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Certain Indefinite Integrals Involving Gegenbauer Polynomials

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I. Introduction and Preliminaries

Gegenbauer polynomials

Gegenbauer polynomials or ultraspherical polynomials $C_n^{\alpha}(x)$ are orthogonal polynomials on the interval [-1,1] with respect to the weight function $(1-x^2)^{\alpha-\frac{1}{2}}$. They generalize Legendre polynomials and Chebyshev polynomials, and are special cases of Jacobi polynomials. They are named for Leopold Gegenbauer.
Gegenbauer polynomials are particular solutions of the Gegenbauer differential equation

$$(1-x^2)y'' - (2\alpha+1)xy' + n(n+2\alpha)y = 0 \tag{1.1}$$

When $\alpha = \frac{1}{2}$, the equation reduces to the Legendre equation, and the Gegenbauer polynomials reduce to the Legendre polynomials.
They are given as Gaussian hypergeometric series in certain cases where the series is in fact finite:

$$C_n^{\alpha}(z) = \frac{(2\alpha)_n}{n!}\,{}_2F_1\left(-n, (2\alpha+n); \alpha+\frac{1}{2}; \frac{1-z}{2}\right) \tag{1.2}$$

They are special cases of the Jacobi polynomials

$$C_n^{\alpha}(z) = \frac{(2\alpha)_n}{(\alpha+\frac{1}{2})_n} P_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x) \tag{1.3}$$

One therefore also has the Rodrigues formula

$$C_n^{\alpha}(z) = \frac{(-2)^n}{n!}\frac{\Gamma(n+\alpha)\Gamma(n+2\alpha)}{\Gamma(\alpha)\Gamma(2n+2\alpha)}(1-x^2)^{-\alpha+\frac{1}{2}}\frac{d^n}{dx^n}\left[(1-x^2)^{n+\alpha-\frac{1}{2}}\right] \tag{1.4}$$

The Pochhammer's symbol is defined by

$$(b,\ k) = (b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } k = 1,2,3,\cdots \\ 1 & ; \text{ if } k = 0 \\ k! & ; \text{ if } b = 1,\ k = 1,2,3,\cdots \end{cases} \tag{1.5}$$

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where b is neither zero nor negative integer and the notation Γ stands for Gamma function.

Polylogarithm function

The polylogarithm (also known as Jonquire's function) is a special function$Li_s(z)$ that is defined by the infinite sum, or power series:

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \tag{1.6}$$

It is in general not an elementary function, unlike the related logarithm function. The above definition is valid for all complex values of the order s and the argument z where$|\ z\ |< 1$.

Generalized Gaussian Hypergeometric Function

Generalized ordinary hypergeometric function of one variable is defined by

$${}_AF_B\left[\begin{array}{ll} a_1, a_2, \cdots, a_A \ ; & \\ & z \\ b_1, b_2, \cdots, b_B \ ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}$$

or

$${}_AF_B\left[\begin{array}{ll} (a_A) \ ; & \\ & z \\ (b_B) \ ; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{ll} (a_j)_{j=1}^{A} \ ; & \\ & z \\ (b_j)_{j=1}^{B} \ ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1.7}$$

where denominator parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers.

II. Main Indefinite Integrals

$$\int \frac{\cosh x\ C_2(x)}{\sqrt{(1-\cosh x)}}\,\mathrm{dx} = -\frac{1}{\sqrt{1-\cosh x}}\ 2\sinh\frac{x}{2}\Big[-8x\ \mathrm{Li}_2(-e^{\frac{-x}{2}}) + 8x\ \mathrm{Li}_2(e^{\frac{-x}{2}}) -$$

$$-16\ \mathrm{Li}_3(-e^{\frac{-x}{2}}) + 16\ \mathrm{Li}_3(e^{\frac{-x}{2}}) - 2x^2\log(1-e^{-\frac{x}{2}}) + 2x^2\log(1+e^{-\frac{x}{2}}) - 4x^2\cosh\frac{x}{2} +$$

$$+16x\sinh\frac{x}{2} - 30\cosh\frac{x}{2} + \log\left(\tanh\frac{x}{4}\right)\Big] + Constant \tag{2.1}$$

$$\int \frac{\cosh x\ C_2(x)}{\sqrt{(1-\cos x)}}\,\mathrm{dx} = -\frac{1}{\sqrt{1-\cos x}}\left(\frac{22}{125} - \frac{4\iota}{125}\right)e^{(-1-\frac{\iota}{2})x}\sin\frac{x}{2}\times$$

$$\times\Big[(16-8\iota)e^{2x}x\ {}_3F_2\left(-\frac{1}{2}-\iota, -\frac{1}{2}-\iota, 1; \frac{1}{2}-\iota, \frac{1}{2}-\iota; e^{\iota x}\right) +$$

$$+(16-8\iota)e^{\iota x}x\ {}_3F_2\left(\frac{1}{2}+\iota, \frac{1}{2}+\iota, 1; \frac{3}{2}+\iota, \frac{3}{2}+\iota; e^{\iota x}\right) -$$

