On Semi 3-Crossed Module by Using Simplicial Algebra
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On Semi 3–Crossed Module by Using Simplicial Algebra

Ali Mutlu$^\alpha$ & Berrin Mutlu$^\sigma$

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1. Introduction

Simplicial algebras play an important role in homological algebra, homotopy theory and algebraic $K$–theory. In each theory the internal structures has been studied relatively. The present article intends to study the 4–types of a simplicial algebra.

Crossed module was initially defined by J.H.C. Whitehead in [10] as a model for 2–types (homotopy) and used it in various contexts, especially in his investigation into the algebraic structure of second relative homotopy groups. We use the definition and elementary theory of crossed module of a commutative algebra given by [9].

Higher dimensional analogues of crossed modules of groups and commutative algebras have been defined respectively: [4] has defined a 2–crossed module of groups as model for 2–types. A 2–crossed module of algebras was given by [5].

In this paper, we extend the crossed module to 4–types by using simplicial method. We also give the description of semi 3–crossed module of commutative algebras and present some applications of Peiffer elements on Moore complex of a simplicial algebra. In particular we investigate Moore complex sequence for $i \geq k \in \{0,1,\ldots,n+2\}$. Let $A$ be a simplicial algebra and $NA_i = 0$ where $NA_i = \bigcap_{i=0}^{n-1} \text{Ker} d_i$ is a Moore complex of $A$. We examine the simplicial long sequence and the Moore long sequence as follows respectively.

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\[ \begin{array}{cccccc}
A_n & A_{n-1} & A_2 & A_1 & A_0 \\
\downarrow d_n & \downarrow d_{n-1} & \downarrow d_2 & \downarrow d_1 & \downarrow d_0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\downarrow s_0 & \downarrow s_1 & \downarrow s_0 & \downarrow s_0 & \\
& & & & \\
\end{array} \]

and

\[ \cdots \to 0 \to NA_n \to \cdots \to NA_2 \to NA_1 \to NA_0. \]

Also we iterate relation between the Moore long exact sequence consists of crossed complex, 2-crossed module, square complex, 2-crossed complex which is defined in [6]. We describe semi 3-crossed module of a commutative algebra, using by \( C_{\alpha,\beta} \) Peiffer elements are defined in [2]. Our aim is given relation between algebraic topology constructions in this article. Observe that Moore complex is the relation between structure of algebraic topology and a simplicial algebra.

II. Construction of Semi 3-Crossed Module

Before giving definition of semi 3-crossed module it will be helpful to have notion of a pre-crossed module and introduce description of pre 2-crossed module.

Throughout this article we denote an action of \( c_0 \in C_0 \) on \( c_1 \in C_1 \) by \( c_0 \cdot c_1 \).

**Definition 2.1** Let \( C_0 \) be a \( k \)-algebra with identity. A pre-crossed module of commutative algebras is a \( C_0 \)-algebra, \( C_1 \) together with a \( C_0 \)-algebra morphism

\[ \partial : C_1 \to C_0, \]

such that for all \( c_1 \in C_1, \ c_0 \in C_0 \) \( \partial(c_0 \cdot c_1) = c_0 \partial(c_1) \).

Now we may describe the definition of a pre 2-crossed module and semi 3-crossed module of commutative algebras.

**Definition 2.2** A pre-2-crossed module of \( k \)-algebras consists of complex of \( C_0 \)-algebra

\[ C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \]

with \( \partial_2, \partial_1 \) morphisms of \( C_0 \)-algebra, where the algebra \( C_0 \) acts on itself by multiplication such that

\[ C_2 \xrightarrow{\partial_2} C_1 \]

is pre-crossed module in which \( C_1 \) acts on \( C_2 \), (we require that for all \( x \in C_2, \ y \in C_1 \) and \( z \in C_0 \) \( (xy)z = x(yz) \)) further, there is a \( C_0 \)-bilinear function giving

\[ \{ \otimes \} : C_1 \otimes C_1 \to C_2 \]
called Peiffer lifting, which satisfies the following axioms:

\[
\begin{align*}
2CM_1^p & \quad \partial_2\{y_0 \otimes y_1\} = y_0y_1 - y_0\partial_1(y_1) \\
2CM_2^p & \quad \{y_0 \otimes y_1y_2\} = \{y_0y_1 \otimes y_2\} + \partial_1(y_2)\{y_0 \otimes y_1\} \\
2CM_3^p & \quad \{y_0 \otimes y_1\} \cdot z = \{y_0 \otimes y_1 \cdot z\}
\end{align*}
\]

for all \(x, x_1, x_2 \in C_2, \ y, y_0, y_1, y_2 \in C_1\) and \(z \in C_0\).

