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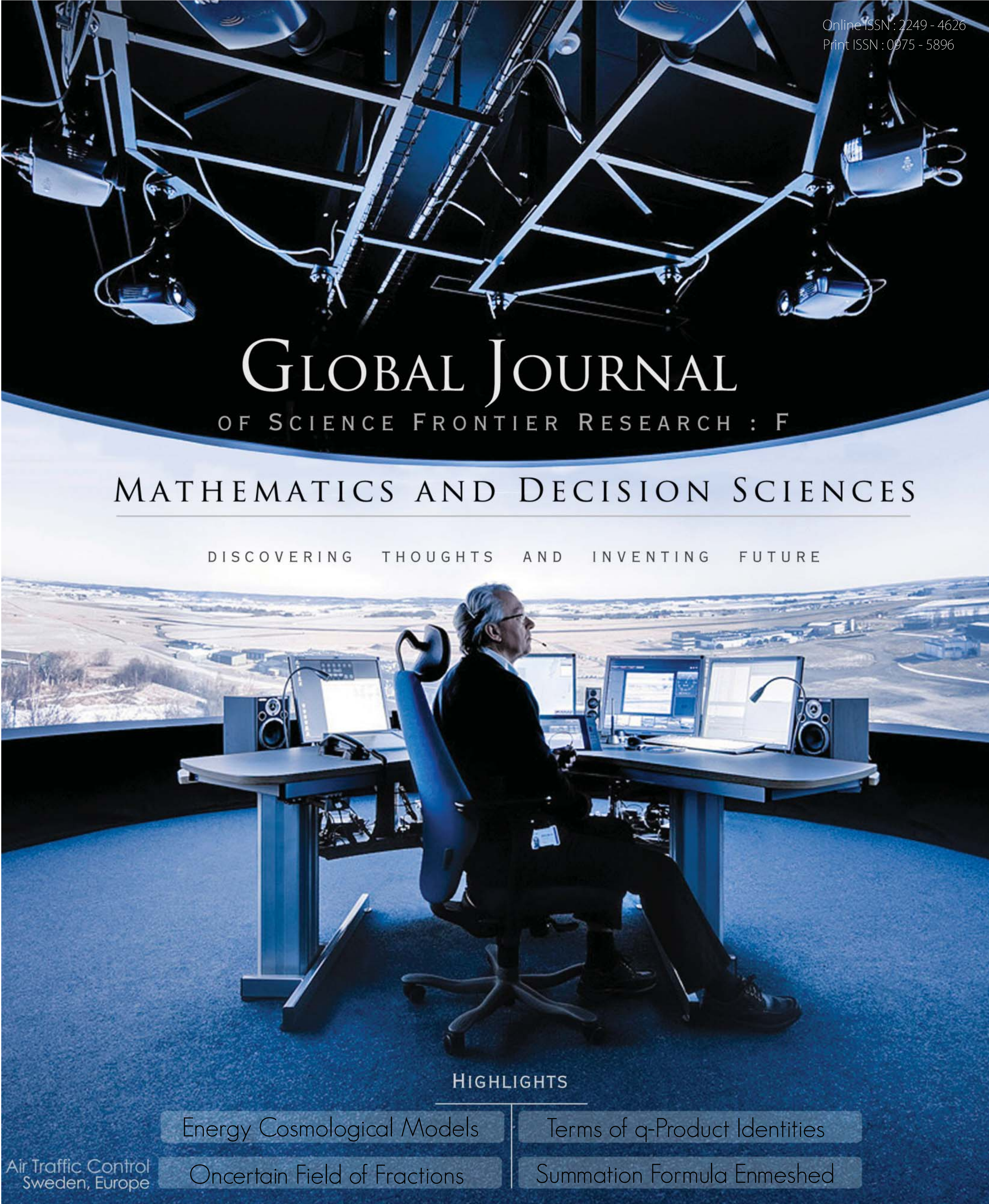

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HIGHLIGHTS

- Energy Cosmological Models
- Terms of q-Product Identities
- Oncertain Field of Fractions
- Summation Formula Enmeshed

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An Efficient Class of Ratio-Cum-Dual to Product Estimator of Finite Population Mean in Sample Surveys

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Abstract - We consider a class of ratio-cum-dual to product estimator for estimating a finite population mean of the study variate. The bias and mean square error of the proposed estimator have been obtained. The asymptotically optimum estimator (AOE) in this class has also been identified along with its approximate bias and mean square error. Theoretical and empirical studies have been done to demonstrate the superiority of the proposed estimator over the other estimators.

Keywords : Finite population mean; ratio-cum-dual to product estimator; Bias; Mean square error; Efficiency.

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Notes

An Efficient Class of Ratio-Cum-Dual to Product Estimator of Finite Population Mean in Sample Surveys

Sanjib Choudhury [α] & Bhupendra Kumar Singh [σ]

Abstract - We consider a class of ratio-cum-dual to product estimator for estimating a finite population mean of the study variate. The bias and mean square error of the proposed estimator have been obtained. The asymptotically optimum estimator (AOE) in this class has also been identified along with its approximate bias and mean square error. Theoretical and empirical studies have been done to demonstrate the superiority of the proposed estimator over the other estimators.

Keywords : Finite population mean; ratio-cum-dual to product estimator; Bias; Mean square error; Efficiency.

I. Introduction

In sample surveys, auxiliary information is used at both selections as well as estimation stages to improve the efficiency of the estimators. The use of auxiliary information at the estimation stage appears to have started with the work of Cochran (1940). When the correlation between study variate and auxiliary variate is positive (high), the ratio method of estimation is used for estimating the population mean. The ratio method is most effective if $\rho C_y/C_x > 1/2$, where C_y, C_x and ρ are coefficient of variation of y, coefficient of variation of x and correlation coefficient between y and x respectively. On the other hand, if the correlation is negative, the product method of estimation is used and this is most effective if $\rho C_y/C_x < -1/2$, suggested by Murthy (1964). Srivenkataramana (1980) first proposed dual to ratio estimator and Bandyopadhyay (1980) proposed dual to product estimator. Singh and Tailor (2005), Singh and Espejo (2003), Tailor and Sharma (2009) worked on ratio-cum-product estimators. Sharma and Tailor (2010), Choudhury and Singh (2012) worked on ratio, dual to ratio and dual to product estimators to estimate the study variable. These motivated authors to propose a new ratio-cum-dual to product estimators for estimating the population mean.

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Consider a finite population $U=(u_1,u_2,...,u_N)$ of size N units. Let y and x denotes the study and auxiliary variates respectively. A sample of size n $(n<N)$ is drawn using simple random sampling without replacement (SRSWOR) to estimate the population mean $\bar{Y}=(1/N)\sum_{i=1}^{N}y_i$ of the study variate y. Let the sample mean $(\bar{x},\ \bar{y})$ are the unbiased estimator of $(\bar{X},\bar{Y})$ based on n observations.

The usual ratio and product estimators for $\bar{Y}$ are

$$\bar{y}_R=\bar{y}\left(\bar{X}/\bar{x}\right)$$

and

$$\bar{y}_P=\bar{y}\left(\bar{x}/\bar{X}\right)\quad \text{respectively,}$$

where $\bar{y}=(1/n)\sum_{i=1}^{n}y_i$ and $\bar{x}=(1/n)\sum_{i=1}^{n}x_i$.

Let $x_i^*=(1+g)\bar{X}-gx_i$, $i=1,2,...,N$, where $g=n/(N-n)$.

Then clearly $\bar{x}^*=(1+g)\bar{X}-g\bar{x}$ is also unbiased estimator for $\bar{X}$ and $Corr\left(\bar{y},\bar{x}^*\right)=-\rho$.

Using the transformation $x_i^*=(1+g)\bar{X}-gx_i$, Srivenkataramana (1980) obtained dual to ratio estimator as

$$\bar{y}_R^*=\bar{y}\left(\bar{x}^*/\bar{X}\right)$$

and Bandyopadhyay (1980) obtained dual to product estimator as

$$\bar{y}_P^*=\bar{y}\left(\bar{X}/\bar{x}^*\right).$$

In this paper, we have proposed a class of ratio-cum-dual to product type estimator for estimating population mean $\bar{Y}$. Numerical illustrations are given in the support of the present study.

II. The Proposed Estimator

For estimating population mean $\bar{Y}$, we propose an estimator as

$$\bar{y}_{RdP}=\bar{y}\left[\alpha\left(\frac{\bar{X}}{\bar{x}}\right)+(1-\alpha)\left(\frac{\bar{X}}{\bar{x}^*}\right)\right] \qquad (1)$$

where α is a suitably chosen scalar.

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To obtain the bias and mean square error (MSE) of $\bar{y}_{RdP}$ to a first degree of approximation, we write

$$e_0 = \left(\bar{y} - \bar{Y}\right)/\bar{Y} \text{ and } e_1 = \left(\bar{x} - \bar{X}\right)/\bar{X}$$

Such that

$$\left.\begin{aligned} E(e_0) = E(e_1) = 0, \;\; E(e_0^2) = \frac{1-f}{n}C_y^2, \\ E(e_1^2) = \frac{1-f}{n}C_x^2, \;\; E(e_0 e_1) = \frac{1-f}{n}CC_x^2, \end{aligned}\right\} \quad (2)$$

where $f = n/N$ is the sampling fraction, $C_y^2 = S_y^2/\bar{Y}^2$, $C_x^2 = S_x^2/\bar{X}^2$, $C = \rho C_y/C_x$ and defined as $\rho = S_{xy}/S_x S_y$, $S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \bar{X}\right)^2$, $S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \bar{Y}\right)^2$ and $S_{xy} = \frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \bar{Y}\right)\left(x_i - \bar{X}\right)$.

Expressing $\bar{y}_{RdP}$ in terms of e's, we obtain

$$\bar{y}_{RdP} = \bar{Y}\left(1+e_0\right)\left\{\alpha\left(1+e_1\right)^{-1} + \left(1-\alpha\right)\left(1-ge_1\right)^{-1}\right\}.$$

We now assume that $|e_1| < 1$ and $|ge_1| < 1$, so that we may expand $\left(1+e_1\right)^{-1}$ and $\left(1-ge_1\right)^{-1}$ as a series in powers of e_1 and ge_1 respectively. Expanding, multiplying out and retaining terms of e's to the second degree, we obtain

$$\bar{y}_{RdP} - \bar{Y} \cong \bar{Y}\left[e_0 + g\left(e_1 + e_0 e_1 + g\,e_1^2\right) + \alpha\left(1+g\right)\left\{-e_1^2 + \left(1-g\right)e_1 - e_0 e_1\right\}\right] \quad (3)$$

Taking the expectation of both sides in equation (3) and using the results of equation (2) we get the bias of $\bar{y}_{RdP}$ as

$$B\left(\bar{y}_{RdP}\right) = \frac{1-f}{n}\bar{Y}C_x^2\left[\left\{g^2 - \alpha\left(g^2 - 1\right)\right\} + C\left\{g - \alpha\left(g+1\right)\right\}\right] \quad (4)$$

The bias, $B\left(\bar{y}_{RdP}\right)$ in (4) is 'zero' if $\alpha = \frac{g\left(C+g\right)}{\left(1+g\right)\left(1-g-C\right)}$. Thus the estimator $\bar{y}_{RdP}$ with $\alpha = \frac{g\left(C+g\right)}{\left(1+g\right)\left(1-g-C\right)}$ is almost unbiased.

Squaring and taking expectations of both the sides of equation (3) and using the results of equation (2), we obtain the MSE of $\bar{y}_{RdP}$ to the first degree of approximation as

$$M\left(\bar{y}_{RdP}\right) = \frac{1-f}{n}\bar{Y}^2\left[C_y^2 + C_x^2\left\{g - \alpha\left(1+g\right)\right\}\left\{2C + g - \alpha\left(1+g\right)\right\}\right] \quad (5)$$

which is minimized when

$$\alpha = \frac{1}{1+g}\left(g+C\right) = \alpha_{opt.}\text{(say)} \tag{6}$$

Substituting equation (6) in equation (1) yield the 'asymptotically optimum estimator' (AOE) as

$$\bar{y}_{RdP}^{opt.} = \bar{y}\left[\left(\frac{g+C}{1+g}\right)\frac{\bar{X}}{\bar{x}} + \left(\frac{1-C}{1+g}\right)\frac{\bar{x}^*}{\bar{X}}\right]$$

Thus the resulting bias and MSE of $\bar{y}_{PdP}^{opt.}$ respectively as

$$B\left(\bar{y}_{RdP}^{opt.}\right) = \frac{1-f}{n}\bar{Y}C_x^2\left(1-C\right)\left(g+C\right) \tag{7}$$

and

$$M\left(\bar{y}_{RdP}^{opt.}\right) = \frac{1-f}{n}\bar{Y}^2C_y^2\left(1-\rho^2\right) \tag{8}$$

which is the same as the MSE of the linear regression estimator $\bar{y}_{reg.} = \bar{y} + b_{yx}\left(\bar{X}-\bar{x}\right)$, where b_{yx} is the sample regression coefficient of y on x.

From equation (7), we note that the bias of AOE $\bar{y}_{RdP}^{opt.}$ is 'zero' if either $C=1$ or $C=-g$.

Remark 2.1.

To the first degree of approximation, the proposed strategy $\bar{y}_{RdP}$ under optimality condition (6), is equal to linear regression estimator.

Remark 2.2.