$$-16e^{2x}{}_4F_3\left(-\frac{1}{2}-\iota, -\frac{1}{2}-\iota, -\frac{1}{2}-\iota, 1; \frac{1}{2}-\iota, \frac{1}{2}-\iota, \frac{1}{2}-\iota; e^{\iota x}\right) +$$

$$+16e^{\iota x}{}_4F_3\left(\frac{1}{2}+\iota, \frac{1}{2}+\iota, \frac{1}{2}+\iota, 1; \frac{3}{2}+\iota, \frac{3}{2}+\iota, \frac{3}{2}+\iota; e^{\iota x}\right) -$$

$$-(6-8\iota)e^{2x}x^2\ {}_2F_1\left(-\frac{1}{2}-\iota, 1; \frac{1}{2}-\iota; e^{\iota x}\right) + (6-8\iota)e^{\iota x}x^2\ {}_2F_1\left(\frac{1}{2}+\iota, 1; \frac{3}{2}+\iota; e^{\iota x}\right) +$$

Notes

$$+(3-4\iota)e^{2x}\ {}_2F_1\left(-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota;e^{\iota x}\right)-(3-4\iota)e^{\iota x}\ {}_2F_1\left(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^{\iota x}\right)+$$

$$+(6-8\iota)e^{2x}x^2-(16-8\iota)e^{2x}x+(13+4\iota)e^{2x}\Bigg]+Constant \quad (2.2)$$

$$\int\frac{\sinh x\ C_2(x)}{\sqrt{(1-\cos x)}}\ \mathrm{dx}=-\frac{1}{\sqrt{1-\cos x}}\left(\frac{22}{125}-\frac{4\iota}{125}\right)e^{(-1-\frac{\iota}{2})x}\sin\frac{x}{2}\times$$

$$\times\Bigg[(16-8\iota)e^{2x}x\ {}_3F_2\left(-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota,\frac{1}{2}-\iota;e^{\iota x}\right)-$$

$$-(16-8\iota)e^{\iota x}x\ {}_3F_2\left(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^{\iota x}\right)-$$

$$-16e^{2x}{}_4F_3\left(-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota,\frac{1}{2}-\iota,\frac{1}{2}-\iota;e^{\iota x}\right)-$$

$$-16e^{\iota x}{}_4F_3\left(\frac{1}{2}+\iota,\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^{\iota x}\right)-$$

$$-(6-8\iota)e^{2x}x^2\ {}_2F_1\left(-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota;e^{\iota x}\right)-(6-8\iota)e^{\iota x}x^2\ {}_2F_1\left(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^{\iota x}\right)+$$

$$+(3-4\iota)e^{2x}\ {}_2F_1\left(-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota;e^{\iota x}\right)+(3-4\iota)e^{\iota x}\ {}_2F_1\left(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^{\iota x}\right)+$$

$$+(6-8\iota)e^{2x}x^2-(16-8\iota)e^{2x}x+(13+4\iota)e^{2x}\Bigg]+Constant \quad (2.3)$$

$$\int\frac{\cosh x\ C_1(x)}{\sqrt{(1-\cos x)}}\ \mathrm{dx}=-\frac{1}{\sqrt{1-\cos x}}\left(\frac{16}{25}-\frac{12\iota}{25}\right)e^{(-1-\frac{\iota}{2})x}\sin\frac{x}{2}\times$$

$$\times\Bigg[2e^{2x}\ {}_3F_2\left(-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota,\frac{1}{2}-\iota;e^{\iota x}\right)+2e^{\iota x}{}_3F_2\left(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^{\iota x}\right)-$$

$$-(2-\iota)e^{2x}\ x{}_2F_1\left(-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota;e^{\iota x}\right)+(2-\iota)e^{\iota x}{}_2F_1\left(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^{\iota x}\right)+$$

$$+(2-\iota)e^{2x}x-2e^{2x}\Bigg]+Constant \quad (2.4)$$

$$\int\frac{\sinh x\ C_1(x)}{\sqrt{(1-\cos x)}}\ \mathrm{dx}=-\frac{1}{\sqrt{1-\cos x}}\left(\frac{16}{25}-\frac{12\iota}{25}\right)e^{(-1-\frac{\iota}{2})x}\sin\frac{x}{2}\times$$

$$\times\Bigg[2e^{2x}\ {}_3F_2\left(-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota,\frac{1}{2}-\iota;e^{\iota x}\right)-2e^{\iota x}{}_3F_2\left(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^{\iota x}\right)-$$

$$-(2-\iota)e^{2x}\ x{}_2F_1\left(-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota;e^{\iota x}\right)-(2-\iota)e^{\iota x}{}_2F_1\left(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^{\iota x}\right)+$$

$$+(2-\iota)e^{2x}x-2e^{2x}\Bigg]+Constant \quad (2.5)$$

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III. Derivation of The Integrals

Involving the same parallel method of ref[3] , one can derive the integrals.

IV. Applications

The integrals which are presented here are very special integrals.These are applied in the field of engineering and other allied sciences.

V. Conclusion

Notes

In our work we have established certain indefinite integrals involving Gegenbauer polynomials and Hypergeometric function . However, one can establish such type of integrals which are very useful for different field of engineering and sciences by involving these integrals.Thus we can only hope that the development presented in this work will stimulate further interest and research in this important area of classical special functions.

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