Let \(A\) be a simplicial algebra with the Moore complex \(NA\). Then the complex of algebras

\[
NA_2 \xrightarrow{\partial_2} NA_1 \xrightarrow{\partial_1} NA_0
\]

is a pre 2-crossed module of algebras, where the Peiffer map is defined as follows:

\[
\{ \otimes \} : \quad NA_1 \otimes NA_1 \longrightarrow NA_2 \\
(x_0 \otimes x_1) \longmapsto s_1(x_0)(s_1(x_1) - s_0(x_1)).
\]

It is obvious that the pre-crossed module condition is obviously satisfied. Indeed it is sufficient to show that \(\partial_2, \partial_1\) are pre-crossed modules and pre 2-crossed module axioms are verified. That is \(NA_0\) acts on \(NA_1\) via \(s_0\) and \(NA_1\) acts on \(NA_2\) and also \(s_1\) and \(NA_0\) acts on \(NA_2\) via \(s_1s_0\). Thus

\[
\partial_1(x_0 \cdot x_1) = \partial_1(s_0(x_0)x_1) = x_0\partial_1(x_1)
\]

\[
\partial_2(x_1 \cdot x_2) = \partial_2(s_1(x_1)x_2) = x_1\partial_2(x_2)
\]

2CM_1^p:

\[
\partial_2\{x_0 \otimes x_1\} = \partial_2(s_1(x_0)(s_1(x_1) - s_0(x_1))),
\]

\[
= x_0x_1 - x_0\partial_1(x_1).
\]

Other two conditions are clear where \(\partial_1, \partial_2\) are restrictions of \(d_1, d_2\) respectively.

Now we can give the definition of a semi 3-crossed module of commutative algebras.

**Definition 2.3** A semi 3-crossed module of \(k\)-algebras consists of a complex \(C_0\)-algebra

\[
C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0
\]

with \(\partial_3, \partial_2, \partial_1\) are morphisms of \(C_0\)-algebra, where the algebra \(C_0\) acts on itself by multiplication, such that

\[
C_3 \xrightarrow{\partial_3} C_2
\]

is a crossed module and

\[
C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0
\]

is a pre 2-crossed module. Thus \(C_2\) acts on \(C_3\) and we require that for all \(w \in C_3, \ x \in C_2, \ y \in C_1\) and \(z \in C_0\) that
(wx)(yz) = \left(w(x(yz))\right).

Furthermore there is also a $C_0$-equivalent function defined as
\[
\{ \otimes \} : C_2 \otimes C_2 \to C_3
\]

Mutlu-Arvasi mapping may be defined as follows
\[
\{x_2 \otimes x'_2\} = H(x_2 \otimes x'_2) = s_1(x_2)s_0(x'_2) - s_1(x_2)s_1(x'_2) + s_2(x_2)s_2(x'_2)
\]
if the following conditions are verified.

3CM1. $\partial_2, \partial_1$ are pre-crossed modules, \( \partial_3 \) is a crossed module
3CM2. $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ is a pre $2-$crossed module
3CM3. $H(x_2 \otimes x'_2) = s_1d_2(x_2)s_0d_2(x'_2) - s_1d_2(x_2)s_1d_2(x'_2) + x_2x'_2$
3CM4. (a) $H(x_2 \otimes \partial_3(y_3)) = s_2(x_2)y_3$
   (b) $H(\partial_3(y_3) \otimes x_2) = s_2(x_2)y_3$
3CM5. $H(x_2 \otimes \partial_3(y_3))H(\partial_3(y_3) \otimes x_2) = 0$
3CM6. $H(\partial_3(y_3) \otimes \partial_3(y'_3)) = y_3y'_3$

where $x_2, x'_2 \in C_2$ and $y_3, y'_3 \in C_3$.