For $\alpha=1$, the estimator $\bar{y}_{RdP}$ in equation (1) boils down to the usual ratio estimator $\bar{y}_R$. The bias and MSE of $\bar{y}_R$ can be obtained by putting $\alpha=1$ in equations (4) and (5) respectively as

$$B\left(\bar{y}_R\right) = \frac{1-f}{n}\bar{Y}C_x^2\left(1-C\right)$$

$$M\left(\bar{y}_R\right) = \frac{1-f}{n}\bar{Y}^2\left\{C_y^2 + C_x^2\left(1-2C\right)\right\} \tag{9}$$

Remark 2.3.

For $\alpha=0$, the estimator $\bar{y}_{RdP}$ in equation (1) boils down to the dual to product estimator $\bar{y}_p^*$, proposed by Bandyopadhyay (1980). The bias and MSE of $\bar{y}_p^*$ can be obtained by putting $\alpha=0$ in equations (4) and (5) respectively as

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$$B\left(\bar{y}_p^*\right)=\frac{1-f}{n}\bar{Y}C_x^2 g\left(g+C\right)$$

and

$$M\left(\bar{y}_p^*\right)=\frac{1-f}{n}\bar{Y}^2\left\{C_y^2+gC_x^2\left(g+2C\right)\right\} \tag{10}$$

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Thus, we see that this study provides unified treatment towards the properties of different estimators.

III. Efficiency Comparisons

a) Comparison of $\bar{y}_{RdP}$

In this section, firstly, we compare MSE of conventional estimators $\bar{y}$, $\bar{y}_R$ and $\bar{y}_P$ with MSE of proposed estimator $\bar{y}_{RdP}$.

The MSE of sample mean $\bar{y}$ under SRSWOR sampling scheme is given by

$$M\left(\bar{y}\right)=\frac{1-f}{n}\bar{Y}^2C_y^2. \tag{11}$$

From equations (5) and (11), it is found that the proposed estimator $\bar{y}_{RdP}$ is more efficient than $\bar{y}$ if

$$\left\{-g+\alpha\left(1+g\right)\right\}\left\{2C+g-\alpha\left(1+g\right)\right\}>0$$

This condition holds if

$$\text{either } \frac{g}{1+g}>\alpha \text{ and } \frac{1}{1+g}\left(2C+g\right)<\alpha,$$

$$\text{or } \frac{g}{1+g}<\alpha \text{ and } \frac{1}{1+g}\left(2C+g\right)>\alpha.$$

Therefore, the range of α for which the proposed estimator $\bar{y}_{RdP}$ is more efficient than $\bar{y}$ is

$$\left[\min\left\{\frac{g}{1+g},\ \frac{1}{1+g}\left(2C+g\right)\right\},\ \max\left\{\frac{g}{1+g},\ \frac{1}{1+g}\left(2C+g\right)\right\}\right].$$

From equations (5) and (9), we note that the estimator $\bar{y}_{RdP}$ has smaller MSE than that of the usual ratio estimator $\bar{y}_R$ if

$$\left\{1+g-\alpha\left(1+g\right)\right\}\left\{1-2C-g+\alpha\left(1+g\right)\right\}>0$$

This condition holds if

$$\text{either } 1>\alpha \text{ and } \frac{1}{1+g}(2C+g-1)<\alpha,$$

$$\text{or } 1<\alpha \text{ and } \frac{1}{1+g}(2C+g-1)>\alpha.$$

Therefore, the range of α for which the proposed estimator $\bar{y}_{RdP}$ is better than $\bar{y}_R$ is

$$\left[\min\left\{1,\ \frac{1}{1+g}(2C+g-1)\right\},\ \max\left\{1,\ \frac{1}{1+g}(2C+g-1)\right\}\right].$$

To compare the usual product estimator $\bar{y}_P$, we write the bias and MSE of $\bar{y}_P$ to the first degree of approximation respectively as

$$B(\bar{y}_P)=\frac{1-f}{n}\bar{Y}CC_x^2$$

$$M(\bar{y}_P)=\frac{1-f}{n}\bar{Y}^2\left\{C_y^2+C_x^2(1+2C)\right\} \qquad (12)$$

We note from equations (5) and (12) that the estimator $\bar{y}_{RdP}$ will dominate over usual product estimator $\bar{y}_P$ if

$$\left\{-(g-1)+\alpha(g+1)\right\}\left\{(2C+1+g)-\alpha(g+1)\right\}>0$$

This condition holds if

$$\text{either } \frac{g-1}{1+g}>\alpha \text{ and } 1+\frac{2C}{1+g}<\alpha$$

$$\text{or } \frac{g-1}{1+g}<\alpha \text{ and } 1+\frac{2C}{1+g}>\alpha.$$

Hence, the range of α in which the proposed estimator $\bar{y}_{PdP}$ is better than $\bar{y}_P$ is

$$\left\{\min\left(\frac{g-1}{1+g},\ 1+\frac{2C}{1+g}\right),\ \max\left(\frac{g-1}{1+g},\ 1+\frac{2C}{1+g}\right)\right\}.$$

Secondly, comparing the MSE between the proposed estimator and dual to ratio estimator $\bar{y}_R^*$, proposed by Srivenkataramana (1980).

The bias and MSE of $\bar{y}_R^*$ to the first degree of approximation respectively as

$$B(\bar{y}_R^*)=-\bar{Y}\frac{1-f}{n}gCC_x^2$$

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and

$$M\left(\bar{y}_R^*\right)=\frac{1-f}{n}\bar{Y}^2\left\{C_y^2+gC_x^2\left(g-2C\right)\right\}. \tag{13}$$

From equations (5) and (13), it is found that the proposed estimator $\bar{y}_{RdP}$ will dominate over Srivenkataramana (1980) estimator $\bar{y}_R^*$ if

$$\left\{2g-\alpha\left(g+1\right)\right\}\left\{-2C+\alpha\left(g+1\right)\right\}>0$$

This condition exist if

$$\text{either } \frac{2g}{1+g}>\alpha \text{ and } \frac{2C}{1+g}<\alpha,$$

$$\text{or } \frac{2g}{1+g}<\alpha \text{ and } \frac{2C}{1+g}>\alpha.$$

Therefore, the range of α in which the proposed estimator $\bar{y}_{RdP}$ is more efficient than dual to ratio estimator $\bar{y}_R^*$ is

$$\left\{\min\left(\frac{2g}{1+g},\ \frac{2C}{1+g}\right),\ \max\left(\frac{2g}{1+g},\ \frac{2C}{1+g}\right)\right\}.$$

Lastly, we compare MSE of the proposed estimator $\bar{y}_{RdP}$ with dual to product estimator $\bar{y}_p^*$.

We note from equations (5) and (10) that

$$M\left(\bar{y}_p^*\right)>M\left(\bar{y}_{RdP}\right) \text{ if}$$

$$\alpha\left(1+g\right)\left\{2C+2g-\alpha\left(1+g\right)\right\}>0$$

This condition exist if

$$\text{either } 0<\alpha<\frac{2}{1+g}\left(C+g\right),$$

$$\text{or } \frac{2}{1+g}\left(C+g\right)<\alpha<0.$$

Therefore, the range of α in which the proposed estimator $\bar{y}_{RdP}$ is more efficient than dual to product estimator $\bar{y}_P^*$ is

$$\left[\min\left\{\frac{2\left(C+g\right)}{1+g},\ 0\right\},\ \max\left\{\frac{2\left(C+g\right)}{1+g},\ 0\right\}\right].$$

Thus, it seems from the above results that the proposed estimator $\bar{y}_{RdP}$ may be made better than other estimators by making a suitable choice of the value of α within the respective ranges.

b) Comparison of 'AOE' of $\bar{y}_{RdP}^{opt.}$

From equations (8)-(13), it is found that the 'AOE' $\bar{y}_{RdP}^{opt.}$ is more efficient than the other existing estimators like $\bar{y}$, $\bar{y}_R$, $\bar{y}_P$, $\bar{y}_R^*$ and $\bar{y}_P^*$. Since

$$M(\bar{y})-M\left(\bar{y}_{RdP}^{opt.}\right)=\frac{1-f}{n}\bar{Y}^2\rho^2C_y^2>0.$$

$$M(\bar{y}_R)-M\left(\bar{y}_{RdP}^{opt.}\right)=\frac{1-f}{n}\bar{Y}^2C_x^2(1-C)^2>0.$$

$$M(\bar{y}_P)-M\left(\bar{y}_{RdP}^{opt.}\right)=\frac{1-f}{n}\bar{Y}^2C_x^2(1+C)^2>0.$$

$$M\left(\bar{y}_R^*\right)-M\left(\bar{y}_{RdP}^{opt.}\right)=\frac{1-f}{n}\bar{Y}^2C_x^2(C-g)^2>0.$$

$$M\left(\bar{y}_p^*\right)-M\left(\bar{y}_{RdP}^{opt.}\right)=\frac{1-f}{n}\bar{Y}^2C_x^2(C+g)^2>0.$$

Hence, we conclude that the proposed estimator '$\bar{y}_{RdP}$' is the best (in the sense of having optimum MSE).

IV. Numerical Illustrations

To examine the merits of the constructed estimator over its competitors numerically, we consider eight sets of data. The source of the population, the nature of the variates y and x and the values of the various parameters are listed in Table 1.

To reflect the gain in the efficiency of the proposed estimator $\bar{y}_{RdP}$ over the estimators $\bar{y}$, $\bar{y}_R$, $\bar{y}_P$, $\bar{y}_R^*$ and $\bar{y}_P^*$, the effective ranges along with the optimum value of α are presented in Table 2 with respect to the population data sets.

The percent relative efficiencies (PREs) of the different estimators with respect to usual unbiased estimator $\bar{y}$ computed by the formula

$$PRE(\mathrm{E},\bar{y})=\frac{M(\bar{y})}{M(\mathrm{E})}\times 100$$

where $\mathrm{E}=\bar{y},\ \bar{y}_R,\ \bar{y}_P,\ \bar{y}_R^*,\ \bar{y}_P^*$ and $\bar{y}_{RdP}$ or $\bar{y}_{RdP}^{opt.}$

and are presented in Table 3.

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Table 1 : Description of the populations

Population	Source	Study variate y	Auxiliary variate x	N	n	ρ	C_y	C_x	$\bar{Y}$
1	Steel and Torrie (1960)	Log of leaf burn in secs	Chlorine percentage	30	6	-0.4996	0.7001	0.7493	0.6860
2	Pandey and Dubey (1988)	---	---	20	8	-0.9199	0.3552	0.3943	19.55
3	Kadilar and Cingi (2006) pp. 1054	Level of apple production	Number of apple trees	106	20	0.82	4.18	2.02	15.37
4	Sukhatme and Sukhatme (1970)	Number of villages in the circles.	A circle consisting more than five villages	89	12	0.766	0.604	2.1901	3.36
5	Maddala (1977)	Consumption per capita.	Deflated prices of veal	30	6	-0.6823	0.2278	0.0986	7.6375
6	Murthy (1967)	Output	Fixed capital	80	20	0.9413	0.3542	0.7507	51.8264
7	Murthy (1967)	Output	Number of workers	80	20	0.9150	0.3542	0.9484	51.8264
8	Kadilar and Cingi (2006) pp. 78	---	---	106	20	0.86	5.22	2.1	2212.59

Table 2 : Effective ranges and optimum value of α of $\bar{y}_{RdP}$.

Population	Ranges of α in which the proposed estimator $\bar{y}_{RdP}$ is better than					Optimum value
	$\bar{y}$	$\bar{y}_R$	$\bar{y}_P$	$\bar{y}_R^*$	$\bar{y}_P^*$	$\alpha_{opt.}$
1	(-0.55,0.20)	(-1.35, 1.00)	(-0.60, 0.25)	(-0.75, 0.40)	(-0.35, 0.00)	-0.1734
2	(-0.59, 0.40)	(-1.19, 1.00)	(-0.20, 0.01)	(-0.99, 0.80)	(-0.19, 0.00)	-0.0972
3	(0.19, 2.94)	(1.00, 2.13)	(-0.62, 3.75)	(0.38, 2.75)	(0.00, 3.13)	1.5654
4	(0.13, 0.50)	(-0.36, 1.00)	(-0.73, 1.37)	(0.27, 0.37)	(0.00, 0.64)	0.3176
5	(-2.32, 0.20)	(-3.12, 1.00)	(-1.52, -0.60)	(-2.52, 0.40)	(-2.12, 0.00)	-1.0611
6	(0.25, 0.92)	(0.17, 1.00)	(-0.50, 1.67)	(0.50, 0.67)	(0.00, 1.17)	0.5831
7	(0.25, 0.76)	(0.01, 1.00)	(-0.50, 1.51)	(0.50, 0.51)	(0.00, 1.01)	0.5063
8	(0.19, 3.66)	(1.00, 2.85)	(_0.62, 4.47)	(0.38, 3.47)	(0.00, 3.85)	1.9231

Table 3 : Percentage relative efficiency of $\bar{y}$, $\bar{y}_R$, $\bar{y}_P$, $\bar{y}_R^*$, $\bar{y}_P^*$ and $\bar{y}_{RdP}$ or $\bar{y}_{RdP}^{opt.}$ with respect to $\bar{y}$.

