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Ref.

[5] Grandjeán A.R. & Vale, M.J., 2-Modulos cruzados en la cohomologia de André-Quillen, *Memorias de la Real Academia de Ciencias* **22**

On Semi 3−Crossed Module by Using Simplicial Algebra

Ali Mutlu[α] & Berrin Mutlu[σ]

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I. Introduction

Simplicial algebras play an important role in homological algebras homotopy theory and algebraic $K-$theory. In each theory the internal structures has been studied relatively. The present article intends to study the $4-$types of a simplicial algebra.

Crossed module was initially defined by J.H.C. Whitehead in [10] as a model for $2-$types (homotopy) and used it in various contexts, especially in his investigation into the algebraic structure of second relative homotopy groups. We use the definition and elementary theory of crossed module of a commutative algebra given by [9].

Higher dimensional analogues of crossed modules of groups and commutative algebras have been defined respectively: [4] has defined a $2-$crossed module of groups as model for $2-$types. A $2-$crossed module of algebras was given by [5].

In this paper, we extend the crossed module to $4-$types by using simplicial method. We also give the description of semi $3-$crossed module of commutative algebras and present some applications of Peiffer elements on Moore complex of a simplicial algebra. In particular we investigate Moore complex sequence for $i \geq k \in \{0, 1, \ldots, n+2\}$. Let $\mathbf{A}$ be a simplicial algebra and $NA_i = 0$ where $NA_i = \bigcap\limits_{i=0}^{n-1} Kerd_i$ is a Moore complex of $\mathbf{A}$. We examine the simplicial long sequence and the Moore long sequence as follows respectively.

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$$\cdots A_n \underset{s_{n-1},\ldots s_0}{\overset{d_n,\ldots,d_0}{\rightleftarrows}} A_{n-1} \cdots A_2 \underset{s_0,s_1}{\overset{d_0,d_1,d_2}{\rightleftarrows}} A_1 \underset{s_0}{\overset{d_0,d_1}{\rightleftarrows}} A_0$$

and

$$\cdots 0 \longrightarrow 0 \longrightarrow NA_n \longrightarrow \cdots \longrightarrow NA_2 \longrightarrow NA_1 \longrightarrow NA_0 \,.$$

Also we iterate relation between the Moore long exact sequence consists of crossed complex, $2-$crossed module, square complex, $2-$crossed complex which is defined in [6]. We describe semi $3-$crossed module of a commutative algebra, using by $C_{\alpha,\beta}$ Peiffer elements are defined in [2]. Our aim is given relation between algebraic topology constructions in this article. Observe that Moore complex is the relation between structure of algebraic topology and a simplicial algebra.

II. Construction of Semi 3-Crossed Module

Before giving definition of semi $3-$crossed module it will be helpful to have notion of a pre-crossed module and introduce description of pre $2-$crossed module.
Throughout this article we denote an action of $c_0 \in C_0$ on $c_1 \in C_1$ by $c_0 \cdot c_1$.

Definition 2.1 *Let C_0 be a $k-$algebra with identity. A* pre-crossed module *of commutative algebras is a C_0-algebra, C_1 together with a C_0-algebra morphism.*

$$\partial : C_1 \longrightarrow C_0,$$

such that for all $c_1 \in C_1,\ c_0 \in C_0$ $\partial(c_0 \cdot c_1) = c_0 \partial(c_1)$.

Now we may describe the definition of a pre $2-$crossed module and semi $3-$crossed module of commutative algebras.

Definition 2.2 *A* pre$-2-$crossed module *of $k-$algebras consists of complex of C_0-algebra*

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

with $\partial_2,\ \partial_1$ morphisms of C_0-algebra, where the algebra C_0 acts on itself by multiplication such that

$$C_2 \xrightarrow{\partial_2} C_1$$

is pre-crossed module in which C_1 acts on C_2, (we require that for all $x \in C_2,\quad y \in C_1$ and $z \in C_0$ $(xy)z = x(yz)$) further, there is a C_0-bilinear function giving

$$\{\otimes\} : C_1 \otimes C_1 \to C_2$$

Ref.

