1

Let H_n be the set of all n-dimensional integer hearty rhotrices. If $H_n^* = H_n \cup \{O_n\}$ then H_n^* is an integral domain, where O_n is an n-dimensional zero rhotrix.

Proof

Notes

It suffices to show that H_n^* is isomorphic to an integral domain $\mathbb{Z}$. That is $H_n^* \cong \mathbb{Z}$.

Define a mapping $\tau : \mathbb{Z} \to H_n^*$ by $\tau(c) = C_n$.

For homomorphism, let $c, d \in \mathbb{Z}$, then

(i) $\tau(c+d) = C_n + D_n = \tau(c) + \tau(d)$ (ii) $\tau(cd) = C_n \circ D_n = \tau(c)\tau(d)$

Therefore τ is a homomorphism

Since $\forall\, \tau(c) \in H_n^*$ there exists $c \in \mathbb{Z}$ such that $\tau(c) = C_n$ then τ is onto.

Now let $\tau(c), \tau(d) \in H_n^*$ such that $\tau(c) = \tau(d)$.

$\tau(c) = \tau(d) \Rightarrow C_n = D_n \Rightarrow c = d$.

Thus τ is one to one.

Hence $H_n^* \cong \mathbb{Z}$.

Definition 2.1

Let H_n^* and H_n be as in Theorem 2.1 above. Then a relation ~ on H_n^*x H_n defined by cross multiplication as $(C_{n1}, D_{n1}) \sim (C_{n2}, D_{n2})$ if $C_{n1} \circ D_{n2} = C_{n2} \circ D_{n1}$, $\ni C_{n1}, C_{n2} \in H_n^*$; $D_{n1}, D_{n2} \in H_n$.

Proposition 2.1

The relation ~ as defined in Definition (2.1) is an equivalence relation.

Proof

Reflexivity and Symmetry of the relation are obvious.

For transitivity, let $C_{n1} \circ D_{n2} = C_{n2} \circ D_{n1}$ and $C_{n2} \circ D_{n3} = C_{n3} \circ D_{n2}$. To show that $C_{n1} \circ D_{n3} = C_{n3} \circ D_{n1}$ we have

$$(C_{n1} \circ D_{n3}) \circ D_{n2} = (C_{n1} \circ D_{n2}) \circ D_{n3}$$

$$= (C_{n2} \circ D_{n1}) \circ D_{n3}$$

$$= (C_{n2} \circ D_{n3}) \circ D_{n1}$$

$$= (C_{n3} \circ D_{n2}) \circ D_{n1}$$

$$= (C_{n3} \circ D_{n1}) \circ D_{n2}$$

$\therefore (C_{n1} \circ D_{n3}) = (C_{n3} \circ D_{n1})$ by cancellation law.

We denote by $\dfrac{C_n}{D_n}$ the equivalence class of (C_n, D_n) in H_n^*x H_n and define $H_n^*[H_n^{\ 1}]$ to be the set of all the equivalenceclasses $\dfrac{C_n}{D_n}$, where $C_n \in H_n^*$ and $D_n \in H_n$.

For all $\frac{C_{n1}}{D_{n1}}, \frac{C_{n2}}{D_{n2}} \in H_n^*[H_n{}^1]$ we define addition and multiplication on $H_n^*[H_n{}^1]$ as follows: $\frac{C_{n1}}{D_{n1}} +' \frac{C_{n2}}{D_{n2}} = \frac{C_{n1} \circ D_{n2} + C_{n2} \circ D_{n1}}{D_{n1} \circ D_{n2}}$ and $\frac{C_{n1}}{D_{n1}} \circ' \frac{C_{n2}}{D_{n2}} = \frac{C_{n1} \circ C_{n2}}{D_{n1} \circ D_{n2}}$.

Proposition2.2

The operations $(+')$, $(\circ')$ as defined above are well-defined.

