Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 13 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On Darboux Helices in Euclidean 3-Space

By Evren Ziplar, Ali Senol & Yusuf Yayli
Ankara University, Turkey

Abstract - In this paper, we introduce a Darboux helix to be a curve in 3-space whose Darboux vector makes a constant angle with a fixed straight line. We completely characterize Darboux helices in terms of κ & τ and thus prove that the class of Darboux helices coincide with the class of slant helices. In special, if we take $t^2 + \kappa^2 =$ constant, the curves are curve of constant precession.

Keywords and phrases : *helices, slant helices, curves of constant precession, darboux vector.*

GJSFR-F Classification : *MSC 2000: 53C040, 53A05*

ON DARBOUXHELICESINEUCLIDEAN3-SPACE

Strictly as per the compliance and regulations of :

Notes

On Darboux Helices in Euclidean 3-Space

Evren Ziplar [α], Ali Senol [σ] & Yusuf Yayli [ρ]

Abstract - In this paper, we introduce a Darboux helix to be a curve in 3-space whose Darboux vector makes a constant angle with a fixed straight line. We completely characterize Darboux helices in terms of κ & τ and thus prove that the class of Darboux helices coincide with the class of slant helices. In special, if we take $t^2 + \kappa^2$ = constant, the curves are curve of constant precession.

Keywords and phrases : helices, slant helices, curves of constant precession, darboux vector.

I. Introduction

In differential geometry, a curve of constant slope or general helix in Euclidean 3-space R^3 is defined by the property that tangent makes a constant angle with a fixed straight line (the axis of general helix). Due to a classical result proved by M.A. Lancert in 1802 in R^3 is a general helix if and only if the ratio $\frac{\kappa}{\tau}$ is constant along curve, where κ and $\tau \neq 0$ denote the curvature and torsion, respectively. Using killing vector field along a curve, Barros gave a similar result for curves in 3-dimensional real space forms [3]. Several authers introduced different types of helices and investigated their properties. For instance, Izumiya and Takeuchi defined in [1] slant helices by the property that the principal normal makes a constant angle with a fixed direction. Moreover, they showed that α is a slant helix in R^3 if and only if the geodesic curvature of the principal normal of a space curve α is a constant function. Kula &Yaylı investigated spherical images of tangent indicatrix of binormal indicatrix of slant helix and they have shown that spherical images are spherical helix [2]. On the other hand the second and the third auther introduced in [6] LC helices in 3-dimensional real space forms and study their main properties.

The purpose of this paper is to introduce and study Darboux helices in R^3.We give a characterization of Darboux helices in terms of $\kappa\&\tau$. We give the relations between darboux helices and slant helices. As a consequence, we observe that Darboux helices coincide with slant helices. Finally, we show that curves of constant precession are darboux helices.

II. Preliminaries

We now recall some basic concepts on classical differantial geometry of space curves in Euclidean space. Let $\alpha : I \subset R \to R^3$ be a curve parameterized by arc lenght and let $\{T, N, B\}$ denote the Frenet frame of the curve α.

Author α : Ankara University, Faculty of Science, Department of Mathematics, 06100 Ankara, Turkey. E-mail : evrenziplar@yahoo.com
Author σ : Cankiri Karatekin University, Faculty of Science, Department of Mathematics, 18100 Cankiri, Turkey.
E-mail : asenol@karatekin.edu.tr
Author ρ : Ankara University, Faculty of Science, Department of Mathematics, 06100 Ankara, Turkey.
E-mail : yyayli@science.ankara.edu.tr

$T(s) = \alpha'(s)$ is a unit tangent vector of α at s.We define the curvature of α by $\kappa(s) = \left\| \alpha''(s) \right\|$. For the derivatives of the frenet-serret formulae hold:

$$\begin{aligned} T'(s) &= \kappa(s).N(s) \\ N'(s) &= -\kappa(s).T(s) + \tau(s).B(s) \\ B'(s) &= -\tau(s).N(s) \end{aligned}$$

where $\tau(s)$ is the torsion of α at s.

For any unit speed curve $\alpha : I \subset R \to R^3$ defined a vector field $C = \dfrac{(\tau T + \kappa B)}{\sqrt{\tau^2 + \kappa^2}}$ along α under the condition that $\kappa(s) \neq 0$ and called it the modified Darboux vector field of α [1].