**Theorem 2.4** (a) If $NA_i = 0$ for $\forall i \geq 1$ in the Moore long sequence, then the Moore long sequence become only an algebra i.e., $A_0$ be an algebra.

(b) If $NA_i = 0$ for $\forall i \geq 2$ in the Moore long sequence, then the Moore long sequence be a crossed module i.e., $\cdots 0 \to 0 \to NA_1 \to NA_0$ is a crossed module.

(c) If $NA_i = 0$ for $\forall i \geq 3$ in the Moore long sequence, then the Moore long sequence become a $2-$crossed module i.e., $\cdots 0 \to 0 \to NA_2 \to NA_1 \to NA_0$ is a $2-$crossed module.

(d) If $NA_i = 0$ for $\forall i \geq 4$ in the Moore long sequence, then the Moore long sequence be semi $3-$crossed module i.e., $\cdots 0 \to 0 \to NA_3 \to NA_2 \to NA_1 \to NA_0$ is a $3-$semi crossed module.

(e) If $NA_i = 0$ for $\forall i \geq n + 1$ in the Moore long sequence, then the Moore long sequence become an $n-$crossed complex i.e., $\cdots 0 \to 0 \to NA_n \to NA_{n-1} \to \cdots \to NA_3 \to NA_2 \to NA_1 \to NA_0$ is an $n-$crossed complex.

(f) If $NA_i = 0$ for $\forall i \geq n + 2$ in the Moore long sequence, then the Moore long sequence be a $T-$complex.

(g) If $C_{a,b}(x_\alpha, y_\beta) = 0$ hypercrossed complex pairings are described in $[2]$ and $[3]$, then the Moore long sequence be a crossed complex.
Proof: (a) Suppose that \( NA_i = 0 \) for \( \forall i \geq 1 \) and so the Moore long sequence obtains as follows. \( \cdots 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow NA_0 = A_0. \) Therefore \( NA_1 = \text{Ker} d_0^1 \) is an ideal of \( A_0. \) On the other hand, if \( a \in \text{Ker} d_0^1, \) then \( NA_1 = 0 \text{ since } d_0(a) = 0. \)

(b) If \( NA_i = 0, \) for \( \forall i \geq 2, \) then \( \cdots 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0 \rightarrow NA_1 \rightarrow NA_0 = A_0 \) be a crossed module (see [2, 9]). On other word, recall that \( C_{\alpha,\beta}(x_{\alpha} \otimes y_{\beta}) = 0 \) in [2], then for \( \alpha = (1), \beta = (0) \)

\[
\begin{align*}
C_{(1),(0)}(x_1 \otimes y_1) & = NA_1 \times NA_1 \rightarrow NA_2 \\
C_{(1),(0)}(x_1 \otimes y_1) & = s_1(x_1)(s_1(y_1) - s_0(y_1)) = 0
\end{align*}
\]

since \( NA_1 \rightarrow NA_0 \) be crossed module i.e, \( NA_0 \) acts on \( NA_1 \) together with \( x_1 \cdot y_1 = s_1(x_1)s_0(y_1) \) and so crossed axioms are verified indeed,

\[
\partial_1(x_1)y_1 = x_1\partial_1(y_1)
\]

and

\[
\partial_1(x_1)y_1 = x_1s_0d_1(y_1) = x_1d_1(y_1) = x_1y_1 \quad (\partial_1 \text{ is defined by restriction } d_1).
\]

(see [2]) Also 1–truncated hypercrossed complex, 1–hypercrossed complex and 1–crossed complex (see [3]).

