Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 9 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On ∅-Recurrent Generalized Sasakian-Space-Forms

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AMS Subject Classification (2000) : *53C15, 53C25*

ON -RECURRENT GENERALIZED SASAKIAN-SPACE-FORMS

Strictly as per the compliance and regulations of :

Ref.

1. Alfonso Carriazo, David. E. Blair and Pablo Alegre, proceedings of the Ninth International Workshop on Differential Geometry, 9 (2005), 31-39.

On ∅-Recurrent Generalized Sasakian-Space-Forms

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I. Introduction

A Riemannian manifold with constant sectional curvature C is known as real-space-form and its curvature tensor is given by

$$R(X,Y)Z = C\{g(Y,Z)X - g(X,Z)Y\}.$$

A Sasakian manifold (M,ϕ,ξ,η,g) is said to be a Sasakian-space-form [1], if all the ϕ-sectional curvatures $K(X\wedge\phi X)$ are equal to a constant C, where $K(X\wedge\phi X)$ denotes the sectional curvature of the section spanned by the unit vector field X orthogonal to ξ and ϕX. In such a case, the Riemannian curvature tensor of M is given by,

$$\begin{aligned} R(X,Y)Z &= (C+3)/4\{g(Y,Z)X - g(X,Z)Y\} \\ &+ (C-1)/4\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ (C-1)/4\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \quad (1.1)$$

As a natural generalization of these manifolds, P. Alegre, D. E. Blair and A. Carriazo [1] [2] introduced the notion of generalized Sasakian-space-form . It is defined as almost contact metric manifold with Riemannian curvature tensor satisfying an equation similar to (1.1), in which the constant quantities $(C+3)/4$ and $(C-1)/4$ are replaced by differentiable functions, i.e.,

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$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned}$$

for any vector fields X, Y, Z on M.

Generalized Sasakian-space-forms and Sasakian-space-forms have been studied by several authors, viz., [1], [3], [4], [6], [10]. The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetric, T.Takahashi [11] introduced the notion of locally ϕ-symmetry on a Sasakian manifold. Generalizing the notion of ϕ-symmetry, U.C.De and co-authors [9] introduced the notion of ϕ-recurrent Sasakian manifold. Further ϕ-recurrent Kenmotsu manifold, ϕ-recurrent LP-Sasakian manifold, concircular ϕ-recurrent LP-Sasakian manifold, pseudo-projectively ϕ-recurrent Kenmotsu manifold were studied by U.C. De and co-authors and Venkatesha and C.S. Bagewadi in their papers [7], [12], [13].

In the present paper we have studied ϕ-recurrent generalized Sasakian-space-form and proved that a ϕ-recurrent generalized Sasakian-space-form is an η-Einstein manifold and a locally ϕ-recurrent generalized Sasakian-space-form is a manifold of constant curvature. Further it is shown that if a ϕ-recurrent generalized Sasakian-space-form has a non zero constant sectional curvature, then it reduces to a locally ϕ-symmertic manifold.

II. Preliminaries

An odd-dimensional Riemannian manifold (M, g) is called an almost contact manifold if there exists on M, a $(1,1)$ tensor field ϕ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\phi^2(X) = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any vector fields X, Y on M.

In particular, in an almost contact metric manifold we also have

$$\phi\xi = 0, \qquad \eta\circ\phi = 0. \tag{2.3}$$

Ref.

3. P. Alegre and A. Cariazo, Structures on generalized Sasakian-space-form, Differential Geom. and its application 26 (2008), 656-666.

$$(\nabla_X \xi) = X - \eta(X)\xi, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.5}$$

Given an almost contact metric manifold (M, ϕ, ξ, η, g), we say that M is an generalized Sasakian-space-form, if there exists three functions f_1, f_2 and f_3 on M such that

Notes

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{2.6}$$

for any vector fields X, Y, Z on M, where R denotes the curvature tensor of M. This kind of manifold appears as a natural generalization of the well-known Sasakian-space-forms $M(C)$, which can be obtained as particular cases of generalized Sasakian-space-forms by taking $f_1 = (C+3)/4$ and $f_2 = f_3 = (C-1)/4$. Further in a $(2n+1)$-dimensional generalized Sasakian-space-form, we have [2]