[2] Arvasi Z. , Applications in Commutative Algebra of The Moore Complex of a Simplicial Algebra, *Ph.D. Thesis*, University of Wales, BANGOR, (1994).

called Peiffer lifting, which satisfies the following axioms:

$$\begin{array}{llcl} 2CM1_p & \partial_2\{y_0 \otimes y_1\} & = & y_0y_1 - y_0\partial_1(y_1) \\ 2CM2_p & \{y_0 \otimes y_1y_2\} & = & \{y_0y_1 \otimes y_2\} + \partial_1(y_2)\ \{y_0 \otimes y_1\} \\ 2CM3_p & \{y_0 \otimes y_1\} \cdot z & = & \{y_0 \otimes y_1 \cdot z\} \end{array}$$

for all $x, x_1, x_2 \in C_2, \quad y, y_0, y_1, y_2 \in C_1$ *and,* $z \in C_0$.

Notes

Let $\mathbf{A}$ be a simplicial algebra with the Moore complex $\mathbf{NA}$. Then the complex of algebras

$$NA_2 \xrightarrow{\partial_2} NA_1 \xrightarrow{\partial_1} NA_0$$

is a pre 2−crossed module of algebras, where the Peiffer map is defined as follows:

$$\begin{array}{rccl} \{\ \otimes\ \}: & NA_1 \otimes NA_1 & \longrightarrow & NA_2 \\ & (x_0 \otimes x_1) & \longmapsto & s_1(x_0)(s_1(x_1) - s_0(x_1)). \end{array}$$

It is obvious that the pre-crossed module condition is obviously satisfied. Indeed it is sufficient to show that ∂_2, ∂_1 are pre-crossed modules and pre 2−crossed module axioms are verified. That is NA_0 acts on NA_1 via s_0 and NA_1 acts on NA_2 and also s_1 and NA_0 acts on NA_2 via s_1s_0. Thus

$$\partial_1(x_0 \cdot x_1) = \partial_1(s_0(x_0)x_1) = x_0\partial_1(x_1)$$

$$\partial_2(x_1 \cdot x_2) = \partial_2(s_1(x_1)x_2) = x_1\partial_2(x_2)$$

2CM1$_p$:

$$\begin{array}{rcl} \partial_2\{x_0 \otimes x_1\} & = & \partial_2(s_1(x_0)(s_1(x_1) - s_0(x_1))), \\ & = & x_0x_1 - x_0\partial_1(x_1). \end{array}$$

Other two conditions are clear where ∂_1, ∂_2 are restrictions of d_1, d_2 respectively.

Now we can give the definition of a semi 3−crossed module of commutative algebras.

Definition 2.3 *A* semi 3−crossed module *of k−algebras consists of a complex C_0−algebra*

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

with $\partial_3, \partial_2, \partial_1$ are morphisms of C_0−algebra, where the algebra C_0 acts on itself by multiplication, such that

$$C_3 \xrightarrow{\partial_3} C_2$$

is a crossed module and

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

is a pre 2−crossed module. Thus C_2 acts on C_3 and we require that for all $w \in C_3, \quad x \in C_2, \quad y \in C_1$ *and* $z \in C_0$ *that*

$$(wx)(yz) = \Big(w(x(yz))\Big).$$

Furthermore there is also a C_0-*equivalent function defined as*

$$\{\ \otimes\ \} : C_2 \otimes C_2 \to C_3$$

Mutlu-Arvasi mapping may be defined as follows

$$\{x_2 \otimes x_2'\} = H(x_2 \otimes x_2') = s_1(x_2)s_0(x_2') - s_1(x_2)s_1(x_2') + s_2(x_2)s_2(x_2')$$

if the following conditions are verified.