(c) If \( NA_i = 0 \) for \( \forall i \geq 3, \) then the Moore long sequence \( \cdots 0 \rightarrow 0 \rightarrow NA_2 \rightarrow NA_1 \rightarrow NA_0 \) is a 2–crossed module and \( C^{(3)}_{\alpha,\beta}(x_{\alpha} \otimes y_{\beta}) = 0 \) for \( \alpha, \beta \in P(3). \) (see [2]) So the Peiffer lifting is defined as follows.

\[
\{ \otimes \} : NA_1 \otimes NA_1 \rightarrow NA_2
\]

\[
\{ x \otimes y \} \mapsto s_1(x_1)(s_1(y_1) - s_0(y_1)) = 0
\]

and 2–crossed module conditions are also satisfied. On the other hand, 2CM2, 2CM4 (a) and (b) of 2–crossed module axioms give us \( C^{(3)}_{\alpha,\beta}(x_{\alpha} \otimes y_{\beta}) = 0, \) which implies \( NA_3 = 0. \) (see [2])

(d) Let \( A \) be a simplicial algebra with the Moore complex \( NA. \) Then the complex of algebras

\[
NA_3/\partial_4(NA_4 \cap I_4) \xrightarrow{\partial_3} NA_2 \xrightarrow{\partial_2} NA_1 \xrightarrow{\partial_1} NA_0
\]

is a semi 3–crossed module of algebras, where and also \( I_4 \) is the ideal generated by the degenerate elements. Now we can define Muthu-Arvasi map as follows:

\[
\{ \otimes \} : NA_2 \otimes NA_2 \rightarrow NA_3/\partial_4(NA_4 \cap I_4)
\]

\[
(x_2 \otimes y_2) \mapsto s_1(x_2)s_0(y_2) - s_1(x_2)s_1(y_2) + s_2(x_2)s_2(y_2)
\]

here the right hand side denotes an ideal in \( NA_3/\partial_4(NA_4 \cap I_4) \) represented by the corresponding element in \( NA_3. \)

3CM1a (a) \( \partial_2, \partial_1 \) are pre-crossed modules that is \( NA_1 \) acts on \( NA_2 \) via \( s_1 \) and \( NA_0 \) acts on \( NA_1 \) via \( s_0. \) Thus \( \partial_1(x_0 \cdot y_1) = \partial_1(s_0(x_0)y_1) = x_0\partial_1(y_1) \) and \( \partial_2(y_1 \cdot y_2) = \partial_2(s_1(y_1)y_2) = y_1\partial_2(y_2) = y_1\partial_2(y_2). \)
(b) It is readily checked that the morphism \( \partial_3 : \text{NA}_4/\partial_4(\text{NA}_4 \cap D_4) \to \text{NA}_2 \) is a crossed module i.e., \( \text{NA}_2 \) acts on \( \text{NA}_4/\partial_4(\text{NA}_4 \cap D_4) \) via \( s_2 \) and we have \( \partial_4 C_{(3)(2)}(x_3 \otimes y_3) = d_4(s_3x_3s_2y_3 - s_3y_3) = 0 \) via mod \( \partial_4(\text{NA}_4 \cap D_4) \) from [2]. Thus \( \partial_4 C_{(3)(2)}(x_3 \otimes y_3) = x_3(s_2\partial_2(y_3) - y_3) \) mod \( \partial_4(\text{NA}_4 \cap D_4) \) so \( \partial_3(x_3 \cdot y_3) = \partial_3(s_3(x_3))y_3 = x_3\partial_3(y_3) \) and \( (\partial_3(x_3))y_3 = x_3s_2\partial_3(y_3) = x_3y_3 \) is obtained.

3CM2s \( \text{NA}_2 \to \text{NA}_1 \to \text{NA}_0 \) is a pre 2–crossed module, where Peiffer map is defined as follows:
\[ \{x_0 \otimes x_1\} \mapsto s_1(x_0)(s_1(x_1) - s_0(x_1)). \]

3CM3s
\[ \partial_1 H(x_2 \otimes y_2) = s_1d_2(x_2)s_0d_2(y_2) - s_1d_2(x_2)s_1d_2(y_2) + x_2y_2 \]
\[ = s_1d_2(x_2)(s_0d_2(y_2) - s_1d_2(y_2)) + x_2y_2 \]

3CM4s (a) Using the hypercrossed complex parings are defined in [2] and then
\[ 0 \equiv \partial_4 C_{(3,1)(0)}(x_2 \otimes y_3) = s_1(x_2)s_0d_3(y_3) - s_1(x_2)s_2d_3(y_3) + s_2(x_2)s_2d_3(y_3) - s_2(x_2)y_3 \]
mod \( \partial_4(\text{NA}_4 \cap I_4) \).