Proof

Suppose $\frac{C'_{n1}}{D'_{n1}} = \frac{C_{n1}}{D_{n1}}$ and $\frac{C'_{n2}}{D'_{n2}} = \frac{C_{n2}}{D_{n2}}$; then $C'_{n1} \circ D_{n1} = D'_{n1} \circ C_{n1}$ and $C'_{n2} \circ D_{n2} = D'_{n2} \circ C_{n2}$, so that $(C'_{n1} \circ D'_{n2} + C'_{n2} \circ D'_{n1}) D_{n1} D_{n2} = C'_{n1} \circ D_{n1} \circ D'_{n2} \circ D_{n2} + C'_{n2} \circ D_{n2} \circ D'_{n1} \circ D_{n1}$

$$= C_{n1} \circ D'_{n1} \circ D'_{n2} \circ D_{n2} + C_{n2} \circ D'_{n2} \circ D'_{n1} \circ D_{n1}$$

$$= (C_{n1} \circ D_{n2} + C_{n2} \circ D_{n1}) D'_{n1} \circ D'_{n2}$$

implying that $\frac{C'_{n1}}{D'_{n1}} +' \frac{C'_{n2}}{D'_{n2}} = \frac{C_{n1}}{D_{n1}} +' \frac{C_{n2}}{D_{n2}}$.

Similarly $(C'_{n1} \circ C'_{n2}) D_{n1} \circ D_{n2} = (C_{n1} \circ C_{n2}) D'_{n1} \circ D'_{n2}$ implies that $\frac{C'_{n1}}{D'_{n1}} \circ' \frac{C'_{n2}}{D'_{n2}} = \frac{C_{n1} \circ C_{n2}}{D_{n1} \circ D_{n2}}$.

By definition (1.3) the equivalence class $\frac{C_n}{D_n} = C_n \circ D_n^{-1}$ since $D_n \neq 0_n \in H_n$. Therefore, for all $D_n \in H_n$, $0_n \in H_n^*$, $\frac{0_n}{D_n} = 0_n \circ D_n^{-1} = 0_n = 0_n \circ I_n = \frac{0_n}{I_n}$. Thus $\frac{0_n}{I_n} = \frac{0_n}{D_n}$ is the additive identity and $-\frac{C_n}{D_n} = \frac{-C_n}{D_n}$ is the additive inverse. Similarly, $\frac{I_n}{I_n} = \frac{D_n}{D_n}$ is the multiplicative identity.

Theorem 2.2

With the above definitions and the definitions of the operations $(+')$ and $(\circ')$, the set of the equivalence classes $H_n^*[H_n{}^1]$ is a commutative ring.

Proof

One should check that the properties of a ring are fulfilled. But the proof follows from the fact that addition and multiplication are the regular addition and multiplication of fractions.

Proposition 2.3

The function $\psi : H_n^* \to H_n^*[H_n{}^1]$ defined by $\psi(C_n) = \frac{C_n}{I_n}$ is a ring homomorphism whose kernel is $\{C_n \in H_n^* : C_n \circ' D_n = 0 \text{ for some } D_n \in H_n^*[H_n{}^1]\}$.

Notes

Proof

Let $C_{n1}, C_{n2} \in H_n^*$, then

$$\psi(C_{n1}+C_{n2}) = \frac{C_{n1}+C_{n2}}{I_n} = (C_{n1}+C_{n2})\circ I_n = C_{n1}\circ I_n + C_{n2}\circ I_n = \frac{C_{n1}}{I_n}+\frac{C_{n2}}{I_n} = \psi(C_{n1})+\psi(C_{n2})$$

$$\psi(C_{n1}\circ' C_{n2}) = \frac{C_{n1}\circ' C_{n2}}{I_n} = (C_{n1}\circ' C_{n2})\circ I_n = C_{n1}\circ I_n \circ' C_{n2}\circ I_n = \frac{C_{n1}}{I_n}\circ'\frac{C_{n2}}{I_n} = \psi(C_{n1})\circ'\psi(C_{n2})$$

$$\psi(I_n) = \frac{I_n}{I_n}.$$

Now $C_n \in \ker\psi$ if and only if $\frac{C_n}{I_n} = \frac{0_n}{I_n}$, if and only if $C_n \circ I_n = 0_n \circ I_n = 0_n$, which imply that $\ker\psi = \{0_n\}$.

Recall from [1] that, the set $M = \{nI : n \in \mathbb{Z}\}$ where I is the unity element of the commutative ring of 3-dimensionalrhotrices R is a subring and submonoid of R under multiplication $(\circ)$. Thus the set $M_n = \{nI_n : n \in \mathbb{Z}\}$ is a subring and submonoid of the commutative ring R_n^* of n-dimensional rhotrices. Therefore any submonoid, H_n of R_n^* with property that for all $Q_n \neq 0 \in R_n^*$ and $S_n \in H_n$, $Q_n \circ S_n \neq 0$ can serve in the above construction for the generalization of proposition 2.3 as stated in the following proposition.