$3CM1_s$ ∂_2, ∂_1 *are pre-crossed modules,* ∂_3 *is a crossed module*

$3CM2_s$ $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ *is a pre* $2-$*crossed module*

$3CM3_s$ $\partial_3 H(x_2 \otimes x_2') = s_1d_2(x_2)s_0d_2(x_2') - s_1d_2(x_2)s_1d_2(x_2') + x_2x_2'$

$3CM4_s$ (a) $H(x_2 \otimes \partial_3(y_3)) = s_2(x_2)y_3$
(b) $H(\partial_3(y_3) \otimes x_2) = s_2(x_2)y_3$

$3CM5_s$ $H(x_2 \otimes \partial_3(y_3))H(\partial_3(y_3) \otimes x_2) = 0$

$3CM6_s$ $H(\partial_3(y_3) \otimes\ \partial_3(y_3')) = y_3y_3'$

where $x_2, x_2' \in C_2$ *and* $y_3, y_3' \in C_3$.

Theorem 2.4 ***(a)*** *If* $NA_i = 0$ *for* $\forall i \geq 1$ *in the Moore long sequence, then the Moore long sequence become only an algebra i.e.,* A_0 *be an algebra.*

(b) *If* $NA_i = 0$ *for* $\forall i \geq 2$ *in the Moore long sequence, then the Moore long sequence be a crossed module i.e.,* $\cdots 0 \to 0 \to NA_1 \to NA_0$ *is a crossed module.*

(c) *If* $NA_i = 0$ *for* $\forall i \geq 3$ *in the Moore long sequence, then the Moore long sequence become a* $2-$*crossed module i.e.,* $\cdots 0 \to 0 \to NA_2 \to NA_1 \to NA_0$ *is a* $2-$*crossed module.*

(d) *If* $NA_i = 0$ *for* $\forall i \geq 4$ *in the Moore long sequence, then the Moore long sequence be semi* $3-$*crossed module i.e.,* $\cdots 0 \to 0 \to NA_3 \to NA_2 \to NA_1 \to NA_0$ *is a* $3-$*semi crossed module.*

(e) *If* $NA_i = 0$ *for* $\forall i \geq n+1$ *in the Moore long sequence, then the Moore long sequence become an* $n-$*crossed complex i.e.,* $\cdots 0 \to 0 \to NA_n \to NA_{n-1} \to \cdots \to NA_3 \to NA_2 \to NA_1 \to NA_0$ *is an* $n-$*crossed complex.*

(f) *If* $NA_i = 0$ *for* $\forall i \geq n+2$ *in the Moore long sequence, then the Moore long sequence be a* $T-$*complex.*

(g) *If* $C_{\alpha,\beta}(x_\alpha, y_\beta) = 0$ *hypercrossed complex pairings are described in [2] and [3], then the Moore long sequence be a crossed complex.*

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue XI Version I Year 2012

Proof: **(a)** Suppose that $NA_i = 0$ for $\forall i \geq 1$ and so the Moore long sequence obtains as follows. $\cdots 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to NA_0 = A_0$. Therefore $NA_1 = \mathrm{Ker} d_0^1$ is an ideal of A_0.
On the other hand, if $a \in \mathrm{Ker} d_0^1$, then $NA_1 = 0$ since $d_0(a) = 0$.
(b) If $NA_i = 0$, for $\forall i \geq 2$ $(1 \leq i \leq n+2)$, then $\cdots 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to NA_1 \to NA_0 = A_0$ be a crossed module (see [2, 9]).
On other word, recall that $C_{\alpha,\beta}(x_\alpha \otimes y_\beta) = 0$ in [2], then for $\alpha = (1),\ \ \beta = (0)$

$$\begin{aligned} C_{(1),(0)}(x_1 \otimes y_1) &= NA_1 \times NA_1 \to NA_2 \\ C_{(1),(0)}(x_1 \otimes y_1) &= s_1(x_1)(s_1(y_1) - s_0(y_1)) = 0 \end{aligned}$$

