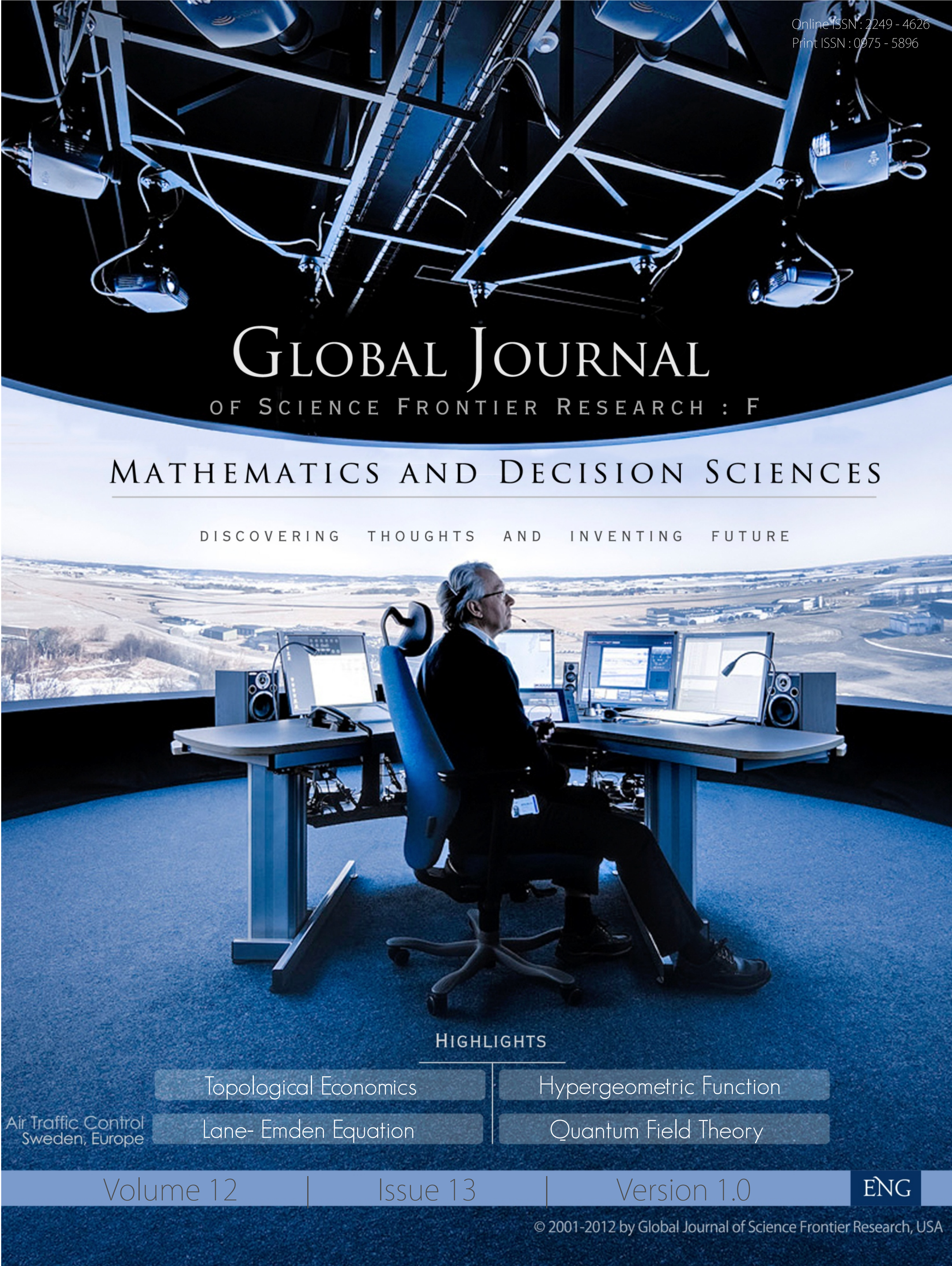
Online ISSN : 2249 - 4626
Print ISSN : 0975 - 5896



Global Journal of Science Frontier Research : F

Mathematics and Decision Sciences

DISCOVERING THOUGHTS AND INVENTING FUTURE

Highlights

- Topological Economics
- Hypergeometric Function
- Lane- Emden Equation
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Volume 12 | Issue 13 | Version 1.0

ENG

Global Journal of Science Frontier Research: F
Mathematics & Decision Sciences

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Mathematics & Decision Sciences
Volume 12 Issue 13 (Ver. 1.0)

Open Association of Research Society

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USA Tax ID: 98-0673427

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Offset Typesetting

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200 USD (B/W) & 250 USD (Color)

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Contents of the Volume

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 13 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Multiply Connected Topological Economics, Nonlinear Theory of Economic Growth and Its Three Laws, and Four Theorems on Knowledge Economic Theory

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Abstract - Using the similar formulas of the preference relation and the utility function, we propose the confidence relations and the corresponding influence functions that represent various interacting strengths of different families, cliques and systems of organization. Since they can affect products, profit, prices, and so on in an economic system, and are usually independent of economic results, therefore, the system can produce a multiply connected topological economics. If the political economy is an economy chaperoned polity, it will produce consequentially a binary economy. When the changes of the product and the influence are independent one another, they may be a node or saddle point. When the influence function large enough achieves a certain threshold value, it will form a wormhole with loss of capital. Various powers produce usually the economic wormhole and various corruptions. Further, we propose new nonlinear theory of economic growth and its three laws: Economic takeoff-growth-stagnancy law, social conservation and economic decay law, and economic growth mode transition and new developed period law. A corresponding figure is represented. The social open-reform is a necessary and sufficient condition for further economic development. Based on the main characteristics of knowledge economy, the four theorems on the knowledge economic theory are proposed, and the production function and basic equations are expounded. Some possible directions of the development on the knowledge economy and a sustainable development theory of new economics are discussed.

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GJSFR-F Classification : *MSC 2010: 54A05; 34A34*

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I. Introduction

Now the economic crisis and the crisis of economics must face an indisputable reality for world. We hope a stable economic growth, but there is often market failure or government failure [1]. Schiller said [1]: "Macroeconomic theory is supposed to explain the business cycle and show policymakers how to control it. But something is obviously wrong." "We have not consistently achieved the goals of full employment, price stability, and vigorous economic growth. All too often, either unemployment or inflation jumps unexpectedly or economic growth slows down."

Generally, theory of economic growth and the sustainable development of economy, etc., all warrant research. Moreover, various corruptions of many higher managers appear again and again.

In mathematical economics the fixed-point theorems of topology are used to prove the Nash equilibrium for n-person games. Arrow and Debreu presented a general model of

Ref.

1. B.R.Schiller, Essentials of Economics(Fourth Edition). McGraw-Hill Companies, Inc. 2002.

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Walrasian equilibrium theory, and proved the existence theorem of equilibrium for a competitive economy by topology [2]. Then McKenzie [3], Debreu [4], et al., developed the competitive equilibrium theory. Differential topology is introduced into economics, and Debreu [5] discussed two detailed questions.

Usual economic theories include only some interpretation on pure market process. The public choice theory is a great economics that intersects the two disciplines: the institutions are those of political science, and the method is that of economic theory [6-8]. It applies and develops scientific economic methods to other social regions. The public choice theory emphasizes comparative institutional analysis and, in particular, by their concentration on the necessary relationship between economic and political institutions. But alternative institutions may also have defects.

II. The Confidence Relations, the Influence Function and Multiply Connected Topological Economy

In the classical economics various quantities may be classified to two types: 1).Quantitative quantities, for example, capital, labor, product and profit, etc; 2).Qualitative quantities, for example, management, policy and preference, etc. The later possesses some human subjective factors. The two respects intersect usually each other, such the economic systems often show more complex social phenomena.

In the microeconomic theory of consumer behavior either a utility function or a binary relation can describe the preferences of an individual. The strict equivalence of these two primitive concepts, ordinal utility functions and preference relations, was first axiomatized by Debreu [9,10]. He studies the concept of cardinal utility in three different situations by means of the same mathematical result that gives a topological characterization of three families of parallel straight lines in a plane [9], and discussed that for every continuous complete and transitive binary relation ≥ defined on an arbitrary subset X of the commodity space R, there is a continuous utility representation; that is, there is a continuous function u of X into R such that $u(y) \geq u(x)$ if and only if $y \geq x$. Therefore, the more basic concept of preferences is applied instead of utility by means of a topology or a metric on the space of preferences. Undoubtedly, it is a great contribution for economics.

The topological structure on the space of preferences is very useful. For example, Hildenbrand used the structure to describe an exchange economy by its distribution of agents characteristics: preferences and endowments.

Moreover, the solution to the static maximization problem in Bellman equation yields a policy function that gives the optimal value of the current control as a function $g_t(x_t)$ of time and the current state. So the tomorrow state in is given by $x_{t+1} = m_t[g_t(x_t), x_t]$, and a solution to a similar problem then yields tomorrow's optimal control [11].

In microeconomics we introduce the confidence relations that represent various interacting strengths of different families, cliques and systems of organization. It is an important human relation in economics, even is independent of economic results. The confidence relation can be defined by a similar method with the preference relation in consumer theory [12,13].

The confidence relation ≥ defined on the choice set X is a complete preordering, continuous and strictly monotone. This requires [11]

Ref.

2. K.J.Arrow and G.Debreu, Existence of an equilibrium for a competitive economy. Econometrica. 22, 265-290(1954).

(1). Reflexivity: $\forall x \in X, x \geq x$;

(2). Completeness: $\forall x, y \in X$, either $x \geq y$ or $y \geq x$ or both;

(3). Transitivity: $\forall x, y, z \in X$, $[x \geq y$ and $y \geq x] \Rightarrow x \geq z$.

Then $\geq$ can be represented by a real-valued, continuous and increasing payoff function.

Further, the definition of the influence function I is similar with the utility function: A real-valued function $I^i : X^i \to R$ represents a confidence preordering $\{\geq_i\}$ defined on the choice set X^i of agent i if $\forall x, y \in X^i, x \geq_i y \Leftrightarrow I^i(x) \geq I^i(y)$. The influence function that represents a confidence preorder is not uniquely defined. Any monotonically increasing transformation $\varphi(\ )$ of I( ) will represent exactly the same confidences, because with $\varphi(\ )$ strictly increasing, we have

$$I(x) \geq I(y) \text{ if and only if } \varphi[I(x)] \geq \varphi[I(y)], \tag{1}$$

for all $\forall x, y \in X$. Hence I( ) is an ordinal influence function. The sign of the difference I(x)-I(y) is important because it tells us which outcome is confided, but the value of this difference is meaningless, as it will change with any nontrivial increasing transformation $\varphi(\ )$. It is also a basic characteristic of topology, where those concrete spacing values are meaningless. Although the influence function is similar to the utility function that obeys the law of diminishing marginal utility, but the influence function seems to obey the law of augmenting lust for power.

The confidence relation, the corresponding influence function I( ) and the function $\varphi(\ )$ can affect products Q, profit and prices, etc., in an economic system. But, they are usually independent of economic results, and sometimes are stochastic, even change suddenly. In a continuous topological manifold of economics they break easily original structure, and form a new hole or branch region. This will construct a multiply connected topological manifold. In an image the economic structure is a cup, while the influence function is a handle.

In a multiply connected region of topology there is a famous Euler-Poincare formula

$$\sum_{m=1}^{n} (-1)^m a_m = \sum_{m=1}^{n} (-1)^m p_m . \tag{2}$$

For a convex polyhedron, a_0, a_1, a_2 denote the number of vertices, edges, and faces, respectively; p_m is the mth Betti number of complex K. This may be considered intuitively as the numbers of m-dimensional holes in K, or is the number of (m+1)-dimensional chains that must be added to K so that every free m-cycle on K is a boundary [14]. The number $\sum_{m=1}^{n} (-1)^m a_m$ is called the Euler characteristic of the complex K. In the polyhedron $p_0 = p_2 = 1, p_1 = 2p$, p is the deficiency of a curved surface. In 2-dimensional curved surface, $a_0 = a_1 + 1 - 2p$. Assume that vertices represent the number of

Ref.

14. J.G.Hocking and G.S.Young, Topology. Addison-Wesley Publishing Company, Inc. 1961.

market, which is direct proportional to the sales volume y and the profit, and edges represent the market network. But the multiply connected economy brings the profit decrease. In this case there is a defective profit due to the deficiency p.

In some systems of organization the profit maximization and the confidence relations are inseparable. The aim of a pure producer is the profit maximization

$$\pi(y,w) = \max_{x,y}\{py - wx\}, \tag{3}$$

where y and x are output and input, p and w are output and input prices. For a social system with the influence function I, we should define an aim function as

$$A = \pi + I. \tag{4}$$

The objective function $f(x;\alpha)$ gives his payoff, when he faces environment α and chooses action x [11]. Usual profit is

$$\pi(Q) = TR(Q) - TC(Q), \tag{5}$$

where $TR(Q)$ and $TC(Q)$ are the total revenue and the total cost, the three quantities all are the functions of products $Q=f(K,L)$. Now the maximum principle is the aim function maximization. The initial condition of the profit maximization is

$$\frac{d\pi}{dQ} = \frac{dTR}{dQ} - \frac{dTC}{dQ} = 0, \tag{6}$$

$$\therefore \frac{dTR}{dQ} = \frac{dTC}{dQ}, \tag{7}$$

i.e., $MR=MC$, the marginal revenue equals to the marginal cost. While now we derive a new result:

$$\frac{dA}{dQ} = \frac{dTR}{dQ} - \frac{dTC}{dQ} + \frac{dI}{dQ} = 0, \tag{8}$$

$$\therefore MR - MC = -\frac{dI}{dQ} < 0. \tag{9}$$

In the social system the aim function deviates the profit maximization, which is defective usually since $MR - MC < 0$.

The production function in the traditional theory of the firm expresses output Q as a function of two inputs: capital K and labor L,

$$Q=Q(K,L). \tag{10}$$

The profit maximization of the firm is

$$\pi = pQ - mK - nL, \tag{11}$$

Notes

where p, m and n are the prices of output, capital and labor flows respectively [15]. If the influence function regards as a condition of the economic system, the economic meaning of the influence function will also be able to be discussed using the Lagrange method of conditional maximization.

III. Political Economy, Singularity and Wormhole

Notes

A complete market economy should be a simplex free economy. But, a non-market economy, and any oligarch economy must be a multiply connected topological economy, which changes to a higher dimension, for example, it will add a new dimension with man-rule. This economy may be dismembered and comminuted, and has various holes and be mangled easily for distortions.

According to <Oxford Advanced Learner Dictionary of Current English> the political economy is study of the political problems of government. It as an early title is very famous, for example, David Ricardo's <The Principles of Political Economy and Taxation>(1817), Thomas Robert Malthus' <Principles of Political Economy>(1820) and <Definitions of Political Economy>(1827), James Mill's <Elements of Political Economy>(1821), and Karl Marx's <The Critique of Political Economy> and so on. The political economy now sounds old-fashioned but usefully emphasizes the importance of choice between alternatives in economics which remains, despite continuing scientific progress [16].

If the political economy is an economy chaperoned polity, it will produce consequentially a binary economy. Its basic group is different with a complete market economy in the algebraic topology.

A general change of the supply-demand function is

$$\frac{dQ_d(Q_s)}{dt} = f(Q_d, Q_s, p) + V + S. \tag{12}$$

Here V is a governmental potential, and S is a stochastic factor. The equation (12) has the outside force and the potential V, such the economic results change along with different V. If V is inequitable and factitious, the results will possess bigger stochasticity.

Various powers are some different attractors, and produce the economic wormhole and various corruptions. In particular, if the multifarious confidence relations exist, a whole economic system and corresponding topological manifold will be covered with many big and small holes like bruises and scars. In this society the highest economic aim is only the confidence relation for families, cliques and systems of organization. Some big or small powers cling to an economic system, and form the multiply connected topological economy, and have a series of corruption with the self-similarity. It is a special type of the fractal economy. There is a binary economic function of power-business. Both is usually asymmetry, i.e., is inequality. Under this system various aspects tend spontaneously to the breaking of symmetry. An imperium is an economic black hole, which will derive the huge corruption, and finally the system dies out.

The political economy should be a pair coupling equations on polity and economy. Assume that a potential is $U = 2aX^2Y$. Here X is a confidence relation, Y is an economic benefit, and a is a coefficient. From this the difference between a theoretic value and a practice value will be estimated.

When the politics is put in command, the economy and its equation will be neglected. But, when oligarch notices the social crisis, the economic rules will be obeyed more. Both alternation exhibits a periodicity. If they conflict, the result will be reform for the economics bigger than politics, or be retrogression for the economics smaller than politics.

The political economy is usually imperfect economic question, even completely is not an economic question for some particular cases. It is not a strict economic rule, because in this case economy is only an appendage of polity. The economy will change along with polity.

The multiply connected topological economy may be extended to various relations between economy and other politics, family, religion, etc. Further, it may be developed to many regions of without direct relations with economy, for example, welfare, environment, and full employment, etc.

If the influence function changes as time, the system will be more complex. Assume that the economic system and its change are linear [11]:

$$\frac{dQ}{dt} = a_{11}Q + a_{12}I \,, \tag{13}$$

$$\frac{dI}{dt} = a_{21}Q + a_{22}I \,. \tag{14}$$

Their characteristic matrix is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{15}$$

The corresponding characteristic equation is

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = \lambda^2 - T\lambda + D = 0 \,. \tag{16}$$

From this we may discuss the general cases. As the simplest example, if the changes of the product and the influence are independent one another, i.e., $a_{12} = a_{21} = 0$, the solutions of the equations will be $Q = Q_0 \exp(a_{11}t), I = I_0 \exp(a_{22}t)$. $\Delta = T^2 - 4D = (a_{11} - a_{22})^2 \geq 0$ for a real domain.

When a_{11}, a_{22} are real numbers of the same signs, $D>0$, the state (Q_0, I_0) of the system is a node point, which is stable for $a_{11}, a_{22} < 0$, and is unstable for $a_{11}, a_{22} > 0$ (Fig.1).

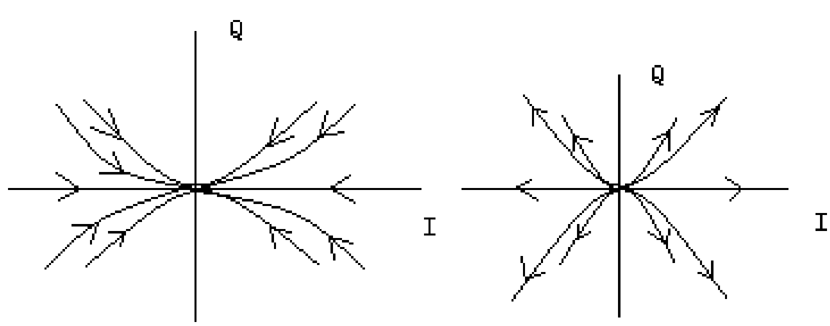

Fig.1 : Stable and unstable node points

Ref.

11. A.de la Fuente, Mathematical Methods and Models for Economists. Cambridge University Press. 2000.

When a_{11}, a_{22} are real numbers of opposite signs, *D<0*, the state (Q_0, I_0) of the system is a saddle point (Fig.2).

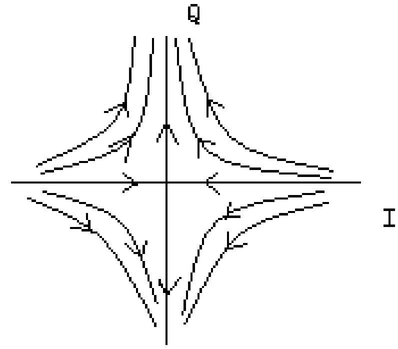

Fig.2 : Saddle point

It represents that product increases and the conference decreases. If the two changes of the product and the influence intersect one another, the states of the economic system will be able also to be the spiral (focal) point, or center, etc.

The form of the influence function can be an unrestricted function, even a stochastic function. Perfect competition prevails that each producer and consumer regards the prices paid and received as independent of his own choices [2]. An economy with the confidence relations and the influence functions is a type of imperfect competitive economic systems, and break the symmetries in economic topology. They are not homeomorphic spaces. Usually this structure will hinder the economic development. If the confidence relations and the influence functions have p -levels or p-types, i.e., $\sum_{i=1}^{p} I_i(Q)$, they will construct a multiply connected normal curved surface with the deficiency p. When the influence function large enough achieves a certain threshold value, the economic elasticity of topological structure will be broken, and a new hole will appear. Unified market economy will be riddled with holes. This will form a new multiply connected topological manifold. As an example, using the concept of general relativity a large influence as mass of general relativity forms a pit in the economic system. According to Fuller-Wheeler theory [17], a very strong pit can construct a wormhole, sometimes called the Einstein-Rosen bridge [18]. Therefore, some capital will pass through a throat into another topological space, or from a region to another region in the same space (Fig.3). This model will may describe a loss of capital (including waste, and corruption).

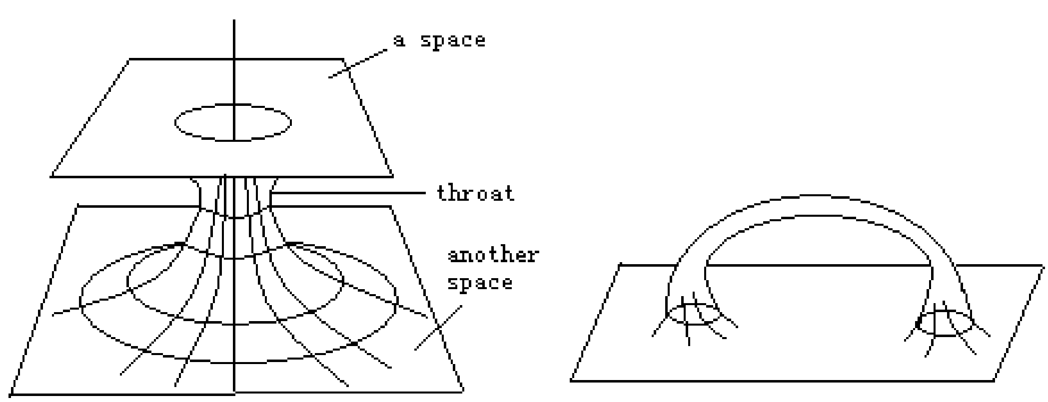

Fig. 3 : The wormhole model in social economics from a space into another topological space, or from a region to another region in the same space

Ref.

17. R.W.Fuller and J.A.Wheeler, Causality and multiply-connected space-time. Phys.Rev., 128,919- 929(1962).

In a word, the confidence relation and the influence function provide the useful tools for a description of human activity in economic system. This method of the multiply connected topological economy can be extended to various aspects on polity-law, on polity-education, on government-people and so on.

IV. Nonlinear Theory of Economic Growth and its three Laws

The theory of economic growth is very important in modern economics [19]. Solow's classical growth model of dynamic economy [20] and its extensions, for example, the theory of endogenous growth and so on [21]. These form the neoclassical growth theories [22].

I think, growth in any economic system with large and increases speed is a linear growth theory, but this is completely impossible for longer time, no matter what for circumstance, resources, markets, or populations. It has been proved time after time by the economic growth of many countries in world, and does not agree with a universal rule of scientific development.

At present, the theories of economic growth not only cannot forecast various economic crisis, and possess many theoretical questions [22-26], for example, some suppositions [25] and the basic equation in Harrod-Domar theory of economic growth [27,28,25]:

$$\dot{k} = sf(k) - nk \,. \tag{17}$$

Then various extensions of the neoclassical economic growth model are proposed, for example, Cobb-Douglas production function [29,30]:

$$Y = AK^{\alpha}L^{\beta} \,, \tag{18}$$

and technological change and innovation, etc. [19,31]. Beine, et al., discussed the relations between brain drain and economic growth [32].

Because the development and change of whole society are very complex and should be nonlinear, the static and stable growth all is impossible. In the economic system much complex and nonlinear processes exist. For instance, Boldrin and Montrucchio proposed that the optimum growth method is possibly nonlinear chaos [33]. Therefore, we propose a nonlinear theory of economic growth. Assume that the evolution and development equation for a corporation is [34]:

$$dF/dt = EF^{m} - BF^{n} + \Gamma(t) \,. \tag{19}$$

Here F is the selling price with a certain gain, and $\Gamma(t)$ is a stochastic term. Its change with time should be direct proportion with m power (force) of F and the fact throughput E, but will diminish along more increase market (assume it direct proportion with n power of F). For the corporation the two aspects are respectively beneficial and unfavourable, or are called common promotion and common restrain [13]. Let m=1, n=2 and $\Gamma(t)$=0, Eq.(19) will be simplified to an equation:

$$dF/dt = F(E - BF) \,. \tag{20}$$

Ref.

20. R.M.Solow, A contribution to the theory of economic growth. Quarterly J.Economics. 70,1,65-94 (1956).

Its solution is:

$$F = \frac{E}{B(1 + Ce^{-Et})}. \tag{21}$$

Since the fact throughput E may express as following:

$$E = E_0 - ST, \tag{22}$$

in which E_0 is an immanence ability of corporation (it includes personnel, equipments and fund, etc., of corporation), the extensive entropy S is a disorder scale in corporation (it shows technique, managed level, rationalization of combination on personnel and so on). The extensive temperature T is defined the drain of corporation, which includes operating costs, laborage, welfare and revenue, etc. From this we obtained seven conclusions [34].

We think that the nonlinear evolution is a universal rule for economic growth. While the development of corporation is also an important base of economic growth, therefore, these conclusions may be extended to apply to the economic growth theory, and propose the three laws:

First law: Economic takeoff-growth-stagnancy law. Any output and corresponding economic development all must pass a general nonlinear evolutional process from takeoff to growth and stagnancy, no matter what for various merchandises or any country. It is unreasonable and impossible that anybody requests a persistent linear growth of economy.

Some concrete statements are: Since the economic development is related with the social throughput, whose quantity is connected with this social immanence ability E_0. When the social immanence ability E_0 is invariant, according to (22) only the higher order may decrease S amount, or decrease drain T, the both methods can increase the social throughput E and the developed level F. But, the decreasing S and T cannot be infinite, both all have the minimum. It corresponds to the maximum of the social effective throughput. This is a maximum F=E/B of developed limit when time t increase continuously. It is a stagnancy dates of economic growth.

Second law: Social conservation and economic decay law. For any society, since the original throughput outmoded gradually, the social ageing, the saturated marketplace; contrarily, employment, laborage, welfare, operating costs, etc., will increase continuously, S and T reach the minimum then will raise, and add the resources consumed, the wastes increased, the environmental largeness press and so on. Such the corresponding social effective throughput E and the original economy develop to a certain extremum, and will descend inevitably.