Proposition 2.4

$R_n^*[H_n{}^1]$ as constructed above is a ring, and there is a homomorphism $\psi : R_n^* \to R_n^*[H_n{}^1]$ given by $\psi(Q_n) = \frac{Q_n}{I_n}$.

Proof

The proof follows from propositions 2.2 and 2.3.

As defined in [1], a diagonal rhotrix of dimension 3 is a rhotrix whose two non-diagonal entries are all zero. Let D be the set of all n-dimensional diagonal rhotrices then it is easy for someone to verify that $W = D \cup \{O_n, I_n\}$, where O_n is the n-dimensional additive identity; I_n is the n-dimensional multiplicative identity is a group and is normal in R_n^* under multiplication.

Proposition 2.5

Let R_n^* be a commutative ring of n-dimensional rhotrices, and let H_n be a submonoid of R_n^* such that $Q_n \circ S_n \neq 0_n$ for every $Q_n \neq 0_n \in R_n^*$ and $S_n \in H_n$. Then every ideal of $R_n^*[H_n{}^1]$ has the form $W[H_n{}^1]$, for suitable W normal in R_n^*.

Proof

Since all the elements of $W[H_n{}^1]$ are also elements of $R_n^*[H_n{}^1]$ and $I_n \in H_n$ then obviously $W[H_n{}^1]$ is an additive subgroup of $R_n^*[H_n{}^1]$.

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For all $\frac{C_n}{D_n} \in R_n^*[H_n{}^1]$; $\frac{W_n}{D_n'} \in W[H_n{}^1]$; $\frac{C_n}{D_n} \circ' \frac{W_n}{D_n'} = \frac{C_n \circ W_n}{D_n \circ D_n'} \in W[H_n{}^1]$ since $C_n \circ W_n \in W, D_n \circ D_n' \in H_n$.

W is normal in R_n^* implies that $\frac{W_n}{D_n'} \circ \frac{C_n}{D_n} \in W[H_n{}^1]$.

Proposition 2.6

$H_n^*[H_n{}^1]$ is an integral domain.

Proof

Suppose $\frac{C_{n1}}{D_{n1}} \circ' \frac{C_{n2}}{D_{n2}} = 0_n \in H_n^*[H_n{}^1]$, that is $\frac{C_{n1} \circ C_{n2}}{D_{n1} \circ D_{n2}} = \frac{0_n}{I_n}$

$\Rightarrow (C_{n1} \circ C_{n2}, D_{n1} \circ D_{n2}) \tilde{} (0_n, I_n)$ and $C_{n1} \circ C_{n2} \circ D_n = 0$ for some $D_n \in H_n$.

$C_{n1} \circ C_{n2} \circ D_n = 0_n \in H_n^*$, which is an integral domain, and $D_n \neq 0_n$, thus $C_{n1} \circ C_{n2} = 0_n$.

So either C_{n1} or C_{n2} is 0_n and consequently either $\frac{C_{n1}}{D_{n1}}$ or $\frac{C_{n2}}{D_{n2}}$ is 0_n.

Theorem 2.3

The set $H_n^*[H_n{}^1]$ of all equivalence classes $\frac{C_n}{D_n}$ is a field.

Proof

From Theorem 2.2, $H_n^*[H_n{}^1]$ is a commutative ring with unity $\frac{I_n}{I_n}$. So we just need to show that every non zero element of $H_n^*[H_n{}^1]$ has multiplicative inverse.

Suppose $\frac{C_n}{D_n} \neq \frac{0_n}{I_n}$, then $C_n \neq 0_n$, so $C_n \in H_n$ which implies that $\frac{C_n}{D_n} \in H_n^*[H_n{}^1]$.

Clearly, $\frac{C_n}{D_n} \circ' \frac{D_n}{C_n} = \frac{C_n \circ D_n}{D_n \circ C_n} = \frac{I_n}{I_n}$. Thus $\frac{D_n}{C_n}$ is the multiplicative inverse of $\frac{C_n}{D_n}$.

III. Conclusion

In this short note, amendment concerning some definitions in [2] and [8] with their generalizations were provided. The steps observed in the construction of field of fractions illustrated in [8] were also amended respectively.

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