III. Darboux Helices

Let $\boldsymbol{\alpha}$ be a curve in E^3 with $\dfrac{\tau}{\kappa} \neq 0$ everywhere with nonzero curvature and torsion κ and τ in E^3. We say that α is a Darboux helix if its Darboux vector makes a constant angle with a fixed direction d, that is $\langle W, d \rangle$ =constant along the curve, where d is a unit vector field in E^3.

$$W = \tau T + \kappa B$$

The direction of the vector d is axis of the Darboux helix. We can identify Darboux helices by the condition torsion and curvature. If $\tau^2 + \kappa^2$ =constant, the darboux helices are the curves of constant precession. So, our curves are more general than the curves of constan precession. Although every general helice is a slant helice, the general helices are not darboux helices. Moreover, there is a relation between darboux helice and the surface of constant precession. The following result describes the relation between darboux helice and the surface of constant precession.

Theorem 1. *A normal conical surface is constant angle if and only if Generating curve α is a Darboux helix* [5].

Theorem 2. *Let $\boldsymbol{\alpha}$ be a curve constant precession. If the conical surfaces construct involving the normal lines to the curve α,then the surface is a constant angle surface with the axis of $d = W + \mu n$* [5].

Theorem 3. *α is a Darboux helix if and only if* $\sigma^*(s) = \dfrac{(\tau^2 + \kappa^2)^{\frac{3}{2}}}{\kappa^2} \dfrac{1}{\left(\frac{\tau}{\kappa}\right)'}$ *function is constant.*

Proof. If the spherical indicatrix of the darboux vector W is a circle or a part of circle, then the curve α is a darboux helis. Let the parameter of the curve (c) be s_c and let T_c be the unit tanget vector of (c). Let κ_c be the geodesic curvature of(c) in E^3.

$$\alpha(s_c) = c(s) = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}} T + \frac{\kappa}{\sqrt{\tau^2 + \kappa^2}} B$$

$$\alpha(s_c) = \sin \Phi T + \cos \Phi B$$

$$\frac{d\alpha}{ds_c} = \frac{dc}{ds} \frac{ds}{ds_c}$$

$$\frac{d\alpha}{ds_c} = \left(\Phi' \cos \Phi T - \Phi' \sin \Phi B + \kappa \sin \Phi N - \tau \cos \Phi N \right) \frac{ds}{ds_c}$$

Ref.

[5] Özkaldi S, Yaylı Y. *Constant angle surfaces and curves in E^3*.International Electronic Journal of Geometry., **4(1)**, 70-78 (2011).

Notes

$$T_c = \frac{d\alpha}{ds_c} = (\Phi' \cos\Phi T - \Phi' \sin\Phi B)\frac{ds}{ds_c}$$

$$\|T_c\| = \left\| \left(\Phi^{'} \cos\Phi T - \Phi^{'} \sin\Phi B\right)\frac{ds}{ds_c} \right\|$$

$$1 = \Phi^{'}\frac{ds}{ds_c}$$

$$\frac{ds}{ds_c} = \frac{1}{\Phi'}$$

(1) $$T_c = \cos\Phi T - \sin\Phi B$$

$$D_{T_c}^{T_c} = \frac{dT_c}{dS_c}\frac{ds}{ds_c}$$

$$D_{T_c}^{T_c} = \left(-\Phi^{'}\sin\Phi T - \Phi^{'}\cos\Phi B + \kappa\cos\Phi N + \tau\sin\Phi N\right)\frac{1}{\Phi'}$$

(2) $$D_{T_c}^{T_c} = \left(-\sin\Phi T - \cos\Phi B + \frac{\|w\|}{\Phi\cdot}N\right)$$

Hence, from the equation (2), the geodesic curvature of (c) are computed as the following.

$$\kappa_c = \left\|D_{T_c}^{T_c}\right\| = \left\|-\sin\Phi T - \cos\Phi B + \frac{\|w\|}{\Phi'}N\right\|$$

(3) $$\kappa_c = \left\|D_{T_c}^{T_c}\right\| = \sqrt{1 + \left(\frac{\|w\|}{\Phi'}\right)^2}$$

Therefore, we obtain

$$D_{T_c}^{T_c} = \nabla_{T_c}^{T_c} - c(s)$$

(4) $$\kappa_c^2 = \kappa_g^2 + 1$$

by using the Gauss map

$$D_{T_c}^{T_c} = \nabla_{T_c}^{T_c} - \langle s(T_c), T_c\rangle\, c(s).$$

and from the equations (3) and (4), we have:

$$1 + \left(\frac{\|w\|}{\Phi'}\right)^2 = \kappa_g^2 + 1$$

(5) $$\kappa_g = \frac{\|w\|}{\Phi'}$$

On the other hand, taking the derivative of $\tan\Phi = \dfrac{\tau}{\kappa}$,

$$\Phi^{'}.\left(1 + \tan^2\Phi\right) = \left(\frac{\tau}{\kappa}\right)^{'}$$

(6) $$\Phi^{'} = \left(\frac{\kappa^2}{\kappa^2 + \tau^2}\right)\left(\frac{\tau}{\kappa}\right)^{'}.$$