since $NA_1 \to NA_0$ be crossed module i.e, NA_0 acts on NA_1 together with $x_1 \cdot y_1 = s_1(x_1)s_0(y_1)$ and so crossed axioms are verified indeed,

$$\partial_1(x_1)y_1 = x_1\partial_1(y_1)$$

and

$\partial_1(x_1)y_1 = x_1 s_0 d_1(y_1) = x_1 d_1(y_1) = x_1 y_1$ (∂_1 is defined by restriction d_1). (see [2]) Also $1-$truncated hypercrossed complex, $1-$hypercrossed complex and $1-$crossed complex (see [3]).
(c) If $NA_i = 0$ for $\forall i \geq 3$, then the Moore long sequence $\cdots 0 \to 0 \to NA_2 \to NA_1 \to NA_0$ is a $2-$crossed module and $C^{(3)}_{\alpha,\beta}(x_\alpha \otimes y_\beta) = 0$ for $\alpha, \beta \in P(3)$. (see [2]) So the Peiffer lifting is defined as follows.

$$\{\otimes\} : NA_1 \otimes NA_1 \to NA_2$$

$$\{x \otimes y\} \mapsto s_1(x_1)(s_1(y_1) - s_0(y_1)) = 0$$

and $2-$crossed module conditions are also satisfied. On the other hand, 2CM2, 2CM4 (a) and (b) of $2-$crossed module axioms give us $C^{(3)}_{\alpha,\beta}(x_\alpha \otimes y_\beta) = 0$, which implies $NA_3 = 0$. (see [2])
(d) Let $\mathbf{A}$ be a simplicial algebra with the Moore complex $N\mathbf{A}$. Then the complex of algebras

$$NA_3/\partial_4(NA_4 \cap I_4) \xrightarrow{\partial_3} NA_2 \xrightarrow{\partial_2} NA_1 \xrightarrow{\partial_1} NA_0$$

is a semi $3-$crossed module of algebras, where and also I_4 is the ideal generated by the degenerate elements. Now we can define Mutlu-Arvasi map as follows:

$$\begin{aligned} \{\ \ \otimes\ \ \} :\ \ NA_2 \otimes NA_2 &\longrightarrow NA_3/\partial_4(NA_4 \cap I_4) \\ (x_2 \otimes y_2) &\longmapsto s_1(x_2)s_0(y_2) - s_1(x_2)s_1(y_2) + s_2(x_2)s_2(y_2) \end{aligned}$$

here the right hand side denotes an ideal in $NA_3/\partial_4(NA_4 \cap I_4)$ represented by the corresponding element in NA_3.
3CM1$_\mathbf{s}$ (a) $\partial_2,\ \ \partial_1$ are pre-crossed modules that is NA_1 acts on NA_2 via s_1 and NA_0 acts on NA_1 via s_0. Thus $\partial_1(x_0 \cdot y_1) = \partial_1(s_0(x_0)y_1) = x_0\partial_1(y_1)$ and $\partial_2(y_1 \cdot y_2) = \partial_2(s_1(y_1)y_2) = y_1\partial_2(y_2) = y_1\partial_2(y_2)$.

Ref.

[2] Arvasi Z. , Applications in Commutative Algebra of The Moore Complex of a Simplicial Algebra, *Ph.D. Thesis*, University of Wales, BANGOR, (1994).