Third law: Economic growth mode transition and new developed period law. Further development of social economy must exploits new merchandise and market, and adjust output configuration, and reform technique, and train personnel, so that boost up the immanence ability of social development and the international competitiveness. At the same time, the social framework and various personnel must readjust combination, and the management level raises up to follow the social development and new talented persons, new equipments, new outputs, new techniques and new capital introduced. Such the society should reform continuously to achieve a higher seedtime. This is namely to search new economic growth point for microeconomics. It corresponds to a development of the paradigms in science.

Ref.

34. Yi-Fang Chang, Social Synergetics. Science Press.2000.138-140.155-161.

The three laws on economic growth may be represented by Figure 4, in which CA expresses the first law, AA' expresses the second law, and AB expresses the third law. It should be a medium-time mode of economic growth, and is also three developed phases of social economy. Point A and dotted line are related with the limits to growth [35]. The third law connects to "the quality ladder"[22], which expresses a new period of development. The second law expresses a seasonal recession. They agree with Lotka-Volterra model in ecology, and are two different foregrounds of economic evolution, and are two-bifurcation phenomena of nonlinear system. An infinite clone of the same developed mode will derive a disorder competition, and finally reach necessarily to chaos and economic crisis. Therefore, the nonlinear chaos economics is possibly related with the crisis economics.

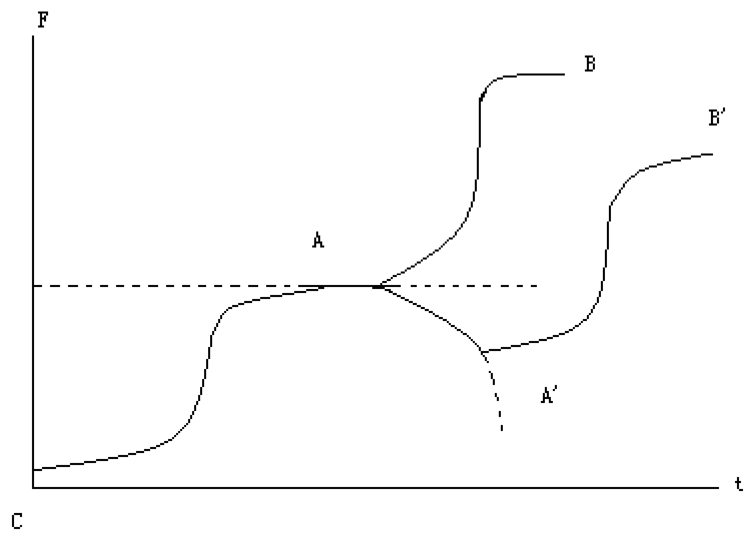

Fig. 4 : The three laws of economic evolution

It may combine the theorem of transformation from energy to quality on social development [34,36]. Rivera-Batiz discussed the relation among democracy, governance and economic growth [37].

In a word, social open is a necessary condition for economic further development, but it must add corresponding social reform as a sufficient condition of economic development [26].

V. Economic Theory and Its Four Theorems on Knowledge Economy

A production function in the classical economy is $Y = F(K, L, X_i)$. A well-known economic theory on the industrial economy is the input-output model, whose mathematical basic is matrix and linear algebra. New epoch of knowledge economy shows a new paradigm of economic growth.

Based on the main characteristics of knowledge economy and its similarity with the information theory, we proposed the four theorems of the knowledge economic theory [38,39]:

1) The innovation theorem by talented persons. The knowledge economy is innovative economy, in which talented persons are the most important. Labor and capital will fall to second roles.
2) From zero to things theorem. This is a process of information translated into substance and wealth. Its mathematical representation is $\int 0\, dT = C$.

Ref.

35. D.L.Meadows, et al., The Limits to Growth. New York: Universe Books. 1974.

3) The increment theorem by cooperation. A main character is networking in knowledge economy, which must emphasize cooperation in a system. For the economic development it includes an exponential change law $F = Ce^{at}$, here the innovative index $a > 0$. This may mathematically apply the Haken's synergetics [40].
4) The continuous cycle theorem. The output of knowledge economy possesses very high scientific and technological content, so it is light and corresponding waste is also little. This theorem includes two aspects: (1)Since the capital is smaller, so that the required natural resource and corresponding waste are also very little, therefore, it is a model of sustainable development. (2)Much riches may be created due to talented persons, and capital can attract more talented persons, such it will enter a fine cycle. This can use Eigen's hypercycle.

These theorems are also a developed process, in which theorem 1 is basic, which corresponds the human capital investment in neoclassical growth model, and other theorems are some results of innovation and development.

For the epoch of knowledge economy, knowledge is first in various bases, talented person is first in various resources, innovation is first in various developments, and cooperation is first in various managements. Its precondition is a right decision-making, which requires confirming a developed mode and a choice function. The talented person is only an order parameter for the new epoch. The production function will be simplified to an approximate single variable function Y=F(T). It is the most important mathematical character on knowledge economic theory. The talented person is a mostly stanchion, and knowledge and information are the most important and the essential production factors. The worth of knowledge is a scale of developed level on the microscopic knowledge economics. The innovation is a core and spirit, and is not a simple clone and expanded reproduction.

The basic mathematical model for the knowledge economy is nonlinear theory. In this case, the Cobb-Douglas production function [29,30] (18) will become to $Y = BT^{\alpha}$, in which α is an index on the talented person, and it includes the amount and quality of talented person, and α=0 is a point of phase transformation. Assume that a change equation of output is:

$$dY / dt = a(T)Y . \tag{23}$$

Its solution is $Y = C\exp[\int a(T)dt]$. When $a > 0$, the economy will show an exponential growth [38]. We think that topology and its tools in this economy will exhibit larger function due to networking of the epoch.

Further, the knowledge economic theory should develop a model of the simultaneous algebraic or differential equations, which are probably applied to describe the macroeconomic configuration of the large system. The epoch of knowledge economy will really realize Francis Bacon's well-known maxim: Knowledge is power!

In economic topology, the economic equilibrium states are some stationary equilibrium regions in the static economics. Based on the east thinking-system, especially, I Ching (Yi) and Lao-Zhuang philosophy, we proposed a sustainable development theory of new economics and its three principles: the common restraint or common promotion principle of the Yin-Yang and the Five-Elements, the whole principle of heaven-humanity-earth, and the cycle principle of some elements. Its major characteristics are entirety, balance and harmony. The highest aim is the principle of unified nature-human-

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society harmony [13,41]. From this we may research the corresponding mathematical theories and some concrete applications, and Chinese traditional agriculture and farm are a classical type of the complete recycling economy [41].

The economic theory of knowledge economy combined new economics of sustainable development and the nonlinear theory of economic growth will be able to form the nonlinear whole economics, which may apply a similar method of the nonlinear whole biology [42,43].

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 13 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

An Indefinite Integral in the Form of Hypergeometric Function

By Salahuddin
P.D.M College of Engineering, India

Abstract - In this paper we have established an indefinite integral involving Hypergeometric function and it's particular cases.

Keywords : *pochhammer symbol; gaussian hypergeometric function.*

GJSFR-F Classification : *MSC 2010: 33C05,33C45,33C15,33D50,33D60*

AN INDEFINITE INTEGRAL IN THE FORM OF HYPERGEOMETRIC FUNCTION

Strictly as per the compliance and regulations of :

Notes

An Indefinite Integral in the Form of Hypergeometric Function

Salahuddin

Year 2012

15

Global Journal of Science Frontier Research (F) Volume XII Issue XIII Version I

Abstract - In this paper we have established an indefinite integral involving Hypergeometric function and it's particular cases.

Keywords and Phrases : *pochhammer symbol; gaussian hypergeometric function.*

I. Introduction

a) Generalized Hypergeometric Functions

A generalized hypergeometric function ${}_pF_q(a_1,...a_p;b_1,...b_q;z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k}=\frac{P(k)}{Q(k)}=\frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(K+b_2)...(k+b_q)(k+1)}\,z. \tag{1}$$

Where $k+1$ in the denominator is present for historical reasons of notation, and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{array}{ccc} a_1,a_2,\cdots,a_p & ; & \\ & & z \\ b_1,b_2,\cdots,b_q & ; & \end{array}\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_kz^k}{(b_1)_k(b_2)_k\cdots(b_q)_kk!} \tag{2}$$

or

$${}_pF_q\left[\begin{array}{ccc} (a_p) & ; & \\ & & z \\ (b_q) & ; & \end{array}\right]\equiv {}_pF_q\left[\begin{array}{ccc} (a_j)_{j=1}^p & ; & \\ & & z \\ (b_j)_{j=1}^q & ; & \end{array}\right]=\sum_{k=0}^{\infty}\frac{((a_p))_kz^k}{((b_q))_kk!} \tag{3}$$

where the parameters $b_1,b_2,\cdots,b_q$ are neither zero nor negative integers and $p,\ q$ are non-negative integers.

The ${}_pF_q$ series converges for all finite z if $p\leq q$, converges for $|\ z\ |<1$ if $p\neq q+1$, diverges for all z , $z\neq 0$ if $p>q+1$.

The ${}_pF_q$ series absolutely converges for $|\ z\ |=1$ if $R(\zeta)<0$, conditionally converges for $|\ z\ |=1, z\neq 0$ if $0\leq R(\zeta)<1$, diverges for $|\ z\ |=1$, if $1\leq R(\zeta)$, $\zeta=\sum_{i=1}^{p}a_i-\sum_{i=0}^{q}b_i$.

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The function ${}_2F_1(a,b;c;z)$ corresponding to $p=2, q=1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function (Gauss 1812, Barnes 1908). To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

Notes

The hypergeometric functions are solutions of Gaussian hypergeometric linear differential equation of second order

$$z(1-z)y'' + [c-(a+b+1)z]y' - aby = 0 \tag{4}$$

The solution of this equation is

$$y = A_0\Big[1 + \frac{ab}{1!\ c}z + \frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2 + \cdots\cdots\Big] \tag{5}$$

This is the so-called regular solution,denoted

$${}_2F_1(a,b;c;z) = \Big[1 + \frac{ab}{1!\ c}z + \frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2 + \cdots\cdots\Big] = \sum_{k=0}^{\infty}\frac{(a)_k\ (b)_k z^k}{(c)_k k!} \tag{6}$$

which converges if c is not a negative integer for all of $\mid z \mid < 1$ and on the unit circle $\mid z \mid = 1$ if $R(c-a-b) > 0$.

It is known as Gauss hypergeometric function in terms of Pochhammer symbol $(a)_k$ or generalized factorial function.

Many of the common mathematical functions can be expressed in terms of the hypergeometric function, or as limiting cases of it. Some typical examples are

$$(1-z)^{-a} = z\ {}_2F_1(1,1;2;-z) \tag{7}$$

$$\sin^{-1} z = z\ {}_2F_1(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2) \tag{8}$$

The special case of (1.3.4) when $a=c$ and $b=1$, or $a=1$ and $b=c$, yields the elementary geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots \tag{9}$$

Hence the term "Hypergeometric" is given. The term hypergeometric was first used by Wallis in his work "Arithmetrica Infinitorum". Hypergeometric series or more precisely Gauss series is due to Carl Friedrich Gauss(1777-1855) who in year 1812 introduced and studied this series in his thesis presented at Gottingen and gave the F-notation for it.

Here z is a real or complex variable. If c is zero or negative integer, the series (6) does not exist and hence the function ${}_2F_1(a,b;\ c;\ z)$ is not defined unless one of the parameters a or b is also a negative integer such that $-c < -a$. If either of the parameters a or b is a negative integer, say $-m$ then in this case (6) reduce to the hypergeometric polynomial defined as

$$ {}_2F_1(-m, b; c; z) = \sum_{n=0}^{m} \frac{(-m)_n (b)_n \, z^n}{(c)_n \, n!} \tag{10} $$

Notes

b) Generalized Ordinary Hypergeometric Function of One Variable

The generalized Gaussian hypergeometric function of one variable is defined as follows

$$ {}_AF_B \left[\begin{array}{lc} a_1, a_2, a_3, \ldots, a_A \; ; & \\ & z \\ b_1, b_2, b_3, \ldots, b_B \; ; & \end{array} \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \, (a_2)_n \, (a_3)_n \cdots (a_A)_n \, z^n}{(b_1)_n \, (b_2)_n \, (b_3)_n \cdots (b_B)_n \, n!} \tag{11} $$

or,

$$ {}_AF_B \left[\begin{array}{lc} (a_A) \; ; & \\ & z \\ (b_B) \; ; & \end{array} \right] = \sum_{n=0}^{\infty} \frac{[(a_A)]_n \, z^n}{[(b_B)]_n \, n!} \tag{12} $$

where for the sake of convenience (in the contracted notation), (a_A) denotes thearray of "A" number of parameters given by $a_1, a_2, a_3, \ldots, a_A$. The denominatorparameters are neither zero nor negative integers. The numerator parameters may be zero and negative integers. A and B are positive integers or zero. Empty sum is to be interpreted as zero and empty product as unity.

$$ \sum_{n=a}^{b} \text{ and } \prod_{n=a}^{b} \text{ are empty if } b < a. $$

$$ [(a_A)]_{-n} = \frac{(-1)^{nA}}{[1-(a_A)]_n} \tag{13} $$

$$ [(a_A)]_n = (a_1)_n (a_2)_n (a_3)_n \cdots (a_A)_n = \prod_{m=1}^{A} (a_m)_n = \prod_{m=1}^{A} \frac{\Gamma(a_m + n)}{\Gamma(a_m)} \tag{14} $$

where $a_1, a_2, a_3, \ldots, a_A; b_1, b_2, b_3, \ldots, b_B$ and z may be real and complex numbers.

$$ {}_3F_2 \left[\begin{array}{lc} a, b, 1 \; ; & \\ & z \\ c, 2 \; ; & \end{array} \right] = \frac{(c-1)}{(a-1)(b-1)\,z} \times $$

$$ \times \left\{ {}_2F_1 \left[\begin{array}{lc} a-1, b-1 \; ; & \\ & z \\ c-1 \; ; & \end{array} \right] - 1 \right\} \tag{15} $$

The convergence conditions of ${}_AF_B$ are given below

Suppose that numerator parameters are neither zero nor negative integers (otherwise question of convergence will not arise).

(i) If $A \leq B$, then series ${}_AF_B$ is always convergent for all finite values of z(real or complex) i.e., $|z| < \infty$.

(ii) If $A = B + 1$ and $|z| < 1$, then series ${}_AF_B$ is convergent.

(iii) If $A = B + 1$ and $|z| > 1$, then series ${}_AF_B$ is divergent.

(iv) If $A = B + 1$ and $|z| = 1$, then series ${}_AF_B$ is absolutely convergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} > 0$$

(v) If $A = B + 1$ and $z = 1$, then series ${}_AF_B$ is convergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} > 0$$

(vi) If $A = B + 1$ and $z = 1$, then series ${}_AF_B$ is divergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} \leq 0$$

(vii) If $A = B + 1$ and $z = -1$, then series ${}_AF_B$ is convergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} > -1$$

(viii) If $A = B + 1$ and $|z| = 1$, but $z \neq 1$, then series ${}_AF_B$ is conditionally convergent, when

$$-1 < \mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} \leq 0$$

(ix) If $A > B + 1$, then series ${}_AF_B$ is convergent, when $z = 0$.

(x) If $A = B + 1$ and $|z| \geq 1$, then it is defined as an analytic continuation of this series.

(xi) If $A = B + 1$ and $|z| = 1$, then series ${}_AF_B$ is divergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} \leq -1$$

(xii) If $A > B + 1$, then a meaningful independent attempts were made to define MacRobert's E-function, Meijer's G-function, Fox's H-function and its related functions.

Notes

(xiii) If one or more of the numerator parameters are zero or negative integers, then series ${}_AF_B$ terminates for all finite values of z i.e., ${}_AF_B$ will be a hypergeometric polynomial and the question of convergence does not enter the discussion.

Notes

II. Main Indefinite Integral

$$\int \frac{\cos x \cos \frac{\pi x}{2a}}{1-\sin x}dx =$$

$$= \Big(-\frac{1}{4}-\frac{\iota}{4}\Big)\Big[\Big(\cos(x-\frac{\pi x}{2a})-\iota \sin\frac{(\pi-2a)x}{2a}\Big)\Big\{-1+(1+\iota)_2F_1\Big(1,1-\frac{\pi}{2a};2-\frac{\pi}{2a};\sin x-\iota\cos x\Big)\Big\}+$$
$$+\Big(\cos(x-\frac{\pi x}{2a})+\iota \sin\frac{(\pi-2a)x}{2a}\Big)\Big\{-1+(1+\iota)_2F_1\Big(1,\frac{\pi}{2a}-1;\frac{\pi}{2a};\sin x-\iota\cos x\Big)\Big\}+$$
$$+\Big(\cos(x+\frac{\pi x}{2a})-\iota \sin(x+\frac{\pi x}{2a})\Big)\Big\{-1+(1+\iota)_2F_1\Big(1,-\frac{\pi}{2a}-1;-\frac{\pi}{2a};\sin x-\iota\cos x\Big)\Big\}+$$
$$+\Big(\cos(x+\frac{\pi x}{2a})+\iota \sin(x+\frac{\pi x}{2a})\Big)\Big\{-1+(1+\iota)_2F_1\Big(1,\frac{\pi}{2a}+1;2+\frac{\pi}{2a};\sin x-\iota\cos x\Big)\Big\}-$$
$$-\frac{(2-2\iota)\sin\frac{x}{2}\cos(x-\frac{\pi x}{2})}{\cos\frac{x}{2}-\sin\frac{x}{2}}-\frac{(2-2\iota)\sin\frac{x}{2}\cos(x+\frac{\pi x}{2})}{\cos\frac{x}{2}-\sin\frac{x}{2}}\Big]+Constant \tag{16}$$

III. Particular Cases of the Integral

If $a=1$ then the following results are derived

$$(i)\int_{-0.1}^{0.1}\frac{\cos x\cos\frac{\pi x}{2a}}{1-\sin x}dx=\int_{-0.1}^{0.1}\frac{\cos x\cos\frac{\pi x}{2}}{1-\sin x}dx=0.19951$$

and the corresponding graph is shown in Figure 1.

$$(ii)\int_{-0.5}^{0.5}\frac{\cos x\cos\frac{\pi x}{2a}}{1-\sin x}dx=\int_{-0.5}^{0.5}\frac{\cos x\cos\frac{\pi x}{2}}{1-\sin x}dx=0.936762$$

and the corresponding graph is shown in Figure 2.

$$(iii)\int_{-1}^{1}\frac{\cos x\cos\frac{\pi x}{2a}}{1-\sin x}dx=\int_{-1}^{1}\frac{\cos x\cos\frac{\pi x}{2}}{1-\sin x}dx=1.42099$$

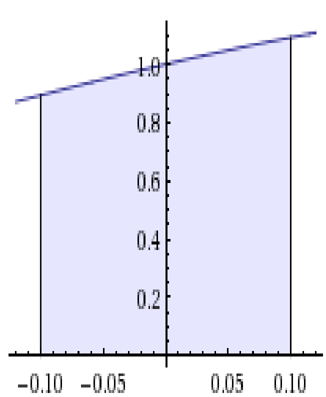

Figure 1:

Global Journal of Science Frontier Research (F) Volume XII Issue XIII Version I Year 2012

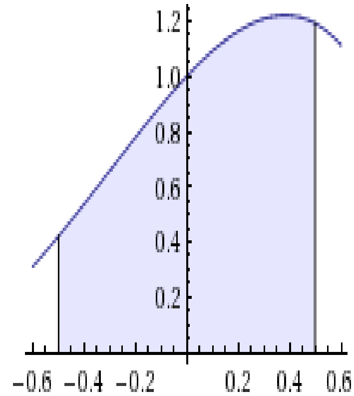

Figure 2:

and the corresponding graph is shown in Figure 3.

$$(iv)\int_{-1.5}^{1.5}\frac{\cos x\cos\frac{\pi x}{2a}}{1-\sin x}dx=\int_{-1.5}^{1.5}\frac{\cos x\cos\frac{\pi x}{2}}{1-\sin x}dx=-0.624575$$

and the corresponding graph is shown in Figure 4.

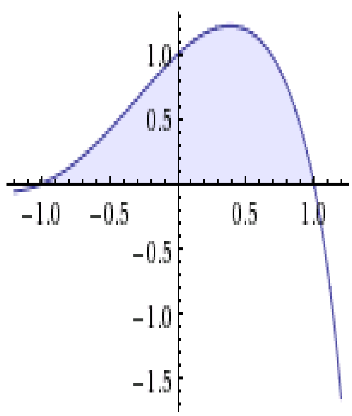

Figure 3:

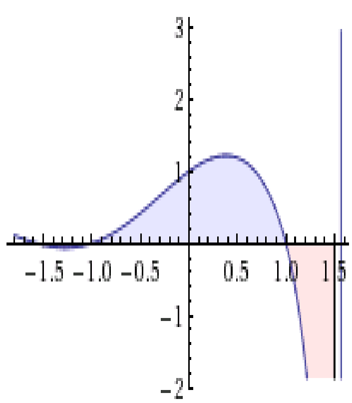

Figure 4:

Notes

IV. Derivation of the Integrals

Involving the same parallel method of ref[4] , one can derive the integral.

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 13 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

An Integral Associated With H-Function of Several Complex Variables

By Rakeshwar Purohit, Ashok Singh Shekhawat & Jyoti Shaktawat
Mohan Lal Sukhadia University, India

Abstract - The object of present paper is to derive an integral pertaining to a product of Fox's H-function [1], generalized polynomials Srivastava [7], general class of multivariable polynomials Srivastava and Garg [8] and H-function of multivariables given by Srivastava and Panda [9] with general arguments of quadratic nature.This paper is capable of yielding numerous result involving classical orthogonal polynomials hitherto scattered in the literature.

Keywords : fox's H-function, general polynomials, general class of multivariable polynomials, multivariable Hfunction, generalized Lauricella function, G-function.

GJSFR-F Classification : MSC 2010: 33C65, 26B12

AN INTEGRALASSOCIATEDWITHH-FUNCTIONOFSEVERALCOMPLEXVARIABLES

Strictly as per the compliance and regulations of :

An Integral Associated With H-Function of Several Complex Variables

Rakeshwar Purohit [α], Ashok Singh Shekhawat [σ] & Jyoti Shaktawat [ρ]

Abstract - The object of present paper is to derive an integral pertaining to a product of Fox's H-function [1], generalized polynomials Srivastava [7], general class of multivariable polynomials Srivastava and Garg [8] and H-function of multivariables given by Srivastava and Panda [9] with general arguments of quadratic nature.This paper is capable of yielding numerous result involving classical orthogonal polynomials hitherto scattered in the literature.

Keywords : fox's H-function, general polynomials, general class of multivariable polynomials, multivariable H-function, generalized Lauricella function, G-function.

I. Introduction

The H-function of multivariable is defined by Srivastava and Panda [9] as:

$$H[x_1,\dots,x_r] = H^{0,\lambda:(u',v');\dots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left[\begin{matrix}[(p):\theta';\dots;\theta^{(r)}]:[(q'):\Delta'];\dots;[(q^{(r)}):\Delta^{(r)}];\\ (s):\psi';\dots;\psi^{(r)}]:[(t'):\delta'];\dots;[(t^{(r)}):\delta^{(r)}];\end{matrix}\; x_1,\dots,x_r\right] \tag{1.1}$$

The Fox's H-function [1]:

$$H^{L,R}_{P,Q}\left[x\left|\begin{matrix}(m_P,M_P)\\(n_Q,N_Q)\end{matrix}\right.\right] = \sum_{G=0}^{\infty}\sum_{g=1}^{L}\frac{(-1)^G}{G!N_g}\phi_{(\eta_G)}\, x^{\eta_G}, \tag{1.2}$$

where

$$\phi_{(\eta_G)} = \frac{\prod\limits_{\substack{j=1\\ j\neq g}}^{L}\Gamma(n_j - N_j\eta_G)\prod\limits_{j=1}^{R}\Gamma(1-m_j+M_j\eta_G)}{\prod\limits_{j=L+1}^{Q}\Gamma(1-n_j+N_j\eta_G)\prod\limits_{j=R+1}^{P}\Gamma(m_j-M_j\eta_G)}$$

and

$$\eta_G = \frac{(\eta_g+G)}{\eta_g}$$

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Ref.

9. H.M.Srivastava and R.Panda, Some Bilateral Generating Function for a Class of Generalized Hypergeometric Polynomials, J. Raine Angew. Math., 283/284(1996), 265-274.

The H-function of multivariable in (1.1) converges absolutely if

$$\left|\arg(x_i)\right| < \frac{1}{2}\pi T_i \,, \tag{1.3}$$

where

$$T_i = -\sum_{j=1+\lambda}^{A}\theta_j^{(i)} + \sum_{j=1}^{v^{(i)}}\Delta_j^{(i)} - \sum_{j=1+v^{(i)}}^{B^{(i)}}\Delta_j^{(i)} - \sum_{j=1}^{C}\Psi_j^{(i)} + \sum_{j=1}^{u^{(i)}}\delta_j^{(i)} - \sum_{j=1+u^{(i)}}^{D^{(i)}}\delta_j^{i} > 0,$$

$$\forall\, i \in (1,\ldots,r) \tag{1.4}$$

Srivastava and Garg introduced and defined a general class of multivariable polynomials [8] as follows

$$S_E^{F_1,\ldots,F_s}[z_1,\ldots,z_s] = \sum_{k_1,\ldots,k_s=0}^{F_1k_1+\ldots+F_sk_s\le E}(-E)_{F_1k_1+\ldots+F_sk_s}\,A(E;k_1,\ldots,k_s)\frac{(z_1)^{k_1}}{k_1!}\cdots\frac{(z_s)^{k_s}}{k_s!} \tag{1.5}$$

where $F_1,\ldots,F_s$ are imaginary positive integer and the coefficients A $(E;k_1,\ldots,k_s)$, $(E;\ k_i \ge 0,\ i = 1,\ldots,s)$ are arbitrary constants, real or complex.
Srivastava has defined and introduced the general polynomials ([7], p.185, eq.(7))

$$S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}[z_1,\ldots,z_r] = \sum_{\beta_1=0}^{[N_1/M_1]}\cdots\sum_{\beta_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1\beta_1}}{\beta_1!}\cdots\frac{(-N_s)_{M_s\beta_s}}{\beta_s!}$$

$$.B[N_1,\beta_1;\ldots;N_s,\beta_s]z_1^{\beta_1}\ldots z_s^{\beta_s}\,, \tag{1.6}$$

where $N_i = 0,1,2,\ldots,\ \forall\ i = (1,\ldots,s)$; $M_1,\ldots,M_s$ are arbitrary positive integers and the coefficients B $[N_1,\beta_1;\ldots;N_s,\beta_s]$ are arbitrary constants, real or complex.