Population	$\bar{y}$	$\bar{y}_R$	$\bar{y}_P$	$\bar{y}_R^*$	$\bar{y}_P^*$	$\bar{y}_{RdP}$ or $\bar{y}_{RdP}^{opt.}$
1	100.00	†	†	†	124.34	133.26
2	100.00	†	526.50	†	537.23	650.26
3	100.00	226.76	†	120.73	†	305.25
4	100.00	†	†	220.46	†	241.99
5	100.00	†	167.59	†	115.73	187.10
6	100.00	†	†	591.38	†	877.54
7	100.00	†	†	612.44	†	614.34
8	100.00	212.82	†	117.95	†	384.02

†Relative efficiency less than 100%.

V. Conclusion

Table 2 provides the wide ranges along with the optimum value of α for which the proposed estimators $\bar{y}_{RdP}$ or $\bar{y}_{RdP}^{opt.}$ are more efficient than all other estimators considered in this paper. It is also observed from Table 2 that there is a scope for choosing α to obtain better estimators than $\bar{y}$, $\bar{y}_R$, $\bar{y}_P$, $\bar{y}_R^*$ and $\bar{y}_P^*$.

Table 3 shows that there is a substantial gain in efficiency by using proposed estimator $\bar{y}_{RdP}$ (or $\bar{y}_{RdP}^{opt.}$) over $\bar{y}$, $\bar{y}_R$, $\bar{y}_P$, $\bar{y}_R^*$ and $\bar{y}_P^*$. This shows that even if the scalar α deviates from its optimum value$\left(\alpha_{opt.}\right)$, the suggested estimator $\bar{y}_{RdP}$ will yield better estimates then $\bar{y}$, $\bar{y}_R$, $\bar{y}_P$, $\bar{y}_R^*$ and $\bar{y}_P^*$. Thus it is preferred to use the proposed estimators $\bar{y}_{RdP}$ or $\bar{y}_{RdP}^{opt.}$ in practice.

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Proof of 'J is a Radical Class' Using Amitsur Theorem

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Abstract - The aim of this paper is to study radical class of rings, right quasi-regular rings and finally, to prove that J , the class of all right quasi-regular rings is a radical class. Amitsur gives a theorem of radical class for the sufficient condition that a class of rings would be a radical class. This paper represents, the proof of, J is a radical class using the theorem of radical class given by Amitsur.

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GJSFR-F Classification : MSC 2010: 16D60, 16N80

PROOF OF J IS A RADICAL CLASSUSING AMITSUR THEOREM

Strictly as per the compliance and regulations of :

Ref.

1. S. Amitsur, “A general theory of radicals I”, Amer. J. Math. 74,774, 1952.

Proof of ‘J is a Radical Class’ Using Amitsur Theorem

Raju Chowdhury [α], Dewan Ferdous Wahid [σ] & Md. Anowar Hossain [ρ]

Abstract - The aim of this paper is to study radical class of rings, right quasi-regular rings and finally, to prove that J, the class of all right quasi-regular rings is a radical class. Amitsur gives a theorem of radical class for the sufficient condition that a class of rings would be a radical class. This paper represents, the proof of, J is a radical class using the theorem of radical class given by Amitsur.

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I. Introduction

The concept of a radical was introduced by J. H. M. Wedderburn [10] in 1908, for the determination of structures of algebras and later on various radicals have been proposed by Artin [14], Baer [11], Jacobson [9], Brown-McCoy [12], Levitzki [7] etc. for the study of rings in the forties. The general theory of radicals was initiated by Kurosh [6] (1953) and Amitsur [1] in the early fifties. Andrunakievic [4], Sulinski [15], Divinsky [8] and many others have followed up the works of Kurosh and Amitsur.

Radical properties based on the notion of nilpotence do not seem to yield fruitful results for rings without chain conditions. The notion of quasi-regularity was introduced by Perlis [16]. In 1945, Jacobson [9] used it and the significant “chainless” results were obtained.

In this paper, the general ring theory covering elementary definition of rings and its ideals, homomorphism, theorem related to homomorphism and some definitions related to radical class has been discussed in preliminaries. Also, we will introduce radical class of rings and some theorems related to radical class. Amitsur gives a theorem of radical class, which works as a sufficient condition of a class of rings that would be a radical class. We will know about this theorem and also right quasi-regular ring, right quasi-regular right ideal and some lemmas related to right quasi regular rings. Finally, we will prove that J, the class of all right quasi-regular rings is a radical class. It has already been proved by using the definition of radical class. But, here we will prove this using Amitsur theorem of radical class.

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II. Preliminaries

2.1. Definition

A ***ring*** is an ordered triple $(R, +, \cdot)$ such that R is a nonempty set and + and $\cdot$ are two binary operations on R satisfying the following axioms:

a) R is an additive abelian group. i.e.

i) $a + b \in R$ for all $a, b \in R$ [closure law]

ii) $(a + b) + c = a + (b + c)$ for all $a, b, c \in R$. [associative law]

iii) there exists an element $0 \in R$ such that $a + 0 = 0 + a = a$, for all $a \in R$. [identity law]

iv) for every non-zero element $a \in R$ there exists an element $-a \in R$ such that $a + (-a) = (-a) + a = 0$. [inverse law]

v) $a + b = b + a$ for all $a, b \in R$. [commutative law]

b) $(R, \cdot)$ is a semi group. i.e.

i) $a \cdot b \in R$ for all $a, b \in R$. [closure law]

ii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$ [associative law]

c) Distributive laws are true in R. i.e. for all $a, b, c \in R$,

i) $a \cdot (b + c) = a \cdot b + a \cdot c$

ii) $(a + b) \cdot c = a \cdot c + b \cdot c$

Example

i) $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ are rings.

ii) The residue class of modulo 6,

$\mathbb{Z}_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ is a ring.

iii) $[x]$, the set of all polynomials in x with real coefficients, is a ring.

2.2. Definition

A non-empty subset I of a ring R is called a ***left (right) ideal*** of R if

i) I is an additive subgroup of R

ii) $\forall\, r \in R$ and $\forall\, i \in I$, $ri \in I$, $(ir \in I)$.

2.3. Definition

A non-empty subset I of a ring R is called an ***ideal*** of R if I is both a left ideal and a right ideal of R. For a commutative ring all left and right ideals are ideals.

Example:

1. $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$.

2. The set of integers $\mathbb{Z}$ is only a subring but not an ideal of the ring of rational numbers $\mathbb{Q}$. As $3 \in \mathbb{Z}$, $\frac{2}{5} \in \mathbb{Q}$ but $3 \cdot \frac{2}{5} = \frac{6}{5} \notin \mathbb{Z}$.

2.4. Definition

Let R be a ring and I be an ideal of R then the ***quotient ring*** or ***factor ring*** $\frac{R}{I}$ is the set $\{ r + I : r \in R \}$, where addition and multiplication of two elements $r_1 + I$, $r_2 + I \in \frac{R}{I}$ are given by

i) $(r_1 + I) + (r_2 + I) = (r_1 + r_2) + I$.

ii) $(r_1 + I)(r_2 + I) = r_1 r_2 + I$.

Example:

$\frac{\mathbb{Z}}{2\mathbb{Z}}$ is a quotient ring.

Notes

2.5. Definition

Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be two rings. A mapping $f: R \to R'$ is called a ring ***homomorphism*** if $\forall\ a, b \in R$

i) $f(a+b) = f(a) + f(b)$ and

ii) $f(ab) = f(a)\ f(b)$.

Notes

2.5.1. Theorem

Every factor ring of a ring is the homomorphic image of that ring.

Proof: Let R be a ring and I be an ideal of R, then we have to show that $\frac{R}{I}$ is a homomorphic image of R. Let us define a map $f: R \to \frac{R}{I}$ by $f(r) = r + I$ for all $r \in R$. We need to show that f is a onto homomorphism.

Clearly f is well defined.

Now $f(r_1 + r_2) = (r_1 + r_2) + I = (r_1 + I) + (r_2 + I) = f(r_1) + f(r_2)$

and $f(r_1 r_2) = r_1 r_2 + I = (r_1 + I)\ (r_2 + I) = f(r_1)\ f(r_2)$.

Thus f is a homomorphism.

Let $r + I \in \frac{R}{I}$ where $r \in R$. Then by definition, $f(r) = r + I$. i.e. $r + I = f(r)$. This implies that every element of $\frac{R}{I}$ is the image of some element in R. Thus f is onto. Hence the theorem is proved.

2.6. Definition

A ring R is said to have the ***ascending chain condition (A.C.C.)*** on left (right) ideals, if every ascending sequence of left (right) ideals $L_1 \subseteq L_2 \subseteq L_3 \subseteq \ldots\ldots \subseteq L_n \subseteq \ldots\ldots$, terminates after a finite number of steps, i.e. there exists a positive integer n such that $L_n = L_{n+1} = \ldots\ldots\ldots$.

2.7. Definition

A ring R is said to have the ***descending chain condition (D.C.C.)*** on left (right) ideals, if every descending sequence of left (right) ideals $R \supseteq L_1 \supseteq L_2 \supseteq L_3 \supseteq \ldots\ldots \supseteq L_n \supseteq \ldots\ldots\ldots$, terminates after a finite number of steps, i.e. there exists a positive integer n such that $L_n = L_{n+1} = \ldots\ldots\ldots$.

III. Radical Class of Rings

3.1. Definition

Let $\mathfrak{R}$ be a nonempty class of rings with a certain property. A ring A is said to be an ***$\mathfrak{R}$-ring*** if $A \in \mathfrak{R}$.

Example:

Let $\mathfrak{R}$ be the class of all nilpotent ring and A be an idempotent ring. Then A is not nilpotent ring and hence $A \notin \mathfrak{R}$. Therefore A is not an $\mathfrak{R}$-ring.

3.2. Definition

An ideal I of a ring A is said to be an $\mathfrak{R}$-ideal if I is an $\mathfrak{R}$-ring. i.e. $I \in \mathfrak{R}$.

Example:

Let $\mathfrak{R}$ be the class of all nilpotent ring and I be an ideal of a nilpotent ring A. Then $I \in \mathfrak{R}$. Therefore I is an $\mathfrak{R}$-ideal.

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3.3. Definition

A ring A is said to be $\mathfrak{R}$-semi-simple if A has no non-zero $\mathfrak{R}$-ideal.

3.4. Definition

Let $\mathfrak{R}$ be a non-empty class of rings with a certain property. Then $\mathfrak{R}$ is said to be a ***radical property*** or ***radical class*** if the following conditions are hold:

A) $\mathfrak{R}$ is homomorphically closed. i.e. every homomorphic image of an $\mathfrak{R}$-ring A is an $\mathfrak{R}$-ring. i.e. if $A\in\mathfrak{R}$ and $I\triangleleft A$, then $\frac{A}{I}\in\mathfrak{R}$.

B) Every ring $A\in\mathfrak{R}$ contains a non-zero $\mathfrak{R}$-ideal $R(A)$ which contains every other $\mathfrak{R}$-ideals of A.

C) $\frac{A}{R(A)}$ has no non-zero $\mathfrak{R}$-ideal. i.e. $\frac{A}{R(A)}$ is $\mathfrak{R}$-is semi-simple.

A radical class is simply called a radical.

3.5. Definition

Let $\mathfrak{R}$ be a radical class. The $\mathfrak{R}$-ideal $R(A)$ of a ring A is called the ***$\mathfrak{R}$ - radical*** of the ring A.

3.6. Definition

Let $\mathfrak{R}$-be a radical class. Then a ring A is said to be an ***$\mathfrak{R}$-radical ring*** if $R(A) = A$, where $R(A)$ is the radical of A.

3.7. Definition

Let $\mathfrak{R}$ be a radical class. Then a ring A is said to be an $\mathfrak{R}$-semi-simple ring if $R(A)$ = 0, where $R(A)$ is the radical of A.

0 is the only ring which is both an $\mathfrak{R}$-radical ring and an $\mathfrak{R}$-semi-simple ring.

3.7.1. Theorem [8]

Let $\mathfrak{R}$ be a non-empty class $\mathfrak{R}$ of rings. Then $\mathfrak{R}$ is said to be a radical class if and only if

A) $\mathfrak{R}$ is homomorphically closed.

D) If every non-zero homomorphic image of a ring A contains a non-zero $\mathfrak{R}$-ideal, then A is in $\mathfrak{R}$. i.e. $\forall\ I\triangleleft A$, if $\frac{A}{I} \supset \frac{B}{I} \in \mathfrak{R}$ then $A\in\mathfrak{R}$, where $B\triangleleft A$.

This theorem is known as Kurosh's Theorem.

3.7.2.1. Lemma (Zorn's Lemma)

Let A be a nonempty partially ordered set in which every totally ordered subset has an upper bound in A. Then A contains at least one maximal element.

3.7.2. Theorem (Amitsur) [3]

Let $\mathfrak{R}$ be a nonempty class of rings. Then $\mathfrak{R}$ is a radical class if and only if

A′) $\mathfrak{R}$ is homomorphically closed.

B′) For any ring A and an ideal I of A if both I and $\frac{A}{I}$ is in $\mathfrak{R}$, then $A\in\mathfrak{R}$. i.e. $\mathfrak{R}$ is closed under extension.

C′) If $I_1 \subseteq I_2 \subseteq I_3 \subseteq$ ………, is an ascending chain of $\mathfrak{R}$-ideals of a ring A, then $\bigcup_\alpha I_\alpha$ is an $\mathfrak{R}$-ideal.

IV. Right Quasi-Regular Rings

4.1. Definition

Let R be a ring and $x\in R$. Then x is called ***right quasi-regular*** if there exists an

Ref.