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.7}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.8}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.9}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi,X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{2.11}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)), \tag{2.12}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \tag{2.13}$$

$$\begin{aligned} (\nabla_W R)(X,Y)\xi &= (df_1(W) - df_3(W))\{\eta(Y)X - \eta(X)Y\} \\ &+ (f_1 - f_2)\{g(Y,W)X - g(X,W)Y\} - R(X,Y)W. \end{aligned} \tag{2.14}$$

Definition 2.1. *A generalized Sasakian - space - form is said to be locally ϕ-symmetric if*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0. \tag{2.15}$$

Definition 2.2. *A generalized Sasakian-space-form is said to be ϕ-recurrent if there exist a non zero 1-form A such that*

Global Journal of Science Frontier Research (F) Volume XII Issue IX Version I Year 2012
65

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z, \tag{2.16}$$

for any arbitrary vector field X, Y, Z and W.

III. ∅-Recurrent Generalized Sasakian - Space - Forms

Two vector fields X and Y are said to be co-directional, if $X = fY$ where f is a non-zero scalar. i.e.,

$$g(X,Z) = fg(Y,Z), \tag{3.1}$$

for all X.

Now from (2.1) and (2.16), we have

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z. \tag{3.2}$$

From (3.2) and the Bianchi identity we get

$$A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) = 0. \tag{3.3}$$

By virtue of (2.12) we obtain from (3.3) that

$$(f_1 - f_3)[A(W)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} + A(X)\{g(W,Z)\eta(Y) \tag{3.4}$$
$$-g(Y,Z)\eta(W)\} + A(Y)\{g(X,Z)\eta(W) - g(W,Z)\eta(X)\}] = 0.$$

Putting $Y = Z = e_i$ in (3.4) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$2n(f_1 - f_3)[A(W)\eta(X) - A(X)\eta(W)] = 0.$$

If $(f_1 - f_3) \neq 0$, then

$$A(W)\eta(X) = A(X)\eta(W), \tag{3.5}$$

for all vector fields X,W.

Replacing X by ξ in (3.5), we get

$$A(W) = \eta(\rho)\eta(W), \tag{3.6}$$

where $A(X) = g(X,\rho)$ and ρ is the vector field associated to the 1-form A. From (3.1) and (3.6) we can state the following:

Theorem 3.1. *In a ϕ-recurrent generalized Sasakian-space-form, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by (3.6), provided $f_1 - f_3 \neq 0$.*

Notes

Let us consider a ϕ-recurrent generalized Sasakian space form. Then from (3.2) we have,

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z,$$

from which it follows that

$$-g((\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)\eta(U) = A(W)g(R(X,Y)Z,U). \tag{3.7}$$

Notes

Let $\{e_i\}, i = 1,2,...,2n+1$ be an orthonormal basis of the tangent space at any point of the space form. Then putting $X = U = e_i$, in (3.7) and taking summation over i, $1 \leq i \leq 2n+1$, we get

(3.8) $$-(\nabla_W S)(Y,Z) = A(W)S(Y,Z).$$

Replacing Z by ξ in (3.8) we get,

$$(\nabla_W S)(Y,\xi) = -A(W)S(Y,\xi). \tag{3.9}$$

Now we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi). \tag{3.10}$$

Using (2.4), (2.5) and (2.13) in (3.10), we get

$$(\nabla_W S)(Y,\xi) = 2n(f_1 - f_3)g(Y,W) - S(Y,W). \tag{3.11}$$

Now using (3.11) in (3.10), we obtain

$$S(Y,W) = 2n(f_1 - f_3)g(Y,W) + A(W)S(Y,\xi). \tag{3.12}$$

Using (2.13) and (3.6) in (3.12), we get

$$S(Y,W) = 2n(f_1 - f_3)g(Y,W) + 2n(f_1 - f_3)\eta(\rho)\eta(W)\eta(Y). \tag{3.13}$$

This leads to the following theorem:

Theorem 3.2. *A ϕ-recurrent generalized Sasakian-space-form is an η-Einstein manifold provided $f_1 - f_3 \neq 0$.*