II. The Main Result

We shall establish the following result:

$$\int_0^\infty y^{1-\beta}(p+qy+sy^2)^{\beta-3/2}\,H_{P,Q}^{L,R}\left[\left(\frac{y}{p+qy+sy^2}\right)^{\sigma}\middle|\begin{matrix}(m_P,M_P)\\(n_Q,N_Q)\end{matrix}\right]$$

$$.S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}\left[z_1\left(\frac{y}{p+qy+sy^2}\right)^{n_1},\ldots,z_s\left(\frac{y}{p+qy+sy^2}\right)^{n_s}\right]$$

Ref.

8. H.M. Srivastava and M. Garg, Some integrals involving a general class of polynomials and the multivariable H-function, Rev. Roumaine Phys., 32 (1987), 685-692.

Notes

$$.S_E^{F_1,\dots,F_s}\left[z_1\left(\frac{y}{p+qy+sy^2}\right)^{n_1},\dots,z_s\left(\frac{y}{p+qy+sy^2}\right)^{n_s}\right]$$

$$.H\left[x_1\left(\frac{y}{p+qy+sy^2}\right)^{\sigma_1},\dots,x_r\left(\frac{y}{p+qy+sy^2}\right)^{\sigma_r}\right]dy$$

$$=\sqrt{\frac{\pi}{c}}\sum_{G=0}^{\infty}\sum_{g=1}^{L}\sum_{\beta_1=0}^{[N_1/M_1]}\dots\sum_{\beta_s=0}^{[N_s/M_s]}\sum_{k_1,\dots,k_s=0}^{F_1K_1+\dots+F_sK_s\le E}\frac{(-1)^G}{G!F_g}\frac{(-N_1)_{M_1\beta_1}}{\beta_1!}\dots\frac{(-N_s)_{M_s\beta_s}}{\beta_s!}\Phi(\eta_G)$$

$$(-E)_{F_1k_1+\dots+F_sk_s}\;B[N_1,\beta_1;\dots;N_s\beta_s]A[E;k_1,\dots,k_s]\frac{z_1^{(\beta_1+k_1)}}{k_1!}\dots\frac{(z_s)^{(\beta_s+ks)}}{k_s!}$$

$$(q+2\sqrt{sp})^{\beta-\sigma\eta_G-\sum_{i=1}^{s}\eta_i(\beta_i+k_i)-1}$$

$$.\,H^{0,\lambda+1:(u',v');\dots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left[\begin{array}{c|c} x_1(q+2\sqrt{sp})^{-\sigma_1} & [\beta-\sigma\eta_G-\sum_{i=1}^{s}n_i(\beta_i+k_i):\sigma_1;\dots;\sigma_r], \\ \vdots & \\ x_r(q+2\sqrt{sp})^{-\sigma_r} & [(s):\Psi';\dots;\Psi^{(r)}], \end{array}\right.$$

$$\left.\begin{array}{ll} [(p):\theta';\dots;\theta^{(r)}] & :[(q'):\Delta'];\dots;[(q^{(r)}):\Delta^{(r)}] \\ [\beta-\sigma\eta_G-\sum_{i=1}^{s}n_i(\beta_i+k_i)-\frac{1}{2}:\sigma_1;\dots;\sigma_r]: & [(t'):\delta'];\dots;[(t^{(r)}):\delta^{(r)}] \end{array}\right] \qquad (2.1)$$

provided that Re(p) > 0, Re(q) > 0, s > 0 and

$$\sigma\min\left[\mathrm{Re}\left(\frac{n_j}{N_j}\right)\right]+\sum_{i'=1}^{r}\sigma_i'\min\left[\mathrm{Re}\left(\frac{t_{j'}^{(i')}}{\delta_{j'}^{(i')}}\right)\right]>\beta-2,\ j=1,\dots,M\text{ and }j'=1,\dots,u^{(i')}.$$

Proof:

In order to prove (2.1) first we express the Fox H-function and a general polynomials in form of series and the H-function of multivariable in terms of Mellin-Barnes contour integrals. Now interchanging the order of summations and integration which is permissible under the stated condition, we obtain

$$\sum_{G=0}^{\infty}\sum_{g=1}^{L}\sum_{\beta_1=0}^{[N_1/M_1]}\cdots\sum_{\beta_s=0}^{[N_s/M_s]}\sum_{k_1,\dots,k_s=0}^{F_1K_1+\dots+F_sK_s\le E}\frac{(-1)^G(-N_1)_{M_1\beta_1}}{\beta_1!}\cdots\frac{(-N_s)_{M_s\beta_s}}{\beta_s!}\phi(\eta_G)$$

$$.\,(-E)_{F_1k_1+\dots+F_sk_s}B[N_1,\beta_1;\dots;N_s\beta_s]A[E;k_1,\dots,k_s]\frac{(z_1)^{(\beta_1+k_1)}}{k_1!}\cdots\frac{(z_s)^{(\beta_s+k_s)}}{k_s!}$$

$$.\,\frac{1}{(2\pi i)^r}\int_{I_1}\cdots\int_{I_r}\psi(\xi\gamma_1,\dots,\gamma_r)\Delta_1(\nu_1)\dots\Delta_r(\gamma_r)x_1^{\gamma_1}\dots x_r^{\gamma_r}$$

$$.\left\{\int_0^\infty y^{1-[\beta-\sigma\eta_G-\sum_{i=1}^{s} n_i(\beta_i+k_i)-\sigma_1\gamma_1-\dots-\sigma_r\gamma_r)}\right.$$

$$\left..(p+qy+sy^2)^{(\beta-\sigma\eta_G-\sum_{i=1}^{s} n_i(\beta_i+k_i)-\sigma_1\xi\gamma_1-\dots-\sigma_r\gamma_r)-\frac{3}{2}}dy\right\}d\gamma_1\dots d\gamma_r, \qquad (2.2)$$

On solving above y-integral with the help of known theorem (Saxena [6]) and reinterpreting the result obtained in terms of H-function of r variable, we reached at the desired result.

III. Special Cases

(a) If $\lambda = A$, $u^{(i)} = 1$, $v^{(i)} = B^{(i)}$ and $D^{(i)} = D^{(i)} + 1$, $\forall\ i \in (1,\dots,r)$ the result in (2.1) reduces to the following integral transformation:

$$\int_0^\infty y^{1-\beta}(p+qy+sy^2)^{\beta-3/2}H_{P,Q}^{L,R}\left[\left(\frac{y}{p+qy+sy^2}\right)^{\sigma}\middle|\begin{matrix}(m_P,M_P)\\(n_Q,N_Q)\end{matrix}\right]$$

$$.S_{N_1,\dots,N_s}^{M_1,\dots,M_s}\left[z_1\left(\frac{y}{p+qy+sy^2}\right)^{n_1},\dots,z_s\left(\frac{y}{p+qy+sy^2}\right)^{n_s}\right]$$

$$.S_E^{F_1,\dots,F_s}\left[z_1\left(\frac{y}{p+qy+sy^2}\right)^{n_1},\dots,z_s\left(\frac{y}{p+qy+sy^2}\right)^{n_s}\right.$$

Ref.

6. R.K.Saxena, An Integral Involving G-Function, Proc. Nat. Inst. Sci. India, 26A(1960), 661-664.

Notes

$$.E^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left[-x_1\left(\frac{y}{p+qy+sy^2}\right)^{\sigma_1},\ldots,-x_r\left(\frac{y}{p+qy+sy^2}\right)^{\sigma_s}\middle|\begin{matrix}[1-(p):\theta';\ldots;\theta^{(r)}]\\ [1-(s):\Psi';\ldots;\Psi^{(r)}]\end{matrix}\right.$$

$$\left.\begin{matrix}:[1-(q'):\Delta'];\ldots;[1-(b^{(r)}):\Delta^{(r)}]\\ [1-(t'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}]\end{matrix}\right]dy$$

$$=\sqrt{\frac{\pi}{c}}\sum_{G=0}^{\infty}\sum_{g=1}^{L}\sum_{\beta_1=0}^{[N_1/M_1]}\cdots\sum_{\beta_s=0}^{[N_s/M_s]}\sum_{k_1,\ldots,k_s=0}^{F_1K_1+\ldots+F_sK_s\le E}\frac{(-1)^G}{G!F_g}\frac{(-N_1)_{M_1\beta_1}}{\beta_1!}\cdots\frac{(-N_s)_{M_s\beta_s}}{\beta_s!}\phi(\eta_G)$$

$$.\ (E)_{F_1k_1+\ldots+F_sk_s}\ B(N_1,\beta_1;\ldots;N_s\beta_s)$$

$$.\ A[E;k_1,\ldots,k_s]\frac{z_1^{(\beta_1+k_1)}}{k_1!}\cdots\frac{(z_s)^{(\beta_s+k_s)}}{k_s!}(q+2\sqrt{sp})^{\beta-\sigma\eta_G-\sum_{i=1}^{s}n_i(\beta_i+k_i)-1}$$

$$\frac{\Gamma(1-\beta+\sigma\eta_G+\sum_{i=1}^{s}n_i(\beta_i+k_i))}{\Gamma(\frac{3}{2}-\beta+\sigma\eta_G+\sum_{i=1}^{s}n_i(\beta_i+k_i))}E^{A+1:B';\ldots;B^{(r)}}_{C+1:D';\ldots;D^{(r)}}$$

$$.\left[-x_1(q+2\sqrt{sp})^{-\sigma_1},\ldots,-x_r(q+2\sqrt{sp})^{-\sigma_r}\middle|\begin{matrix}[1-\beta+\sigma\eta_G+\sum_{i=1}^{s}n_i(\beta_i+k_i):\sigma_1;\ldots;\sigma_r],\\ [1-(s):\Psi';\ldots;\Psi^{(r)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[1-(p):\theta';\ldots;\theta^{(r)}]:[1-(q'):\Delta'];\ldots;[1-(q^{(r)}):\Delta^{(r)}]\\ [\frac{3}{2}-\beta+\sigma\eta_G+\sum_{i=1}^{s}n_i(\beta_i+k_i):\sigma_1;\ldots;\sigma_r]:[1-(t'):\delta'];\ldots;[1-(t^{(r)});\delta^{(r)}]\end{matrix}\right]\tag{3.1}$$

provided that Re(p) > 0, Re(q) > 0,s > 0, the series on the right side exists.

(b) If $\theta',\ldots,\theta^{(r)}=\Delta',\ldots,\Delta^{(r)}=\psi',\ldots,\psi^{(r)}=\delta',\ldots,\delta^{(r)}=\sigma_1,\ldots,\sigma_r=\beta',\ldots,\beta^{(r)}$ in (2.1), we get the following integral transformation:

$$\int_0^{\infty}y^{1-\beta}(p+qy+sy^2)^{\beta-3/2}H^{L,R}_{P,Q}\left[\left(\frac{y}{p+qy+sy^2}\right)^{\sigma}\middle|\begin{matrix}(m_P,M_P)\\(n_Q,N_Q)\end{matrix}\right]$$

$$.S^{M_1,\dots,M_s}_{N_1,\dots,N_s}\left[z_1\left(\frac{y}{p+qy+sy^2)}\right)^{n_1},\dots,z_s\left(\frac{y}{p+qy+sy^2}\right)^{n_s}\right]$$

$$.S^{F_1,\dots,F_s}_{E}\left[z_1\left(\frac{y}{p+qy+sy^2}\right)^{n_1},\dots,z_s\left(\frac{y}{p+qy+sy^2}\right)^{n_s}\right.$$

$$.F^{0,\lambda:(u',v');\dots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left[x_1^{1/\beta'}\left(\frac{y}{p+qy+sy^2}\right),\dots,x_r^{1/\beta^r}\left(\frac{y}{p+qy+sy^2}\right),\left|\begin{matrix}(p):(q');\dots;(q^{(r)})\\(s):(t');\dots;(t^{(r)})\end{matrix}\right.\right]dy$$

$$=\sqrt{\frac{\pi}{c}}\sum_{G=0}^{\infty}\sum_{g=1}^{L}\sum_{\beta_1=0}^{[N_1/M_1]}\dots\sum_{\beta_s=0}^{[N_s/M_s]}\sum_{k_1,\dots,k_s=0}^{F_1K_1+\dots+F_sK_s\le E}\frac{(-1)^G}{G!\,F_g}\frac{(-N_1)_{M_1\beta_1}}{\beta_1!}\dots\frac{(-N_s)_{M_s\beta_s}}{\beta_s!}\phi(\eta_G)$$

$$.\ (-E)_{F_1k_1+\dots+F_sk_s}\ B(N_1,\beta_1;\dots;N_s\beta_s)$$

$$.\ A[E;k_1,\dots,k_s]\frac{z_1^{(\beta_1+k_1)}}{k_1!}\dots\frac{(z_s)^{(\beta_s+k_s)}}{k_s!}(q+2\sqrt{sp})^{\beta-\sigma\eta_G-\sum_{i=1}^{s}n_i(\beta_i+k_i)-1}$$

$$\frac{\Gamma(1-\beta+\sigma\eta_G+\sum_{i=1}^{s}n_i\alpha_i)(\beta_i+k_i)}{\Gamma(\frac{3}{2}-\beta+\sigma\eta_G+\sum_{i=1}^{s}n_i(\beta_i+k_i)}E^{A+1:B';\dots;B^{(r)}}_{C+1:D';\dots;D^{(r)}}$$

$$.F^{0,\lambda+1\ :(u',v');\dots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left[x_1^{1/\beta'}(q+2\sqrt{sp})^{-1},\dots,x_r^{1/\beta^r}(q+2\sqrt{sp})^{-1}\right|$$

$$\left.\begin{matrix}\left[\beta-\sigma\eta_G-\sum_{i=1}^{s}n_i(\beta_i+k_i)\right],(p):(q');\dots;q^{(r)}\\(s),\left[\beta-\sigma\eta_G-\sum_{i=1}^{s}n_i(\beta_i+k_i)-\frac{1}{2}\right]:(t');\dots;(t^{(r)})\end{matrix}\right]\qquad(3.2)$$

provided that Re (p) > 0, Re (q) > 0, s > 0; $\beta^{(i)}>0$ (i = 1,…,r), $2(u^{(i)}+v^{(i)})>(A+C+B^{(i)}+D^{(i)})$

$$|\arg(z_i)|<\left[u^{(i)}+v^{(i)}-\frac{A}{2}-\frac{C}{2}-\frac{B^{(i)}}{2}-\frac{D^{(i)}}{2}\right]\pi\ \text{and}$$

Notes

$$\sigma\left\{\min_{1\le j\le M}[\mathrm{Re}(n_j/N_j)]\right\}+\sum_{i=1}^{r}\left\{\min_{1\le j\le u^{(i)}}[\mathrm{Re}(t_j^{(i)})]\right\}>\beta-2.$$

(c) When we put $\lambda = A = C = 0$ in (2.1), we get the following transformation

Notes

$$\int_0^\infty y^{1-\beta}(p+qy+sy^2)^{\beta-3/2}H_{P,Q}^{L,R}\left[\left(\frac{y}{p+qy+sy^2}\right)^{\sigma}\middle|\begin{matrix}(m_P,M_P)\\(n_Q,N_Q)\end{matrix}\right]$$

$$.S_{N_1,\dots,N_s}^{M_1,\dots,M_s}\left[z_1\left(\frac{y}{p+qy+sy^2)}\right)^{n_1},\dots,z_s\left(\frac{y}{p+qy+sy^2}\right)^{n_s}\right]$$

$$..S_E^{F_1,\dots,F_s}\left[z_1\left(\frac{y}{p+qy+sy^2}\right)^{n_1},\dots,z_s\left(\frac{y}{p+qy+sy^2}\right)^{n_s}\right.$$

$$.\prod_{i=1}^{r}H_{B^{(i)},D^{(i)}}^{u^{(i)},v^{(i)}}\left[x_i\left(\frac{y}{p+qy+sy^2}\right)^{\sigma_1}\middle|\begin{matrix}[(b^{(i)}):\phi^{(i)}]\\ [d^{(i)}\ :\delta^{(i)}]\end{matrix}\right]dy$$

$$=\sqrt{\frac{\pi}{c}}\sum_{G=0}^{\infty}\sum_{g=1}^{L}\sum_{\beta_1=0}^{[N_1/M_1]}\cdots\sum_{\alpha_s=0}^{[N_s/M_s]}\sum_{k_1,\dots,k_s=0}^{F_1K_1+\dots+F_sK_s\le E}\frac{(-1)^G}{G!F_g}\frac{(-N_1)_{M_1\beta_1}}{\beta_1!}\cdots\frac{(-N_s)_{M_s,\beta_s}}{\beta_s!}\phi(\eta_G)$$

$$.(-E)_{F_1k_1+\dots+F_sk_s}B[N_1,\beta_1;\dots;N_s,\beta_s]A[E;k_1,\dots,k_s]\frac{z_1^{(\beta_1+k_1)}}{k_1!}\cdots\frac{(z_s)^{(\beta_s+k_s)}}{k_s!}$$

$$(q+2\sqrt{sp})^{\beta-\sigma\eta_G-\sum_{i=1}^{s}n_i(\beta_i+k_i)-1}H_{1,1,:[B',D'];\dots;[B^{(r)},D^{(r)}]}^{0,1:(u',v');\dots;(u^{(r)},v^{(r)})}$$

$$\left[\begin{matrix}x_1(q+2\sqrt{sp})^{-\sigma_1}\\ \vdots\\ x_r(q+2\sqrt{sp})^{-\sigma_r}\end{matrix}\middle|\begin{matrix}[\beta-\sigma\eta_G-\sum_{i=1}^{s}n_i(\beta_i+k_i):\sigma_1;\dots;\sigma_r]:[(q'):\Delta'];\dots;[(q^{(r)}):\Delta^{(r)}]\\ \\ [\beta-\sigma\eta_G-\sum_{i=1}^{s}n_i(\beta_i+k_i)-\frac{1}{2}:\sigma_1;\dots;\sigma_r]:[(t'):\delta'];\dots;[(t^{(r)}):\delta^{(r)}]\end{matrix}\right]$$

(3.3)

valid and the same condition which is obtained from (2.1).

(d) Taking $N_i \to 0$, $(i = 1,\ldots,s)$, $E \to 0$, $p = 0$, $s = 1$, the result in (2.1) reduces to the known result after a slight simplification obtained by Goyal and Mathur [4].

(e) If $r = 1$ and M_i, $N_i \to 0$ $(i = 2,\ldots,s)$, $E \to 0$ the result in (2.1) reduces to the known result with a slight modification recently derived by Gupta and Jain [5].

(f) Taking $E \to 0$, the result in (2.1) reduces to the known result given in [3],after a little simplification.

Notes

IV. Acknowledgement

The authors are grateful to Professor H.M. Srivastava, University of Victoria, Canada for his kind help and valuable suggestions in the preparation of this paper.

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 13 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Homology Invariant Functions for Lane- Emden Equation of Finite Polytropic Index

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Abstract - The present paper is devoted to establish a general Mathematica module to determine if a function is homology invariant or not. The module is described through its basic points, propose, input, output and computational steps. Applications of the module are also given.

Keywords : *lane-emden equation, cosmology, homology theorem.*

GJSFR-F Classification : MSC 2010: 55N35, 55P20, 83F05

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Homology Invariant Functions for Lane-Emden Equation of Finite Polytropic Index

M. A. Sharaf [α] & L.A.Alaqal [σ]

Abstract - The present paper is devoted to establish a general *Mathematica* module to determine if a function is homology invariant or not. The module is described through its basic points, propose, input, output and computational steps. Applications of the module are also given.

Keywords : lane-emden equation, cosmology, homology theorem.

I. Introduction

The reduction of the differential equations is probably the most challenging problem in dynamics and physics. A general interpretation of reducibility includes various transformations and changes the original problem not only along mathematical lines but also in a physical sense.

What concerns us in the present paper is the reduction of the second order Lane-Emden equation of stellar structure into a first order through what is known as homology theorem. The important consequence of the use of homology theorem.

is that, if we can find two independent homology invariant functions, say u and v, then the Lane-Emden equation transformed to u and v variables is of order one.

Due to the important role of homology invariant functions in the reduction of Lane-Emden equation, the present paper is devoted to establish a general *Mathematica* module to determine if a function is homology invariant or not. The module is described through its basic points, propose, input, output and computational steps. Applications of the module are also given.

II. The Homology Theorem and Homology Invariant Functions

The Lane-Emden equation of finite polytropic index $n \neq -1, \pm\infty$ is given as (Prialnik 2007)

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \quad . \tag{1}$$

This equation is subject to the initial conditions

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$$\text{at} \quad \xi = 0 \;\; ; \;\; \theta = 1, \quad \frac{d\theta}{d\xi} = 0 \; . \qquad (2)$$

a) Theorem

If $\theta(\xi)$ *is a solution of the Lane-Emden equation of index n then* $A^{\frac{2}{n-1}}\theta(A\xi)$ *, where* A *is an arbitrary real constant is also a solution of the equation* .The proof of this theorem is given by Horedt (2004)

Thus, if one solution $\theta = \theta(\xi)$ of the Lane-Emden equation is known, we can derive a whole homologous family $\{\theta(\xi)\}$ of solutions. In particular, if θ is just the Lane-Emden function defined by the initial conditions of Equation (2), then its homologous family $\{\theta(\xi)\}$ defines a whole set of solutions that are all finite at the origin $\xi = 0$. Solutions that are finite at the origin are called *E*-solutions and denoted by θ_E. The Lane-Emden function defined by the initial conditions from Equation (2) is just a particular member of the set $\{\theta(\xi)\}$ of *E*-solutions. All *E*-solutions can be found from the Lane-Emden function through the homology transformation

$$\theta(\xi) \to A^{2/(n-1)} \; \theta(A\xi) \qquad n \neq -1, \pm\infty \, . \qquad (3)$$

It should also be noted that, any solution $\theta_E = \theta_E(\xi)$ that is finite at the origin $\xi = 0$ is an *E*-solution, and its derivative is zero $(d\theta_E / d\xi)_{\xi=0} = 0$.

The general solution of the second order Lane-Emden equation must characterized by two integration constants. According to the homology theorem one of the two constants must be "trivial" in the sense that it defines merely the scale factor A of the homology transformation, and we should be able to transform the second order Lane-Emden equation into a first order differential equation(Chandrasekhar (1957).

b) Homology invariant functions

1-A function Q (say) is said to *homology invariant* if it is invariant to the homologous transformation:

$$\theta^*(\xi) = A^{\omega}\,\theta(A\xi) \;\; \text{or} \;\; \theta^*\!\left(\frac{\xi}{A}\right) = A^{\omega}\,\theta(\xi) \; ; \; \omega = 2/(n-1) \; ; n \neq -1, \pm\infty, \qquad (4)$$

So, to prove that, Q is homology invariant function, we have to prove that

$$Q^*(\xi) = Q(A\xi) \;\; \text{or} \;\; Q^*\!\left(\frac{\xi}{A}\right) = Q(\xi) \qquad (5)$$

2-The homology transformation for the derivatives are:

$$\left|\frac{d^k\theta^*(\xi)}{d\xi^k}\right|_{\xi=\frac{\xi}{A}} = A^{\omega+k}\,\frac{d^k\theta(\xi)}{d\xi^k} \;\; ; \;\; n \neq -1, \pm\infty, \qquad (6)$$

Notes

III. Mathematica Module "HomoInvFunPol"

In this section we shall develop a *Mathematica Module* called "**HomoInvFunPol**".

In what follows, the module will be described through its basic points, propose, input, output and computational steps. Applications of the module are also given

- *Propose*

To determine if a function is homology invariant or not.

- *Input*

F: A function of ξ, which we required to known if it is homology invariant or not

n: The polytropic index, such that $n \neq -1, \pm\infty$

k1, k2, k3, k4 : Nonnegative integers which represent the order of the derivatives that may exist in the function $F(\xi)$

- *Output*

1 - Message informing that, the function is homology invariant or not homology invariant

2 - Full proof of the result

- *Module List*

Notes

Appendix A : *Mathematica* Module HomoInvFunPol

```
HomoInvFunPol[F_, n_, k1_, k2_, k3_, k4_] := Module[{},
  P1 = s; 2/(n - 1);
  P2 = {k1 s^k1, k2 s^k2,
        k3 s^k3, k4 s^k4};
  W1 = s A A ;
  W2 = {k1 s A^k1 A^k1 k1,
        k2 s A^k2 A^k2 k2,
        k3 s A^k3 A^k3 k3,
        k4 s A^k4 A^k4 k4};
  T = (z_/r_)^m_ z^m r^m, h_ g_^m_ h^m g^m,
      h_^u_^m_ h^m u;
  Fs = F P1 P2;
  H = Fs . A; G H W1 W2
    FullSimplify; S G T FullSimplify;
  TF = F S;
  Print["The Given function F", F]; Which[TF == True,
    Print[" is Homology invariant function "],
    Print[" The proof "],
    Print[" F", Fs],
    Print[" F A", H],
    Print[" Appling the general rules for homology
invariant functions we get "],
    Print[" F A", S],
    Print[" So we find that F A equal to F ,
this is the required to be proved. "],
   TF == False , Print["The function F is not
 Homology invariant function"], Print[" The proof "],
    Print[" F", Fs],
    Print[" F A", H],
    Print[" Appling the general rules for homology
invariant functions we get "],
    Print[" F A", S FullSimplify],
    Print[" So we find that F A not equal to F ,
this is the required to be proved. "]]];
```

Notes

Notes

a) Applications

1-Is the function $F_1(\xi) = -\xi\psi^n(\xi)/\psi'(\xi)$ is homology invariant or not?.

Appling the module with $k1 = 1, k2 = k3, = k4 = 0$, we get the following message, and the proof of the result

```
The Given function F

is Homology invariant function

The proof

Appling the general rules for homology
invariant functions we get

So we find that F equal to F ,
this is the required to be proved.
```

The above function $F_1(\xi) = -\xi\psi'(\xi/\psi(\xi)$ was introduced by Milne(1930) ,it plays an important role in fitting up solutions at the surface of the composite stellar models (Menzel et al 1963).

$$F_1(\xi) = 3 \times \frac{\text{local densiy } \rho(r)}{\text{mean densiy } \overline{\rho}(r) \text{ within radius r}}$$

2-Is the function $F_2(\xi) = -\xi\psi'(\xi)/\psi(\xi)$ is homology invariant or not.

Appling the module with $k1 = 1, k2 = k3, = k4 = 0$, we get the following message, and the proof of the result

```
The Given function F

is Homology invariant function

The proof

Appling the general rules for homology
invariant functions we get

So we find that F equal to F ,
this is the required to be proved.
```

Also, the function $F_2(\xi) = -\xi\psi'(\xi)/\psi(\xi)$ was introduced by Milne (1932), and plays the same important role as the function $F_1(\xi)$ in fitting up solutions at the surface of the composite stellar models (Menzel et al 1963).

$$F_2(\xi) = \frac{1}{n+1}\,\frac{\left(\frac{1}{\gamma-1}\right)\text{grav.itational energy}}{\text{internal energy}}\ \text{per unit mass at r}$$

where $\gamma = C_p / C_v$, C_p and C_v are the specific heats at constant pressure and constant volume respectively.