3. S. Tumurbat & R. Wisbauer, "Radicals With The α-Amitsur Property", W. Sc. P. C., Vol. 7, No. 3,347–361, 2008.

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element $y \in R$ such that $x + y + xy = 0$.

We often write $x + y + xy$ by $x \circ y$. When $x \circ y = 0$ then the element y is called ***right quasi-inverse*** of x.

4.2. Definition

A ring R is said to be ***right quasi-regular*** if every element in R is right quasi-regular.

4.2.1. Lemma

If R is a ring with 1, then $(1+x)$ has right inverse $(1 + y)$ iff x is right quasi-regular.

Proof: Let us consider a ring with unity element 1. Let $(1 + y)$ be the right inverse of $(1 + x)$. Then we have,

$(1 + x)\ (1 + y) = 1$

$\Rightarrow 1 + y + x + xy = 1$

$\Rightarrow x + y + xy = 0$

$\Rightarrow x$ is right quasi-regular.

Conversely, let x be right quasi-regular. Then there is a right quasi-inverse y such that $x + y + xy = 0$

$\Rightarrow 1 + y + x + xy = 1$

$\Rightarrow 1(1 + y) + x(1 + y) = 1$

$\Rightarrow (1 + x)\ (1 + y) = 1$

i.e. $(1 + y)$ is right inverse of $(1 + x)$.

4.2.2. Lemma

Let R be a ring. Then for any element x in R, x is a right quasi-regular if and only if $\{r + xr\} = R, \forall\ r \in R$.

Proof: Let R be a ring and $x \in R$. Consider $\{r + xr\}$, the set of all elements $r + xr$, $\forall\ r \in R$. Then $\{r + xr\}$ is clearly a right ideal of R. Now suppose that $\{r + xr\} = R$. We are to show that x is right quasi-regular. Since $\{r + xr\} = R$, then $x = r + xr$ for some r in R. This implies that $x + (-r) + x\ (-r) = 0$. This implies that x is right quasi-regular for some $r \in R$.

Conversely, suppose that x is right quasi-regular element of R. We have to show that $R = \{r + xr\}$. Since x is right quasi-regular then $\exists$ an element $y \in R$ such that $x + y + xy = 0 \Rightarrow x = (-y) + x\ (-y) \in \{r + xr\}$. Then $xr \in \{r + xr\}$ and therefore $r \in \{r + xr\}$ for every $r \in R$. Hence $\{r + xr\} = R$.

4.3. Definition

Let R be a ring and I be a right ideal of R. Then I is called a right quasi-regular right ideal if every element of I is right quasi-regular.

4.3.1. Lemma[8]

If x is right quasi-regular and if y belongs to a right quasi-regular right ideal I, then $x + y$ is right quasi-regular.

Proof: Since x is right quasi-regular, then there exists an element x' such that, $x + x' + xx' = 0$. Now, consider the element $y + yx'$. Then $y + yx'$ is in I and thus is right quasi-regular. Let z be the right quasi-inverse of $y + yx'$ then, $(y + yx') + z + (y + yx')z = 0$.

Ref.

8. N. Divinsky, "Rings and Radicals (University of Toronto press)", 1965.

Now we will show that $x' + z + x'z$ is a right quasi-inverse of $x + y$.

Therefore, $(x + y) + (x' + z + x'z) + (x + y)(x' + z + x'z)$

$= x + y + x' + z + x'z + xx' + xz + xx'z + yx' + yz + yx'z$

$= (x + x' + xx') + (y + yx') + z + (y + yx')z + (x + x' + xx')z$

$= 0$

Hence $x + y$ is right quasi-regular.

4.3.2. Lemma

The sum of two right quasi-regular right ideals of a ring is also a right quasi-regular right ideal.

Proof: Let I_1 and I_2 be two right quasi-regular right ideals of a ring R. We have to show that $I_1 + I_2$ is also a right quasi-regular right ideal of R. Let $p \in I_1 + I_2$ then $p = x + y$ for some $x \in I_1$ and $y \in I_2$. Since x is right quasi-regular and $y \in I_2$ then we have $x + y$ is also right quasi-regular (by Lemma 4.3.1). i.e. p is right quasi-regular.

Hence every element of $I_1 + I_2$ is right quasi-regular.

Hence $I_1 + I_2$ is right quasi-regular right ideal of R.

4.3.3. Lemma

The sum of any finite number of right quasi-regular right ideals of a ring is again a right quasi-regular right ideal.

Proof: Let $I_1, I_2 \ldots, I_n$ are right quasi-regular right ideals of a ring R. We have to show that $I_1 + I_2 + \ldots\ldots + I_n$ is right quasi-regular right ideal. We shall prove this by the method of induction on n.

If $n = 1$ then the proof is obvious. Now suppose $n = 2$, then, $I_1 + I_2$ is right quasi-regular right ideal (by Lemma 4.3.1).

Now, let $I = I_1 + I_2 + \ldots\ldots + I_{n-1}$ a right quasi-regular right ideal of R. We show that $I + I_n$ is right quasi-regular right ideal of R.

Let $p \in I + I_n$ then $p = x' + y'$ for some $x' \in I$ and $y' \in I_n$. Then x' is right quasi-regular and y' belongs to a right quasi-regular right ideal I_n. Therefore $x' + y'$ is right quasi-regular (by Lemma 4.3.1). Hence $I + I_n$ is right quasi-regular right ideal. i.e. $I_1 + I_2 + \ldots\ldots + I_{n-1} + I_n$ is a right quasi-regular right ideal of R.

4.3.4. Lemma

Sum (Union) of all right quasi-regular right ideals of a ring R is a right quasi-regular right ideal of R.

4.3.5. Lemma[8]

$J(R)$, the sum of all right quasi-regular right ideals of a ring R is a two sided ideal of R.

Proof: Let x be any element in $J(R)$ and r any element of R. We have to show that rx is in $J(R)$ i.e. $J(R)$ is a left ideal. We know that $J(R)$ is a right quasi-regular right ideal. Hence $xr \in J(R)$ is a right quasi-regular. Then there exists an element w such that

$xr + w + xrw = 0$. Then

$rx + (-rx - rwx) + rx(-rx - rwx)$

$= rx - rx - rwx - rx \cdot rx - rx \cdot rwx$

$= -r(w + xr + xrw)x$

$= -r \cdot 0 \cdot x$

$= 0$

Ref.

8. N. Divinsky, "Rings and Radicals (University of Toronto press)", 1965.

Therefore, rx is right quasi-regular.

Next consider the right ideal generated by rx. This is the set of all $rxi + rxs$, where i is an integer and s is in R. The element $xi + xs$ is in $J(R)$ and, as above, $r(xi + xs)$ is right quasi-regular. Therefore, $\{rxi + rxs\}$ is a right quasi-regular right ideal. It is thus in $J(R)$ and then, in particular, rx is in $J(R)$. Therefore $J(R)$ is a two-sided ideal of R.

Notes

4.3.6. Lemma

Every homomorphic image of a right quasi-regular ring R is right quasi-regular.

Proof: Let R be a right quasi-regular ring and I be any ideal of R, then we have to show that $\frac{R}{I}$ is right quasi-regular. Let $x \in \frac{R}{I}$ then $x = r + I$ for some $r \in R$.

Since R is right quasi-regular, then r is right quasi-regular. Then there exists an element $r' \in R$ such that $r + r' + rr' = 0$.

Now $(r + I) + (r' + I) + (r + I)(r' + I) = r + r' + I + rr' + I$

$= r + r' + rr' + I$

$= 0 + I$

$= I$

But I is the zero element of $\frac{R}{I}$. Therefore $r' + I$ is right quasi-inverse of $r + I$. Hence $r + I$ is right quasi-regular i.e. x is right quasi-regular. Therefore $\frac{R}{I}$ is right quasi-regular.

4.3.7. Lemma

Let R be a ring and I be an ideal of R. If I and $\frac{R}{I}$ are right quasi-regular then R is right quasi-regular.

Proof: Since $\frac{R}{I}$ is right quasi-regular, then for every $x \in \frac{R}{I}$, there exists $y \in \frac{R}{I}$ such that

$(x + I) + (y + I) + (x + I)(y + I) = I$

$\Rightarrow x + I + y + I + xy + I = I$

$\Rightarrow x + y + xy + I = I$

$\Rightarrow x + y + xy \in I$

Since I is right quasi-regular then there exists $w \in I$ such that

$x + y + xy + w + (x + y + xy)w = 0$

$\Rightarrow x + y + xy + w + xw + yw + xyw = 0$

$\Rightarrow x + (y + w + yw) + x(y + w + yw) = 0$

This implies that $y + w + yw$ is a right quasi-inverse of x and thus x is a right quasi-regular. Hence R is right quasi-regular.

V. Conclusions

From the above discussions, we can prove the following theorem.

5.1. Theorem

The class of all right quasi-regular rings is a radical class.

Proof: Let J be the class of all right quasi-regular rings. We shall prove this using Amitsur theorem.

By Lemma 4.3.6, A′) holds. i.e. J is homomorphically closed.

By Lemma 4.3.7, B′) holds. i.e. J is closed under extension.

To prove C′), let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots\ldots$ be an ascending chain of right quasi-regular right ideals of a ring R. We have to show that $\bigcup_\alpha I_\alpha$ is right quasi-regular. Let $x \in \bigcup_\alpha I_\alpha$ then $x \in I_\alpha$ for some α. Since each I_α is right quasi-regular right ideal then $\exists$ an element x' such that $x + x' + xx' = 0$. i.e. x is right quasi-regular. Hence every element of $\bigcup_\alpha I_\alpha$ is right quasi-regular. i.e. $\bigcup_\alpha I_\alpha$ is right quasi-regular. Hence J is a radical class.

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Positive Solutions for Systems of Three-Point Nonlinear Boundary Value Problems on Time Scales

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GJSFR-F Classification : *MSC 2010: 34B15, 39B10, 34B18*

POSITIVE SOLUTIONS FOR SYSTEMS OF THREE-POINT NONLINEAR BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Positive Solutions for Systems of Three-Point Nonlinear Boundary Value Problems on Time Scales

A Kameswara Rao [α] & S. Nageswara Rao [σ]

Abstract - Values of λ are determined for which there exist positive solutions of the system of dynamic equations, $u^{\Delta\Delta}(t)+\lambda p(t)f(v(\sigma(t)))=0,\ v^{\Delta\Delta}(t)+\lambda q(t)g(u(\sigma(t)))=0,$ for $t\in[a,b]_{\mathbb{T}}$ Satisfying the three - point boundary conditions, $\alpha u(a)-\beta u^{\Delta}(a)=0,\ u(\sigma^2(b))-\delta u(\eta)=0,\ \alpha v(a)-\beta v^{\Delta}(a)=0,\ v(\sigma^2(b))-\delta v(\eta)=0,$ where $\mathbb{T}$ is a time scales. A Guo-Krasnosel'skii fixed point theorem is applied.

Keywords : Time scales, three-point boundary value problems, dynamic equations, system of equations, positive solution, eigenvalue problem.

I. Introduction

Let $\mathbb{T}$ be a time scale with $a,\sigma^2(b)\in\mathbb{T}$. Given an interval J of $\mathbb{R}$, we will use the interval notation

$$J_{\mathbb{T}}=J\cap\mathbb{T}.$$

We are concerned with determining intervals of the parameter λ (eigenvalues) for which there exist positive solutions for the system of dynamic equations,

$$\begin{aligned} u^{\Delta\Delta}(t)+\lambda p(t)f(v(\sigma(t)))&=0,\quad t\in[a,b]_{\mathbb{T}},\\ v^{\Delta\Delta}(t)+\lambda q(t)g(u(\sigma(t)))&=0,\quad t\in[a,b]_{\mathbb{T}}, \end{aligned}\tag{1.1}$$

satisfying the boundary conditions

$$\begin{aligned} \alpha u(a)-\beta u^{\Delta}(a)&=0,\quad u(\sigma^2(b))-\delta u(\eta)=0,\\ \alpha v(a)-\beta v^{\Delta}(a)&=0,\quad v(\sigma^2(b))-\delta v(\eta)=0, \end{aligned}\tag{1.2}$$

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where $\alpha, \beta \geq 0$, $\alpha + \beta > 0$, $\lambda > 0$, $0 < \delta < 1$, $\eta \in [a, \sigma^2(b)]$, and

(A1) $f, g \in C([0, \infty), [0, \infty))$,

(A2) $p, q \in C([a, \sigma(b)]_{\mathbb{T}}, [0, \infty))$, and each does not vanish identically on any closed subinterval of $[a, \sigma(b)]_{\mathbb{T}}$,

(A3) All of

$$f_0 := \lim_{x \to 0^+} \frac{f(x)}{x}, \quad g_0 := \lim_{x \to 0^+} \frac{g(x)}{x},$$

$$f_\infty := \lim_{x \to \infty} \frac{f(x)}{x}, \quad g_\infty := \lim_{x \to \infty} \frac{g(x)}{x}$$

exist as positive real numbers.

On a larger scale, there has been a great deal of activity in studying positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from a theoretical sense [9, 10, 12, 15] and as applications for which only positive solutions are meaningful. These considerations are caste primarily for scalar problems, but good attention has been given to boundary value problems for systems of differential equations [13, 14, 19, 21, 22].

The main tool in this paper is an application of the Guo-Krasnoselskii fixed point theorem for operators leaving a Banach space cone invariant [10]. A Green's function plays a fundamental role in defining an appropriate operator on a suitable cone.