Global Journal of Science Frontier Research (F) Volume XII Issue XIII Version I Year 2012

Hence, by using the equations (5) and (6) , we get:

$$\kappa_g = \frac{\sqrt{\kappa^2+\tau^2}}{\left(\frac{\kappa^2}{\kappa^2+\tau^2}\right)\left(\frac{\tau}{\kappa}\right)'}$$

$$\kappa_g = \frac{\left(\kappa^2+\tau^2\right)\frac{3}{2}}{\kappa^2}\frac{1}{\left(\frac{\tau}{\kappa}\right)'},$$

where $\|w\| = \sqrt{\kappa^2+\tau^2}$.The spherical indicatrix of (c) is a circle or a part of circle. Since the first curvature of a circle is constant, we obtain κ_c =constant. So, κ_g =constant. If we denote κ_g with $\sigma^*(s)$,

$$\kappa_g = \frac{\left(\kappa^2+\tau^2\right)\frac{3}{2}}{\kappa^2}\frac{1}{\left(\frac{\tau}{\kappa}\right)'} = \sigma^*(s)$$

and so, we have

$$\frac{\left(\kappa^2+\tau^2\right)\frac{3}{2}}{\kappa^2}\frac{1}{\left(\frac{\tau}{\kappa}\right)'} = \sigma^*(s)$$

which is constant function.

Theorem 4. *Let $\alpha : I \to E^3$ be a curve in E^3. We assume that $\frac{\kappa}{\tau}$ is not constant, where κ and τ are curvature of α. Then,*

α is a slant helice if and only if α is a Darboux helice

Proof. we assume that α is a slant helice. So we can write:

(7) $$\sigma(s) = \frac{\kappa^2}{\left(\kappa^2+\tau^2\right)\frac{3}{2}}\left(\frac{\tau}{\kappa}\right)'.$$

Similarly, if the curve α is a darboux helice

(8) $$\sigma^*(s) = \frac{\left(\tau^2+\kappa^2\right)^{\frac{3}{2}}}{\kappa^2}\frac{1}{\left(\frac{\tau}{\kappa}\right)'}.$$

Consequently, we obtain:

$$\sigma(s)\sigma^*(s) = sbt$$

$$\sigma(s) = sbt \ \Leftrightarrow \sigma^*(s) = sbt$$

From the previous Theorem, firstly we are going to find the axis of the slant helices since a slant helice is also a darboux helice.

3.1. **The axis of Darboux helice.** We first assume that α is a slant helix. Let d be the vector field such that the function $\langle N, d\rangle$ =cosθ=constant. There exists a_1 and a_3 such that

(9) $$d = a_1T + a_3B + \cos\theta N.$$

Then, if we take the derivative of the equation (9) and by using frenet equation, we have:

$$d' = (a' - \cos\theta.\kappa)T + (a_1\kappa - \tau a_3)N + (a_3' + \cos\theta.\tau)B$$

since the system $\{T, N, B\}$ is linear independent, we get:

$$a_1' - cos\theta.\kappa = 0$$

Notes

Notes

(10) $$a_1\kappa - \tau a_3 = 0$$

(11) $$a_3' + \cos\theta.\tau = 0$$

and from (10) and (9), respectively

(12) $$a_1 = \left(\frac{\tau}{\kappa}\right).a_3$$

(13) $$\langle d, d\rangle = a_1^2 + a_3^2 + \cos^2\theta = \text{constant}$$

By using the equalities (12) and (13), we obtain:

(14) $$\left(\frac{\tau}{\kappa}\right)^2 a_1^2 + a_3^2 + \cos^2\theta = \text{constant}$$

and from the equation (14) we have

$$\left(\left(\frac{\tau}{\kappa}\right)^2 + 1\right) a_3^2 = m^2$$

where m^2 is constant. So,

(15) $$a_3 = \frac{m}{\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}},$$

Taking the derivative in each part of the equation (15) and by using (13), we get:

(16) $$\frac{\kappa^2}{(\tau^2 + \kappa^2)^{\frac{3}{2}}}.\left(\frac{\tau}{\kappa}\right)' = \text{constant}$$

We deduce from that the curve α is slant helice when we have d . Conversely, assume that the condition (16) is satisfied. In order to simplify the computations, we assume that the function (16) is constant. Define

(17) $$d = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}}T + \frac{\kappa}{\sqrt{\tau^2 + \kappa^2}}B + \cos\theta N$$

A differentiation of (17) together the frenet equations gives $d' = 0$, that is, d is a constant vector. It can easily be seen that $d' = 0$, that is d is a constant. On the other hand, $\langle N, d\rangle$ =cosθ and this means that α is a slant helix.

Now, we are going to show that the darboux vector $W = \tau T + \kappa B$ makes a constant angle with the constant direction

$$d = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}}T + \frac{\kappa}{\sqrt{\tau^2 + \kappa^2}}B + cos\theta N.$$

The constant direction d is the axis of both the slant helice α and the darboux helice α.These axises coincide but the making angles of these helices with d are different.

Since α is a slant helice, $\langle N, d\rangle$ =cosθ =constant

$$d = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}}T + \frac{\kappa}{\sqrt{\tau^2 + \kappa^2}}B + \cos\theta N$$

$$
\begin{aligned}
d &= \frac{W}{\|W\|} + \cos\theta N \\
\langle d, W\rangle &= \|d\| . \|W\| .cos\lambda \\
\langle d, W\rangle &= \sqrt{1+\cos^2\theta}. \|W\| .cos\lambda \\
\frac{\langle W, W\rangle}{\|W\|} &= \sqrt{1+\cos^2\theta}. \|W\| .cos\lambda \\
\cos\lambda &= \frac{1}{\sqrt{1+\cos^2\theta}}
\end{aligned}
$$

Since $\cos\theta$ =constant , $\cos\lambda$ is constant.

3.2. **Curves of constant precession.** A unit speed curve of constant precession is defined by the property that its (Frenet) Darboux vector revolves about a fixed line in space with angle and constant speed. A curve of constant precession is characterized by having

$$
\begin{aligned}
\kappa(s) &= \varpi \sin(\mu(s)), \\
\tau(s) &= \varpi \cos(\mu(s)),
\end{aligned}
$$

where $\varpi \rangle 0$, μ and are constant[4].

If α is a curve of constant precession , α is a slant helix [?]

From the axis of the Darboux helice,

$$d = \frac{\tau}{\sqrt{\tau^2+\kappa^2}}T + \frac{\kappa}{\sqrt{\tau^2+\kappa^2}}B + \cos\theta N$$

and

$$d = \frac{W}{\|W\|} + \cos\theta N \tag{18}$$

where $W = \tau T + \kappa B$.From (18),

$$\sqrt{\tau^2+\kappa^2}.d = W + \sqrt{\tau^2+\kappa^2}.\cos\theta N$$

By taking $\varpi = \|W\| = \sqrt{\tau^2+\kappa^2}$, $\varpi.d = A$ and $\varpi.\cos\theta = \mu$:

$$A = W + \mu.N$$

If $\|W\|$ =constant, the darboux helice α a curve of constant precession. We deduce from that [4] is true.

Remark 1. *All characterizations given for these slant helices can be given for these darboux helices.*

Theorem 5. *Let $\boldsymbol{\alpha}$ be a unit speed curve in E^3 and let α be a slant helice (darboux helice). The curvatures κ, τ of the curve α satisfy the following non-linear equation system.*

$$\left(\frac{\tau}{\sqrt{\tau^2+\kappa^2}}\right)' - \mu\kappa = 0, \left(\frac{\kappa}{\sqrt{\tau^2+\kappa^2}}\right)' - \mu\tau = 0$$

Proof. Since α is a slant helice (darboux helice), the axis of α :

$$d = \frac{\tau}{\sqrt{\tau^2+\kappa^2}}T + \frac{\kappa}{\sqrt{\tau^2+\kappa^2}}B + cos\theta N, \tag{19}$$

Notes

where κ, τ are curvatures of α.Taking the derivative in each part of the equation (19), we get

$$\acute{d} = \left(\frac{\tau}{\sqrt{\tau^2+\kappa^2}}\right)^{\acute{}} T + \left(\frac{\kappa}{\sqrt{\tau^2+\kappa^2}}\right)^{\acute{}} B + \mu(-\kappa T + \tau B) = 0$$

since the system$\{T, B\}$ is linear independent,

$$\left(\frac{\tau}{\sqrt{\tau^2+\kappa^2}}\right)^{\acute{}} - \mu\kappa = 0, \left(\frac{\kappa}{\sqrt{\tau^2+\kappa^2}}\right)^{\acute{}} + \mu\tau = 0$$

Conclusion 1. *If we take $\tau^2+\kappa^2$ =constant, then the curve α is a curve of constant precession* [4].