Appendix B : Applications of the *Mathematica* Module HomoInvFunPol

```
F3 □ □ □ 2/(n□1) □ □ □ ;
F4 □ □ □ (1□n)/(□1□n) □ □ ' □ ;
F5 □ □ □k □ □ ' □ □ □ □ □ □1□1/2 □1□k □1□n ;
F6 □ □ □k □ □ □ k n□1 n□1 2 □ ' □ ;
F7 □ □ □k□1 □ □ □1□1/2 k □1□n □ □ ' □ ;
F8 □ □ □k□1 □ □ ' □ □1□k□ 2 k/(1□n) □ □ ' ' □ ;
F9 □ □ □k □ □ □ k□ ;
F10 □ □ □k □ □ ' □ k n□1 n□1
```

We apply our module with appropriate values of to the above new functions and we get homology

F3: k1=k2=k3=k4=0

The Given function F □ □ □ 2/(□1□n) □ □

is Homology invariant function

□ The proof

□ F□ □ □ □ 2/(□1□n) □s □ □

□ F□ □ □A □ (□/A)^(2/(□1□n)) □s □ □/A

□ Appling the general rules for homology invariant functions we get

□ F□ □ □A □ □ 2/(□1□n) □ □

□ So we find that F□ □ □A □ equal to F □ □ ,

this is the required to be proved.

F4: k1=1,k2=k3=k4=0

The Given function F □ □ □ (1□n)/(□1□n) □□ □ □

is Homology invariant function

□ The proof

□ F□ □ □ □ (1□n)/(□1□n) □s□ □ □

□ F□ □ □A □ (□/A)^((1□n)/(□1□n)) □s□ □ □/A

□ Appling the general rules for homology invariant functions we get

□ F□ □ □A □ □ (1□n)/(□1□n) □□ □ □

□ So we find that F□ □ □A □ equal to F □ □ ,

this is the required to be proved.

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue XIII Version I Year 2012

Notes

F5: k1=1,k2=k3=k4=0

The Given function $F\square\square\square\square^{k}\ \square\square\square^{1\square\frac{1}{2}\,\square 1\square k\square\square 1\square n\square}\ \square\square\square$

is Homology invariant function

□ The proof

□ $F^{\square}\square\square\square\square^{k}\ \square_{s}\square\square^{1\square\frac{1}{2}\,\square 1\square k\square\square 1\square n\square}\ \square_{s}\square\square$

□ $F^{\square}\square\square A\square\left(\frac{\square}{A}\right)^{k}\ \square_{s}\frac{\square}{A}^{1\square\frac{1}{2}\,\square 1\square k\square\square 1\square n\square}\ \square_{s}\frac{\square}{A}$

□ Appling the general rules for homology invariant functions we get

□ $F^{\square}\square\square A\square\square\square^{k}\ \square\square\square^{1\square\frac{1}{2}\,\square 1\square k\square\square 1\square n\square}\ \square\square\square$

□ So we find that $F^{\square}\square\square A\square$ equal to $F\square\square$, this is the required to be proved.

F6 : k1=1,k2=k3=k4=0

The Given function $F\square\square\square\frac{\square^{k}\ \square\square\square^{\frac{1}{2}\,\square 1\square k\square 1\square n\square n\square}}{\square^{\square}\square\square}$

is Homology invariant function

□ The proof

□ $F^{\square}\square\square\square\frac{\square^{k}\ \square_{s}\square\square^{\frac{1}{2}\,\square 1\square k\square 1\square n\square n\square}}{\square_{s}{}^{\square}\square\square}$

□ $F^{\square}\square\square A\square\frac{\frac{\square}{A}^{k}\ \square_{s}\frac{\square}{A}^{\frac{1}{2}\square 1\square k\square 1\square n\square n\square}}{\square_{s}{}^{\square}\frac{\square}{A}}$

□ Appling the general rules for homology invariant functions we get

□ $F^{\square}\square\square A\square\frac{\square^{k}\ \square\square\square^{\frac{1}{2}\,\square 1\square k\square 1\square n\square n\square}}{\square^{\square}\square\square}$

□ So we find that $F^{\square}\square\square A\square$ equal to $F\square\square$, this is the required to be proved.

F7: k1=1,k2=k3=k4=0

The Given function $F\square\square\square\square^{1\square k}\ \square\square\square^{1\square\frac{1}{2}\,k\,\square 1\square n\square}\ \square\square\square$

is Homology invariant function

□ The proof

□ $F^{\square}\square\square\square\square^{1\square k}\ \square_{s}\square\square^{1\square\frac{1}{2}\,k\,\square 1\square n\square}\ \square_{s}{}^{\square}\square\square$

□ $F^{\square}\square\square A\square\left(\frac{\square}{A}\right)^{1\square k}\ \square_{s}\frac{\square}{A}^{1\square\frac{1}{2}\,k\,\square 1\square n\square}\ \square_{s}\frac{\square}{A}$

□ Appling the general rules for homology invariant functions we get

□ $F^{\square}\square\square A\square\square\square^{1\square k}\ \square\square\square^{1\square\frac{1}{2}\,k\,\square 1\square n\square}\ \square^{\square}\square\square$

□ So we find that $F^{\square}\square\square A\square$ equal to $F\square\square$, this is the required to be proved.

F8: k1=1,k2=2,k3=k4=0

The Given function $F\square\square\square\square^{1\square k}\ \square^{\square}\square\square^{1\square k\square\frac{2k}{1\square n}}\ \square^{\square\square}\square\square$

is Homology invariant function

□ The proof

□ $F^{\square}\square\square\square\square^{1\square k}\ \square_{s}{}^{\square}\square\square^{1\square k\square\frac{2k}{1\square n}}\ \square_{s}{}^{\square\square}\square\square$

□ $F^{\square}\square\square A\square\left(\frac{\square}{A}\right)^{1\square k}\ \square_{s}{}^{\square}\frac{\square}{A}^{1\square k\square\frac{2k}{1\square n}}\ \square_{s}{}^{\square\square}\frac{\square}{A}$

□ Appling the general rules for homology invariant functions we get

□ $F^{\square}\square\square A\square\square\square^{1\square k}\ \square^{\square}\square\square^{1\square k\square\frac{2k}{1\square n}}\ \square^{\square\square}\square\square$

□ So we find that $F^{\square}\square\square A\square$ equal to $F\square\square$, this is the required to be proved.

F9: k1=k2=k3=k4=0

The Given function $F\square\square\square\square^{k}\ \square\square\square^{\frac{1}{2}\,k\,\square 1\square n\square}$

is Homology invariant function

□ The proof

□ $F^{\square}\square\square\square\square^{k}\ \square_{s}\square\square^{\frac{1}{2}\,k\,\square 1\square n\square}$

□ $F^{\square}\square\square A\square\left(\frac{\square}{A}\right)^{k}\ \square_{s}\frac{\square}{A}^{\frac{1}{2}\,k\,\square 1\square n\square}$

□ Appling the general rules for homology invariant functions we get

□ $F^{\square}\square\square A\square\square\square^{k}\ \square\square\square^{\frac{1}{2}\,k\,\square 1\square n\square}$

□ So we find that $F^{\square}\square\square A\square$ equal to $F\square\square$, this is the required to be proved.

F10: k1=1,k2=k3=k4=0

The Given function $F\square\square\square\square^{k}\ \square^{\square}\square\square^{\frac{k\,\square 1\square n\square}{1\square n}}$

is Homology invariant function

□ The proof

□ $F^{\square}\square\square\square\square^{k}\ \square_{s}{}^{\square}\square\square^{\frac{k\,\square 1\square n\square}{1\square n}}$

□ $F^{\square}\square\square A\square\left(\frac{\square}{A}\right)^{k}\ \square_{s}{}^{\square}\frac{\square}{A}^{\frac{k\,\square 1\square n\square}{1\square n}}$

□ Appling the general rules for homology invariant functions we get

□ $F^{\square}\square\square A\square\square\square^{k}\ \square^{\square}\square\square^{\frac{k\,\square 1\square n\square}{1\square n}}$

□ So we find that $F^{\square}\square\square A\square$ equal to $F\square\square$, this is the required to be proved.

In concluded the present paper, a general *Mathematica* module was established to determine if a function is homology invariant or not. The module is described through its basic points, propose, input, output and computational steps. Applications of the module are also given

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Notes

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 13 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Negative Energies and Time Reversal in Quantum Field Theory

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Abstract - The theoretical and phenomenological status of negative energies is reviewed in Quantum Field Theory leading to the conclusion that hopefully their rehabilitation might only be completed in a modified general relativistic model.

GJSFR-F Classification : MSC 2010: 81T20, 81T10

NEGATIVEENERGIESANDTIMEREVERSALINQUANTUMFIELDTHEORY

eJournal Print Journal epaper open standard 3d virtual journal

Ref.

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Negative Energies and Time Reversal in Quantum Field Theory

Frederic Henry-Couannier

Abstract - The theoretical and phenomenological status of negative energies is reviewed in Quantum Field Theory leading to the conclusion that hopefully their rehabilitation might only be completed in a modified general relativistic model.

I. Introduction

With recent cosmological observations related to supernovae, CMB and galactic clustering the evidence is growing that our universe is undergoing an accelerated expansion at present. Though the most popular way to account for this unexpected result has been the reintroduction of a cosmological constant or a new kind of dark matter with negative pressure, scalar fields with negative kinetic energy, so-called phantom fields, have recently been proposed [1] [2] [3] as new sources leading to the not excluded possibility that the equation of state parameter be less than minus one. Because such models unavoidably lead to violation of positive energy conditions, catastrophic quantum instability of the vacuum is expected and one has to impose an ultraviolet cutoff to the low energy effective theory in order to keep the instability at unobservable rate. Stability is clearly the challenge for any model trying to incorporate negative energy fields interacting with positive energy fields. But before addressing this crucial issue, it is worth recalling and analyzing how and why Quantum Field Theory discarded negative energy states. We shall find that this was achieved through several not so obvious mathematical choices, often in close relation with the well known pathologies of the theory, vacuum and UV loop divergences. Following another approach starting from the orthogonal alternative mathematical choices, the crucial link between negative energies, time reversal and the existence of discrete symmetry conjugate worlds will appear.

II. Negative Energy and Classical Fields

a) *Extremum action principle*

Let us first address the stability of paths issue. Consider the path r(t) of a material point of mass m with fixed endpoints at time t_1 and t_2 in the potential U(r,t). The action S is:

$$S = \int_{t_1}^{t_2} (1/2\, mv^2 \; - U(r,t))dt$$

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The extremum condition (δS=0) is all we need to establish the equation of motion:

$$m\dot{v} = -\frac{\partial U}{\partial r}$$

S has no maximum because of the kinetic term positive sign. The extremum we find is a minimum. Let us try now a negative kinetic term:

$$S = \int_{t_1}^{t_2} (-1/2\, m\, v^2 \; - U(r,t))dt$$

The extremum condition (δS=0) is all we need to establish the equation of motion:

$$\text{-}m\dot{v} = -\tfrac{\partial U}{\partial r}$$

S has no minimum because of the kinetic term negative sign. The extremum we find is a maximum. Eventually, it appears that the fundamental principle is that of stationary (δS=0) action, the extremum being a minimum or a maximum depending on the sign of the kinetic term. In all cases we find stable trajectories.

b) *Classical relativistic fields*

We can also check that negative kinetic energy terms (ghost terms) in a free field action are not problematic. When we impose the extremum action condition the negative energy field solutions simply maximize the action. Now, in special relativity for a massive or mass-less particle, two energy solutions are always possible:

$$E = \pm\sqrt{p^2+m^2},\; E = \pm\,|p|$$

In other words, the Lorentz group admits, among others, negative energy representations $E^2 - p^2 = m^2 > 0,\; E < 0,\; E^2 - p^2 = 0,\; E < 0$. Thus, not only can we state that negative energy free field terms are not problematic but also that negative energy field solutions are expected in any relativistic field theory. For instance the Klein-Gordon equation:

$$\left(\partial^\mu \partial_\mu + m^2\right) \overset{(\sim)}{\phi}(x) = 0$$

admits when $m^2 > 0$ (we shall not try to understand here the physical meaning of tachyonic ($m^2 < 0$) and vacuum ($E = p = m = 0$) representations) positive $\phi(x)$ and negative $\tilde{\phi}(x)$ energy free field solutions. Indeed, the same Klein-Gordon equation results from applying the extreme action principle to either the 'positive' scalar action:

$$\int d^4x\, \phi(x) \left(\partial^\mu \partial_\mu + m^2\right) \phi(x)$$

or the 'negative' scalar action:

$$-\int d^4x\, \tilde{\phi}(x) \left(\partial^\mu \partial_\mu + m^2\right) \tilde{\phi}(x)$$

From the former a positive conserved Hamiltonian is derived through the Noether theorem:

$$\int d^3x \left(\frac{\partial \phi^\dagger(\mathbf{x},t)}{\partial t}\frac{\partial \phi(\mathbf{x},t)}{\partial t} + \sum_{i=1,3}\frac{\partial \phi^\dagger(\mathbf{x},t)}{\partial x_i}\frac{\partial \phi(\mathbf{x},t)}{\partial x_i} + m^2\phi^\dagger(\mathbf{x},t)\phi(\mathbf{x},t)\right)$$

Notes

while a negative one is derived from the latter:

$$-\int d^3x(\frac{\partial\tilde{\phi}^\dagger(\mathbf{x},t)}{\partial t}\frac{\partial\tilde{\phi}(\mathbf{x},t)}{\partial t}+\sum_{i=1,3}\frac{\partial\tilde{\phi}^\dagger(\mathbf{x},t)}{\partial x_i}\frac{\partial\tilde{\phi}(\mathbf{x},t)}{\partial x_i}+m^2\tilde{\phi}^\dagger(\mathbf{x},t)\tilde{\phi}(\mathbf{x},t))$$

III. Negative Energy in Relativistic Quantum Field Theory (QFT)

a) Creating and annihilating negative energy quanta

At first sight it would seem that the negative frequency terms appearing in the plane wave Fourier decomposition of any field naturally stand for the negative energy solutions. But as soon as we decide to work in a self-consistent quantization theoretical framework, that is the second quantization one, the actual meaning of these negative frequency terms is clarified. Operator solutions of field equations in conventional QFT read:

$$\phi(x)=\phi_+(x)+\phi_-(x)$$

with $\phi_+(x)$ a positive frequency term creating **positive** energy quanta and $\phi_-(x)$ a negative frequency term annihilating **positive** energy quanta. So negative energy states are completely avoided thanks to the mathematical choice of creating and annihilating only positive energy quanta and $\phi(x)$ built in this way is just the positive energy solution. This choice would be mathematically justified if one could argue that there are strong reasons to discard the 'negative action' we introduced in the previous section. But there are none and as we already noticed the Klein-Gordon equation is also easily derived from such action and the negative energy field solution:

$$\tilde{\phi}(x)=\tilde{\phi}_+(x)+\tilde{\phi}_-(x)$$

(with $\tilde{\phi}_+(x)$ a positive frequency term annihilating **negative** energy quanta and $\tilde{\phi}_-(x)$ a negative frequency term creating **negative** energy quanta) is only coherent with the negative Hamiltonian derived from the negative action through the Noether theorem (in the same way it is a standard QFT result that the usual positive energy quantum field $\phi(x)$ is only coherent with the above positive Hamiltonian [6] [7]). Therefore, it is mathematically unjustified to discard the negative energy solutions. Neglecting them on the basis that negative energy states remain up to now undetected is also very dangerous if we recall that antiparticles predicted by the Dirac equation were considered unphysical before they were eventually observed. If negative (or tachyonic) energy states are given a profound role to play in physics, this must be fully understood otherwise we might be faced with insurmountable difficulties at some later stage.

There is a widespread belief that the negative energy issue were once and for all understood in terms of antiparticles. Indeed, because charged fields are required not to mix operators with different charges, the charge conjugate creation and annihilation operators (antiparticles) necessarily enter into the game. Following Feynman's picture, such antiparticles can as well be considered as negative energy particles propagating backward in time. According S.Weinberg [8], it is only in relativistic (Lorentz transformation do not leave invariant the order of events separated by space-like intervals) quantum mechanics (non negligible probability for a particle to get from x_1 to x_2 even if x_1 - x_2 is space-like) that antiparticles are a necessity to avoid the logical paradox of a particle being absorbed before it is emitted. However, these antiparticles have nothing to do with genuine negative energy states propagating forward in time, whose quanta are by construction of the conventional QFT fields never created nor annihilated. Therefore, our deep understanding of the actual meaning of field negative frequency terms in QFT does not "solve" the negative energy issue

Ref.

[6] S. Weinberg, *Quantum Field Theory.*

since the corresponding solutions were actually neglected from the beginning. As we shall see, there is a heavy price to pay for having neglected the negative energy solutions: all those field vacuum divergences that unavoidably arise after quantization and may be an even heavier price are the ideas developed to cancel such infinities without reintroducing negative energy states.

b) A unitary time reversal operator

In a classical relativistic framework, one could not avoid energy reversal under time reversal simply because energy is the time component of a four-vector. But, when one comes to establish in Quantum Field Theory the effect of time reversal on various fields, nobody wants to take this simple picture serious anymore mainly because of the unwanted negative energy spectrum it would unavoidably bring into the theory. It is argued that negative energy states remain undetected and that their existence would necessarily trigger catastrophic decays of particles and vacuum: matter could not be stable. To keep energies positive, the mathematical choice of an anti-unitary time reversal operator comes to the rescue leading to the idea that the time-mirrored system corresponds to 'running the movie backwards' interchanging the roles of initial and final configurations. We shall come back to the stability issue later. But for the time being, let us stress that the running backward movie picture is not self–evident. In particular, the interchange of initial and final state under time reversal is very questionable. To see this, let us first recall that there are two mathematical possibilities for a time reversal operator; either it must be unitary or anti-unitary. These lead to two quite different, both mathematically coherent time reversal conjugate scenarios:

The process $i \to f$ being schematized as:

$$|i\rangle = a+(E_{i1})...a+(E_{in})\,|0\rangle \quad \overset{TIMEARROW}{\Rightarrow} \times \quad \langle f| = \langle 0|\, a(E_{f1})...a(E_{fp})$$

$$-\infty \leftarrow t \qquad\qquad t \to +\infty$$

the time reversed coordinate is $t_{rev} = -t$ and:

The conventional QFT anti-unitary time reversal scenario interchanges initial and final states:

$$i \to f \overset{T}{\Rightarrow} T^A(f) \to T^A(i)$$

$$|f\rangle = a+(E_{f1})...a+(E_{fp})\,|0\rangle \quad \overset{TIMEARROW}{\Rightarrow} \quad \langle i| = \langle 0|\, a(E_{i1})...a(E_{in})$$

$$-\infty \leftarrow t_{rev} \qquad\qquad t_{rev} \to +\infty$$

The unitary one does not interchange initial and final state but reverses energies

$$i \to f \overset{T}{\Rightarrow} T^U(i) \to T^U(f)$$

$$\left\langle \tilde{f} \right| = \langle 0|\, a(-E_{f1})...a(-E_{fp}) \quad \overset{TIMEARROW}{\Leftarrow} \quad \left|\tilde{i}\right\rangle = a^+(-E_{i1})...a^+(-E_{in})\,|0\rangle$$

$$-\infty \leftarrow t_{rev} \qquad\qquad t_{rev} \to +\infty$$

Our common sense intuition then tells us that the interchange of initial and final state, hence the anti-unitary picture stands to reason. This is because we naively require that in the time reverted picture the initial state (the ket) must come 'before' the final state (the bra) i.e for a lower value of t_{rev}. However, paying careful attention to the issue we realize that the time arrow, an underlying concept of time flow which here influences our intuition is linked to a specific property of the time coordinate which is not relevant for a spatial coordinate,

Notes

namely its irreversibility or causality. But as has been pointed out by many authors, there are many reasons to suspect that such irreversibility and time arrow may only be macroscopic scale (or statistical physics) valid concepts not making sense for a microscopic time, at least before any measurement takes place. We believe that our microscopic time coordinate, before measurement takes place, should be better considered as a spatial one, i.e possessing no property such as an arrow. Then, the unitary picture is the most natural one as a time reversal candidate process simply because it is the usual choice for all other discrete and continuous symmetries.

But if neither t nor t_{rev} actually stand for the genuine flowing time which we experiment and measure, the latter must arise at some stage and it is natural to postulate that its orientation corresponding to the experimented time arrow is simply defined in such a way that, as drawn in the previous pictures, the initial state (the creator) always comes before the final state (the annihilator) in this flowing time. This clearly points toward a theoretical framework where the time will be treated as a quantum object undergoing radical transformations from the microscopic to the macroscopic time we measure. Let us anticipate that the observable velocities will be better understood in term of this new macroscopic flowing time variable which arrow (orientation) keeps the same under reversal of the unflowing space–like t coordinate.

Therefore, the interchange of initial and final states is only justified under the assumption that time coordinate reversal implies time arrow reversal. But this is not at all obvious and thus there is no more strong reason to prefer and adopt the QFT anti-unitary choice. At the contrary, we can now list several strong arguments in favor of the unitary choice:

- The mathematical handling of an anti-unitary operator is less trivial and induces unusual complications when applied for instance to the Dirac field.
- The QFT choice leads to momentum reversal, a very surprising result for a mass-less particle, since in this case it amounts to a genuine wavelength reversal and not frequency reversal, as one would have expected.
- Its anti-unitarity makes T really exceptional in QFT. As a consequence, not all basic four-vectors transform the same way under such operator as the reference space-time four-vector. In our mind, a basic four-vector is an object involving the parameters of a one particle state such as for instance its energy and three momentum components. The one particle state energy is the time component of such an object but does not reverses as the time itself if T is taken anti-unitary. This pseudo-vector behavior under time reversal seems nonsense and leads us to prefer the unitary scenario. At the contrary, we can understand why (and accept that) the usual operator four-vectors, commonly built from the fields, behave under discrete transformations such as unitary parity differently than the reference space-time four-vector. This is simply because, as we shall see, they involve in a nontrivial way the parity-pseudo-scalar 3-volume.
- Time irreversibility at macroscopic scale allows us to define unambiguously our time arrow. But, as we already noticed, the arrow of time at the microscopic scale or before any measurement process takes place may be not so well defined. The statement that the time arrow is only a macroscopic scale (or may be statistical physics) valid concept is not so innovative. We know from Quantum Mechanics that all microscopic quantum observables acquire their macroscopic physical status through the still enigmatic measurement process. Guessing that the time arrow itself only becomes meaningful at macroscopic scale, we could reverse our microscopic time coordinate t as an arrowless spatial coordinate. Reverting the time arrow is more problematic since this certainly raises the well known time reversal and causality paradoxes. But the good new is that reversing the

time coordinate does not necessarily imply reversing the arrow of time, i.e interchanging initial and final state. In the unitary picture, you do not actually go backward in time since you just see the same succession (order) of events counting the t_{rev} time “à rebours”, with only the signs of the involved energies being affected and you need not worry anymore about paradoxes. Therefore, in a certain sense, the running backward movie picture was may be just a kind of entropy reversal picture, a confusing and inappropriate macroscopic scale concept which obscured our understanding of the time coordinate reversal and led us to believe that the anti-unitary scenario was obviously the correct one.

- Charge and charge density are invariant while current densities get reversed under a unitary time reversal (see section VI).
- Negative energy fields are natural solutions of all relativistic equations.
- The instability issue might be solved in a modified general relativistic model as we shall show in [5].