is calculated. Thus, we have
\[ H(x_2 \otimes \partial_3(y_3)) = s_1(x_2)s_0d_3(y_3) - s_1(x_2)s_2d_3(y_3) + s_2(x_2)s_2d_3(y_3) \mod \partial_4(\text{NA}_4 \cap I_4) \]
and therefore we obtain
\[ H(x_2 \otimes \partial_3(y_3)) = s_2(x_2)y_3 \mod \partial_4(\text{NA}_4 \cap I_4). \]

(b) Again using the hypercrossed complex parings in [2] then
\[ 0 \equiv \partial_4 C_{(1,0,3)}^{(4)}(y_3 \otimes x_2) = s_1d_3(y_3)s_0(x_2) - s_1d_3(y_3)s_1(x_2) + s_1d_3(y_3)s_0(x_2) - s_1d_3(y_3)s_1(x_2) + y_3s_2(x_2) \]
mod \( \partial_4(\text{NA}_4 \cap I_4) \).

is found. This equality also holds
\[ H(\partial_3(y_3) \otimes x_2) = s_1d_3(y_3)s_0(x_2) - s_1d_3(y_3)s_1(x_2) + s_2d_3(y_3)s_2(x_2) \mod \partial_4(\text{NA}_4 \cap I_4). \]

and so we acquire
\[ H(\partial_3(y_3) \otimes x_2) = -s_2(x_2)y_3 \mod \partial_4(\text{NA}_4 \cap I_4) \]
is commutated. Thus, the result of is given (a) and (b) of 3CM4s as above.

3CM5s
\[ H(x_2 \otimes \partial_3(y_3)) + H(\partial_3(y_3) \otimes x_2) = s_2(x_2)y_3 - s_2(x_2)y_3 = 0. \]
3CM6. By [2] we may also be written this equation as.

\[ 0 \equiv \partial_4 C^{(4)}_{(1)(0)}(y_3 \otimes y_3) = s_1d_3(y_3)s_0d_3(y'_3) - s_1d_3(y_3)s_1d_3(y'_3) + s_2d_3(y_3)s_2d_3(y'_3) - y'_3y_3 \mod \partial_4(NA_4 \cap I_4). \]

Using the equation is obtained as

\[ H(\partial_3(y_3) \otimes \partial_3(y'_3)) = s_1d_3(y_3)s_0d_3(y'_3) - s_1d_3(y_3)s_1d_3(y'_3) + s_2d_3(y_3)s_2d_3(y'_3) \mod \partial_4(NA_4 \cap I_4). \]

Hence, we yield

\[ H(\partial_3(y_3) \otimes \partial_3(y'_3)) \equiv y'_3y_3 \mod \partial_4(NA_4 \cap I_4) \]

(e) If \( NA_i = 0 \) for \( \forall i \geq n + 1 \) in the Moore long sequence, then the Moore long sequence be an \( n \)-crossed complex with \( C^{(n+1)}_{(\alpha)(\beta)}(x \otimes y) = 0 \). Recall that from [2] we have the trivial map as follows.

\[ C^{(n+1)}_{(\alpha)(\beta)}(x \otimes y) = NA_{(n+1)}-\alpha \otimes NA_{(n+1)}-\beta \rightarrow NA_{n+1}. \]

And so this \( NA_n \) also be a commutative algebra for \( n \geq 2 \) since

\[ 0 = \partial_{n+1}C^{(n+1)}_{(n-1),(n)}(x \otimes y) = s_{n-1}d_n(x)y - xy = (\phi^{(n+1)}_{n-1}d_n(x))y - xy = yx. \]

Here \( NA \) is a simplicial chain complex where \( NA_n \) is commutative for \( n \geq 2 \), \( \phi^{(n+1)}_{n-1} \) is an action of \( NA_0 \) on \( NA_n \) for each \( n \geq 1 \) and \( \partial_n \) is \( NA_0 \)-algebra morphism defined as

\[ \cdots \rightarrow NA_n/\partial_{n+1}K_{n+1} \rightarrow NA_{n-1}/\partial_nK_n \rightarrow \cdots \rightarrow NA_2/\partial_3K_3 \rightarrow NA_1/\partial_2K_2 \rightarrow NA_0 \]

this is obviously a crossed complex, where \( K_i = NA_i \cap I_i \).