(b) It is readily checked that the morphism $\partial_3 : NA_3/\partial_4(NA_4 \cap D_4) \to NA_2$ is a crossed module i.e., NA_2 acts on $NA_3/\partial_4(NA_4 \cap D_4)$ via s_2 and we have $\partial_4 C_{(3)(2)}(x_3 \otimes y_3) = d_4(s_3x_3(s_2y_3 - s_3y_3) = 0$ via mod $\partial_4(NA_4 \cap D_4)$ from [2]. Thus $\partial_4 C_{(3)(2)}(x_3 \otimes y_3) = x_3(s_2\partial_2(y_3) - y_3)$ mod $\partial_4(NA_4 \cap D_4)$ so $\partial_3(x_3 \cdot y_3) = \partial_3(s_3(x_3))y_3 = x_3\partial_3(y_3)$ and $(\partial_3(x_3))y_3 = x_3 s_2 \partial_3(y_3) = x_3y_3$ is obtained

3CM2$_s$ $NA_2 \to NA_1 \to NA_0$ is a pre 2−crossed module, where Peiffer map is defined as follows:
$\{x_0 \otimes \ x_1\} \longmapsto s_1(x_0)(s_1(x_1) - s_0(x_1))$.

3CM3$_s$

$$\begin{aligned}\partial_4 H(x_2 \otimes y_2) &= s_1d_2(x_2)s_0d_2(y_2) - s_1d_2(x_2)s_1d_2(y_2) + x_2y_2 \\ &= s_1d_2(x_2)(s_0d_2(y_2) - s_1d_2(x_2)) + x_2y_2\end{aligned}$$

3CM4$_s$ (a) Using the hypercrossed complex parings are defined in [2] and then

$$\begin{aligned}0 \equiv \partial_4 C^{(4)}_{(3,1)(0)}(x_2 \otimes y_3) \ &= s_1(x_2)s_0d_3(y_3) - s_1(x_2)s_2d_3(y_3) + s_2(x_2)s_2d_3(y_3) - s_2(x_2)y_3 \\ &\quad \bmod \partial_4(NA_4 \cap I_4).\end{aligned}$$

is calculated. Thus, we have

$$H(x_2 \otimes \partial_3(y_3)) \ = s_1(x_2)s_0d_3(y_3) - s_1(x_2)s_2d_3(y_3) + s_2(x_2)s_2d_3(y_3) \ \bmod \partial_4(NA_4 \cap I_4)$$

and therefore we obtain

$$H(x_2 \otimes \partial_3(y_3)) \ = \ s_2(x_2)y_3 \ \bmod \partial_4(NA_4 \cap I_4).$$

(b) Again using the hypercrossed complex parings in [2] then

$$\begin{aligned}0 \equiv \partial_4 C^{(4)}_{(1)(0,3)}(y_3 \otimes x_2) \ &= \ s_1d_3(y_3)s_0(x_2) - s_1d_3(y_3)s_1(x_2) + \\ &\quad s_1d_3(y_3)s_0(x_2) - s_1d_3(y_3)s_1(x_2) + y_3s_2(x_2) \\ &\quad \bmod \partial_4(NA_4 \cap I_4).\end{aligned}$$

is found. This equality also holds

$$\begin{aligned}H(\partial_3(y_3) \otimes x_2) \ &= \ s_1d_3(y_3)s_0(x_2) - s_1d_3(y_3)s_1(x_2) + s_2d_3(y_3)s_2(x_2) \\ &\quad \bmod \partial_4(NA_4 \cap I_4).\end{aligned}$$

and so we acquire

$$H(\partial_3(y_3) \otimes x_2) \ = -s_2(x_2)y_3 \bmod \partial_4(NA_4 \cap I_4)$$

is commutated. Thus, the result of is given (a) and (b) of $3CM4_s$ as above.

3CM5$_s$

$$H(x_2 \otimes \partial_3(y_3)) + H(\partial_3(y_3) \otimes x_2) \ = \ s_2(x_2)y_3 - s_2(x_2)y_3 = 0.$$

Ref.

[2] Arvasi Z. , Applications in Commutative Algebra of The Moore Complex of a Simplicial Algebra, *Ph.D. Thesis*, University of Wales, BANGOR, (1994).