IV. Negative Energy Quantum Fields, Time Reversal and Vacuum Energies

We shall now explicitly build the QFT neglected solutions, e.g. the usual bosonic and fermionic negative energy fields, show how these are linked to the positive ones through time reversal and how vacuum divergences cancel from the Hamiltonians.

a) *The neutral scalar field*

The positive energy scalar field solution of the Klein-Gordon equation is:

$$\phi(x,t)=\int\frac{d^3p}{(2\pi)^{3/2}(2E)^{1/2}}\left[a(p,E)e^{i(Et-px)}+a^\dagger(p,E)e^{-i(Et-px)}\right]$$

with $E=\sqrt{p^2+m^2}$. The negative energy scalar field solution of the same Klein-Gordon equation is:

$$\tilde{\phi}(x,t)=\int\frac{d^3p}{(2\pi)^{3/2}(2E)^{1/2}}\left[\tilde{a}^\dagger(-p,-E)e^{i(Et-px)}+\tilde{a}(-p,-E)e^{-i(Et-px)}\right]$$

We just required this field to create and annihilate negative energy quanta. Assuming T is anti-unitary, it is well known that a scalar field is transformed according

$$T\phi(x,t)T^{-1}=\phi(x,-t)$$

where, for simplicity, an arbitrary phase factor was chosen unity. Then it is straightforward to show that:

$$Ta^\dagger(p,E)T^{-1}=a^\dagger(-p,E)$$

We do not accept this result because we want time reversal to flip energy, not momentum. If instead, the T operator is chosen unitary like all other discrete transformation operators (P, C) in Quantum Field Theory we cannot require $T\phi(x,t)T^{-1}=\phi(x,-t)$, but rather:

$$T\phi(x,t)T^{-1}=\tilde{\phi}(x,-t)$$

The expected result is then obtained as usual through the change in the variable p→-p:

$$Ta^\dagger(p,E)T^{-1}=\tilde{a}^\dagger(p,-E)$$

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue XIII Version I Year 2012

Notes

This confirms that a unitary T leads to energy reversal of scalar field quanta. Momentum is invariant. For a massive particle this may be interpreted as mass reversal coming along with velocity reversal. But in the unitary time reversal scenario it is not at all obvious that the velocity is built out of the time coordinate which gets reversed. Instead, as soon as this velocity is measured it seems more natural to build it out of the (as well measured) flowing time which never gets reversed. In this case, neither velocity nor mass get reversed. The Hamiltonian for our free neutral scalar field reads:

$$H = +\frac{1}{2}\int d^3x\,[\,(\frac{\partial\phi(x,t)}{\partial t})^2 + (\frac{\partial\phi(x,t)}{\partial x})^2 + m^2\phi^2(x,t)]$$

The Hamiltonian for the corresponding negative energy field is:

$$\tilde{H} = \tilde{P}^0 = -\frac{1}{2}\int d^3x\,[\,(\frac{\partial\tilde{\phi}(x,t)}{\partial t})^2 + (\frac{\partial\tilde{\phi}(x,t)}{\partial x})^2 + m^2\tilde{\phi}^2(x,t)]$$

The origin of the minus sign under time reversal of H will be investigated in sections VI. After replacing the scalar fields by their expressions, the computation then follows the same line as in all QFT books, leading to:

$$H = \frac{1}{2}\int d^3p\,p^0(a^\dagger(p,E)a(p,E) + a(p,E)a^\dagger(p,E))$$

$$\tilde{H} = \;-\frac{1}{2}\int d^3p\,p^0(\tilde{a}^\dagger(-p,-E)\tilde{a}(-p,-E) + \tilde{a}(-p,-E)\tilde{a}^\dagger(-p,-E))$$

With $p^0 = \sqrt{p^2+m^2}$ and the usual commutation relations,

$$[a_p^\dagger, a_{p'}] = \delta^4(p-p'), [\tilde{a}_p^\dagger, \tilde{a}_{p'}] = \delta^4(p-p')$$

vacuum divergences cancel (as we shall see, in a general relativistic framework, these only cancel as gravitational sources), and for the total Hamiltonian we get:

$$H_{total} = \int d^3p\,p^0\left\{a^\dagger(p,E)a(p,E) - \tilde{a}^\dagger(-p,-E)\tilde{a}(-p,-E)\right\}$$

It is straightforward to check that the energy eigenvalue for a positive (resp negative) energy ket is positive (resp negative), as it should. For a vector field, the infinities would cancel in the same way assuming as well the usual commutation relations.

b) The Dirac field

Let us investigate the more involved case of the Dirac field. The Dirac field is solution of the free equation of motion:

$$(i\gamma^\mu\partial_\mu - m)\quad(x,t) = 0$$

When multiplying this Dirac equation by the unitary T operator from the left, we get:

$$(iT\gamma^\mu T^{-1}\partial_\mu - TmT^{-1})T\psi(x,t) = (iT\gamma^\mu T^{-1}\partial_\mu - TmT^{-1})\tilde{\boldsymbol{\psi}}(x,-t) = 0$$

If the rest energy term m is related to the Higgs field value at its minimum (or another dynamical field) its transformation under time reversal is more involved than that of a pure number. Rather, we have:

$$m = g\phi_0(x,t) \rightarrow \tilde{m} = TmT^{-1} = g\tilde{\phi}_0(x,-t)$$

Making the replacement, $\partial_0 = -\partial^0, \partial_i = \partial^i$ and requiring that the T conjugate Dirac and scalar fields at its minimum $\tilde{\ }(x,-t) = T\,\psi(x,t)T^{-1}$, $\tilde{\phi}_0(x,-t) = T\,\phi_0(x,t)T^{-1}$ together should obey the same equation, e.g.

$$(i\gamma^\mu\partial^\mu - g\tilde{\phi}_0(x,-t))\tilde{\boldsymbol{\psi}}(x,-t) = 0$$

as $\ (x,t)$ and $\phi_0(x,t)$, leads to:

$$T\gamma^i T^{-1} = \gamma^i,\, T\gamma^0 T^{-1} = -\gamma^0$$

The T operator is then determined to be $T = \gamma^1\gamma^2\gamma^3$. Now assuming also that $\tilde{\phi}_0(x,t) = -\phi_0(x,t)$, the Dirac equation satisfied by $\tilde{\ }(x,t)$ reads:

$$(i\gamma^\mu\partial_\mu + m)\tilde{\boldsymbol{\psi}}(x,t) = 0$$

γ^0, γ^i being a particular gamma matrices representation used in equation $(i\gamma^\mu\partial_\mu - m)\ (x,t) = 0$, then $(i\gamma^\mu\partial_\mu + m)\tilde{\boldsymbol{\psi}}(x,t) = 0$ can simply be obtained from the latter by switching to the new gamma matrices representation $-\gamma^0$, $-\gamma^i$ and the negative energy Dirac field $\tilde{\ }(x,t)$. As is well known, all gamma matrices representations are unitary equivalent and here γ^5 is the unitary matrix transforming the set γ^0, γ^i into $-\gamma^0$, $-\gamma^i$ $(\gamma^5\gamma^\mu\left(\gamma^5\right)^{-1} = -\gamma^\mu)$. Thus $\tilde{\ }(x,t)$ satisfies the same Dirac equation as $\gamma^5\boldsymbol{\psi}(x,t)$. The physical consequences will be now clarified. Let us write down the positive (resp negative) energy Dirac field solutions of their respective equations.

$$(x,t) = \frac{1}{(2\pi)^{3/2}} \sum_{\sigma=\pm 1/2} \int_p \frac{d^3p}{(2E)^{1/2}} \{u(-E,m,-p,-\sigma)a_c(E,m,p,\sigma)e^{i(Et-px)}$$

$$+u(E,m,p,\sigma)a^\dagger(E,m,p,\sigma)e^{-i(Et-px)}\}$$

$$\tilde{\ }(x,t) = \frac{1}{(2\pi)^{3/2}} \sum_{\sigma=\pm 1/2} \int_p \frac{d^3p}{(2E)^{1/2}} \{u(-E,-m,-p,-\sigma)\tilde{a}^\dagger(-E,-m,-p,-\sigma)e^{i(Et-px)}$$

$$+u(E,-m,p,\sigma)\tilde{a}_c(-E,-m,-p,-\sigma)e^{-i(Et-px)}\}$$

with $E = \sqrt{p^2+m^2}$. Classifying the free Dirac waves propagating in the x direction, we have as usual for the positive energy field spinors:

$$u(E,m,p_x,+1/2) = \begin{bmatrix} 1 \\ 0 \\ \frac{\sigma_x p_x}{m+E} \\ 0 \end{bmatrix},\ u(-E,m,-p_x,-1/2) = \begin{bmatrix} 1 \\ 0 \\ \frac{\sigma_x p_x}{m-E} \\ 0 \end{bmatrix}$$

$$u(E,m,p_x,-1/2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{\sigma_x p_x}{m+E} \end{bmatrix},\ u(-E,m,-p_x,+1/2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{\sigma_x p_x}{m-E} \end{bmatrix}$$

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Notes

The negative energy field spinors are also easily obtained through the replacement m → -m

$$u(-E,-m,-p_x,-1/2) = \begin{bmatrix} 1 \\ 0 \\ \frac{\sigma_x p_x}{-m-E} \\ 0 \end{bmatrix},\ u(E,-m,p_x,1/2) = \begin{bmatrix} 1 \\ 0 \\ \frac{\sigma_x p_x}{-m+E} \\ 0 \end{bmatrix}$$

$$u(-E,-m,-p_x,+1/2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{\sigma_x p_x}{-m-E} \end{bmatrix},\ u(E,-m,p_x,-1/2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{\sigma_x p_x}{-m+E} \end{bmatrix}$$

We demand that:

$$T\psi(x,t)T^{-1} = \tilde{\boldsymbol{\psi}}(x,-t)$$

This implies:

$$Ta^{\dagger}(E,m,p,\sigma)T^{-1}u(E,m,p,\sigma) = u(-E,-m,-p \to p,-\sigma)\tilde{a}^{\dagger}(-E,-m,p,-\sigma)$$

Hence:

$$Ta^{\dagger}(E,m,p,\sigma)T^{-1} = \tilde{a}^{\dagger}(-E,-m,p,-\sigma)$$

Thus, upon time reversal, energy, rest energy and spin are reversed. Because momentum is invariant helicity also flips its sign. Without having reverted the rest energy term in the negative energy Dirac field equation we could not have obtained this simple link through time reversal between the positive and negative energy creation operators. The rest energy reversal in the spinor expressions also reveals the difference between a true negative energy spinor $u(-E,-m,.,.)$ and a negative frequency spinor $u(-E,m,.,.)$. The Hamiltonian for (x,t) is:

$$H = P^0 = \int d^3x[\,\bar{\ }(x,t)(-i\gamma^i.\partial_i + m)\boldsymbol{\psi}(x,t)] \ + \ h.c$$

The negative energy field Hamiltonian will be built out of negative energy fields explicitly different from those entering in H. Hence, it is hopeless trying to obtain such kind of simple transformation relations such as $P^0 \Rightarrow \pm P^0$. On the other hand we can build the negative energy Hamiltonian and check that it provides the correct answer when applied to a given negative energy ket. We know that $T\gamma^i T^{-1} = \gamma^i$, $T\gamma^0 T^{-1} = -\gamma^0$, so that:

$$T\bar{\boldsymbol{\psi}}(x,t)T^{-1} = T\psi^{\dagger}(x,t)\gamma^0 T^{-1} = -T\psi^{\dagger}(x,t)T^{-1}\gamma^0$$
$$= -(T\psi(x,t)T^{-1})^{\dagger}\gamma^0 = -\bar{\tilde{\boldsymbol{\psi}}}(x,-t)$$

This will produce an extra minus sign in the negative energy Dirac field Hamiltonian. The origin of the other minus sign is the same as for the scalar field Hamiltonian and will be clarified later. The Hamiltonian for $\tilde{\ }(x,t)$ is then:

$$\tilde{H} = \tilde{P}^0 = - - \int d^3x[\bar{\tilde{\boldsymbol{\psi}}}(x,t)(-i\gamma^i\partial_i - m)\tilde{\boldsymbol{\psi}}(x,t)] \ + \ h.c$$

Because the positive (resp negative) energy spinor satisfies $(i\gamma^{\mu}\partial_{\mu} - m)\boldsymbol{\psi}(x,t) = 0$, (resp $(i\gamma^{\mu}\partial_{\mu} + m)\tilde{\boldsymbol{\psi}}(x,t) = 0$) we have $(-i\gamma^i\partial_i + m)\ (x,t) = i\gamma^0\partial_0\boldsymbol{\psi}(x,t)$, (resp $(-i\gamma^i\partial_i - m)\tilde{\boldsymbol{\psi}}(x,t) = i\gamma^0\partial_0\tilde{\boldsymbol{\psi}}(x,t)$). The Hamiltonians then read:

$$H = P^0 = i\int d^3x[\boldsymbol{\psi}^{\dagger}(x,t)\partial_0\boldsymbol{\psi}(x,t)] \ + \ h.c$$

$$\tilde{H} = \tilde{P}^0 = i\int d^3x[\tilde{\boldsymbol{\psi}}^{\dagger}(x,t)\partial_0\tilde{\boldsymbol{\psi}}(x,t)] \ + \ h.c$$

Assuming for simplicity that we are dealing with a neutral field, the computation proceeds as usual for the positive energy Hamiltonian. With $p^0 = \sqrt{p^2 + m^2}$:

$$H = \frac{1}{2} \sum_{\sigma=\pm 1/2} \int d^3p \, p^0 (a^\dagger(E,p,\sigma) a(E,p,\sigma) - a(E,p,\sigma) a^\dagger(E,p,\sigma))$$

Negative energy spinors possessing the same orthogonality properties as positive energy spinors, the negative energy Hamiltonian is then obtained by the simple replacements $a^\dagger(E,p,\sigma) \to \tilde{a}(-E,-p,-\sigma)$; $a(E,p,\sigma) \to \tilde{a}^\dagger(-E,-p,-\sigma)$:

$$\tilde{H} = \frac{1}{2} \sum_{\sigma=\pm 1/2} - \int d^3p \, p^0 (\tilde{a}^\dagger(-E,-p,-\sigma) \tilde{a}(-E,-p,-\sigma)$$
$$-\tilde{a}(-E,-p,-\sigma) \tilde{a}^\dagger(-E,-p,-\sigma))$$

Infinities cancel as for the boson fields when we apply the fermionic anti-commutation relations $\{a^\dagger_{p,\sigma}, a_{p',\sigma'}\} = \delta^4(p-p')\,\delta_{\sigma,\sigma'}$, $\{\tilde{a}^\dagger_{p,\sigma}, \tilde{a}_{p',\sigma'}\} = \delta^4(p-p')\,\delta_{\sigma,\sigma'}$, leading to:

$$H_{total} = \sum_{\sigma=\pm 1/2} \int d^3p \, p^0 \left\{ a^\dagger(p,E,\sigma) a(p,E,\sigma) - \tilde{a}^\dagger(-p,-E,-\sigma) \tilde{a}(-p,-E,-\sigma) \right\}$$

It is also easily checked that the energy eigenvalue for a positive (resp negative) energy ket is positive (resp negative), as it should. When we realize how straightforward are the cancellation of vacuum divergences for all fields it is very tempting to state that such infinities appeared only because half of the field solutions were neglected! We shall show in [5] that actually, in a general relativity context, our vacuum divergences only vanish as a source for gravitation. But the Casimir effect should still survive.

V. Phenomenology of the Uncoupled Positive and Negative Energy Worlds

We shall now show that the uncoupled positive and negative energy worlds are both perfectly viable: no stability issue arises and in both worlds the behavior of matter and radiation is completely similar so that the negative signs may just appear as a matter of convention [9] [10]. Consider a gas made with negative energy matter particles (fermions) and negative energy photons. The interaction between two negative energy fermions is going on through negative energy photons exchange. Because the main result will only depend on the bosonic nature of the considered interaction field, let us compute and compare the simpler propagator of the positive and negative energy scalar fields.

-For a positive energy scalar field:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}(2p^0)^{1/2}} \left[a(p)e^{ipx} + a_c^\dagger(p)e^{-ipx}\right]$$

we get as usual:

$$\langle 0| T(\phi(x)\phi^\dagger(y)) |0\rangle = \langle 0| \phi(x)\phi^\dagger(y) |0\rangle \; \theta(x_0 - y_0) + \langle 0| \phi^\dagger(y)\phi(x) |0\rangle \; \theta(y_0 - x_0)$$

$$= \langle 0| \int \frac{d^3p}{(2\pi)^3 2p^0} a(p)a^\dagger(p)e^{ip(x-y)} |0\rangle \, \theta(x_0 - y_0)$$

$$+ \langle 0| \int \frac{d^3p}{(2\pi)^3 2p^0} a_c(p)a_c^\dagger(p)e^{-ip(x-y)} |0\rangle \, \theta(y_0 - x_0)$$

Ref.

[5] F. Henry-Couannier, gr-qc/0410055.

Notes

$$= \int \frac{d^3p}{(2\pi)^3 2p^0} e^{ip(x-y)} \theta(x_0 - y_0) + \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip(x-y)} \theta(y_0 - x_0)$$

$$= \Delta(y-x)\theta(x_0 - y_0) + \Delta(x-y)\theta(y_0 - x_0)$$

-For a negative energy scalar field:

$$\tilde{\phi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}(2p^0)^{1/2}} \left[\tilde{a}^\dagger(p) e^{ipx} + \tilde{a}_c(p) e^{-ipx}\right]$$

we obtain:

$$\langle 0| T(\tilde{\phi}(x)\tilde{\phi}^\dagger(y)) |0\rangle = \langle 0| \tilde{\phi}(x)\tilde{\phi}^\dagger(y) |0\rangle \; \theta(x_0 - y_0) + \langle 0| \tilde{\phi}^\dagger(y)\tilde{\phi}(x) |0\rangle \; \theta(y_0 - x_0)$$

$$= \langle 0| \int \frac{d^3p}{(2\pi)^3 2p^0} \tilde{a}_c(p)\tilde{a}_c^\dagger(p) e^{-ip(x-y)} |0\rangle \, \theta(x_0 - y_0)$$

$$+ \langle 0| \int \frac{d^3p}{(2\pi)^3 2p^0} \tilde{a}(p)\tilde{a}^\dagger(p) e^{ip(x-y)} |0\rangle \, \theta(y_0 - x_0)$$

$$= \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip(x-y)} \theta(x_0 - y_0) + \int \frac{d^3p}{(2\pi)^3 2p^0} e^{ip(x-y)} \theta(y_0 - x_0)$$

$$= \Delta(x-y)\theta(x_0 - y_0) + \Delta(y-x)\,\theta(y_0 - x_0)$$

Summing the two propagators, the theta functions cancel:

$$\langle 0| T(\tilde{\phi}(x)\tilde{\phi}^\dagger(y)) |0\rangle + \langle 0| T(\phi(x)\phi^\dagger(y)) |0\rangle$$

$$= (\Delta(x-y) + \Delta(y-x))(\theta(x_0 - y_0) + \theta(y_0 - x_0)\,)$$

$$= \Delta(x-y) + \Delta(y-x) \propto \int (\delta\left(E - p^0\right) + \delta\left(E + p^0\right)) e^{-iE(x_0 - y_0)} dE$$

Therefore, if the two propagators could contribute with the same coupling to the interaction between two currents, the virtual particle terms would cancel each other. Only on-shell particles could still be exchanged between the two currents provided energy momentum conservation does not forbid it. For a photon field as well the two off-shell parts of the propagators would be found opposite. Hence the coulomb potential derived from the negative energy photon field propagator would be exactly opposite to the coulomb potential derived from the positive energy photon field propagator: as a consequence, the 1/r Coulomb potential and electromagnetic interactions would simply disappear. The interesting point is that in our negative energy gas, where we assume that only the exchange of negative energy virtual photons takes place, the coulomb potential is reversed compared to the usual coulomb potential generated by positive energy virtual photons exchange. However in this repulsive potential between oppositely charged fermions, these still attract each other, as in the positive energy world, because of their negative inertial terms in the equation of motion (as deduced from their negative terms in the action). The equation of motion for a given negative energy matter particle in this Coulomb potential is:

$$-m\dot{v} = - - \frac{\partial U_c}{\partial r}$$

or

$$m\dot{v} = -\frac{\partial U_c}{\partial r}$$

We find ourselves in the same situation as that of a positive energy particles gas interacting in the usual way e.g through positive energy photons exchange. Hence negative energy atoms will form and the main results of statistical physics apply: following Boltzman law, our particles will occupy with the greatest probabilities states with minimum $\frac{1}{2}m\dot{v}^2$, thus with maximum energy$-\frac{1}{2}m\dot{v}^2$. Temperatures are negative. This result can be extended to all interactions propagated by bosons as are all known interactions. The conclusion is that the non-coupled positive and negative energy worlds are perfectly stable, with positive and negative energy particles minimizing the absolute value of their energies:

VI. Actions and Hamiltonians Under Time Reversal and Parity

a) Negative integration volumes?

Starting from the expression of the Hamiltonian density for a positive energy neutral scalar field:

$$T^{00}(x,t) = \left(\frac{\partial\phi(x,t)}{\partial t}\right)^2 + \sum_{i=1,3}\left(\frac{\partial\phi(x,t)}{\partial x_i}\right)^2 + m^2\phi^2(x,t)$$

and applying time reversal we get:

$$\left(\frac{\partial\tilde{\phi}(x,-t)}{\partial t}\right)^2 + \sum_{i=1,3}\left(\frac{\partial\tilde{\phi}(x,-t)}{\partial x_i}\right)^2 + m^2\tilde{\phi}^2(x,-t)$$

with $T\phi(x,t)T^{-1} \equiv \tilde{\phi}(x,-t)$ From such expression, a naive free Hamiltonian density for the scalar field$\tilde{\phi}(x,t)$ may be proposed:

$$\tilde{T}^{00}(x,t) = \left(\frac{\partial\tilde{\phi}(x,t)}{\partial t}\right)^2 + \sum_{i=1,3}\left(\frac{\partial\tilde{\phi}(x,t)}{\partial x_i}\right)^2 + m^2\tilde{\phi}^2(x,t)$$

It thus happens that $\tilde{T}^{00}(x,t)$ is manifestly positive since it is a sum of squared terms. We of course cannot accommodate negative energy fields with positive Hamiltonian densities so following the procedure used to obtain negative kinetic energy terms for a phantom field, we just assumed in the previous sections a minus sign in front of this expression. But how could we justify this trick if time reversal does not provide us with this desired minus sign? One possible solution appears when we realize that according to general relativity, actually T^{00} is not a spatial energy density but rather$\sqrt{g}T^{00}$ where $g \equiv -Det\, g_{\mu\nu}$. This is also expected to still remain positive because of a rather strange mathematical choice in general relativity: integration volumes such as dt, d^4x, d^3x are not signed and should not flip sign under time reversal or parity transformations. Let us try the more natural opposite way: $t \to -t \Rightarrow dt \to -dt$ and $x \to -x \Rightarrow dx \to -dx$, natural in the sense that this is naively the straightforward mathematical way to proceed and let us audaciously imagine that for instance a negative 3-dimentional volume is nothing else but the image of a 3-dimentional positive volume in a mirror. Then, the direct consequence of working with signed volumes is that the general relativistic integration element $d^4x\sqrt{g}$ is not invariant anymore under coordinate transformations (such as P or T) with negative Jacobian (it is often stated that the absolute value of the Jacobian is imposed by a fundamental theorem of integral calculus[2]. But should not this apply only to change of variables and not general coordinate transformations?). We are then led to choose an invariant integration element under any

Notes

coordinate transformations: this is $d^4x\left|\frac{\partial\xi}{\partial x}\right|$, where $\left|\frac{\partial\xi}{\partial x}\right|$ stands for the Jacobian of the transformation from the inertial coordinate system ξ^α to x^μ. Because $\left|\frac{\partial\xi}{\partial x}\right|$ is not necessarily positive as is $\sqrt{g}$ in general relativity, it will get reversed under P or T transformations affecting Lorentz indices only so that spatial charge density $\left|\frac{\partial\xi}{\partial x}\right|J^0$, scalar charge $Q=\int\left|\frac{\partial\xi}{\partial x}\right|J^0d^3x$, spatial energy-momentum densities $\left|\frac{\partial\xi}{\partial x}\right|T^{\mu 0}$ and energy-momentum four-vector $P^\mu=\int\left|\frac{\partial\xi}{\partial x}\right|T^{\mu 0}d^3x$ should transform accordingly. For instance, it is often stated that a unitary time reversal operator is not allowed because it would produce the not acceptable charge reversal. This analysis is no more valid if the Jacobi determinant flips its sign. Indeed, though J^0, as all four-vector time components, becomes negative, the spatial charge density $\left|\frac{\partial\xi}{\partial x}\right|J^0$ and scalar charge $Q=\int\left|\frac{\partial\xi}{\partial x}\right|J^0d^3x$ remain positive under unitary time reversal. It is also worth checking what is now the effect of unitary space inversion: $P^\mu=\int\left|\frac{\partial\xi}{\partial x}\right|T^{\mu 0}d^3x$ transforms under Parity as $T^{\mu 0}$ times the pseudo-scalar Jacobi determinant $\left|\frac{\partial\xi}{\partial x}\right|$, so that:

$$P^0\Rightarrow -P^0,\ P^i\Rightarrow P^i$$

$Q=\int\left|\frac{\partial\xi}{\partial x}\right|J^0d^3x$ also transforms under Parity as J^0 times the pseudo-scalar Jacobi determinant $\left|\frac{\partial\xi}{\partial x}\right|$, so that:

$$Q\Rightarrow -Q$$

So, if unitary Parity has the same effect on various fields, currents and energy densities as in conventional quantum field theory, it now produces a flip in the energy and charge signs but does not affect momentum! Anyway, we see that the signed Jacobi determinant could do the good job for providing us with the desired minus signs. However, working with negative integration volumes amounts to give up the usual definition of the integral which insures that it is positive definite. If we are not willing to give up this definition, another mechanism should be found to provide us with the necessary minus sign. The issue will be reexamined and a more satisfactory solution described in [5].

VII. Interactions Between Positive and Negative Energy Fields?

Postulate the existence of a new inertial coordinate system $\tilde{\xi}$ such that $\left|\frac{\partial\tilde{\xi}}{\partial x}\right|$ is negative. This can be achieved simply by considering the two time reversal conjugate (with opposite proper times) inertial coordinate systems ξ and $\tilde{\xi}$. We may then define the positive energy quantum $F(x)$ fields (resp negative energy $\tilde{F}(x)$ fields) as the fields entering in the action with positive $\left|\frac{\partial\xi}{\partial x}\right|$ (resp negative $\left|\frac{\partial\tilde{\xi}}{\partial x}\right|$) entering in the integration volume so that the energy $P^0=\int\left|\frac{\partial\xi}{\partial x}\right|T^{00}d^3x$ (resp $\tilde{P}^0=\int\left|\frac{\partial\tilde{\xi}}{\partial x}\right|\tilde{T}^{00}d^3x$) is positive (resp negative). The action for positive energy matter and radiation is then as usual:

$$S=\int d^4x\left|\frac{\partial\xi}{\partial x}\right|\left\{L(\Psi(x),\frac{\partial\xi^\alpha}{\partial x^\mu}(x))+L(A_\mu(x),\frac{\partial\xi^\alpha}{\partial x^\mu}(x))+J_\mu(x)A^\mu(x)\right\}$$

Similarly, the action for negative energy matter and radiation is:

$$\tilde{S}=\int d^4x\left|\frac{\partial\tilde{\xi}}{\partial x}\right|\left\{L(\tilde{\Psi}(x),\frac{\partial\tilde{\xi}^\alpha}{\partial x^\mu}(x))+L(\tilde{A}_\mu(x),\frac{\partial\tilde{\xi}^\alpha}{\partial x^\mu}(x))+\tilde{J}_\mu(x)\tilde{A}^\mu(x)\right\}$$

Notes

Hence positive energy fields move under the influence of the gravitational field $\frac{\partial \xi^\alpha}{\partial x^\mu}$, while negative energy fields move under the influence of the gravitational field $\frac{\partial \tilde{\xi}^\alpha}{\partial x^\mu}$. Then, the mixed coupling in the form $J_\mu(x)\tilde{A}^\mu(x)$ that we might have naively hoped is not possible just because the integration volume must be $d^4x \left|\frac{\partial \xi}{\partial x}\right|$ for $F(x)$ type fields and $d^4x \left|\frac{\partial \tilde{\xi}}{\partial x}\right|$ for $\tilde{F}(x)$ type fields. Indeed coherence requires that in the action the negative Jacobian be associated with negative energy fields $\tilde{F}(x)$ involving negative energy quanta creation and annihilation operators. This is a good new since it is well known that couplings between positive and negative energy fields lead to an unavoidable stability problem due to the fact that energy conservation keeps open an infinite phase space for the decay of positive energy particles into positive and negative energy particles. A scenario with positive and negative energy fields living in different metrics also provides a good way to account for the undiscovered negative energy states. However the two metrics should not be independent if we want to introduce a connection at least gravitational between positive and negative energy worlds, mandatory to make our divergences gravitational effects actually cancel. In [5] we shall explicit this dependency between the two conjugate metrics and the mechanism that gives rise through the extremum action principle to the negative source terms in the Einstein equation. It will be clear that this mechanism only works properly if, as in general relativity, we keep working with Jacobi determinants absolute values and do not give up the usual definition of integrals.