From (2.14), we have

$$\begin{aligned}(\nabla_W R)(X,Y)\xi &= (df_1(W) - df_3(W))\{\eta(Y)X - \eta(X)Y\} \\ &+ (f_1 - f_3)\{g(Y,W)X - g(X,W)Y\} - R(X,Y)W.\end{aligned}$$

By virtue of (2.12), it follows from (2.14) that

$$\eta(\nabla_W R)(X,Y)\xi = 0. \tag{3.14}$$

In view of (3.14) and (3.2), we obtain

$$A(W)R(X,Y)\xi \;=\; -(\nabla_W R)(X,Y)\xi. \tag{3.15}$$

By using (2.14) in (3.15), we get

$$\begin{aligned} &-(df_1(W)-df_3(W))\{\eta(Y)X-\eta(X)Y\} \\ -(f_1-f_3)\{g(Y,W)X-g(X,W)Y\}+R(X,Y)W &= A(W)R(X,Y)\xi. \end{aligned} \tag{3.16}$$

Hence if X and Y are orthogonal to ξ, then we get from (2.10) that

$$R(X,Y)\xi = 0.$$

Thus we obtain

$$R(X,Y)W \;=\; (f_1-f_3)\{g(Y,W)X-g(X,W)Y\}, \tag{3.17}$$

for all X,Y,W.

This leads to the following theorem:

Theorem 3.3. *A locally ϕ-recurrent generalized Sasakian-space-form is a manifold of constant curvature.*

Again let us suppose that a generalized Sasakain-space-form is ϕ-recurrent. Then from (3.2) and (2.14), it follows that

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= [-(df_1(W)-df_3(W))\{g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\} \\ &- (f_1-f_3)\{g(Y,W)g(X,Z)-g(X,W)g(Y,Z)\} \\ &- g(R(X,Y)W,Z)]\xi - A(W)R(X,Y)Z. \end{aligned} \tag{3.18}$$

From (3.18) it follows that

$$\phi^2(\nabla_W R)(X,Y)Z = A(W)R(X,Y)Z - A(W)(f_1-f_3)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}, \tag{3.19}$$

which yields

$$\phi^2(\nabla_W R)(X,Y)Z \;=\; A(W)R(X,Y)Z.$$

Notes

Notes

Hence we state the following:

Theorem 3.4. *A generalized Sasakian-space-form satisfying the relation (3.18) is ϕ-recurrent provided that X and Y are orthogonal to ξ.*

Next, we suppose that in a ϕ-recurrent generalized *Sasakian-space-form* the sectional curvature of a plane $\pi \subset T_pM$ defined by

$$K_p(\pi) \quad = \quad g(R(X,Y)Y,X),$$

is a non zero constant K, where X, Y is any orthonormal basis of π. Then we have

$$g((\nabla_Z R)(X,Y)Y,X) = 0. \tag{3.20}$$

By virtue of (3.20) and (3.2) we obtain

$$g((\nabla_Z R)(X,Y)Y,\xi)\eta(X) \quad = \quad A(Z)g(R(X,Y)Y,X). \tag{3.21}$$

Since in a ϕ-recurrent generalized Sasakian-space-form, the relation (3.18) holds good. Using (3.18) in (3.21) we get

$$\begin{aligned} &[-(df_1(Z) - df_3(Z))\{g(X,Y)\eta(Y) - g(Y,Y)\eta(X)\} \\ &-A(Z)\eta(R(X,Y)Y)]\eta(X) = A(Z)g(R(X,Y)Y,X). \end{aligned} \tag{3.22}$$

If X and Y are orthogonal to ξ,

$$-A(Z)\eta(X)\eta(R(X,Y)Y) = KA(Z). \tag{3.23}$$

Putting $Z = \xi$ in (3.23) we obtain

$$\eta(\rho)[\eta(X)\eta(R(X,Y)Y) + K] = 0.$$

Which implies that

$$\eta(\rho) = 0. \tag{3.24}$$

Hence by (3.6) we obtain from (2.16) that

$$\phi^2((\nabla_W R)(X,Y)Z) = 0. \tag{3.25}$$

This leads to the following theorem:

Theorem 3.5. *If a ϕ-recurrent generalized Sasakian-space-form has a non-zero constant sectional curvature, then it reduces to a locally ϕ-symmetric manifold provided that X and Y are orthogonal to ξ.*

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