To prove the opposite of it let \( NA_n/\partial_{n+1}K_{n+1} \) for \( n \geq 2 \), then

Thus \( C^{(n-1)}_{(\alpha)(\beta)}(x, y) = 0 \) implies that \( NA_{n+1} = 0 \). This is also an \( n \)-truncated complex. (see [3]).

(f) If \( NA_i = 0 \) for \( \forall i \geq n + 2 \), then the Moore long sequence be a \( T \)-complex. To proof see [1] and [3].

(g) If \( C^{(n-1)}_{(\alpha)(\beta)}(x_\alpha, y_\beta) = 0 \), then the Moore long sequence become a crossed complex.

Therefore, we have the following results.
Corollary 2.5 If $NA_3/\partial_3(NA_4 \cap I_4) = 0$, then $NA_2 \rightarrow NA_1 \rightarrow NA_0$ corresponds a 2-crossed module. (see [2])

Proof: Let $A$ be a simplicial algebra with the Moore complex $NA$. Then the complex of algebras

$$
\begin{align*}
NA_2 \xrightarrow{\partial_2} NA_1 \xrightarrow{\partial_1} NA_0
\end{align*}
$$

is a 2-crossed module of algebras, where the Peiffer map is defined as follows:

$$
\{ \otimes \} : \begin{array}{c}
NA_1 \otimes NA_1 \\
(NA_2/\partial_3(NA_3 \cap I_3))
\end{array} \rightarrow \begin{array}{c}
NA_2/\partial_3(NA_3 \cap I_3) \\
(x_1 \otimes y_1) \rightarrow s_1x_1(s_1y_1 - s_0y_1).
\end{array}
$$

Here the right hand side denotes a coset in $NA_2/\partial_3(NA_3 \cap I_3)$ represented by an element in $NA_2$ and $\partial_3(NA_3 \cap I_3) = 0$.

Corollary 2.6 If $NA_0 = 0$, then $NA_3/\partial_4(NA_4 \cap I_4) \xrightarrow{\partial_3} NA_2 \xrightarrow{\partial_2} NA_1$ is a 2-crossed module with defined Peiffer map as

$$
\{ \otimes \} : \begin{array}{c}
NA_2 \times NA_2 \\
(NA_3/\partial_4(NA_4 \cap I_4))
\end{array} \rightarrow \begin{array}{c}
NA_3/\partial_4(NA_4 \cap I_4) \\
(x_2 \otimes y_2) \rightarrow s_2(x_2)s_2(y_2) - s_2(x_2)s_2(y_2).
\end{array}
$$

Proof: Indeed the function is satisfied 2-crossed module axioms.

2CM1: $\partial_3\{x_2 \otimes y_2\} = x_2s_1\partial_2(y_2) - x_2y_2$.

2CM2: $\{\partial_3(x_2)\otimes\partial_3(y_2)\} = x_2y_2$ since $\partial_4C^{(4)}_{(2)(1)}(x_2 \otimes y_2) = s_2d_3(x_2)(s_1d_3(y_3) - s_2d_3(y_3)) + x_3y_3 = 0 \mod \partial_4(NA_4 \cap I_4)$. (see [2])

2CM3: $\{x_2 \otimes y_2y_2\} = \{x_2y_2 \otimes y_2\} + \partial_2y_2\{x_2 \otimes y_2\}$

2CM4: (a) Let $\partial_4C^{(4)}_{(3,2)(1)}(x_2 \otimes y_3) = s_2(x_2)(s_1d_3(y_3) - s_2d_3(y_3) + y_3)$. So $\partial_4C^{(4)}_{(3,2)(1)}(x_2 \otimes y_3) = 0 \mod \partial_4(NA_4 \cap I_4)$. (see [2]) Then $\{x_2 \otimes \partial_3(y_3)\} = s_2(x_2)y_3$ is obtained by the definition of action.

(b) Let $\partial_4C^{(4)}_{(2)(3,1)}(y_3 \otimes x_2) = (s_2d_3(y_3) - y_3)(s_1x_2 - s_2(x_2))$. So $\partial_4C^{(4)}_{(2)(3,1)}(y_3 \otimes x_2) = 0 \mod \partial_4(NA_4 \cap I_4)$. (see [2]) Then $\{\partial_3(y_3) \otimes x_2\} = y_3x_2 - \partial_2(y_3)x_2$ mod $\partial_4(NA_4 \cap I_4)$ is found.