II. Some Preliminaries

In this section, we state some preliminary lemmas and the well-known Guo-Krasnosel'skii fixed point theorem.

Let $G(t, s)$ be the Green's function for the boundary value problem

$$-y^{\Delta\Delta}(t) = 0, \tag{2.1}$$

$$\alpha y(a) - \beta y^{\Delta}(a) = 0, \qquad y(\sigma^2(b)) - \delta y(\eta) = 0, \tag{2.2}$$

which is given by

$$G(t, s) = \frac{1}{d} \begin{cases} G_1(t, s) : & a \leq s \leq \eta \\ G_2(t, s) : & \eta \leq \sigma(s) \leq \sigma^2(b) \end{cases}$$

where

$$G_1(t, s) = \begin{cases} [\beta + \alpha(\sigma(s) - a)]\,[\sigma^2(b) - \delta\eta - t(1 - \delta)], & \sigma(s) \leq t \\ [\beta + \alpha(t - a)]\,[\sigma^2(b) - \delta\eta - \sigma(s)(1 - \delta)], & t \leq s \end{cases}$$

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[9] L. H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.*, **120**(1994), No. 3, 743-748.

$$G_2(t,s) = \begin{cases} [\beta + \alpha(\sigma(s)-a)]\,(\sigma^2(b)-t) + (t-\sigma(s))\,(\eta+\beta-\alpha a)\,\delta, & \sigma(s)\le t \\ [\beta+\alpha(t-a)]\,(\sigma^2(b)-\sigma(s))\,, & t \le s \end{cases}$$

and

$$d := \beta(1-\delta) + \alpha(\sigma^2(b) - a - \delta(\eta - a)).$$

Notes

Lemma 2.1 *For $h(t) \in C[a,\sigma^2(b)]_{\mathbb{T}}$, the BVP*

$$-y^{\Delta\Delta}(t) = h(t), \quad t \in [a,b]_{\mathbb{T}}, \tag{2.3}$$

$$\alpha y(a) - \beta y^{\Delta}(a) = 0, \quad y(\sigma^2(b)) - \delta y(\eta) = 0, \tag{2.4}$$

has a unique solution

$$y(t) = \frac{\beta+\alpha(t-a)}{d}\int_a^{\sigma(b)} (\sigma^2(b)-\sigma(s))h(s)\Delta s$$
$$-\frac{\delta(\beta+\alpha(t-a))}{d}\int_a^{\eta}(\eta-\sigma(s))h(s)\Delta s - \int_a^t (t-\sigma(s))h(s)\Delta s. \tag{2.5}$$

From (2.5) obviously we have that

$$y(t) \le \frac{\beta+\alpha(t-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))h(s)\Delta s, \tag{2.6}$$

and

$$y(\eta) \ge \frac{\beta+\alpha(\eta-a)}{d}\int_\eta^{\sigma(b)}(\sigma^2(b)-\sigma(s))h(s)\Delta s. \tag{2.7}$$

Lemma 2.2 *Let $0<\delta<1$. If $h(t)\in C[a,\sigma^2(b)]_{\mathbb{T}}$, and $h \ge 0$, then the unique solution y of the problem* (2.3), (2.4) *satisfies*

$$y(t) \ge 0, \quad t \in (a,\sigma^2(b))_{\mathbb{T}}.$$

Proof: From the fact that $y^{\Delta\Delta}(t) = -h(t) \le 0$, we know that the graph of $y(t)$ is concave down on $[a,\sigma^2(b)]_{\mathbb{T}}$ and $y^{\Delta}(t)$ is monotone decreasing. Thus $y^{\Delta}(t) \le y^{\Delta}(a) = \frac{\alpha}{\beta}y(a)$, where $\beta \ne 0$.

Case 1. If $y(a)<0$, then $y^{\Delta}(t)<0$ for $[a,\sigma^2(b)]_{\mathbb{T}}$. Thus y is a monotone decreasing function, that is $y(t) \ge y(\sigma^2(b))$.

1. If $y(\sigma^2(b)) \ge 0$, then $y(t)>0$. So this contradicts the assertion $y(t)$ is a monotone decreasing function.

2. If $y(\sigma^2(b))<0$, then we have that

$$y(\eta) = \frac{1}{\delta} y(\sigma^2(b)) < 0,$$

$$y(\sigma^2(b)) = \delta y(\eta) \geq y(\eta),$$

which contradicts the assertion that $y(t)$ is monotone decreasing.

Case 2. If $y(a) \geq 0$, then $y^{\Delta}(a) \geq 0$. So $y(t)$ is a monotone increasing on $[a, a+\epsilon]$.

1. If $y(\sigma^2(b)) \geq 0$, then $y(t) \geq 0$ on $[a, \sigma^2(b)]_{\mathbb{T}}$.
2. If $y(\sigma^2(b)) < 0$, then we have that

$$y(\eta) = \frac{1}{\delta} y(\sigma^2(b)) < 0,$$

$$y(\sigma^2(b)) = \delta y(\eta) \geq y(\eta),$$

which contradicts the assertion that the graph of $y(t)$ is concave down on $(a, \sigma^2(b))_{\mathbb{T}}$.

If $\beta = 0$, from the boundary conditions we obtain $y(a) = 0$.

1. If $y(\sigma^2(b)) \geq 0$, then the concavity of y implies that $y(t) \geq 0$ for $t \in [a, \sigma^2(b)]_{\mathbb{T}}$.
2. If $y(\sigma^2(b)) < 0$, then

$$y(\eta) = \frac{1}{\delta} y(\sigma^2(b)) < 0,$$

$$y(\sigma^2(b)) = \delta y(\eta) \geq y(\eta).$$

This contradicts with the concavity of y.

Lemma 2.3 *If $y^{\Delta\Delta}(t) \leq 0$, then $\frac{y(\sigma^2(b))}{\sigma^2(b)} \leq \frac{y(t)}{t} \leq \frac{y(\eta)}{\eta}$ for all $t \in [\eta, \sigma^2(b)]_{\mathbb{T}}$.*

Proof: Let $h(t) := y(t) - \frac{t}{\sigma^2(b)-a} y(\sigma^2(b))$. Thus, we have $h(\eta) > 0$ and $h(\sigma^2(b)) = 0$. Since $h^{\Delta\Delta}(t) \leq 0$ then $h(t) \geq 0$ on $[\eta, \sigma^2(b)]_{\mathbb{T}}$. So $\frac{y(\sigma^2(b))}{\sigma^2(b)} \leq \frac{y(t)}{t}$. For the function $h(t)$, since $h(\eta) > 0$, $h(\sigma^2(b)) = 0$ and $h^{\Delta\Delta}(t) \leq 0$ then the function $h(t)$ is decreasing on $[\eta, \sigma^2(b)]_{\mathbb{T}}$. So $\frac{y(t)}{t} \leq \frac{y(\eta)}{\eta}$ for all $t \in [\eta, \sigma^2(b)]_{\mathbb{T}}$.

Lemma 2.4 *Let $0 < \delta < 1$. If $h(t) \in C[a, \sigma^2(b)]_{\mathbb{T}}$, and $h \geq 0$, then the unique solution y of the problem* (2.3), (2.4) *satisfies*

$$\inf_{t \in [\eta, \sigma^2(b)]_{\mathbb{T}}} y(t) \geq \gamma \|y\|,$$

where

$$\gamma := \min\left\{ \frac{\delta(\sigma^2(b) - \eta)}{\sigma^2(b) - \delta\eta - a(1-\delta)}, \frac{\delta\eta}{\sigma^2(b)} \right\}.$$

Notes

Proof: By the second boundary condition we know that $y(\eta) \geq y(\sigma^2(b))$. Pick $t_0 \in (a, \sigma^2(b))_{\mathbb{T}}$ such that $y(t_0) = \|y\|$. If $t_0 < \eta < \sigma^2(b)$, then

$$\min_{t\in[\eta,\sigma^2(b)]_{\mathbb{T}}} y(t) = y(\sigma^2(b)),$$

and

$$\frac{y(\sigma^2(b)) - y(\eta)}{\sigma^2(b) - \eta} \leq \frac{y(\eta) - y(t_0)}{\eta - t_0}.$$

Therefore

$$\min_{t\in[\eta,\sigma^2(b)]_{\mathbb{T}}} y(t) \geq \frac{\delta(\sigma^2(b) - \eta)}{\sigma^2(b) - \delta\eta - a(1-\delta)}\|y\|.$$

If $\eta \leq t_0 < \sigma^2(b)$, again we have $y(\sigma^2(b)) = \min_{t\in[\eta,\sigma^2(b)]_{\mathbb{T}}} y(t)$. From Lemma 2.3, we have $\frac{y(\eta)}{\eta} \geq \frac{y(t_0)}{t_0}$. Combining with the boundary condition $\delta y(\eta) = y(\sigma^2(b))$, we conclude that

$$\frac{y(\sigma^2(b))}{\delta\eta} \geq \frac{y(t_0)}{t_0} \geq \frac{y(t_0)}{\sigma^2(b)} = \frac{\|y\|}{\sigma^2(b)}.$$

This is

$$\min_{t\in[\eta,\sigma^2(b)]_{\mathbb{T}}} y(t) \geq \frac{\delta\eta}{\sigma^2(b)}\|y\|.$$

We note that a pair $(u(t), v(t))$ is a solution of the eigenvalue problem (1.1), (1.2) if, and only if,

$$u(t) = \lambda \int_a^{\sigma(b)} G(t,s)p(s)f\left(\lambda \int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\right)\Delta s,\ a \leq t \leq \sigma^2(b),$$

where

$$v(t) = \lambda \int_a^{\sigma(b)} G(t,s)q(s)g(u(\sigma(s)))\Delta s, \quad a \leq t \leq \sigma^2(b).$$

Values of λ for which there are positive solutions (positive with respect to a cone) of (1.1), (1.2) will be determined via applications of the following fixed point-theorem [17].

Theorem 2.5 (***Krasnosel'skii***) *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that, either

(i) $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2,$ *or*

Ref.

[17] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff Ltd, Groningen, The Netherlands (1964).

(ii) $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then, T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

III. Positive Solutions in a Cone

In this section, we apply Theorem 2.5 to obtain solutions in a cone (i.e., positive solutions) of (1.1), (1.2).

For our construction, let $\mathcal{B} = \{x : [a, \sigma^2(b)]_{\mathbb{T}} \to \mathbb{R}\}$ with supremum norm $\|x\| = \sup\{| x(t) |: t \in [a, \sigma^2(b)]_{\mathbb{T}}\}$ and define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{ x \in \mathcal{B} \mid x(t) \geq 0 \text{ on } [a, \sigma^2(b)]_{\mathbb{T}}, \text{ and } \min_{t \in [\eta, \sigma^2(b)]_{\mathbb{T}}} x(t) \geq \gamma \|x\| \right\}.$$

For our first result, define positive numbers L_1 and L_2 by

$$L_1 := \max \left\{ \left[\gamma \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s)) p(s) \Delta s f_\infty \right]^{-1}, \right.$$
$$\left. \left[\gamma \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s)) q(s) \Delta s g_\infty \right]^{-1} \right\},$$

and

$$L_2 := \min \left\{ \left[\frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(s)) p(s) \Delta s f_0 \right]^{-1}, \right.$$
$$\left. \left[\frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(s)) q(s) \Delta s g_0 \right]^{-1} \right\}.$$

Theorem 3.1 *Assume that conditions* $(A1) - (A3)$ *are satisfied. Then, for each* λ *satisfying*

$$L_1 < \lambda < L_2, \tag{3.1}$$

there exists a pair (u, v) *satisfying* (1.1), (1.2) *such that* $u(x) > 0$ *and* $v(x) > 0$ *on* $(a, \sigma^2(b))_{\mathbb{T}}$.

Proof: Let λ be as in (3.1). And let $\epsilon > 0$ be chosen such that

$$\max \left\{ \left[\gamma \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s)) p(s) \Delta s (f_\infty - \epsilon) \right]^{-1}, \right.$$

Notes

$$\left[\gamma\frac{\beta+\alpha(\eta-a)}{d}\int_{\eta}^{\sigma(b)}(\sigma^2(b)-\sigma(s))q(s)\Delta s(g_\infty-\epsilon)\right]^{-1}\Bigg\}\leq\lambda,$$

and

Notes

$$\lambda\leq\min\Bigg\{\left[\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)\Delta s(f_0+\epsilon)\right]^{-1},$$
$$\left[\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))q(s)\Delta s(g_0+\epsilon)\right]^{-1}\Bigg\},$$

Define an integral operator $T:\mathcal{P}\to\mathcal{B}$ by

$$Tu(t):=\lambda\int_a^{\sigma(b)}G(t,s)p(s)f\left(\lambda\int_a^{\sigma(b)}G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\right)\Delta s,\ u\in\mathcal{P}. \tag{3.2}$$

We seek suitable fixed points of T in the cone $\mathcal{P}$.

By Lemma 2.4, $T\mathcal{P}\subset\mathcal{P}$. In addition, standard arguments show that T is completely continuous.