3CM6$_s$ By [2] we may also be written this equation as.

$$0 \equiv \partial_4 C^{(4)}_{(1)(0)}(y_3 \otimes y_3') = s_1 d_3(y_3) s_0 d_3(y_3') - s_1 d_3(y_3) s_1 d_3(y_3') + s_2 d_3(y_3) s_2 d_3(y_3') - y_3' y_3 \mod \partial_4(NA_4 \cap I_4).$$

Using the equation is obtained as

$$H(\partial_3(y_3) \otimes \partial_3(y_3')) = s_1 d_3(y_3) s_0 d_3(y_3') - s_1 d_3(y_3) s_1 d_3(y_3') + s_2 d_3(y_3) s_2 d_3(y_3') \mod \partial_4(NA_4 \cap I_4).$$

Hence, we yield

$$H(\partial_3(y_3) \otimes \partial_3(y_3')) \equiv y_3 y_3' \mod \partial_4(NA_4 \cap I_4)$$

(e) If $NA_i = 0$ for $\forall i \geq n+1$ in the Moore long sequence, then the Moore long sequence be an $n-$crossed complex with $C^{(n+1)}_{(\alpha)(\beta)}(x \otimes y) = 0$. Recall that from [2] we have the trivial map as follows.

$$C^{(n+1)}_{(\alpha)(\beta)}(x \otimes y) = NA_{(n+1))-\#\alpha} \otimes NA_{(n+1))-\#\beta} \to NA_{n+1}.$$

And so this NA_n also be a commutative algebra for $n \geq 2$ since

$$\begin{aligned} 0 &= \partial_{n+1} C^{(n+1)}_{(n-1),(n)}(x \otimes y) \\ &= s_{n-1} d_n(x) y - xy \\ &= (\phi^{(n+1)}_{n-1} d_n(x)) y - xy \\ &= yx. \end{aligned}$$

Here NA is a simplicial chain complex where NA_n is commutative for $n \geq 2$, $\phi^{(n+1)}_{n-1}$ is an action of NA_0 on NA_n for each $n \geq 1$ and ∂_n is NA_0-algebra morphism defined as

$$\cdots \longrightarrow NA_n/\partial_{n+1}K_{n+1} \longrightarrow NA_{n-1}/\partial_n K_n \longrightarrow \cdots \longrightarrow$$

$$NA_2/\partial_3 K_3 \longrightarrow NA_1/\partial_2 K_2 \longrightarrow NA_0$$

this is obviusly a crossed complex, where $K_i = NA_i \cap I_i$.

To prove the opposite of it let $NA_n/\partial_{n+1}K_{n+1}$ for $n \geq 2$, then

Thus $C^{(n-1)}_{(\alpha)(\beta)}(x, y) = 0$ implies that $NA_{n+1} = 0$. This is also an n-truncated complex. (see [3]).

(f) If $NA_i = 0$ for $\forall i \geq n+2$, then the Moore long sequence be a $T-$complex. To proof see [1] and [3].

(g) If $C^{(n-1)}_{(\alpha)(\beta)}(x_\alpha, y_\beta) = 0$, then the Moore long sequence become a crossed complex.

Therefore, we have the following results.

Ref.

[2] Arvasi Z. , Applications in Commutative Algebra of The Moore Complex of a Simplicial Algebra, *Ph.D. Thesis*, University of Wales, BANGOR, (1994).

Corollary 2.5 *If $NA_3/\partial_4(NA_4 \cap I_4) = 0$, then $NA_2 \to NA_1 \to NA_0$ corresponds a 2−crossed module. (see [2])*

Proof: Let **A** be a simplicial algebra with the Moore complex **NA.** Then the complex of algebras

$$NA_2 \xrightarrow{\partial_2} NA_1 \xrightarrow{\partial_1} NA_0$$

is a 2−crossed module of algebras, where the Peiffer map is defined as follows:

$$\begin{array}{rrcl} \{\quad \otimes \quad\}: & NA_1 \otimes NA_1 & \longrightarrow & NA_2/\partial_3(NA_3 \cap I_3) \\ & (x_1 \otimes y_1) & \longmapsto & s_1x_1(s_1y_1 - s_0y_1). \end{array}$$