2CM5

$$
\{x_2 \otimes y_2\} \cdot z = s_2(x_2)(s_2(y_2) - s_0(y_2)) \cdot z
= s_2s_0(z)s_2(x_2)(s_2(y_2) - s_0(y_2))
= s_2(s_0(zx_2))(s_2(y_2) - s_0(y_2))
= s_2(x_2z)(s_2(y_2) - s_0(y_2))
= \{x_2 \cdot z \otimes y_2\}.
$$

Now we can consider the following diagram of morphism

$$
\begin{array}{c}
\xymatrix{
NA_2 \otimes NA_2 \ar[rr]^\rho & & \\
NA_3/\partial_4(NA_4 \cap I_4) \ar[r]_{\partial_3} & NA_2 \ar[r]_{\partial_2} & NA_1.
}
\end{array}
$$

Ref.

The algebra $NA_2$ acts, in two ways on the algebra $NA_3/\partial_4(NA_4 \cap I_4)$ by multiplication via $s_1$ and via $s_2$ both within $A_3$. The action via $s_1$ will also be denoted by $x \cdot y = s_1(x)y$ and the action via $s_2$ will be denoted by $xy = s_2(x)y$. The action of $NA_1$ on $NA_3$ is given as follows: from equality $(s_1(x) - s_2s_1d_2(x))y \equiv 0 \mod NA_3/\partial_4(NA_4 \cap I_4)$, there is a commutative diagram

$$
\begin{array}{c}
NA_3/\partial_4(NA_4 \cap I_4) \otimes NA_2 \rightarrow NA_3/\partial_4(NA_4 \cap I_4) \\
\end{array}
$$

given by

$$
\begin{array}{c}
(y \otimes x) \rightarrow x \cdot y = s_1(x)y \\
\end{array}
$$

which gives an equality

$$
\partial_2(xy) = s_2s_1d_2(x)y = s_1(x)y.
$$

Let us define the map $\rho$ by $\rho(x \otimes x') = \partial_2(x)x' - xx'$ for $x, x' \in NA_2$, that is the Peiffer element in $NA_2$ which corresponds to $\{x \otimes x'\}$. Thus if the map $\rho$ is the trivial map $\partial_2 : NA_2 \rightarrow NA_1$ is a crossed module.

Now if the Moore long sequence is iterated as follows, so then two results are obtained where $K_i = NA_i \cap I_i$.

$$
\cdots 0 \rightarrow 0 \rightarrow NA_n/\partial_{n+1}K_{n+1} \rightarrow NA_{n-1}/\partial_nK_n \\
\end{array}
$$

Corollary 2.7

$$
\cdots 0 \rightarrow 0 \rightarrow NA_k/\partial_{k+1}K_{k+1} \rightarrow NA_k/\partial_{k-1}NA_{k-2} \\
\end{array}
$$

is a 2-crossed module with defined Peiffer element

$$
\{x_{k-1} \otimes y_{k-1}\} = C_{(1)(0)}(x_{k-1} \otimes y_{k-1}).
$$

So the 2-crossed module conditions are clearly verified.

Corollary 2.8

$$
\cdots 0 \rightarrow 0 \rightarrow NA_k/\partial_{k+1}K_{k+1} \rightarrow NA_k/\partial_{k-1}NA_{k-1} \rightarrow 0
$$
is a semi 3−crossed module, where the Mutlu-Arvasi map is defined as follows:

\[ \{x_{k-1} \otimes y_{k-1}\} = C^{(k)}_{(0)(1)}(x_{k-1} \otimes y_{k-1}) \]

It is clear that semi 3−crossed module conditions are satisfied.

**Corollary 2.9** A 3−truncated complex is a semi 3−crossed module.

We may follow the same procure as we did in Corolarly 2.8 in order to get to results.

**Corollary 2.10** The category of semi 3−crossed modules is equivalent to the category of simplicial algebras with Moore complex of length 3.

**References Références Referencias**