Now, from the definitions of f_0 and g_0, there exists $H_1>0$ such that

$$f(x)\leq(f_0+\epsilon)x\ \text{ and }\ g(x)\leq(g_0+\epsilon)x,\quad 0<x\leq H_1.$$

Let $u\in\mathcal{P}$ with $\|u\|=H_1$. We first have from (2.6) and choice of ϵ,

$$\begin{aligned}\lambda&\int_a^{\sigma(b)}G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\\ &\leq\lambda\frac{\beta+\alpha(t-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)g(u(\sigma(r)))\Delta r\\ &\leq\lambda\frac{\beta+\alpha(t-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)(g_0+\epsilon)u(r)\Delta r\\ &\leq\lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)\Delta r(g_0+\epsilon)\|u\|\\ &\leq\|u\|\\ &=H_1.\end{aligned}$$

As a consequence, we next have from (2.6) and choice of ϵ,

$$Tu(t)=\lambda\int_a^{\sigma(b)}G(t,s)p(s)f\left(\lambda\int_a^{\sigma(b)}G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\right)\Delta s$$

$$\le \lambda\frac{\beta+\alpha(t-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)$$
$$f\left(\lambda\int_a^{\sigma(b)}G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\right)\Delta s$$
$$\le \lambda\frac{\beta+\alpha(t-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)$$
$$(f_0+\epsilon)\lambda\int_a^{\sigma(b)}G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Delta s$$
$$\le \lambda\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)(f_0+\epsilon)H_1\Delta s$$
$$\le H_1$$
$$= \|u\|.$$

So, $\|Tu\|\le\|u\|$. If we set

$$\Omega_1=\{x\in\mathcal{B}:\|x\|<H_1\},$$

then

$$\|Tu\|\le\|u\|,\quad\text{for}\quad u\in\mathcal{P}\cap\partial\Omega_1. \tag{3.3}$$

Next, from the definitions of f_∞ and g_∞, there exists $\overline{H}_2>0$ such that

$$f(x)\ge(f_\infty-\epsilon)x \text{ and } g(x)\ge(g_\infty-\epsilon)x,\quad x\ge\overline{H}_2.$$

Let

$$H_2=\max\left\{2H_1,\frac{\overline{H}_2}{\gamma}\right\}.$$

Let $u\in\mathcal{P}$ and $\|u\|=H_2$. Then,

$$\min_{t\in[\eta,\sigma^2(b)]_{\mathbb{T}}}u(t)\ge\gamma\|u\|\ge\overline{H}_2.$$

Consequently, from (2.7) and choice of ϵ,

$$\lambda\int_a^{\sigma(b)}G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r$$
$$\ge\lambda\frac{\beta+\alpha(\eta-a)}{d}\int_\eta^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)g(u(\sigma(r)))\Delta r$$
$$\ge\lambda\frac{\beta+\alpha(\eta-a)}{d}\int_\eta^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)(g_\infty-\epsilon)u(r)\Delta r$$

Notes

$$\geq \lambda \frac{\beta+\alpha(\eta-a)}{d}\int_{\eta}^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)(g_\infty-\epsilon)\Delta r\gamma\|u\|$$
$$\geq \|u\|$$
$$= H_2.$$

And so, we have from (2.7) and choice of ϵ,

$$Tu(\eta) \geq \lambda \frac{\beta+\alpha(\eta-a)}{d}\int_{\eta}^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s) f\left(\lambda\int_{\eta}^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\right)\Delta s$$
$$\geq \lambda \frac{\beta+\alpha(\eta-a)}{d}\int_{\eta}^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s) (f_\infty-\epsilon)\lambda\int_{\eta}^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Delta s$$
$$\geq \lambda \frac{\beta+\alpha(\eta-a)}{d}\int_{\eta}^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)(f_\infty-\epsilon)H_2\Delta s$$
$$\geq \lambda\gamma \frac{\beta+\alpha(\eta-a)}{d}\int_{\eta}^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)(f_\infty-\epsilon)H_2\Delta s$$
$$\geq H_2$$
$$= \|u\|.$$

Hence, $\|Tu\| \geq \|u\|$. So, if we set

$$\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_2\},$$

then

$$\|Tu\| \geq \|u\|, \quad \text{for} \quad u \in \mathcal{P}\cap\partial\Omega_2. \tag{3.4}$$

Applying Theorem 2.5 to (3.3) and (3.4), we obtain that T has a fixed point $u \in \mathcal{P}\cap(\overline{\Omega}_2\backslash\Omega_1)$. As such, and with v being defined by

$$v(t) = \lambda\int_a^{\sigma(b)} G(t,s)q(s)g(u(\sigma(s)))\Delta s,$$

the pair (u,v) is a desired solution of (1.1), (1.2) for the given λ. The proof is complete.

Prior to our next result, we define positive numbers L_3 and L_4 by

$$L_3 := \max\left\{\left[\gamma\frac{\beta+\alpha(\eta-a)}{d}\int_{\eta}^{\sigma(b)}(\sigma^2(b)-\sigma(s))p(s)\Delta s f_0\right]^{-1}, \left[\gamma\frac{\beta+\alpha(\eta-a)}{d}\int_{\eta}^{\sigma(b)}(\sigma^2(b)-\sigma(s))q(s)\Delta s g_0\right]^{-1}\right\},$$

Notes

and

$$L_4 := \min\left\{ \left[\frac{\beta+\alpha(\sigma^2(b)-a)}{d} \int_a^{\sigma(b)} (\sigma^2(b)-\sigma(s))p(s)\Delta s f_\infty \right]^{-1}, \left[\frac{\beta+\alpha(\sigma^2(b)-a)}{d} \int_a^{\sigma(b)} (\sigma^2(b)-\sigma(s))q(s)\Delta s g_\infty \right]^{-1} \right\}.$$

Theorem 3.2 *Assume that conditions (A1) – (A4) are satisfied. Then, for each λ satisfying*

$$L_3 < \lambda < L_4, \tag{3.5}$$

there exists a pair (u, v) satisfying (1.1), (1.2) such that $u(x) > 0$ and $v(x) > 0$ on $(a, \sigma^2(b))_{\mathbb{T}}$.

Proof: Let λ be as in (3.5). And let $\epsilon > 0$ be chosen such that

$$\max\left\{ \left[\gamma\frac{\beta+\alpha(\eta-a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b)-\sigma(s))p(s)\Delta s(f_0-\epsilon) \right]^{-1}, \left[\gamma\frac{\beta+\alpha(\eta-a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b)-\sigma(s))q(s)\Delta s(g_0-\epsilon) \right]^{-1} \right\} \le \lambda,$$

and

$$\lambda \le \min\left\{ \left[\frac{\beta+\alpha(\sigma^2(b)-a)}{d} \int_a^{\sigma(b)} (\sigma^2(b)-\sigma(s))p(s)\Delta s(f_\infty+\epsilon) \right]^{-1}, \left[\frac{\beta+\alpha(\sigma^2(b)-a)}{d} \int_a^{\sigma(b)} (\sigma^2(b)-\sigma(s))q(s)\Delta s(g_\infty+\epsilon) \right]^{-1} \right\}.$$

Let T be the cone preserving, completely continuous operator that was defined by (3.2).

From the definitions of f_0 and g_0, there exists $H_3 > 0$ such that

$$f(x) \ge (f_0-\epsilon)x \text{ and } g(x) \ge (g_0-\epsilon)x, \quad 0 < x \le H_3.$$

Also, from the definition of g_0 it follows that $g(0) = 0$ and so there exists $0 < H_3 < \overline{H}_3$ such that

$$\lambda g(x) \le \frac{\overline{H}_3}{\frac{\beta+\alpha(\sigma^2(b)-a)}{d}\int_a^{\sigma(b)}(\sigma^2(b)-\sigma(r))q(r)\Delta r}, \quad 0 \le x \le H_3.$$

Notes

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Choose $u \in \mathcal{P}$ with $\|u\| = H_3$. Then

$$\begin{aligned}
&\lambda \int_a^{\sigma(b)} G(\sigma(s), r) q(r) g(u(\sigma(r)))\Delta r \\
&\quad \le \lambda \frac{\beta + \alpha(t-a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(r)) q(r) g(u(\sigma(r)))\Delta r \\
&\quad \le \lambda \frac{\beta + \alpha(\sigma^2(b)-a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(r)) q(r) g(u(\sigma(r)))\Delta r \\
&\quad \le \frac{\frac{\beta+\alpha(\sigma^2(b)-a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(r)) q(r) \overline{H}_3 \Delta r}{\frac{\beta+\alpha(\sigma^2(b)-a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(s)) q(s) \Delta s} \\
&\quad \le \overline{H}_3.
\end{aligned}$$

Then, by (2.7)

$$\begin{aligned}
Tu(\eta) \ge{} & \lambda \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s)) p(s) \\
& f\left(\lambda \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(r)) q(r) g(u(\sigma(r)))\Delta r\right)\Delta s \\
\ge{} & \lambda \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s)) p(s) \\
& (f_0 - \epsilon)\lambda \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(r)) q(r) g(u(\sigma(r)))\Delta r \Delta s \\
\ge{} & \lambda \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s)) p(s) \\
& (f_0 - \epsilon)\lambda\gamma \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(r)) q(r) (g_0 - \epsilon)) \|u\| \Delta r \Delta s \\
\ge{} & \lambda \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s)) p(s) (f_0 - \epsilon) \|u\| \Delta s \\
\ge{} & \lambda\gamma \frac{\beta + \alpha(\eta - a)}{d} \int_\eta^{\sigma(b)} (\sigma^2(b) - \sigma(s)) p(s) (f_0 - \epsilon) \|u\| \Delta s \\
\ge{} & \|u\|.
\end{aligned}$$

So, $\|Tu\| \ge \|u\|$. If we put

$$\Omega_3 = \{x \in \mathcal{B} : \|x\| < H_3\},$$

then

$$\|Tu\| \ge \|u\|, \quad \text{for} \quad u \in \mathcal{P} \cap \partial\Omega_3. \tag{3.6}$$

Next, by definitions of f_∞ and g_∞, there exists $\overline{H}_4$ such that

$$f(x) \leq (f_\infty + \epsilon)x \quad \text{and} \quad g(x) \leq (g_\infty + \epsilon)x, \quad x \geq \overline{H}_4$$

Clearly, since g_∞ is assumed to be a positive real number, it follows that g is unbounded at ∞, and so, there exists $\widetilde{H}_4 > \max\{2H_3, \overline{H}_4\}$ such that $g(x) \leq g(\widetilde{H}_4)$, for $0 < x \leq \widetilde{H}_4$.

Set

$$f^*(t) = \sup_{a \leq s \leq t} f(s), \quad g^*(t) = \sup_{a \leq s \leq t} g(s), \text{ for } t \geq 0.$$

Clearly f^* and g^* are nondecreasing real valued functions for which it holds

$$\lim_{x \to \infty} \frac{f_i^*(x)}{x} = f_\infty, \quad \lim_{x \to \infty} \frac{g_i^*(x)}{x} = g_\infty.$$

Hence, there exists H_4 such that $f^*(x) \leq f^*(H_4)$, $g^*(x) \leq g^*(H_4)$, for $0 < x \leq H_4$.

Choosing $u \in \mathcal{P}$ with $\|u\| = H_4$, we have

$$\begin{aligned}
Tu(t) \leq{}& \lambda \frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(s))p(s) \\
& f\left(\lambda \frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(r))q(r)g(u(\sigma(r)))\Delta r\right)\Delta s \\
\leq{}& \lambda \frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(s))p(s) \\
& f^*\left(\lambda \frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(r))q(r)g(u(\sigma(r)))\Delta r\right)\Delta s \\
\leq{}& \lambda \frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(s))p(s) \\
& f^*\left(\lambda \frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(r))q(r)g^*(u(\sigma(r)))\Delta r\right)\Delta s \\
\leq{}& \lambda \frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(s))p(s) \\
& f^*\left(\lambda \frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(r))q(r)g^*(H_4)\Delta r\right)\Delta s \\
\leq{}& \lambda \frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(s))p(s) \\
& f^*\left(\lambda \frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(r))q(r)(g_\infty + \epsilon)H_4\Delta r\right)\Delta s \\
\leq{}& \lambda \frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma(b)} (\sigma^2(b) - \sigma(s))p(s)f^*(H_4)\Delta s
\end{aligned}$$

Notes

$$\leq \lambda \frac{\beta + \alpha(\sigma^2(b) - a)}{d} \int_a^{\sigma} (\sigma^2(b) - \sigma(s))p(s)(f_\infty + \epsilon)H_4 \Delta s$$
$$\leq H_4$$
$$= \|u\|,$$

and so $\|Tu\| \leq \|u\|$. For this case, if we let

$$\Omega_4 = \{x \in \mathcal{B} : \|x\| < H_4\},$$

then

$$\|Tu\| \leq \|u\|, \quad \text{for} \quad u \in \mathcal{P} \cap \partial\Omega_4. \tag{3.7}$$

Application of part (ii) of Theorem 2.5 yields a fixed point u of T belonging to $\mathcal{P} \cap (\overline{\Omega}_4 \backslash \Omega_3)$, which in turn yields a pair (u, v) satisfying (1.1), (1.2) for the chosen value of λ. The proof is complete.

Notes

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Application of Laplace Transform

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Abstract - The present discounted value equation in finance has a broad range of uses and may be applied to various areas of finance including corporate finance, banking finance and investment finance etc .The basic premise of present discounted value is the time value money .Not many analytic solutions exist for present discounted value problems but by using Laplace transform we can deduce some of the closed form solutions quite easily. In this note we show how present discounted value in finance related to Laplace transforms. Also we discus on the present value rules for the elementary functions and the general properties of the Laplace transform. And we will focus on the application of time derivative property using Laplace transforms to each present value rule.

Keywords : Present discounted value, cash flow, perpetuity, Time derivative, Laplace transform.