VIII. Maximal C, P and Baryonic Asymmetries

One of the most painful concerns in High Energy Physics is related to our seemingly inability to provide a satisfactory explanation for the maximal Parity violation observed in the weak interactions. The most popular model that may well account, through the seesaw mechanism, for the smallness of neutrino masses is quite disappointing from this point of view since parity violation is just put in by hand, as it is in the standard model, in the form of different spontaneous symmetry breaking scalar patterns in the left and right sectors. The issue is just postponed, and we are still waiting for a convincing explanation for this trick. Actually, one gets soon convinced that the difficulty comes from the fact that Parity violation apparently only exists in the weak interaction. Much more easy would be the task to search for its origin if this violation was universal. And yet, quite interestingly, it seems possible to extend parity violation to all interactions, just exploiting the fundamental structure of fermion fields and at the same time explain why this is only detectable and apparent in the weak interactions. There exists four basic degrees of freedom, solutions of the Dirac field equations: these are $\psi_L(x), \psi_R(x), \psi_{cL}(x), \psi_{cR}(x)$ but two of them suffice to create and annihilate quanta of both charges and helicities: for instance the usual $\psi(x) = \psi_L(x) + \psi_R(x)$ may be considered as the most general Dirac solution:

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int_{p,\sigma} u(p,\sigma)a_c(p,\sigma).e^{i(px)} + v(p,\sigma)a^\dagger(p,\sigma)e^{-i(px)}\, d^3p$$

But another satisfactory base, as far as our concern is just to build kinetic_interaction terms and not mass terms, could be the pure left handed $\psi_L(x) + \psi_{cL}(x)$ field making use of the charge conjugate field.

$$\psi_c(x) = \frac{1}{(2\pi)^{3/2}} \int_{p,\sigma} u(p,\sigma)a(p,\sigma)\, e^{i(px)} + v(p,\sigma)a_c^\dagger(p,\sigma)e^{-i(px)}\, d^3p$$

Indeed, from a special relativistic mass-less Hamiltonian such as

$$H_L^0 = \int d^3x[\psi_L^\dagger(-i\alpha.\nabla)\,\psi_L] + \int d^3x[\psi_{cL}^\dagger(-i\alpha.\nabla)\,\psi_{cL}]$$

Notes

the same normal ordered current and physics as the usual one are derived when requiring various global symmetries to become local (this is checked in the Annex).

$$\overline{\Psi}\gamma_\mu \frac{1-\gamma_5}{2}\Psi(x) - \overline{\Psi_c}\gamma_\mu \frac{1-\gamma_5}{2}\Psi_c(x) =: \overline{\Psi}\gamma_\mu \Psi(x) :$$

Assume now that the corresponding general Dirac field built out of only right handed components is not redundant with the previous (as is generally believed because except for a Majorana particle, both create and annihilate quanta of all charges and helicities) but lives in the conjugate metric, an assumption which we shall later justify. This then would be from our world point of view a negative energy density field. This manifestly maximal parity violating framework would not allow to detect any parity violating behavior in those interactions involving only charged Dirac particles in their multiplets, because the charge conjugate left handed field $\psi_{cL}(x)$ can successfully mimic the right handed field $\psi_R(x)$. However, in any interaction involving a completely neutral e.g Majorana fermion, $\psi_{cL}(x)$ could not play this role anymore resulting as in the weak interaction in visible maximal parity and charge violation (we claim that though no symmetry forbids it, the one degree of freedom Majorana field for a neutrino cannot be associated simultaneously with the two degrees of freedom of the Dirac charge field, since this amounts to duplicate the Majorana kinetic term and appears as an awkward manipulation, therefore one has to choose which electron/positron charge is associated with the neutral particle (neutrino) in the multiplet, this resulting in maximal charge violation and making the already present parity violation manifest). Even neutral-less fermion multiplets as in the quark sector of the weak interactions could then have inherited this parity and charge violation provided their particles lived together with neutral fermion particles in higher dimensional groups before symmetry breaking occurred producing their separation into distinct multiplets.

Now what about mass terms? For charged fields, coupling with a positive energy right handed field must take place to produce the chirality flipping mass term. But the right handed field is not there. It may be that no bare mass term is explicitly allowed to appear in an action and that a new mechanism should be found to produce interaction generated massive propagators starting from a completely mass-less action. Let us guess that such scenario is not far from the one which is actually realized in nature, because maximal Charge and Parity violation, and the related bayonic asymmetry of the universe has otherwise all of the characteristics of a not solvable issue.

But why should right handed chiral fields be negative energy density fields? Because this is what the pseudo-vector behavior under Parity of the operator four-momentum told us in section VI.1. Remember however that the unitary parity conjugate field creates positive energy point - like quanta and can be viewed as a positive point-like energy field (this is a standard QFT result). This four-vector behavior under parity of the one particle state four-momentum (an object we called a basic four-vector in section III.2) seems to be in contradiction with the pseudo-vector behavior of the four-momentum field operator. Actually the measured energy of the particle is obtained by acting with the energy operator on the one particle state ket. So there is no contradiction because this measurement is of course performed on a non zero three dimensional volume and we have to admit that the measured energy of the particle we get is negative from our world point of view as a result of the particle being living in an enantiomorphic 3-dimensionnal space. In other words, from our world point of view, the parity conjugate field has a negative energy **density,** which we may consider as a positive energy per negative inertial 3-volume, so that it leads to a negative energy when integrated on a general coordinate 3-volume (as if it was a parity scalar, the behavior of this 3-volume under a parity transformation plays no role in our discussion since this is just one of the general coordinate trans-

Notes

formations). Then the PT fields are again positive integrated energy (energy measured in a finite volume) fields but oppositely charged (charge measured in a finite volume), i.e describing anti-particles (see VI.1) living in our world metric and interacting with their PT symmetric fields describing particles.

In short, we believe that recognizing the universality of Parity violation, i.e the fact that we are living in a left chiral world, is also an interesting approach to the issue. It then suffices to introduce the right chiral parity conjugate world (its action) to plainly restore Parity invariance of the total action. Eventually it may be, as already Sakharov suggested in 1967 [11] , that we are living in a left chiral positive energy world with its particles and antiparticles while the conjugate world is from our world point of view a right chiral negative energy world with its particles and antiparticles.

IX. Synthesis

Let us gather the main information we learned from our investigation of negative energies in Relativistic QFT indicating that the correct theoretical framework for handling them should be found in a modified GR.

- The TheoreticaI Viewpoint

In second quantification, all relativistic field equations admit genuine negative energy field solutions creating and annihilating negative energy quanta. Unitary time reversal links these fields to the positive energy ones. The unitary choice, usual for all other symmetries in physics also allows to avoid the well known paradoxes associated with time reversal. Positive and negative energy fields vacuum divergences we encounter after second quantization are unsurprisingly found to be exactly opposite. The negative energy fields action must be maximised. However there is no way to reach a coherent theory involving negative energies in flat space-time. Indeed, if positive and negative energy scalar fields are time reversal conjugate, their Hamiltonian densities and actions must also be so which we shall find to be only possible in the context of general relativity thanks to the metric transformation under discrete symmetries.

- The Phenomenological Viewpoint

In a mirror negative energy world which fields remain non coupled to our world positive energy fields, stability is insured and the behavior of matter and radiation is as usual. Hence, it's just a matter of convention to define each one as a positive or negative energy world. Only if they could interact, would we expect hopefully promising new phenomenology since many outstanding enigmas, among which are the flat galactic rotation curves, the Pioneer effect, the universe flatness, acceleration and its voids, indicate that repelling gravity might play an important role in physics. On the other hand, negative energy states never manifested themselves up to now, strongly suggesting that a barrier is at work preventing the two worlds to interact except through gravity.

- The Main Issues

A trivial cancellation between vacuum divergences is not acceptable since the Casimir effect shows evidence for vacuum fluctuations. But in our approach, the positive and negative energy worlds will be maximally gravitationally coupled in such a way as to only produce exact cancellations of vacuum energies gravitational effects. Also, a generic catastrophic instability issue arises whenever quantum positive and negative energy fields are allowed to interact. If we restrict the stability issue to our modified gravity we will see that this disastrous scenario is also avoided. At last, allowing both positive and negative energy virtual photons to propagate the electromagnetic interaction simply makes it disappear. The local gravitational interaction will be treated very differently in our modified GR so that this unpleasant feature also be avoided.

Ref.

[11] A. D. Sakharov, *JETP Lett.* **5**, 24 (1967).

- Outlooks

A left-handed kinetic and interaction Lagrangian can satisfactorily describe all known physics except mass terms which anyway remain problematic in modern physics. This strongly supports the idea that the right handed chiral fields might be living in another world (where the 3-volume reversal under parity presumably would make these fields acquire a negative energy density) and may provide an interesting explanation for maximal parity violation observed in the weak interaction.

If the connection between the two worlds is fully reestablished above a given energy threshold, then loop divergences naturally would get cancelled thanks to the positive and negative energy virtual propagators compensation. Such reconnection might take place through a new transformation process allowing particles to jump from one metric to the conjugate one[4] presumably at places where the conjugate metrics meet each other.

Notes

X. Conclusion

Of course, negative energy matter remains undiscovered at present and the stability issue strongly suggests that making it interact with normal matter requires new non standard interaction mechanisms. However, considering the seemingly many related theoretical and phenomenological issues and recalling the famous historical examples of equation solutions that were considered unphysical for a long time before they were eventually observed, we believe it is worth trying to understand how negative energy solutions should be handled in GR. We will propose a special treatment for discrete symmetry transformations in GR. A new gravitational picture will be derived in [5] opening rich phenomenological and theoretical perspectives and making us confident that the approach is on the right way.

XI. Acknowledgments

I above all thank my parents but also my friends and family for their staunch support. I specially thank J.P Petit, J. Mallet, G. D'agostini and P. Midy who accepted to check carefully my computations. It is a pleasure to warmly thank A. Tilquin, J-B. Devivie, T. Schucker, G. Esposito-Farèse and D. Buskulic for many helpful advices, comments and fascinating discussions. I acknowledge my colleagues and friends from the Centre de Physique des Particules de Marseille, the Centre de Physique Théorique and the Faculté de Luminy for their kindness and interest in my work, particularly P. Coyle, E. Nagy, M. Talby, E. Kajfasz, F. Bentosela, J. Busto, O. Leroy, A. Duperrin and my friends from the RENOIR group.

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Annex

Notes

For a purely left-handed kinetic lagrangien,

$$L_{kin} = -\bar{\Psi}_L \gamma^\mu \partial_\mu \Psi_L - \bar{\Psi}_{Lc} \gamma^\mu \partial_\mu \Psi_{Lc}$$

Gauge invariance yields interaction terms :

$$L_{kin} + L_{int} = -\bar{\Psi}_L (\gamma^\mu [\partial_\mu + ieA_\mu]) \Psi_L - \bar{\Psi}_{Lc} (\gamma^\mu [\partial_\mu - ieA_\mu]) \Psi_{Lc}$$

from which follows the QED current :

$$[\bar{\Psi} \gamma_\mu (\frac{1-\gamma_5}{2}) \Psi(x) - \bar{\Psi}_c \gamma_\mu (\frac{1-\gamma_5}{2}) \Psi_c(x)]$$

Useful formula ([6] p219&225)

$$u^+(q,\sigma) = (u^*(q,\sigma))^T = (-\beta C v(q,\sigma))^T = -v(q,\sigma)^T C^T \beta^T \Rightarrow u^+(q,\sigma) = v(q,\sigma)^T C\beta$$

$$v^+(q,\sigma) = (v^*(q,\sigma))^T = (-\beta C u(q,\sigma))^T = -u(q,\sigma)^T C^T \beta^T$$
$$\Rightarrow v^+(q,\sigma) = u(q,\sigma)^T C\beta \ (1)$$

then

$$u^+(q',\sigma')\beta\gamma_\mu \frac{1-\gamma_5}{2} u(q,\sigma) = v(q',\sigma')^T C\gamma_\mu \frac{1-\gamma_5}{2} u(q,\sigma)$$

$$v^+(q,\sigma)\beta\gamma_\mu \frac{1+\gamma_5}{2} v(q',\sigma') = u(q,\sigma)^T C\gamma_\mu \frac{1+\gamma_5}{2} v(q',\sigma')$$

using

$$\begin{aligned} (C\gamma_\mu \tfrac{1-\gamma_5}{2})^T &= \tfrac{1-\gamma_5^T}{2} \gamma_\mu^T C^T = -\tfrac{1-\gamma_5^T}{2} \gamma_\mu^T C \\ &= \tfrac{1-\gamma_5^T}{2} C\gamma_\mu = C\tfrac{1-\gamma_5}{2}\gamma_\mu = C\gamma_\mu \tfrac{1+\gamma_5}{2} \end{aligned}$$

we obtain the first useful formula

$$u^+(q',\sigma')\beta\gamma_\mu \frac{1-\gamma_5}{2} u(q,\sigma) = v^+(q,\sigma)\beta\gamma_\mu \frac{1+\gamma_5}{2} v(q',\sigma')$$

From (1) we get

$$\begin{aligned} v^+(q,\sigma)\beta\gamma_\mu \tfrac{1-\gamma_5}{2} u(q',\sigma') &= u(q,\sigma)^T C\gamma_\mu \tfrac{1-\gamma_5}{2} u(q',\sigma') \\ &= u(q',\sigma')^T C\gamma_\mu \tfrac{1+\gamma_5}{2} u(q,\sigma) \end{aligned}$$

but

$$v^+(q',\sigma')\beta = u(q',\sigma')^T C$$

which leads to the second useful formula

$$v^+(q,\sigma)\beta\gamma_\mu \frac{1-\gamma_5}{2} u(q',\sigma') = v^+(q',\sigma')\beta\,\gamma_\mu \frac{1+\gamma_5}{2} u(q,\sigma)$$

Notes

Computation of the left-handed current

$$\overline{\Psi}\gamma_\mu\frac{1-\gamma_5}{2}\Psi(x) =$$

$$\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} u^*(p',\sigma')\beta\gamma_\mu\frac{1-\gamma_5}{2}u(p,\sigma).e^{i(-p'x+px)}a^\dagger(p',\sigma')a(p,\sigma)d^3pd^3p'$$

$$+\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} v^*(p',\sigma')\beta\gamma_\mu\frac{1-\gamma_5}{2}v(p,\sigma).e^{i(p'x-px)}a_c(p',\sigma')a_c^\dagger(p,\sigma)d^3pd^3p'$$

$$+\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} v^*(p',\sigma')\beta\gamma_\mu\frac{1-\gamma_5}{2}u(p,\sigma).e^{i(p'x+px)}a_c(p',\sigma')a(p,\sigma)d^3pd^3p'$$

$$+\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} u^*(p',\sigma')\beta\gamma_\mu\frac{1-\gamma_5}{2}v(p,\sigma).e^{i(-p'x-px)}a^\dagger(p',\sigma')a_c^\dagger(p,\sigma)d^3pd^3p'$$

$$\overline{\Psi_c}\gamma_\mu\frac{1-\gamma_5}{2}\Psi_c(x) =$$

$$\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} v^*(p,\sigma)\beta\gamma_\mu\frac{1-\gamma_5}{2}v(p',\sigma').e^{i(px-p'x)}a(p,\sigma)a^\dagger(p',\sigma')d^3pd^3p'$$

$$\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} u^*(p,\sigma)\beta\gamma_\mu\frac{1-\gamma_5}{2}u(p',\sigma').e^{i(-px+p'x)}a_c^\dagger(p,\sigma)a_c(p',\sigma')d^3pd^3p'$$

$$\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} u^*(p,\sigma)\beta\gamma_\mu\frac{1-\gamma_5}{2}v(p',\sigma').e^{i(-px-p'x)}a_c^\dagger(p,\sigma)a^\dagger(p',\sigma')d^3pd^3p'$$

$$\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} v^*(p,\sigma)\beta\gamma_\mu\frac{1-\gamma_5}{2}u(p',\sigma').e^{i(px+p'x)}a(p,\sigma)a_c(p',\sigma')d^3pd^3p'$$

$$\overline{\Psi_c}\gamma_\mu\frac{1-\gamma_5}{2}\Psi_c(x) =$$

$$\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} u^*(p',\sigma')\beta\gamma_\mu\frac{1+\gamma_5}{2}u(p,\sigma).e^{i(px-p'x)}a(p,\sigma)a^\dagger(p',\sigma')d^3pd^3p'$$

$$+\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} v^*(p',\sigma')\beta\gamma_\mu\frac{1+\gamma_5}{2}v(p,\sigma).e^{i(-px+p'x)}a_c^\dagger(p,\sigma)a_c(p',\sigma')d^3pd^3p'$$

$$+\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} u^*(p',\sigma')\beta\gamma_\mu\frac{1+\gamma_5}{2}v(p,\sigma).e^{i(-px-p'x)}a_c^\dagger(p,\sigma)a^\dagger(p',\sigma')d^3pd^3p'$$

$$+\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} v^*(p',\sigma')\beta\gamma_\mu\frac{1+\gamma_5}{2}u(p,\sigma).e^{i(px+p'x)}a(p,\sigma)a_c(p',\sigma')d^3pd^3p'$$

$$\overline{\Psi_c}\gamma_\mu\tfrac{1-\gamma_5}{2}\Psi_c(x) =$$
$$-\tfrac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} u^*(p',\sigma')\beta\gamma_\mu\tfrac{1+\gamma_5}{2}u(p,\sigma).e^{i(px-p'x)}a^\dagger(p',\sigma')a(p,\sigma)d^3pd^3p'$$

$$-\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} v^*(p',\sigma')\beta\gamma_\mu\frac{1+\gamma_5}{2}v(p,\sigma).e^{i(-px+p'x)}a_c(p',\sigma')a_c^\dagger(p,\sigma)d^3pd^3p'$$

$$-\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} u^*(p',\sigma')\beta\gamma_\mu\frac{1+\gamma_5}{2}v(p,\sigma).e^{i(-px-p'x)}a^\dagger(p',\sigma')a_c^\dagger(p,\sigma)d^3pd^3p'$$

$$-\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} v^*(p',\sigma')\beta\gamma_\mu\frac{1+\gamma_5}{2}u(p,\sigma).e^{i(px+p'x)}a_c(p',\sigma')a(p,\sigma)d^3pd^3p'$$

$$+\frac{1}{(2\pi)^3}\int\limits_{p,\sigma} u^*(p,\sigma)\beta\gamma_\mu\frac{1+\gamma_5}{2}u(p,\sigma).d^3p$$

$$+\frac{1}{(2\pi)^3}\int\limits_{p,\sigma} v^*(p,\sigma)\beta\gamma_\mu\frac{1+\gamma_5}{2}v(p,\sigma).d^3p$$

$$\overline{\Psi}\gamma_\mu\frac{1-\gamma_5}{2}\Psi(x)-\overline{\Psi_c}\gamma_\mu\frac{1-\gamma_5}{2}\Psi_c(x)=$$

$$\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} u^*(p',\sigma')\beta\gamma_\mu\, u(p,\sigma).e^{i(px-p'x)}a^\dagger(p',\sigma')a(p,\sigma)d^3pd^3p'$$

$$\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} v^*(p',\sigma')\beta\gamma_\mu\, v(p,\sigma).e^{i(-px+p'x)}a_c(p',\sigma')a_c^\dagger(p,\sigma)d^3pd^3p'$$

$$\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} u^*(p',\sigma')\beta\gamma_\mu\, v(p,\sigma).e^{i(-px-p'x)}a^\dagger(p',\sigma')a_c^\dagger(p,\sigma)d^3pd^3p'$$

$$\frac{1}{(2\pi)^3}\int\limits_{p,p',\sigma,\sigma'} v^*(p',\sigma')\beta\gamma_\mu u(p,\sigma).e^{i(px+p'x)}a_c(p',\sigma')a(p,\sigma)d^3pd^3p'$$

$$-\frac{1}{(2\pi)^3}\int\limits_{p,\sigma} v^*(p,\sigma)\beta\gamma_\mu\, v(p,\sigma).d^3p$$

At last:

$$\overline{\Psi}\gamma_\mu\frac{1-\gamma_5}{2}\Psi(x)-\overline{\Psi_c}\gamma_\mu\frac{1-\gamma_5}{2}\Psi_c(x)=:\overline{\Psi}\gamma_\mu\Psi(x):$$

For a Majorana field, $\Psi_c(x)$ is not there and we are left only with a chiral kinetic term:

$$\overline{\Psi}\gamma_\mu\frac{1-\gamma_5}{2}\Psi(x)$$

We believe that such term cannot be duplicated to be found associated in multiplets with both $\Psi(x)$ and $\Psi_c(x)$ of a Dirac field, so that the above chiral kinetic term will necessarily result in a chiral interaction term in which parity and charge violation explicitly manifest themselves.

Notes

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 13 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Computational Algorithm for Gravity Turn Maneuver

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Abstract - In this paper, computational algorithm for gravity turn maneuver is established for variable thrust-to- weight ratio. The applications of the algorithm was illustrated graphically.

Keywords : *descent guidance, trajectory optimization, navigation, control.*

GJSFR-F Classification : *MSC 2010: 83C27*

Computational Algorithm for Gravity Turn Maneuver

M. A. Sharaf [α] & L.A.Alaqal [σ]

Abstract - In this paper, computational algorithm for gravity turn maneuver is established for variable thrust-to- weight ratio. The applications of the algorithm was illustrated graphically.

Keywords : descent guidance, trajectory optimization, navigation, control.

I. Introduction

It is known that (Thomson 1986) the tangent of the optimum thrust attitude φ for placing space vehicle into an orbit is always linear function of time Likewise, the optimum thrust attitude for maximum range can be shown to be $\varphi =$ constant. These conditions may be satisfactory for a rocket traveling in vacuum but, owing to the large angle of attack α (see Fig.1) which results from such trajectories, they are not feasible through the atmosphere. Thus for flight through the atmosphere, a trajectory known as gravity turn or zero –lift turn is generally used.

A gravity turn maneuver is used in launching a spacecraft into, or descending from, an orbit around a celestial body such as a planet or a moon (ShangKristian et al 2011, Mehedi et al 2011). It is a trajectory optimization that uses gravity to steer the vehicle onto its desired trajectory. It offers two main advantages over a trajectory controlled solely through vehicle's own thrust. Firstly, the thrust doesn't need to be used to change the ship's direction so more of it can be used to accelerate the vehicle into orbit. Secondly, and more importantly, during the initial ascent phase the vehicle can maintain low or even zero angle of attack. This minimizes transverse aerodynamic stress on the launch vehicle, allowing for a lighter launch vehicle (Samuel 1965). The term gravity turn can also refer to the use of a planet's gravity to change a spacecraft's direction in other situations than entering or leaving the orbit (Roger 1964).

In a gravity turn, the thrust vector is kept parallel to the velocity vector at all times (see Fig 2) starting with some nonvertical initial velocity vector $\mathbf{v}_0$.

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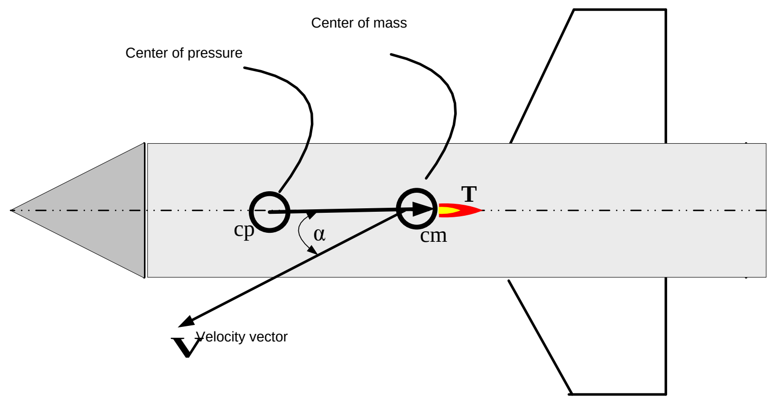

Fig1 : The angle of attack

In this paper, computational algorithm for gravity turn maneuver is established for variable thrust-to-weight ratio. The application of the algorithm was illustrated graphically.

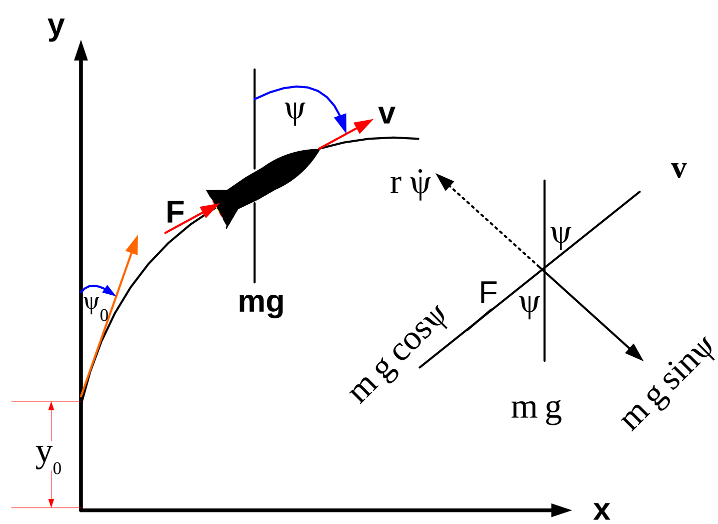

Fig. 2 : Gravity turn trajectory

II. Forces Equations

It is convenient here to measure the angle made by the velocity vector from vertical, as shown in Fig.2. Assuming zero aerodynamic drag and constant gravity field g, we can write the force equations as:

$$\frac{1}{g}\frac{dv}{dt}=\frac{F}{m\,g}-\cos\psi\,, \tag{1}$$

$$\frac{v}{g}\frac{d\psi}{dt}=\sin\psi\ , \tag{2}$$

where F is the magnitude of thrust vector and m is the instantaneous vehicle mass.

Notes

These equations are nonlinear and no analytical solution is known when F/mg varies with time.

III. NUMERICAL SOLUTION FOR VARYING F/MG

When F/mg is constant, Equations (1) and (2) can be solved analytically. For F/mg to be constant, the thrust F must decrease with time, this is because, the mass m decreases with the time t, consequently F should decreases with t so as to keep the ratio constant.

Notes

Let $F/mg = n$ over short increment of the flight path. It could be shown that(Thomson 1986) the solution for gravity turn trajectory when n is constant is represented by the following three equations

$$v = C\, z^{n-1}\,(1+z^2)\ . \tag{3}$$

The constant C can be evaluated from the initial conditions that at $z = z_0$, $v = v_0$ to get:

$$C = \frac{v_0}{z_0^{n-1}\,(1+z_0^2)}\ . \tag{4}$$

$$\Delta t = \frac{C}{g}\left\{ z^{n-1}\left(\frac{1}{n-1} + \frac{z^2}{n+1}\right) - z_0^{n-1}\left(\frac{1}{n-1} + \frac{z_0^2}{n+1}\right)\right\}. \tag{5}$$

To apply Equations (3) (4) and (5) for a varying F/mg, the following algorithm is devoted

a) *Computational algorithm*

▲ Purpose: To compute the coordinates(x,y) and the tangential velocity v of space vehicle along gravity turn path with varying F/mg ratio.