So, the following theorem can be given.

Theorem 6. *A necessary and sufficient condition that a curve be of constant precession is that $\kappa(s) = \varpi \sin(\mu(s), \tau(s) = \varpi \cos(\mu(s))$.up to reflection or phase shift of arclength, for constants ϖ and μ.*

Proof. Since $A' = 0$,

$$\left(\acute{\tau} - \mu\kappa\right) T + \left(\acute{\kappa} + \mu\tau\right) B = 0$$

and uniqueness of solutions of pairs of linear equations imply that $A' = 0$ if and only if $\kappa(s) = \varpi \sin(\mu(s), \tau(s) = \varpi \cos(\mu(s))$.

The following example is related to darboux helices.

Example 1. *Let the curve $\alpha(s)$ be a curve parametrized by the vector function:*

$$\begin{aligned} \alpha(s) \quad = \quad & (\frac{\sqrt{5}+1}{5-\sqrt{5}}\mathrm{Sin}(\frac{\sqrt{5}-1}{2}s) - \frac{\sqrt{5}-1}{5+\sqrt{5}}\mathrm{Sin}(\frac{\sqrt{5}+1}{2}s), \\ & \frac{\sqrt{5}+1}{\sqrt{5}-5}\mathrm{Cos}(\frac{\sqrt{5}-1}{2}s) + \frac{\sqrt{5}-1}{5+\sqrt{5}}\mathrm{Cos}(\frac{\sqrt{5}+1}{2}s), \\ & \frac{4}{\sqrt{5}}\mathrm{Sin}(\frac{s}{2})) \end{aligned}$$

where $s \in [0, 10\pi]$.Then, $\alpha(s)$ is a darboux helix (or a curve of constant precession), where $\kappa(s) = -\mathrm{Sin}\frac{\sqrt{5}}{2}s$ and $\tau(s)$ =$\mathrm{Cos}\frac{\sqrt{5}}{2}s$.The curve is rendered in the following figure.

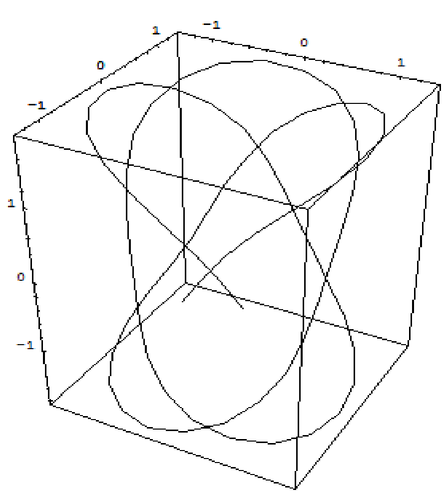

Figure 1. The darboux helix $\alpha\,(s)$

Ref.

[4] Scofield, P.D. *Curves of constant precession.* Am. Math. Montly **102**, 531-537, (1995).

Global Journal of Science Frontier Research (F) Volume XII Issue XIII Version I Year 2012

Conclusion 2. *All helices are slant helices. The slant helices which are not helices are defined as Darboux helices. The Darboux helices are more general than the curves of constant precession.*

References Références Referencias

[1] Izumiya, S and Tkeuchi, N. *New special curves and developable surfaces*, Turk J. Math., **28**, 153-163, (2004).

[2] Kula, L and Yaylı Y. *On slant helix and its spherical indicatrix.* Applied Mathematics and computation, **169**, 600-607, (2005)

[3] Barros, M. *General helices and a theorem Lancert.* Proc. Amer. Math. Soc.,**125(5)**,1503-1509 (1907).

[4] Scofield, P.D. *Curves of constant precession.* Am. Math. Montly **102**, 531-537, (1995).

[5] Özkaldi S, Yaylı Y. *Constant angle surfaces and curves in* E^3.Internatıonal Electronic Journal of Geometry., **4(1)**, 70-78 (2011).

[6] Senol A, Yaylı Y., *LC helices in space forms*, Chaos, Solitons& Fractals, **42 (4)**, 2115-2119 (2009).

Notes