GJSFR-F Classification : MSC 2010: 44A10

Application of Laplace Transform

Dr. N. A. Patil [α] & Vijaya N. Patil [σ]

Abstract - The present discounted value equation in finance has a broad range of uses and may be applied to various areas of finance including corporate finance, banking finance and investment finance etc .The basic premise of present discounted value is the time value money .Not many analytic solutions exist for present discounted value problems but by using Laplace transform we can deduce some of the closed form solutions quite easily. In this note we show how present discounted value in finance related to Laplace transforms. Also we discus on the present value rules for the elementary functions and the general properties of the Laplace transform. And we will focus on the application of time derivative property using Laplace transforms to each present value rule.

Keywords : Present discounted value, cash flow, perpetuity, Time derivative, Laplace transform.

I. Introduction

During the past few decades, methods based on integral transforms, in particular, the Laplace transforms, are being increasingly employed in mathematics, physics, mechanics and other engineering sciences. Laplace transforms have a wide variety of applications in the solution of differential, integral and difference equations. It is much less used in financial engineering. One reason is technical: not many people know that all that they need to do is to make simple calculations in the Laplace domain.

The outline of this note is as follows –

In section 1 we show the relation between present discounted value and Laplace transforms.

In section 2 we identified the present value rules for each of the cash flow.

In section 3 we discussed the general properties of Laplace transforms with present value rules.

In section 4 we show the application of time derivative property to each of the present value rules.

II. Relation Between Present Value and Laplace Transform

The Present value of a series of payments given by,

$$(PV)_t = \sum_{t=1}^{T} \frac{C(t)}{(1+r)^t} \qquad (1)$$

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Where, $(PV)_t$ = Present discounted value at time t

$C(t)$ = Cash flow

r = rate of discount

t = Period

Here we assume the Present value with continuous compounding .It is the current value of a stream of cash flows. In other words, it is the amount that we would be willing to pay today in order to receive a cash flow or a series of them in the future.
Now by using an exponential series we can write equation (1) as,

$$(PV)_t = \sum_{t=1}^{T} e^{-rt} C(t) \quad (2)$$

In the limiting case replacing summation to an integral, equation (2) can be written as

$$(PV)_r = \int_0^T e^{-rt} C(t)\, dt \quad (3)$$

Again here T is some finite quantity. So if we consider as $T \to \infty$, equation (3) will becomes

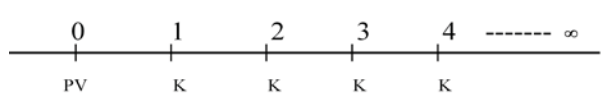
$$(PV)_r = \int_0^\infty e^{-rt} C(t)\, dt \quad (4)$$

This equation is the exact replica of Laplace transform in mathematics.

Therefore, $$(PV)_r = L[C(t)] \quad (5)$$

III. Laplace Transforms and Present Value Rules for Some Cash Flows

Using Present value equation: Consider the case of constant cash payment K made at the end of each year at interest rate r as shown in the following time line,



Here the cash flow is continuous forever Therefore the Present value is given by an infinite geometric series:

$$PV = \frac{K}{1+r} + \frac{K}{(1+r)^2} + \frac{K}{(1+r)^3} + ------ \quad (6)$$

Dividing both sides by (1+r) we get,

$$\frac{PV}{1+r} = \frac{K}{(1+r)^2} + \frac{K}{(1+r)^3} + \frac{K}{(1+r)^4} ------ \quad (7)$$

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Subtracting equation (7) from (6) we get,

$$PV - \frac{PV}{1+r} = \frac{K}{1+r}$$

On solving we get the Present value of perpetuity.

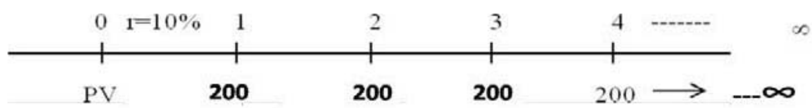
Using Laplace transform equation: If the cash flow is constant say K then the Present discounted value of a stream at interest rate r is given by,

$$L[K] = K\int_0^{\infty} e^{-rt}\, dt = \frac{K}{r}$$

This is the same formula as above.

For example: An insurance company has just launched a security that will pay Rs.200 indefinitely, starting the first payment next year. How much should this security be worth today if the appropriate return is 10% ?

We solve this example by using the time line,



$$PV = \frac{K}{r} = \frac{200}{0.10} = Rs.2000$$

Using Present value equation: Consider the payments in perpetuity increases at a certain growth rate g as shown on the time line:

The Present value of a growing perpetuity can be written as the following infinite series-

$$PV = \frac{K}{1+r} + \frac{K(1+g)}{(1+r)^2} + \frac{K(1+g)^2}{(1+r)^3} + ------- \qquad (8)$$

Multiplying both sides by $\frac{(1+g)}{(1+r)}$. Hence we get,

$$PV\frac{(1+g)}{(1+r)} = \frac{K(1+g)}{(1+r)^2} + \frac{K(1+g)^2}{(1+r)^3} + \frac{K(1+g)^3}{(1+r)^4} ------ \qquad (9)$$

Subtracting equation (9) from (8) we get,

$$PV - \frac{PV(1+g)}{1+r} = \frac{K}{1+r}$$

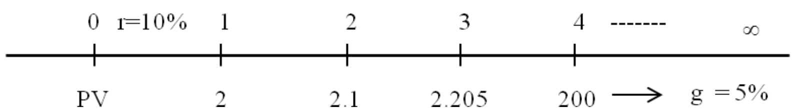
On solving we get the Present value of a growing perpetuity.

Using Laplace transform equation: For an exponential or geometric cash flow the Present discounted value of a stream growing at rate g, is given by:

$$L[C(t)] = K\, L\left[e^{g\,t}\right] = \frac{K}{r-g} \quad if \;\; r > g \qquad (10)$$

This is the geometric growth stream or Present value of growing perpetuity having cash flow after the first period divided by the difference between the discount rate and the growth rate and the growth rate must be less than the interest rate.

For example: A company is expected to pay Rs.2 of dividend per share that wills increases 5% forever. If investors require 10% return on the company's stocks, how much should investors pay for the stocks? The cash flows are as follows:



$$PVG = \frac{K}{r-g} = \frac{2}{0.10-0.05} = Rs.40 \tag{11}$$

For an arithmetic cash flow the Present discounted value of a stream at rate r, is given by:

$$L\left[C(t)\right] = \mathrm{K}\, L\left[t\right] = \frac{K}{r^2} \tag{12}$$

This shows that an arithmetic growth stream is equivalent to receiving one consol per period in perpetuity. This rule is widely used in finance for solving Preset value.

The above rules are commonly used transforms but more useful are the general properties of the Laplace transforms in an algebraic fashion .Let us look at some of the main properties.

Property 1: Linearity: The Laplace Transform is a linear operator. Hence if the Laplace Transforms of the cash flows f(t) and g(t) both exists then we have for any arbitrary constants (a,b) that:

$$L\left\{a\ f\left(t\right) + b\ g\left(t\right)\right\} = a\ L\left\{f\left(t\right)\right\} + b\ \ L\left\{g\left(t\right)\right\} \tag{13}$$

This property allows us to deduce more complex transforms to simple transforms.
Property 2: Geometric scaling:

$$L\left\{e^{\alpha t} c(t)\right\} = V(r-\alpha) \quad for\ \alpha < r \tag{14}$$

This property shows that the scaling a cash flow by geometric term is equivalent to corresponding reduction in the rate of discount.
Property 3: Multiplication by t:

$$\mathrm{L}\left\{\mathrm{t\ C(t)}\right\} = -\mathrm{V}'\left(\mathrm{r}\right) \tag{15}$$

We can confirm this property by using the derivative of exponential function.
Property 4: Time shifting:

$$\left.\begin{matrix} \mathrm{L}\left\{\mathrm{C(a+bt)}\right\} & \text{for } t \geq a/b \\ 0 & \text{for } t < a/b \end{matrix}\right\} = \mathrm{e}^{r\,a/b}\left(\frac{1}{b}\right)\mathrm{P\ (r/b)} \tag{16}$$

This property applies the change of variable theorem of integral calculus and helpful for finding cash flows with altered time schedules.

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Property 5: Time derivative:

$$L\{C'(t)\} = r\,L\{C(t)\} - C(0) \tag{17}$$

This property identifies a fundamental linear relationship between Laplace transform for cash flows and their time derivatives. For the confirmation we use integration by parts:

$$\int u dv = uv - \int v du$$

When we evaluate the integral over relevant range 0 to ∞ for the Laplace transform and impose a standard assumption in present value problems that the marginal present value of the cash flow vanishes as t gets large.

All present value rules of section second can be derived from this time derivative property of Laplace transform and hence having particular significance in finance.
For the confirmation we rewrite the time-derivative property by using notation

$$(PV)_r = L\left[C(t)\right]$$, as:

$$L[C(t)] = \frac{C(0)}{r} + \frac{L\left[C'(t)\right]}{r} \tag{18}$$

Apply the property, for some cash flows.

Ex: For S (t) =K ⇒ S (0) =K & $S'(t)$ =0, we get the consol rule by using property 5.

Alternatively, each asset is valued as if its cash flow were projected at a constant level equal to the current rate plus the present value of the time derivative of the cash flow.
For geometric cash flow (2.I)

$$S(t) = e^{\alpha t} \Rightarrow S(0) = 1 \;\&\; S'(t) = \alpha\, e^{\alpha t}$$, we get

$$L[e^{\alpha t}] = \frac{1}{\theta} + \frac{\alpha L\left[e^{\alpha t}\right]}{\theta} \tag{19}$$

Alternatively, we could combine the consol rule with property 2 to establish the rule for geometric growth.

Similarly we can derive the rule for arithmetic growth by using equation (19) or combining the consol rule and property 3.

IV. Conclusion

In this article we have presented the close relationship between present discounted value in finance and Laplace transform. We can solve the present discounted value examples within a few minutes by using Laplace equation method. The result seems to be new & to have a potential to increase the practical utility of Laplace transform especially in finance. However it is important to notice that frequency domain is possible appreciate also in the real world & applied in the areas like economics or finance. But the Laplace transform is the big source for present discounted value function to illustrate the enhanced problems.

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Formation of a Summation Formula Enmeshed with Hypergeometric Function

By Salahuddin
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Abstract - The main aim of the present paper is to form a summation formula attached with contiguous relation and recurrence relation.

Keywords : Gaussian Hypergeometric function , Contiguous function, Recurrence relation, Bailey summation theorem and Legendre duplication formula.

GJSFR-F Classification : MSC 2010: 65B10, 33D60

FORMATION OF A SUMMATION FORMULA ENMESHED WITH HYPERGEOMETRIC FUNCTION

Strictly as per the compliance and regulations of :

Notes

Formation of a Summation Formula Enmeshed with Hypergeometric Function

Salahuddin

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Keywords : Gaussian Hypergeometric function , Contiguous function, Recurrence relation, Bailey summation theorem and Legendre duplication formula.

I. Introduction

Generalized Gaussian hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{matrix} a_1, a_2, \cdots, a_A & ; & \\ & & z \\ b_1, b_2, \cdots, b_B & ; & \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} $$

or

$$ {}_AF_B\left[\begin{matrix} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{matrix}\right] \equiv {}_AF_B\left[\begin{matrix} (a_j)_{j=1}^{A} & ; & \\ & & z \\ (b_j)_{j=1}^{B} & ; & \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1} $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

Contiguous Relation[E. D. p.51(10), Andrews p.363(9.16)] is defined as follows

$$ (a-b)\ {}_2F_1\left[\begin{matrix} a,\ b\ ; & \\ c\quad ; & z \end{matrix}\right] = a\ {}_2F_1\left[\begin{matrix} a+1,\ b\ ; & \\ c\quad ; & z \end{matrix}\right] - b\ {}_2F_1\left[\begin{matrix} a,\ b+1\ ; & \\ c\quad ; & z \end{matrix}\right] \tag{2} $$

Recurrence relation of gamma function is defined as follows

$$ \Gamma(z+1) = z\ \Gamma(z) \tag{3} $$

Legendre duplication formula[Bells & Wong p.26(2.3.1)] is defined as follows

$$ \sqrt{\pi}\ \Gamma(2z) = 2^{(2z-1)}\ \Gamma(z)\ \Gamma\left(z+\frac{1}{2}\right) \tag{4} $$

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$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{b+1}{2})}{\Gamma(b)} \tag{5}$$

$$= \frac{2^{(a-1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\Gamma(a)} \tag{6}$$

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Bailey summation theorem [Prud, p.491(7.3.7.8)]is defined as follows

$${}_2F_1\left[\begin{matrix} a, 1-a & ; & \frac{1}{2} \\ c & ; & \end{matrix}\right] = \frac{\Gamma(\frac{c}{2})\ \Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})} = \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c-1}\ \Gamma(\frac{c+a}{2})\ \Gamma(\frac{c+1-a}{2})} \tag{7}$$