Here the right hand side denotes a coset in $NA_2/\partial_3(NA_3 \cap I_3)$ represented by an element in NA_2 and $\partial_3(NA_3 \cap I_3) = 0$. □

Corollary 2.6 *If $NA_0 = 0$, then $NA_3/\partial_4(NA_4 \cap I_4) \xrightarrow{\partial_3} NA_2 \xrightarrow{\partial_2} NA_1$ is a 2−crossed module with defined Peiffer map as*

$$\begin{array}{rrcl} \{\quad \otimes \quad\}: & NA_2 \times NA_2 & \longrightarrow & NA_3/\partial_4(NA_4 \cap I_4) \\ & (x_2 \otimes\ y_2) & \longmapsto & s_2(x_2)s_2(y_2) - s_2(x_2)s_2(y_2). \end{array}$$

Proof: Indeed the function is satisfied 2−crossed module axioms.

2CM1: $\partial_3\{x_2 \otimes y_2\} = x_2s_1\partial_2(y_2) - x_2y_2$.

2CM2: $\{\partial_3(x_2)\otimes\partial_3(y_2)\} = x_2y_2$ since $\partial_4 C^{(4)}_{(2)(1)}(x_2\otimes y_2) = s_2d_3(x_3)(s_1d_3(y_3) - s_2d_3(y_3)) + x_3y_3 = 0 \mod \partial_4(NA_4 \cap I_4)$. (see [2])

2CM3: $\{x_2 \otimes y_2y_2'\} = \{x_2y_2 \otimes y_2'\} + \partial_2y_2'\{x_2 \otimes y_2\}$

2CM4: (a) Let $\partial_4 C^{(4)}_{(3,2)(1)}(x_2 \otimes y_3) = s_2(x_2)(s_1d_3(y_3) - s_2d_3(y_3) + y_3)$. So $\partial_4 C^{(4)}_{(3,2)(1)}(x_2 \otimes y_3) = 0 \mod \partial_4(NA_4 \cap I_4)$. (see [2]) Then $\{x_2 \otimes \partial_3(y_3)\} = s_2(x_2)y_3$ is obtained by the definition of action.

(b) Let $\partial_4 C^{(4)}_{(2)(3,1)}(y_3\otimes x_2) = (s_2d_3(y_3) - y_3)(s_1(x_2) - s_2(x_2))$. So $\partial_4 C^{(4)}_{(2)(3,1)}(y_3\otimes x_2) = 0 \mod \partial_4(NA_4 \cap I_4)$. (see [2]) Then $\{\partial_3(y_3) \otimes\ x_2\} = y_3x_2 - \partial_2(y_3)x_2 \mod \partial_4(NA_4 \cap I_4)$ is found.

2CM5

$$\begin{aligned} \{x_2 \otimes y_2\} \cdot z &= s_2(x_2)(s_2(y_2) - s_0(y_2)) \cdot z \\ &= s_2s_0(z)s_2(x_2)(s_2(y_2) - s_0(y_2)) \\ &= s_2(s_0(zx_2))(s_2(y_2) - s_0(y_2)) \\ &= s_2(x_2z)(s_2(y_2) - s_0(y_2)) \\ &= \{x_2 \cdot z \otimes y_2\}. \end{aligned}$$

Now we can consider the following diagram of morphism

$$\begin{tikzcd} & NA_2 \otimes NA_2 \arrow[dl, "\{\ \otimes\ \}"'] \arrow[d, "\rho"] & \\ NA_3/\partial_4(NA_4 \cap D_4) \arrow[r, "\partial_3"'] & NA_2 \arrow[r, "\partial_2"'] & NA_1. \end{tikzcd}$$

Ref.

[2] Arvasi Z. , Applications in Commutative Algebra of The Moore Complex of a Simplicial Algebra, *Ph.D. Thesis*, University of Wales, BANGOR, (1994).