▲ Input: $t_0, \psi_0, v_0, x_0, y_0, n$

▲ Computational steps:

1 - $\Delta\psi_0 = \psi_0/100$

2 - $-z_0 = \tan\frac{1}{2}\psi_0$

3 - $-C = \dfrac{v_0}{z_0^{n-1}(1+z_0)^2}$

4 - $-\psi = \psi_0 + \Delta\psi_0$

5 - $-z = \tan\frac{1}{2}\psi$

6 - $v = Cz^{n-1}(1+z^2)$

7 - $$\Delta t = \frac{C}{g}\left\{ z^{n-1}\left(\frac{1}{n-1}+\frac{z^2}{n+1}\right) - z_0^{n-1}\left(\frac{1}{n-1}+\frac{z_0^2}{n+1}\right)\right\}$$

8 - $$\Delta x = \frac{1}{2}(v_0 \sin\psi_0 + v\sin\psi)\,\Delta t \quad ; \quad \Delta y = \frac{1}{2}(v_0\cos\psi_0 + v\cos\psi)\,\Delta t$$

9 - $$x = x_0 + \Delta x \ ; \quad y = y_0 + \Delta y$$

10 - $$x_0 = x \ ; \quad y_0 = y ; \ \psi_0 = \psi \ ; \quad t_0 = t + \Delta t$$

12 - Go to step 2

The procedure can be repeated up to any time

b) Graphical illustrations

The above algorithm was applied with the initial conditions

$$t_0 = 0 \quad ; \quad \psi_0 = 10^{o} \ ; \quad v_0 = 500\text{ft/sec} \ ; \ x_0 = 0 \quad ; \ y_0 = 3000\text{ft}$$

with n variable according to the formula: $n(t) = 3e^{-5t}$. Note that, the initial and the computed coordinates referred to the geocentric coordinate system. The output are illustrated graphically in the following figures.

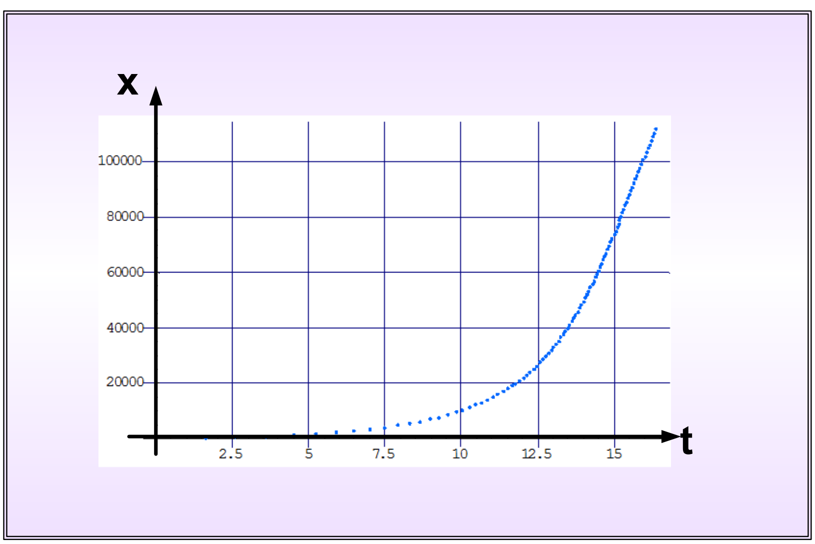

Fig. 3 : The variation of the x coordinate with time along gravity turn path with:

$$t_0 = 0 \quad ; \quad \psi_0 = 10^{o} \ ; \quad v_0 = 500\text{ft/sec} \ ; \ x_0 = 0 \quad ; \ y_0 = 3000\text{ft} \quad ; \ n(t) = 3e^{-5t}$$

Notes

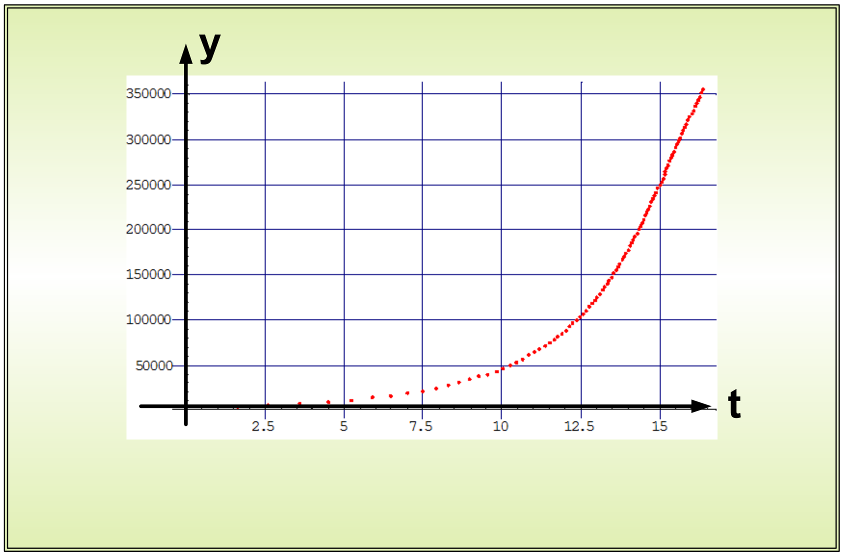

Fig. 4 : The variation of the y coordinate with time along gravity turn path with:

$$t_0 = 0 \quad ; \quad \psi_0 = 10^o \; ; \quad v_0 = 500\text{ft/sec} \; ; \; x_0 = 0 \quad ; \quad y_0 = 3000\text{ft} \quad ; \quad n(t) = 3e^{-5t}$$

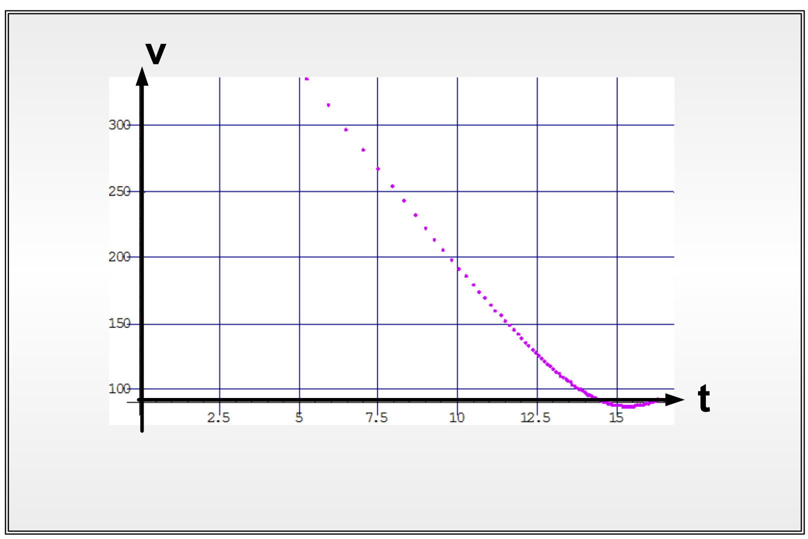

Fig. 5 : The variation of the velocity with time along gravity turn path with:

$$t_0 = 0 \quad ; \quad \psi_0 = 10^o \; ; \quad v_0 = 500\text{ft/sec} \; ; \; x_0 = 0 \quad ; \quad y_0 = 3000\text{ft} \quad ; \quad n(t) = 3e^{-5t}$$

In concluded the present paper, computational algorithm for gravity turn maneuver is established for variable thrust-to-weight ratio. The applications of the algorithm was illustrated graphically.

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 13 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Trigonometric and Wavelet Transforms for Certain Class of Generalized Functions

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Abstract - In this article, we discuss certain class of generalized functions for sine and cosine transforms. we, also obtain a new relationship between Fourier sine transform (Fourier cosine transform) and wavelet transform. Other related theorems are also established in concern.

Keywords and phrases : distribution; tempered ultradistribution; wavelet transform; generalized functions; fourier sine transform.

GJSFR-F Classification : 1991 MSC: Primary 54C40, 14E20; Secondary 46E25, 20C20

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Keywords and phrases : distribution; tempered ultradistribution; wavelet transform; generalized functions; fourier sine transform.

I. Introduction

A function $g(t)$ is said to be a mother wavelet (or simply, a wavelet) if it satisfies the following conditions:

$$(i) \int_{\mathcal{R}} g(t)\, dt = 0; (ii) \int_{\mathcal{R}} |g(t)|^2\, dt = 0; (iii) \int_{\mathcal{R}} \frac{|g(t)|^2}{t} dt < \infty.$$

The continuous wavelet transform of $f(t)$ with respect to a mother wavelet $g(t)$ is defined by [2, 8]

$$\mathcal{W}_g f(b,a) = \int_{\mathcal{R}} f(t) \frac{1}{\sqrt{|a|}} \overline{g} \left(\frac{t-b}{a} \right) dt.$$

Let $\mathcal{S}(\mathcal{R})$ denote the space of rapid descent and $\mathcal{S}'(\mathcal{R})$ its strong dual of tempered distributions over $\mathcal{R}$, the set of real numbers [5]. Due to [6], $\mathcal{S}_+(\mathcal{R})$ is the set of all those functions in $\mathcal{S}(\mathcal{R})$ whose Fourier transform is supported by the positive frequencies and $\mathcal{S}_0(\mathcal{R})$ is the space of all those functions from $\mathcal{S}(\mathcal{R})$ for which all moments vanish. That is, $g \in \mathcal{S}_0(\mathcal{R})$ if $g \in \mathcal{S}(\mathcal{R})$ and $\left|x^\alpha g^{(\beta)}(x)\right| < \infty$ and $\int_{\mathcal{R}} x^n g(x)\, dx = 0$. Further, if g is a wavelet in $\mathcal{S}_0(\mathcal{R})$ and $\Delta = \{(b,a) : b \in \mathcal{R}, a > 0\}$, then the wavelet transform for $f \in \mathcal{S}(\mathcal{R})$ is defined by [6]

$$\mathcal{W}_g f(b,a) = \left\langle f(t), \overline{g_{b,a}(t)} \right\rangle, (b,a) \in \Delta, t \in \mathcal{R}.$$

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where,

$$g_{b,a}(t) = \mathcal{T}_b \mathcal{D}_a g(t) = \frac{1}{a} g\left(\frac{t-b}{a}\right), t, b \in \mathcal{R}, a > 0,$$

$\mathcal{T}_b$ and $\mathcal{D}_a$ stands for the translation and dilation of the wavelet g .

The Fourier cosine transform of a complex-valued absolutely integrable function $f(t)$ over $(-\infty, \infty)$ is the function of the variable ξ given by [9, p.42]

$$\mathcal{F}_{cn}(f(t))(\xi) = \int_{\mathcal{R}} f(t) \cos t\xi dt.$$

Fourier sine transform $\mathcal{F}_{sn}$ has similar representation.

Fourier sine and cosine transforms being the imaginary and real parts of the complex Fourier transform, their properties for sine and cosine transform can be obtained from the known properties of the Fourier transform. Furthermore, the product of different combinations of even and odd functions is an odd function and the product of similar combinations of even and odd functions is an even function, establishes that the Fourier transform of an even function is, indeed, a Fourier cosine transform and similarly, the Fourier transform of an odd function in is a Fourier sine transform.

Our approach in this work is infact a motivation of [8] which describe a relationship between the cited trigonometric transforms, Fourier sine and Fourier cosine transforms, and the wavelet transform in the sense of generalized functions. However, the approach in this paper seems different and interesting.

II. Wavelet Transform of Ultradifferentiable Functions

By an ultradifferentiable function we mean a C^∞ function whose derivatives satisfy certain growth conditions as the order of the derivatives increase $[1, 2, 3, 4]$.

Denote by $\mathcal{U}_o(n_i, n, \mathcal{R})$ and $\mathcal{U}_e(n_i, n, \mathcal{R})$ the spaces of all odd (resp. even) C^∞ functions $\phi(x)$ on $\mathcal{R}$ such that $\left|x^i \mathcal{D}_x^j \phi(x)\right| < N n^i n_j$, $j = 1, 2, 3, \ldots$ for some positive constant N.

Functions in $\mathcal{U}_o(n_i, n, \mathcal{R})$ and $\mathcal{U}_e(n_i, n, \mathcal{R})$ are , indeed, ultradifferentiable functions and decrease to zero faster than every power of $1/x$. Denoting by $\acute{\mathcal{U}_o}(n_i, n, \mathcal{R})$ and $\acute{\mathcal{U}_e}(n_i, n, \mathcal{R})$ the duals of the corresponding spaces. The resulting space is called the odd (resp. even) tempered ultradistribution spaces which contain the space $\mathcal{S}'(\mathcal{R})$ properly [5].

In view of definitions, $\mathcal{U}_o(n_i, n, \mathcal{R}) \subset \mathcal{S}(\mathcal{R})$ and $\mathcal{U}_e(n_i, n, \mathcal{R}) \subset \mathcal{S}(\mathcal{R})$, we write $\mathcal{S}'(\mathcal{R}) \subset \acute{\mathcal{U}_o}(n_i, n, \mathcal{R})$ and $\mathcal{S}'(\mathcal{R}) \subset \acute{\mathcal{U}_e}(n_i, n, \mathcal{R})$, see [9, Theorem 2.1] .

Denote by $\mathcal{U}_o(n_i, n, \mathcal{R}, +)$ and $\mathcal{U}_e(n_i, n, \mathcal{R}, +)$ the subsets of $\mathcal{U}_o(n_i, n, \mathcal{R})$ and $\mathcal{U}_e(n_i, n, \mathcal{R})$ whose Fourier transforms are supported by the positive frequencies. Whereas, $\acute{\mathcal{U}_o}(n_i, n, \mathcal{R}, +)$ and $\acute{\mathcal{U}_e}(n_i, n, \mathcal{R}, +)$ are the respective duals of $\mathcal{U}_o(n_i, n, \mathcal{R}, +)$ and $\mathcal{U}_e(n_i, n, \mathcal{R}, +)$. In notations

$$\mathcal{U}_o(n_i, n, \mathcal{R}, +) = \left\{\phi : \phi \in \mathcal{U}_o(n_i, n, \mathcal{R}),\ \operatorname{supp} \widehat{\phi} \subset \mathcal{R}_+\right\}$$

Ref.

[9] Sneddon, I.N. *The Use of Integral Transforms*, Tata McGraw-Hill Publications Company Ltd, New Delhi, 1974.

Notes

and

$$\mathcal{U}_e(n_i,n,\mathcal{R},+)=\left\{\phi:\phi\in\mathcal{U}_e(n_i,n,\mathcal{R}),\ \operatorname{supp}\widehat{\phi}\subset\mathcal{R}_+\right\}.$$

Denoting by $\mathcal{U}_o(n_i,n,\mathcal{R},*)$ $(\mathcal{U}_e(n_i,n,\mathcal{R},*))$ the set of all $g\in\mathcal{U}_o(n_i,n,\mathcal{R})$ $(\mathcal{U}_e(n_i,n,\mathcal{R}))$ where,

$$\left|x^\alpha g^{(\beta)}(n)\right|<\infty \text{ and } \int_{\mathcal{R}} x^n g(x)\,dx=0,$$

the following propostion holds true:

Proposition 1: *Let* $f\in\mathcal{U}_o'(n_i,n,\mathcal{R})$ $(\mathcal{U}_e'(n_i,n,\mathcal{R}))$ *and* $g\in\mathcal{U}_o(n_i,n,\mathcal{R},*)$ $(\mathcal{U}_e(n_i,n,\mathcal{R},*))$, *then the wavelet transform of* f *is given by*

$$\mathcal{W}_g f(b,a)=\left\langle f(t),\overline{g}_{b,a}(t)\right\rangle,(b,a)\in\Delta,t\in\mathcal{R}, \tag{1.1}$$

where Δ has the usual meaning.

Proof : We have $\mathcal{U}_o(n_i,n,\mathcal{R})\subset\mathcal{S}(\mathcal{R})$ and $\mathcal{U}_e(n_i,n,\mathcal{R})\subset\mathcal{S}(\mathcal{R})$. Also, by the assumption that $\left|x^\alpha g^\beta(x)\right|<\infty$ and $\int_{\mathcal{R}} x^n g(x)\,dx=0$ we deduce that the wavelet $g\in\mathcal{U}_o(n_i,n,\mathcal{R},*)$ and $g\in\mathcal{U}_e(n_i,n,\mathcal{R},*)$ which justifies the proposition.

Theorem 1: *Let* $(b,a)\in\Delta$ *and* $g\in\mathcal{U}_o(n_i,n,\mathcal{R},*)$ *then*

$$\mathcal{F}_{sn}(g_{b,a}(x))(\xi)=0,\ \textit{for}\ \xi\notin\mathcal{R}_+.$$

Proof : Let $g\in\mathcal{U}_o(n_i,n,\mathcal{R},*)$ then

$$\begin{aligned}\mathcal{F}_{sn}(g_{b,a}(x))(\xi) &= \int_{\mathcal{R}} g_{b,a}(x)\sin\xi x dx\\ &= \int_{\mathcal{R}} \frac{1}{a}g\left(\frac{x-b}{a}\right)\sin\xi x dx\\ &= \int_{\mathcal{R}} g(y)\sin\xi(ay+b)\,dy,\ \frac{x-b}{a}=y\\ &= \int_{\mathcal{R}} g(y)(\sin\xi ay\cos b\xi+\sin b\xi\cos\xi ay)\,dy.\end{aligned}$$

In view of the fact that multiplication of different combinations of even and odd functions yield odd functions, we have

$$\mathcal{F}_{sn}(g_{b,a}(x))(\xi)=\cos b\xi\int_{\mathcal{R}} g(y)\sin\xi aydy.$$

Hence, $\mathcal{F}_{sn}(g_{b,a}(x))(\xi)=0$, for $\xi\notin\mathcal{R}_+$.

Theorem 2 : *Let* $(b,a)\in\Delta$ and $g\in\mathcal{U}_e(n_i,n,\mathcal{R},*)$. *Then*

$$\mathcal{F}_{cn}(g_{b,a}(x))(\xi)=0,\ \textit{for}\ \xi\notin\mathcal{R}_+.$$

Proof is similar to the proof employed above and, thus, avoided.

Theorem 3: *Let* $(b,a)\in\Delta$ *and* $g\in\mathcal{U}_o(n_i,n,\mathcal{R},*)$. *Then*

$$\mathcal{F}_{sn}(\mathcal{W}_g\phi)=\mathcal{W}_{\breve{g}_{sn}}\phi,$$

for all $\phi \in \mathcal{U}_o(n_i, n, \mathcal{R})$, *where* $\breve{g}_{sn} = \mathcal{F}_{sn}^{-1}(g_{s,t}(x))$

Proof : We have

$$\begin{aligned}
\mathcal{F}_{sn}(\mathcal{W}_g\phi)(s,t) &= \int_{\mathcal{R}} \left(\int_{\Delta} \phi(x)\, \overline{g_{b,a}}(x)\, dx \right) \sin((b,a).(s,t))\, d(b\times a) \\
&= \int_{\Delta} \phi(x) \left(\int_{\mathcal{R}} \overline{g_{b,a}(x)} \sin((b,a).(s,t))\, d(b\times a) \right) dx \\
&= \int_{\Delta} \phi(x)\, \overline{\breve{g}_{sn}(x)} dx \\
&= \mathcal{W}_{\breve{g}_{sn}}\phi.
\end{aligned}$$

This proves the theorem.

Theorem 4: *Let* $(b,a) \in \Delta$ *and* $g \in \mathcal{U}_e(n_i, n, \mathcal{R}, *)$ *. Then*

$$\mathcal{F}_{cn}(\mathcal{W}_g\phi) = \mathcal{W}_{\breve{g}_{cn}}\phi,$$

for all $\phi \in \mathcal{U}_e(n_i, n, \mathcal{R})$ *,where* $\breve{g}_{cn} = \mathcal{F}_{cn}^{-1}(g_{s,t}(x))$.

Proof being similar to that of Theorem 2 we avoid details.

Theorem 5: *Let* $(b,a) \in \Delta, g \in \mathcal{U}_o(n_i, n, \mathcal{R}, *)$ *and* $\phi, \boldsymbol{\psi} \in \mathcal{U}_o(n_i, n, \mathcal{R})$ *then*

$$\mathcal{W}_g\left(\mathcal{F}_{sn}^{-1}\phi\mathcal{F}_{sn}^{-1}\boldsymbol{\psi}\right) = \mathcal{W}_{\breve{g}_{sn}}(\phi * \boldsymbol{\psi}),$$

where $\breve{g}_{sn} = \mathcal{F}_{sn}^{-1}\left(\overline{g_{b,a}(y)}\right)(x) = \int \overline{g_{b,a}(y)} \sin yx\; dy$, and $*$ *is the convolution product.*

Proof : Let $\phi, \boldsymbol{\psi} \in \mathcal{U}_o(n_i, n, \mathcal{R})$. Then, we have

$$\begin{aligned}
\mathcal{W}_{\breve{g}_{sn}}(\phi * \boldsymbol{\psi})(y) &= \int_{\Delta} (\phi * \boldsymbol{\psi})(y)\, \overline{\breve{g}_{b,a}(y)} dy \\
&= \int_{\Delta} \left(\int_{\mathcal{R}} \phi(y)\, \boldsymbol{\psi}(y-\eta)\, d\eta \right) \overline{\breve{g}_{b,a}(y)} dy
\end{aligned}$$

Let $y - \eta = \xi$ then $dy = d\xi$ .Then ,

$$\begin{aligned}
\mathcal{W}_{\breve{g}_{sn}}(\phi * \boldsymbol{\psi})(y) &= \int_{\Delta} \left(\int_{\mathcal{R}} \phi(y)\, \boldsymbol{\psi}(\xi) \right) \overline{\breve{g}_{b,a}(\xi+\eta)} d\eta d\xi \\
&= \int_{\Delta} \left(\int_{\mathcal{R}} \phi(y)\, \boldsymbol{\psi}(\xi) \right) \left(\int_{\mathcal{R}} \overline{g_{b,a}(x)} \sin(\xi+\eta)\, x dx \right) d\eta d\xi.
\end{aligned}$$

Now, using

$$\sin(\xi+\eta)\, x = \sin \xi x \cos \eta x + \sin \eta x \cos \xi x$$

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and rules of integration for even and odd functions, we have

$$\begin{aligned}\mathcal{W}_{\breve{g}_{sn}}(\phi*\psi)(y) &= \int\limits_{\Delta}\overline{g_{b,a}(x)}\left(\int\limits_{\mathcal{R}}\psi(\xi)\cos\xi x\int\limits_{\mathcal{R}}\phi(\eta)\sin\eta x d\eta d\xi\right)dx \\ &= \int_{\Delta}\overline{g_{b,a}(x)}\left(\int_{\mathcal{R}}\psi(\xi)\cos\xi x d\xi\int_{\mathcal{R}}\phi(\eta)\sin\eta x d\eta\right)dx \\ &= \int\limits_{\Delta}\overline{g_{b,a}(x)}\hat{\psi}_c(\xi)\hat{\phi}_{sn}(\eta)dx \\ &= \mathcal{W}_g\left(\hat{\psi}_{cn}\hat{\phi}_{sn}\right),\end{aligned}$$

where $\hat{\psi}_{cn} = \mathcal{F}_{cn}\psi$ and $\hat{\phi}_{sn} = \mathcal{F}_{sn}\psi$. This completes the proof of the theorem.

Theorem 6: *Let* $(b,a) \in \Delta$ *and* $g \in \mathcal{U}_e(n_i, n, \mathcal{R}, *)$. *Let* ϕ, $\psi \in \mathcal{U}_e(n_i, n, \mathcal{R})$ *then*

$$\mathcal{W}_g\left(\hat{\phi}_{cn}\hat{\psi}_{cn}\right) = \mathcal{W}_{\breve{g}_{cn}}(\phi*\psi),$$

where $*$ *is the convolution product and* $\breve{g}_{cn}$ is the inverse Fourier cosine function of g .

Analysis of the proof of above theorem being similar to that employed for Theorem 5, details are avoided.

Theorem 7: *Given* $\mathcal{T}_1, \mathcal{T} \in \mathcal{U}'_o(n_i, n, \mathcal{R})$ *,we have*

$$\mathcal{F}_{sn}\left((\mathcal{W}_g\mathcal{T})(\mathcal{W}_g\mathcal{T}_1)\right) = (\mathcal{W}_{\breve{g}_{sn}}\mathcal{T}) * (\mathcal{W}_{\breve{g}_{sn}}\mathcal{T}_1).$$

Proof : Let $\phi \in \mathcal{U}_o(n_i, n, \mathcal{R})$. Then in view of Theorem 3, we have

$$\begin{aligned}\langle(\mathcal{W}_g\mathcal{T})(\mathcal{W}_g\mathcal{T}_1), \phi\rangle &= \mathcal{F}_{sn}^{-1}\left(\mathcal{F}_{sn}(\mathcal{W}_g\mathcal{T}_1)\mathcal{F}_{sn}(\mathcal{W}_g\mathcal{T}_1)\right), \phi\rangle \\ &= \mathcal{F}_{sn}^{-1}\left((\mathcal{W}_{\breve{g}_{sn}}\mathcal{T}) * (\mathcal{W}_{\breve{g}_{sn}}\mathcal{T}_1)\right), \phi\rangle.\end{aligned}$$

Hence,

$$(\mathcal{W}_g\mathcal{T})(\mathcal{W}_g\mathcal{T}_1) = \mathcal{F}_{sn}^{-1}\left((\mathcal{W}_{\breve{g}}\mathcal{T}) * (\mathcal{W}_{\breve{g}}\mathcal{T}_1)\right).$$

Equivalently,

$$\mathcal{F}_{sn}(\mathcal{W}_g\mathcal{T})(\mathcal{W}_g\mathcal{T}_1) = (\mathcal{W}_{\breve{g}}\mathcal{T}) * (\mathcal{W}_{\breve{g}}\mathcal{T}_1).$$

The proof of the theorem ,thus, completes. Following theorems are natural consequence of the above theorem and proofs being almost similar, we omit details.

Theorem 8: *Given* $\mathcal{T}_1, \mathcal{T} \in \mathcal{U}'_e(n_i, n, \mathcal{R})$*, we have*

$$\mathcal{F}_{cn}\left((\mathcal{W}_g\mathcal{T})(\mathcal{W}_g\mathcal{T}_1)\right) = (\mathcal{W}_{\breve{g}cn}\mathcal{T}) * (\mathcal{W}_{\breve{g}_{cn}}\mathcal{T}_1).$$

Theorem 9: *Let* $\mathcal{T}_1, \mathcal{T} \in \mathcal{U}'_o(n_i, n, \mathcal{R})$ *.Then*

$$\mathcal{F}_{sn}(\mathcal{W}_g\mathcal{T}) * (\mathcal{W}_g\mathcal{T}_1) = (\mathcal{W}_{\breve{g}_{sn}}\mathcal{T})(\mathcal{W}_{\breve{g}_{sn}}\mathcal{T}_1).$$

Theorem 10: *Let* $\mathcal{T}_1, \mathcal{T} \in \mathcal{U}'_e(n_i, n, \mathcal{R})$*. Then*

$$\mathcal{F}_{cn}(\mathcal{W}_g\mathcal{T}) * (\mathcal{W}_g\mathcal{T}_1) = (\mathcal{W}_{\breve{g}_{cn}}\mathcal{T})(\mathcal{W}_{\breve{g}_{cn}}\mathcal{T}_1).$$

III. Asymptote Properties of the Wavelet Transform

Let $(c_m)_{m=1}$ be a sequence of continuous positive functions defined on $(0, a_m)$, $a_m > 0$ and $\lim\limits_{t\to0^+} \frac{c_{m+1}(t)}{c_m(t)} = 0$. Let $(u_m)_{m=1}^{\infty}$be a sequence in $\acute{\mathcal{U}_o}(n_i, n, \mathcal{R}, +)$ and $\acute{\mathcal{U}_e}(n_i n, \mathcal{R}, +)$, respectively, such that $u_m \neq 0, m = 1, 2, ..., p < \infty$ and $u_m = 0, m > p$.The set of pairs $(c_m(t), u_m)$ is denoted by Ω.

A function $f \in \acute{\mathcal{U}_o}(n_i, n, \mathcal{R}, +)$ and $\acute{\mathcal{U}_e}(n_i, n, \mathcal{R}, +)$, respectively, is said to have a quasiasymptotic behaviour($f \sim h$) at 0^+ related to $c(t)$ if there is a non-zero $h \in \acute{\mathcal{U}_o}(n_i, n, \mathcal{R}, +)$ and $\acute{\mathcal{U}_e}(n_i, n, \mathcal{R}, +)$ such that

$$\lim_{t\to0^+}\left\langle \frac{f(tx)}{c(t)}, \phi(x) \right\rangle = \langle h(x), \phi(x)\rangle, \ \phi \in \mathcal{U}_o(n_i, n, \mathcal{R}), \mathcal{U}_e(n_i, n, \mathcal{R}).$$

A positive function $\mathcal{L}(x)$, which is continuous on $(0, \infty)$, is said to be slowly varying at 0^+ if

$$\lim_{t\to0^+}\frac{\mathcal{L}(tx)}{\mathcal{L}(t)} = 1, x \in (0, \infty).$$

By Ω_1 denote the subset of Ω of all $(c_m(t), u_m) \in \Omega$ such that $u_m \sim v_m$ at 0^+related to $c_m(t)$ and $v_m \neq 0$ when $u_m \neq 0, m = 1, 2, ..., \infty$ and $v_m \in \acute{\mathcal{U}_o}(n_i, n, \mathcal{R}, +)$ and $\acute{\mathcal{U}_e}(n_i, n, \mathcal{R}, +)$. Let $(c_m(t), u_m) \in \Omega_1$.

An ultradistribution $f \in \acute{\mathcal{U}_o}(n_i, n, \mathcal{R}, +)$ (resp. $\acute{\mathcal{U}_e}(n_i, n, \mathcal{R}, +)$) satisfying

$$\lim_{t\to0^+}\left\langle \frac{\left(f(.) - \sum\limits_{i=1}^{m} u_i(.)\right)(tx)}{c_m(t)}, \phi(x) \right\rangle = 0$$

for all $\phi \in \mathcal{U}_o(n_i, n, \mathcal{R})$ resp. $\mathcal{U}_e(n_i, n, \mathcal{R})$, is said to have a quasi-asymptotic expansion at 0^+ related to $(c_m(t), u_m)$. In notations, we write

$$f \sim \sum u_i/c_m(t), \text{ at } 0^+.$$

Next, we state without proof the following theorems, namely, theorem 11,12 and 13. Detailed proof is analoguous to that of Theorem 3.1 and Theorem 3.2 of [8] and hence avoided.

Theorem 11: *Let $a_m, m = 1, 2, ...,$ be a sequence of strictly increasing real numbers and $\mathcal{L}_m, m = 1, 2, ...,$ be a sequence of slowly varying functions at 0^+. Let $f \in \acute{\mathcal{U}_o}(n_i, n, \mathcal{R}, +)$ has a quasiasymptotic expansion at 0^+ with respect to $(t^{a_m}\mathcal{L}_m(t), u_m), t > 0$. Then*

$$Lim_{t\to0^+}\frac{\mathcal{W}_g f(tb, ta) - \sum\limits_{i=1}^{m}\mathcal{W}_g u_i(tb, ta)}{t^{a_m}\mathcal{L}_m(t)} = 0, m = 1, 2, ...$$

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Theorem 12: *Let $a_m, m = 1, 2, ...,$ be a sequence of strictly increasing real numbers and $\mathcal{L}_m, m = 1, 2, ...$be a sequence of slowly varying functions at 0^+. Let $f \in \acute{\mathcal{U}}_e (n_i, n, \mathcal{R}, +)$ has a quasiasymptotic expansion at 0^+ with respect to $(t^{a_m}\mathcal{L}_m(t), u_m), t > 0$. Then*

$$Lim_{t\to 0^+} \frac{\mathcal{W}_g f(tb, ta) - \sum_{i=1}^{m} \mathcal{W}_g u_i(tb, ta)}{t^{a_m}\mathcal{L}_m(t)} = 0, m = 1, 2, ...$$

Theorem 13: *Let $a_m, m = 1, 2...., \infty$ be a sequence of strictly real numbers and $(\mathcal{L}_m)_{m=1}^{\infty}$ be a sequence of slowly varying functions at 0^+. Let $f \in \acute{\mathcal{U}}_{b_0^+} (n_i, n, \mathcal{R})$ has a quasiasymptotic expansion at b_0^+ with respect to $(t^{a_m}\mathcal{L}_m(t), u_m)$. Then*

$$Lim_{t\to 0^+} \frac{\mathcal{W}_g f(b_0, ta) - \sum_{i=1}^{m} \mathcal{W}_g u_i(b_0, ta)}{t^{a_m}\mathcal{L}_m(t)} = 0, m = 1, 2, ...$$

IV. Acknowledgement

Author is thankful to Prof. P. K. Banerji, Emeritus Fellow, UGC (India), and Dr. Deshna Loonker, DST (SERC), India, Young Scientist, Department of Mathematics, JNV University, Jodhpur, India, for revising and formatting the paper.

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 13 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On Darboux Helices in Euclidean 3-Space

By Evren Ziplar, Ali Senol & Yusuf Yayli
Ankara University, Turkey

Abstract - In this paper, we introduce a Darboux helix to be a curve in 3-space whose Darboux vector makes a constant angle with a fixed straight line. We completely characterize Darboux helices in terms of κ & τ and thus prove that the class of Darboux helices coincide with the class of slant helices. In special, if we take $t^2 + \kappa^2 =$ constant, the curves are curve of constant precession.

Keywords and phrases : helices, slant helices, curves of constant precession, darboux vector.

GJSFR-F Classification : MSC 2000: 53C040, 53A05

ON DARBOUXHELICESINEUCLIDEAN3-SPACE

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On Darboux Helices in Euclidean 3-Space

Evren Ziplar [α], Ali Senol [σ] & Yusuf Yayli [ρ]

Abstract - In this paper, we introduce a Darboux helix to be a curve in 3-space whose Darboux vector makes a constant angle with a fixed straight line. We completely characterize Darboux helices in terms of κ & τ and thus prove that the class of Darboux helices coincide with the class of slant helices. In special, if we take $t^2 + \kappa^2$ = constant, the curves are curve of constant precession.

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I. Introduction

In differential geometry, a curve of constant slope or general helix in Euclidean 3-space R^3 is defined by the property that tangent makes a constant angle with a fixed straight line (the axis of general helix). Due to a classical result proved by M.A. Lancert in 1802 in R^3 is a general helix if and only if the ratio $\frac{\kappa}{\tau}$ is constant along curve, where κ and $\tau \neq 0$ denote the curvature and torsion, respectively. Using killing vector field along a curve, Barros gave a similar result for curves in 3-dimensional real space forms [3]. Several authers introduced different types of helices and investigated their properties. For instance, Izumiya and Takeuchi defined in [1] slant helices by the property that the principal normal makes a constant angle with a fixed direction. Moreover, they showed that α is a slant helix in R^3 if and only if the geodesic curvature of the principal normal of a space curve α is a constant function. Kula &Yaylı investigated spherical images of tangent indicatrix of binormal indicatrix of slant helix and they have shown that spherical images are spherical helix [2]. On the other hand the second and the third auther introduced in [6] LC helices in 3-dimensional real space forms and study their main properties.

The purpose of this paper is to introduce and study Darboux helices in R^3.We give a characterization of Darboux helices in terms of $\kappa \& \tau$. We give the relations between darboux helices and slant helices. As a consequence, we observe that Darboux helices coincide with slant helices. Finally, we show that curves of constant precession are darboux helices.

II. Preliminaries

We now recall some basic concepts on classical differantial geometry of space curves in Euclidean space. Let $\alpha : I \subset R \to R^3$ be a curve parameterized by arc lenght and let $\{T, N, B\}$ denote the Frenet frame of the curve α.

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$T(s) = \alpha^{'}(s)$ is a unit tangent vector of α at s.We define the curvature of α by $\kappa(s) = \left\| \alpha^{''}(s) \right\|$. For the derivatives of the frenet-serret formulae hold:

$$\begin{aligned} T^{'}(s) &= \kappa(s).N(s) \\ N^{'}(s) &= -\kappa(s).T(s) + \tau(s).B(s) \\ B^{'}(s) &= -\tau(s).N(s) \end{aligned}$$

where $\tau(s)$ is the torsion of α at s.

For any unit speed curve $\alpha : I \subset R \to R^3$ defined a vector field $C = \dfrac{(\tau T + \kappa B)}{\sqrt{\tau^2 + \kappa^2}}$ along α under the condition that $\kappa(s) \neq 0$ and called it the modified Darboux vector field of α [1].

III. Darboux Helices

Let $\boldsymbol{\alpha}$ be a curve in E^3 with $\dfrac{\tau}{\kappa} \neq 0$ everywhere with nonzero curvature and torsion κ and τ in E^3. We say that α is a Darboux helix if its Darboux vector makes a constant angle with a fixed direction d, that is $\langle W, d \rangle$ =constant along the curve, where d is a unit vector field in E^3.

$$W = \tau T + \kappa B$$

The direction of the vector d is axis of the Darboux helix. We can identify Darboux helices by the condition torsion and curvature. If $\tau^2 + \kappa^2$ =constant, the darboux helices are the curves of constant precession. So, our curves are more general than the curves of constan precession. Although every general helice is a slant helice, the general helices are not darboux helices. Moreover, there is a relation between darboux helice and the surface of constant precession. The following result describes the relation between darboux helice and the surface of constant precession.

Theorem 1. *A normal conical surface is constant angle if and only if Generating curve α is a Darboux helix* [5].

Theorem 2. *Let $\boldsymbol{\alpha}$ be a curve constant precession. If the conical surfaces construct involving the normal lines to the curve α,then the surface is a constant angle surface with the axis of $d = W + \mu n$* [5].

Theorem 3. *α is a Darboux helix if and only if $\sigma^*(s) = \dfrac{(\tau^2+\kappa^2)^{\frac{3}{2}}}{\kappa^2} \dfrac{1}{\left(\frac{\tau}{\kappa}\right)'}$ function is constant.*

Proof. If the spherical indicatrix of the darboux vector W is a circle or a part of circle, then the curve α is a darboux helis. Let the parameter of the curve (c) be s_c and let T_c be the unit tanget vector of (c). Let κ_c be the geodesic curvature of(c) in E^3.

$$\alpha(s_c) = c(s) = \frac{\tau}{\sqrt{\tau^2+\kappa^2}} T + \frac{\kappa}{\sqrt{\tau^2+\kappa^2}} B$$

$$\alpha(s_c) = \sin\Phi T + \cos\Phi B$$

$$\frac{d\alpha}{ds_c} = \frac{dc}{ds}\frac{ds}{ds_c}$$

$$\frac{d\alpha}{ds_c} = \left(\Phi^{'} \cos\Phi T - \Phi^{'} \sin\Phi B + \kappa \sin\Phi N - \tau \cos\Phi N \right) \frac{ds}{ds_c}$$

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Notes

$$T_c = \frac{d\alpha}{ds_c} = (\Phi' \cos\Phi T - \Phi' \sin\Phi B)\,\frac{ds}{ds_c}$$

$$\|T_c\| = \left\| \left(\Phi^{'} \cos\Phi T - \Phi^{'} \sin\Phi B\right) \frac{ds}{ds_c} \right\|$$

$$1 = \Phi^{'}\,\frac{ds}{ds_c}$$

$$\frac{ds}{ds_c} = \frac{1}{\Phi'}$$

(1) $$T_c = \cos\Phi T - \sin\Phi B$$

$$D_{T_c}^{T_c} = \frac{dT_c}{dS_c}\frac{ds}{ds_c}$$

$$D_{T_c}^{T_c} = \left(-\Phi^{'} \sin\Phi T - \Phi^{'} \cos\Phi B + \kappa \cos\Phi N + \tau \sin\Phi N\right)\frac{1}{\Phi'}$$

(2) $$D_{T_c}^{T_c} = \left(-\sin\Phi T - \cos\Phi B + \frac{\|w\|}{\Phi\cdot} N\right)$$

Hence, from the equation (2), the geodesic curvature of (c) are computed as the following.

$$\kappa_c = \left\| D_{T_c}^{T_c} \right\| = \left\| -\sin\Phi T - \cos\Phi B + \frac{\|w\|}{\Phi'} N \right\|$$

(3) $$\kappa_c = \left\| D_{T_c}^{T_c} \right\| = \sqrt{1 + \left(\frac{\|w\|}{\Phi'}\right)^2}$$

Therefore, we obtain

$$D_{T_c}^{T_c} = \nabla_{T_c}^{T_c} - c(s)$$

(4) $$\kappa_c^2 = \kappa_g^2 + 1$$

by using the Gauss map

$$D_{T_c}^{T_c} = \nabla_{T_c}^{T_c} - \langle s(T_c), T_c \rangle\, c(s).$$

and from the equations (3) and (4), we have:

$$1 + \left(\frac{\|w\|}{\Phi'}\right)^2 = \kappa_g^2 + 1$$

(5) $$\kappa_g = \frac{\|w\|}{\Phi'}$$

On the other hand, taking the derivative of $\tan\Phi = \dfrac{\tau}{\kappa}$,

$$\Phi^{'}.\left(1 + \tan^2\Phi\right) = \left(\frac{\tau}{\kappa}\right)^{'}$$

(6) $$\Phi^{'} = \left(\frac{\kappa^2}{\kappa^2 + \tau^2}\right)\left(\frac{\tau}{\kappa}\right)^{'}.$$

Hence, by using the equations (5) and (6) , we get:

$$\kappa_g = \frac{\sqrt{\kappa^2+\tau^2}}{\left(\frac{\kappa^2}{\kappa^2+\tau^2}\right)\left(\frac{\tau}{\kappa}\right)'}$$

$$\kappa_g = \frac{\left(\kappa^2+\tau^2\right)\frac{3}{2}}{\kappa^2}\frac{1}{\left(\frac{\tau}{\kappa}\right)'},$$

where $\|w\| = \sqrt{\kappa^2+\tau^2}$.The spherical indicatrix of (c) is a circle or a part of circle. Since the first curvature of a circle is constant, we obtain κ_c =constant. So, κ_g =constant. If we denote κ_g with $\sigma^*(s)$,

$$\kappa_g = \frac{\left(\kappa^2+\tau^2\right)\frac{3}{2}}{\kappa^2}\frac{1}{\left(\frac{\tau}{\kappa}\right)'} = \sigma^*(s)$$

and so, we have

$$\frac{\left(\kappa^2+\tau^2\right)\frac{3}{2}}{\kappa^2}\frac{1}{\left(\frac{\tau}{\kappa}\right)'} = \sigma^*(s)$$

which is constant function.

Theorem 4. *Let $\alpha : I \to E^3$ be a curve in E^3. We assume that $\frac{\kappa}{\tau}$ is not constant, where κ and τ are curvature of α. Then,*

α is a slant helice if and only if α is a Darboux helice

Proof. we assume that α is a slant helice. So we can write:

(7) $$\sigma(s) = \frac{\kappa^2}{\left(\kappa^2+\tau^2\right)\frac{3}{2}}\left(\frac{\tau}{\kappa}\right)'.$$

Similarly, if the curve α is a darboux helice

(8) $$\sigma^*(s) = \frac{\left(\tau^2+\kappa^2\right)^{\frac{3}{2}}}{\kappa^2}\frac{1}{\left(\frac{\tau}{\kappa}\right)'}.$$

Consequently, we obtain:

$$\sigma(s)\sigma^*(s) = sbt$$

$$\sigma(s) = sbt \;\; \Leftrightarrow \sigma^*(s) = sbt$$

From the previous Theorem, firstly we are going to find the axis of the slant helices since a slant helice is also a darboux helice.

3.1. **The axis of Darboux helice.** We first assume that α is a slant helix. Let d be the vector field such that the function $\langle N, d\rangle$ =cosθ=constant. There exists a_1 and a_3 such that

(9) $$d = a_1T + a_3B + \cos\theta N.$$

Then, if we take the derivative of the equation (9) and by using frenet equation, we have:

$$d' = (a' - \cos\theta.\kappa)T + (a_1\kappa - \tau a_3)N + (a_3' + \cos\theta.\tau)B$$

since the system $\{T, N, B\}$ is linear independent, we get:

$$a_1' - cos\theta.\kappa = 0$$

Notes

(10) $$a_1\kappa - \tau a_3 = 0$$

(11) $$a_3' + \cos\theta.\tau = 0$$

and from (10) and (9), respectively

(12) $$a_1 = \left(\frac{\tau}{\kappa}\right).a_3$$

(13) $$\langle d, d\rangle = a_1^2 + a_3^2 + \cos^2\theta = \text{constant}$$

By using the equalities (12) and (13), we obtain:

(14) $$\left(\frac{\tau}{\kappa}\right)^2 a_1^2 + a_3^2 + \cos^2\theta = \text{constant}$$

and from the equation (14) we have

$$\left(\left(\frac{\tau}{\kappa}\right)^2 + 1\right) a_3^2 = m^2$$

where m^2 is constant. So,

(15) $$a_3 = \frac{m}{\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}},$$

Taking the derivative in each part of the equation (15) and by using (13), we get:

(16) $$\frac{\kappa^2}{(\tau^2+\kappa^2)^{\frac{3}{2}}}.\left(\frac{\tau}{\kappa}\right)' = \text{constant}$$

We deduce from that the curve α is slant helice when we have d . Conversely, assume that the condition (16) is satisfied. In order to simplify the computations, we assume that the function (16) is constant. Define

(17) $$d = \frac{\tau}{\sqrt{\tau^2+\kappa^2}}T + \frac{\kappa}{\sqrt{\tau^2+\kappa^2}}B + \cos\theta N$$

A differentiation of (17) together the frenet equations gives $d' = 0$, that is, d is a constant vector. It can easily be seen that $d' = 0$, that is d is a constant. On the other hand, $\langle N, d\rangle$ =cosθ and this means that α is a slant helix.

Now, we are going to show that the darboux vector $W = \tau T + \kappa B$ makes a constant angle with the constant direction

$$d = \frac{\tau}{\sqrt{\tau^2+\kappa^2}}T + \frac{\kappa}{\sqrt{\tau^2+\kappa^2}}B + cos\theta N.$$

The constant direction d is the axis of both the slant helice α and the darboux helice α.These axises coincide but the making angles of these helices with d are different.

Since α is a slant helice, $\langle N, d\rangle$ =cosθ =constant

$$d = \frac{\tau}{\sqrt{\tau^2+\kappa^2}}T + \frac{\kappa}{\sqrt{\tau^2+\kappa^2}}B + \cos\theta N$$

Notes

$$\begin{aligned} d &= \frac{W}{\|W\|} + \cos\theta N \\ \langle d, W\rangle &= \|d\| \, . \, \|W\| \, .cos\lambda \\ \langle d, W\rangle &= \sqrt{1+\cos^2\theta}. \, \|W\| \, .cos\lambda \\ \frac{\langle W, W\rangle}{\|W\|} &= \sqrt{1+\cos^2\theta}. \, \|W\| \, .cos\lambda \\ \cos\lambda &= \frac{1}{\sqrt{1+\cos^2\theta}} \end{aligned}$$

Since $\cos\theta$ =constant , $\cos\lambda$ is constant.

3.2. **Curves of constant precession.** A unit speed curve of constant precession is defined by the property that its (Frenet) Darboux vector revolves about a fixed line in space with angle and constant speed. A curve of constant precession is characterized by having

$$\begin{aligned} \kappa(s) &= \varpi \sin(\mu(s), \\ \tau(s) &= \varpi \cos(\mu(s)), \end{aligned}$$

where $\varpi\rangle 0$, μ and are constant[4].

If α is a curve of constant precession , α is a slant helix [**?**]

From the axis of the Darboux helice,

$$d = \frac{\tau}{\sqrt{\tau^2+\kappa^2}}T + \frac{\kappa}{\sqrt{\tau^2+\kappa^2}}B + \cos\theta N$$

and

$$d = \frac{W}{\|W\|} + \cos\theta N \tag{18}$$

where $W = \tau T + \kappa B$.From (18),

$$\sqrt{\tau^2+\kappa^2}.d = W + \sqrt{\tau^2+\kappa^2}.\cos\theta N$$

By taking $\varpi = \|W\| = \sqrt{\tau^2+\kappa^2}$, $\varpi.d = A$ and $\varpi.\cos\theta = \mu$:

$$A = W + \mu.N$$

If $\|W\|$ =constant, the darboux helice α a curve of constant precession. We deduce from that [4] is true.

Remark 1. *All characterizations given for these slant helices can be given for these darboux helices.*

Theorem 5. *Let* $\boldsymbol{\alpha}$ *be a unit speed curve in* E^3 *and let* α *be a slant helice (darboux helice). The curvatures* κ, τ *of the curve* α *satisfy the following non-linear equation system.*

$$\left(\frac{\tau}{\sqrt{\tau^2+\kappa^2}}\right)^{'} - \mu\kappa = 0, \left(\frac{\kappa}{\sqrt{\tau^2+\kappa^2}}\right)^{'} - \mu\tau = 0$$

Proof. Since α is a slant helice (darboux helice), the axis of α :

$$d = \frac{\tau}{\sqrt{\tau^2+\kappa^2}}T + \frac{\kappa}{\sqrt{\tau^2+\kappa^2}}B + cos\theta N, \tag{19}$$

Notes

where κ, τ are curvatures of α.Taking the derivative in each part of the equation (19), we get

$$\acute{d} = \left(\frac{\tau}{\sqrt{\tau^2+\kappa^2}}\right)^{\prime} T + \left(\frac{\kappa}{\sqrt{\tau^2+\kappa^2}}\right)^{\prime} B + \mu(-\kappa T + \tau B) = 0$$

since the system$\{T, B\}$ is linear independent,

$$\left(\frac{\tau}{\sqrt{\tau^2+\kappa^2}}\right)^{\prime} - \mu\kappa = 0, \left(\frac{\kappa}{\sqrt{\tau^2+\kappa^2}}\right)^{\prime} + \mu\tau = 0$$

Conclusion 1. *If we take $\tau^2+\kappa^2$ =constant, then the curve α is a curve of constant precession* [4].

So, the following theorem can be given.

Theorem 6. *A necessary and sufficient condition that a curve be of constant precession is that $\kappa(s) = \varpi \sin(\mu(s), \tau(s) = \varpi \cos(\mu(s))$.up to reflection or phase shift of arclength, for constants ϖ and μ.*

Proof. Since $A^{'} = 0$,

$$\left(\acute{\tau} - \mu\kappa\right) T + \left(\acute{\kappa} + \mu\tau\right) B = 0$$

and uniqueness of solutions of pairs of linear equations imply that $A^{'} = 0$ if and only if $\kappa(s) = \varpi \sin(\mu(s), \tau(s) = \varpi \cos(\mu(s))$.

The following example is related to darboux helices.

Example 1. *Let the curve $\alpha(s)$ be a curve parametrized by the vector function:*

$$\begin{aligned} \alpha(s) \quad &= \quad (\frac{\sqrt{5}+1}{5-\sqrt{5}}\mathrm{Sin}(\frac{\sqrt{5}-1}{2}s) - \frac{\sqrt{5}-1}{5+\sqrt{5}}\mathrm{Sin}(\frac{\sqrt{5}+1}{2}s), \\ &\frac{\sqrt{5}+1}{\sqrt{5}-5}\mathrm{Cos}(\frac{\sqrt{5}-1}{2}s) + \frac{\sqrt{5}-1}{5+\sqrt{5}}\mathrm{Cos}(\frac{\sqrt{5}+1}{2}s), \\ &\frac{4}{\sqrt{5}}\mathrm{Sin}(\frac{s}{2})) \end{aligned}$$

where $s \in [0, 10\pi]$.Then, $\alpha(s)$ is a darboux helix (or a curve of constant precession), where $\kappa(s) = -\mathrm{Sin}\frac{\sqrt{5}}{2}s$ and $\tau(s)$ =$\mathrm{Cos}\frac{\sqrt{5}}{2}s$.The curve is rendered in the following figure.

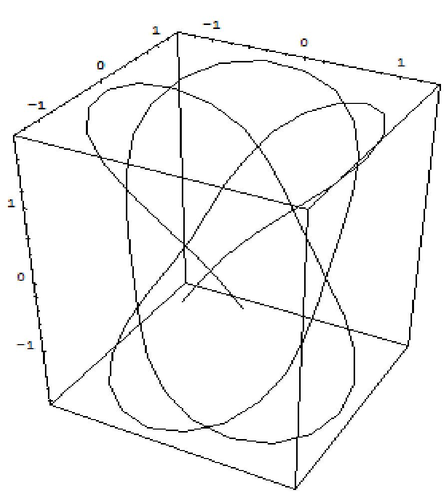

Figure 1. The darboux helix $\alpha(s)$

Ref.

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Conclusion 2. *All helices are slant helices. The slant helices which are not helices are defined as Darboux helices. The Darboux helices are more general than the curves of constant precession.*

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The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.
- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

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- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
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Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
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- In spite of position, each table must be titled, numbered one after the other and complete with heading
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- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

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INDEX

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ISSN 9755896