Notes

The algebra NA_2 acts, in two way on the algebra $NA_3/\partial_4(NA_4 \cap I_4)$ by multiplication via s_1 and via s_2 both within A_3. The action via s_1 will also be denoted by $x \cdot y = s_1(x)y$ and the action via s_2 will be denoted by $xy = s_2(x)y$. The action of NA_1 on NA_3 is given as follows: from equality $(s_1(x) - s_2s_1d_2(x))y \equiv 0 \mod NA_3/\partial_4(NA_4 \cap I_4)$, there is a commutative diagram

$$\begin{tikzcd}
NA_3/\partial_4(NA_4 \cap I_4) \otimes NA_2 \arrow[r] \arrow[d] & NA_3/\partial_4(NA_4 \cap I_4) \arrow[d] \\
NA_3/\partial_4(NA_4 \cap I_4) \otimes NA_1 \arrow[r] & NA_3/\partial_4(NA_4 \cap I_4)
\end{tikzcd}$$

given by

$$\begin{tikzcd}
(y \otimes x) \arrow[r, mapsto] \arrow[d, mapsto] & x \cdot y = s_1(x)y \arrow[d, mapsto] \\
(x \otimes \partial_2(y)) \arrow[r, mapsto] & \partial_2(xy) = s_2s_1d_2(x)y
\end{tikzcd}$$

which gives an equality

$$\partial_2(xy) = s_2s_1d_2(x)y = s_1(x)y.$$

Let us define the map ρ by $\rho(x \otimes x') = \partial_2(x)x' - xx'$ for $x, x' \in NA_2$, that is the Peiffer element in NA_2 which corresponds to $\{x \otimes x'\}$. Thus if the map ρ is the trivial map $\partial_2 : NA_2 \to NA_1$ is a crossed module.

Now if the the Moore long sequence is iterated as follows, so then two results are obtained where $K_i = NA_i \cap I_i$.

$$\cdots 0 \longrightarrow 0 \longrightarrow NA_n/\partial_{n+1}K_{n+1} \longrightarrow NA_{n-1}/\partial_n K_n$$

$$\longrightarrow \cdots \longrightarrow NA_1/\partial_2 K_2 \longrightarrow NA_0$$

Corollary 2.7

$$\cdots 0 \longrightarrow 0 \longrightarrow NA_k/\partial_{k+1}K_{k+1} \xrightarrow{\partial_k} NA_{k-1} \xrightarrow{\partial_{k-1}} NA_{k-2}$$

$$\longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

is a $2-$*crossed module with defined Peiffer element*

$$\{x_{k-1} \otimes \ y_{k-1}\} = C_{(1)(0)}(x_{k-1} \otimes \ y_{k-1}).$$

So the $2-$*crossed module conditions are clearly verified.*

Corollary 2.8

$$\cdots 0 \longrightarrow 0 \longrightarrow NA_k/\partial_{k+1}K_{k+1} \xrightarrow{\partial_k} NA_{k-1} \xrightarrow{\partial_{k-1}} NA_{k-2} \xrightarrow{\partial_{k-2}} NA_{k-3}$$

$$\xrightarrow{\partial_{k-3}} 0 \longrightarrow \cdots \longrightarrow 0$$

is a semi $3-$*crossed module, where the Mutlu-Arvasi map is defined as follows:*

$$\{x_{k-1} \otimes \ y_{k-1}\} = C^{(k)}_{(0)(1)}(x_{k-1} \otimes y_{k-1})$$

It is clear that semi $3-$*crossed module conditions are satisfied.*

Corollary 2.9 *A* $3-$*truncated complex is a semi* $3-$*crossed module.*

We may follow the same procure as we did in Corolarly 2.8 in order to get to results.

Corollary 2.10 *The category of semi* $3-$*crossed modules is equivalent to the category of simplicial algebras with Moore complex of length 3.*

Notes

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