II. Main Result of Summation Formula

$${}_2F_1\left[\begin{matrix} a \ , & -a-48 & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right]$$

$$= \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c+48}}\Bigg[\frac{-363834790475054308503006241456128000 0a}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})} +$$

$$+ \frac{42305344591446351932334390127779840000a^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})} +$$

$$+ \frac{-17735301112690243793611848889783296000a^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})} +$$

$$+ \frac{-260721277882237830279315267675256 32a^5 - 14008285511532735512012072508 2688a^6}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})} +$$

$$+ \frac{1132102670325934198927434006115 20a^7 - 159130849326500108009025283 9184a^8}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})} +$$

$$+ \frac{-24948818014045090594527735 0672a^9 + 189661135473875295822147 7012a^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})} +$$

$$+ \frac{340576268631980322449458500a^{11} + 22323936367213507094680 81a^{12}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})} +$$

$$+ \frac{-2219611899285453926263 92a^{13} - 448893841878895011825 8a^{14}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})} +$$

$$+ \frac{262377425651100055980a^{15} + 172630819314104691 1a^{16} + 16724033867495328a^{17}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})} +$$

$$+ \frac{-67838656895768a^{18} - 2424088437540a^{19} - 16239228929a^{20} - 12952632a^{21} + 295702a^{22}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})} +$$

$$+ \frac{1140a^{23} + a^{24} + 3638347904750568937046801299537920000c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})} +$$

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$$+\frac{-1102386030668985916563291720017510400ac}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{829604772507557871543587735061135360a^{2}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-258906052741514131599991388352479232a^{3}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{366873883396175498327904190658199552a^{4}c-1795799068626468355105179908796825 6a^{5}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-783414259355080511741474195684352a^{6}c+843408734535020024213988436820 48a^{7}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{973370298518103355293671126656a^{8}c-16068714875102192640941298227 2a^{9}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-22489836587671852682139098 88a^{10}c+1491725776263111430616533 92a^{11}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{3615991694923403891801160a^{12}c-402375840588037669160 08a^{13}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-214557912061279534378 4a^{14}c-1584749543138052042 4a^{15}c+294522342618394704a^{16}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{5916283797667056a^{17}c+27173155233840a^{18}c-187374214720a^{19}c-2438354072a^{20}c}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-8088520a^{21}c-1416a^{22}c+24a^{23}c+679332584754532051136613068046336 0000c^{2}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-1213582706891607789420020611861708 8000ac^{2}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{675367521087951027003971459430678528 0a^{2}c^{2}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-162486991180565607096219121559779737 6a^{3}c^{2}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{171251591774737199593026229784064000a^{4}c^{2}-4093063242641347774786447363015680a^{5}c^{2}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-51114992451050204207273969265894 4a^{6}c^{2}+2250406728599404392469230286323 2a^{7}c^{2}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{913186425471360033494807245312a^{8}c^{2}-3380471826275553368210048448 0a^{9}c^{2}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-1295637035432751803115296208a^{10}c^{2}+158908796338920555045321 84a^{11}c^{2}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{1022450363916086376744200a^{12}c^2+560807752073797536308 0a^{13}c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-2786537377716261781 04a^{14}c^2-4752455855653094160a^{15}c^2-5301794887227696a^{16}c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{495348526392720a^{17}c^2+4527490455488a^{18}c^2+8270782520a^{19}c^2-69085016a^{20}c^2}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-326040a^{21}c^2-312a^{22}c^2+5613309455109683097975789723844608000c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-7268957877902293679747953623910318080ac^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{3148476881911767042344731733158526976a^2c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-5917764723955107491535215159989370 88a^3c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{45128923170400422228420079851380736a^4c^3+5555930291075194182761162530816a^5c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-1507165572553265053707102164300 80a^6c^3+22170812806930476641453160250 88a^7c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{2549004306304763398682774069 76a^8c^3-190831988666633701604240 2944a^9c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-255978165939209274666102816a^{10}c^3-13700808854608199360327 20a^{11}c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{113902943230614008414944a^{12}c^3+18705095428997979441 44a^{13}c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-79806558556965468 80a^{14}c^3-425296980132412960a^{15}c^3-3111050730435648a^{16}c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{7889860890912a^{17}c^3+200089649760a^{18}c^3+804243440a^{19}c^3+352352a^{20}c^3-2288a^{21}c^3}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{2768586081553372140200792041508044800c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-2788045206948865839414630121447882752ac^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{962315653870080299754170370182283264a^2c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-141439234805306232175165147182489600a^3c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

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$$+\frac{735056955575796465308774280255 8976a^4c^4+1990687917058352516656713598028 80a^5c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-25294347834251193603125177586944a^6c^4-1474379931817745543033138265 60a^7c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{36251936778359362132532742528a^8c^4+3460298869054577507516395 20a^9c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-24523514138602855095025488a^{10}c^4-447590500623950105662240a^{11}c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{471218506932929667912 0a^{12}c^4+177404400410212015040a^{13}c^4+93229824160563376 0a^{14}c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-14323139972581440a^{15}c^4-193197305325024a^{16}c^4-553302990240a^{17}c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{2501298800a^{18}c^4+15215200a^{19}c^4+16016a^{20}c^4}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{92363648123490815804287285744946380 8c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-747023445605858669273498410297589760ac^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{207668786206542828581701432891146240a^2c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-23663020067855212035887457634222080a^3c^5+7329045676219872303514191113256 96a^4c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{4990335380545033542436854778265 6a^5c^5-257889374699953083193430422425 6a^6c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-6657768824962958405631023001 6a^7c^5+2874347583304554423474949632a^8c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{74463995548879100021824512a^9c^5-1062528271193599593294912a^{10}c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-4280767591218020129427 2a^{11}c^5-124023155193475329792a^{12}c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{7667675875623731968a^{13}c^5+86927587193935488a^{14}c^5+5208901746048a^{15}c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-4594408955136a^{16}c^5-22868541696a^{17}c^5-15951936a^{18}c^5+64064a^{19}c^5}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{2233307458979827095185295489934295 04c^6-1474723588336689738317374900037222 40ac^6}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{33122202957018388352610598824574976a^2c^6 - 2871630447882289896332967060062208a^3c^6}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{34441427528122369258470645260288a^4c^6 + 6831394798548698327794234414080a^5c^6}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{-150155381817771045217714537472a^6c^6 - 8692060148779265084777793024a^7c^6}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{108392221996820991579377664a^8c^6 + 6575336744109767454300480a^9c^6}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{8164199124092113502400a^{10}c^6 - 2080877086126562051328a^{11}c^6}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{-21558516560807978752a^{12}c^6 + 121341101729527680a^{13}c^6 + 2983367975812224a^{14}c^6}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{12147729834240a^{15}c^6 - 32015343360a^{16}c^6 - 273873600a^{17}c^6 - 320320a^{18}c^6}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{408771514970143788397713545515499 52c^7 - 2221950242789970233852001403063500 8ac^7}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{4022878448009248326347020243828736a^2c^7 - 256858653782602724496822439018496a^3c^7}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{-16416479665937631914021066342 40a^4c^7 + 61338808704698279487055193088 0a^5c^7}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{-2367132688592308795171872768a^6c^7 - 65499806740789371096482918 4a^7c^7}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{-15236507461275405703157 76a^8c^7 + 32953952491634442631334 4a^9c^7}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{342564608393461676851 2a^{10}c^7 - 50038536300856097280a^{11}c^7 - 974138332318255104a^{12}c^7}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{-2184451291486464a^{13}c^7 + 45506452992000a^{14}c^7 + 291087413760a^{15}c^7 + 283345920a^{16}c^7}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{-823680a^{17}c^7 + 583259441224305592897332542282137 6c^8}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{-2618050661653803280443469114376192ac^8 + 3794931144588982675024706966650 88a^2c^8}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{-16864207890780906911526743900160a^3c^8 - 4477203210256570938380325765 12a^4c^8}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

$$+\frac{37986129070713604063172198400a^5c^8 + 389769280200184576457945088a^6c^8}{\Gamma\left(\frac{c-a+1}{2}\right)\ \Gamma\left(\frac{c+a+48}{2}\right)}+$$

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$$+\frac{-313723142528224658387755840a^{7}c^{8}-41110917850064461148 0064a^{8}c^{8}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{9363608406942489722880a^{9}c^{8}+187215352358190114816a^{10}c^{8}-1464246134954803 20a^{11}c^{8}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-21752854863387648a^{12}c^{8}-123540894597120a^{13}c^{8}+173781104640a^{14}c^{8}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{2503987200a^{15}c^{8}+3294720a^{16}c^{8}+662317910051181926669156716904448c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-2454001157960715329587654531481 60ac^{9}+281532382552098214900414724177 92a^{2}c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-7810056452760125024627410862 08a^{3}c^{9}-426418956203790186175502745 60a^{4}c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{1612351973725244164720885760a^{5}c^{9}+39662648103197518265810944a^{6}c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-926453550912364548243456a^{7}c^{9}-23555359060666652876800a^{8}c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{1009032254068042547 20a^{9}c^{9}+5299360980595089408a^{10}c^{9}+24692042573406208a^{11}c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-227873864458240a^{12}c^{9}-1996213739520a^{13}c^{9}-2571345920a^{14}c^{9}+5857280a^{15}c^{9}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{60743744451907338804193681997824c^{10}-18518348550706651821965710458880ac^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{1653100889178452682493737828352a^{2}c^{10}-217407216082713760994568437 76a^{3}c^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-26008133294619037943037296 64a^{4}c^{10}+42802618860435796873871360a^{5}c^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{2038210720517206870835200a^{6}c^{10}-12463849823864973467648a^{7}c^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-758892596094305001472a^{8}c^{10}-2575932665689743360a^{9}c^{10}+83354346743635968a^{10}c^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{679546442358784a^{11}c^{10}-290098192384a^{12}c^{10}-13243310080a^{13}c^{10}-19914752a^{14}c^{10}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{4546715789559603603424633946112c^{11}-1133943987191935145013892087808ac^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{76868622553074562239414927360a^{2}c^{11}-6333853945783824633298944a^{3}c^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-112614782541939583740936192a^4c^{11}+383669909202609748246528a^5c^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{675552270870497957731456a^6c^{11}+197559515798431514624a^7c^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-14999874634184294400a^8c^{11}-121790210462269440a^9c^{11}+596723478134784a^{10}c^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{8090331791360a^{11}c^{11}+13534789632a^{12}c^{11}-25346048a^{13}c^{11}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{279714662512116971474262163456c^{12}-56599454327811805687654121472ac^{12}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{2812155919136601281891139584a^2c^{12}+34239457060775837590487040a^3c^{12}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-3555265567506078642012160a^4c^{12}-21202879517144336793600a^5c^{12}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{1500614958719249924096a^6c^{12}+13623195491897671680a^7c^{12}-170239161849888768a^8c^{12}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-2179202261483520a^9c^{12}-1109168406528a^{10}c^{12}+43341742080a^{11}c^{12}+76038144a^{12}c^{12}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{14202828556049274882760900608c^{13}-2306244763812047662339850240ac^{13}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{79549699725335858543329280a^2c^{13}+1970351955192820966359040a^3c^{13}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-81197049931788972195840a^4c^{13}-1121385597257318400000a^5c^{13}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{21460893335812374528a^6c^{13}+318310898510266368a^7c^{13}-703360552796160a^8c^{13}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-20253520035840a^9c^{13}-44148916224a^{10}c^{13}+70189056a^{11}c^{13}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{596172495237225865390587904c^{14}-76594063955412337515560960ac^{14}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{1673197695444638048976896a^2c^{14}+67447668828029406412800a^3c^{14}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-1280086116620532449280a^4c^{14}-28565196119329013760a^5c^{14}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{163144044077973504a^6c^{14}+4184336248995840a^7c^{14}+6613981593600a^8c^{14}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

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$$+\frac{-90493747200a^9c^{14}-190513152a^{10}c^{14}+20667780972974403768811 52c^{15}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-20633688999407340989972 48ac^{15}+236162276343982462074 88a^2c^{15}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{1609625006757485477888a^3c^{15}-119466061670535659 52a^4c^{15}-457413975935680512a^5c^{15}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-10221360827 5968a^6c^{15}+31611720302592a^7c^{15}+91954348032a^8c^{15}-127008768a^9c^{15}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{5895809488243432106229 76c^{16}-446944850785573242142 72ac^{16}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{1365971037141501214 72a^2c^{16}+276913581612741427 20a^3c^{16}-16081668010082304a^4c^{16}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-4728686302986240a^5c^{16}-14549489418240a^6c^{16}+120658329600a^7c^{16}+317521920a^8c^{16}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{137451630843129841582 08c^{17}-76779763199419351 0400ac^{17}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-26970120258752348 16a^2c^{17}+340925659342700544a^3c^{17}+1206869439283200a^4c^{17}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-29915346370560a^5c^{17}-122376683520a^6c^{17}+149422080a^7c^{17}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{259068608527627976704c^{18}-102450616642358476 80ac^{18}-86103626336960512a^2c^{18}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{2895000352849920a^3c^{18}+16701453762560a^4c^{18}-99365683200a^5c^{18}-348651520a^6c^{18}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{3884083664023191552c^{19}-102879574253109248ac^{19}-1204310832578560a^2c^{19}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{15688841297920a^3c^{19}+100631838720a^4c^{19}-110100480a^5c^{19}+45210938613170176c^{20}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-739630552973312ac^{20}-9921293713408a^2c^{20}+46022000640a^3c^{20}+242221056a^4c^{20}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{393630531452928c^{21}-3495365181440ac^{21}-46552580096a^2c^{21}+46137344a^3c^{21}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{2410953048064c^{22}-9164554240ac^{22}-96468992a^2c^{22}+9261023232c^{23}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+$$

$$+\frac{-8388608ac^{23}+16777216c^{24}}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a+48}{2})}+\frac{20007974164906320568399715106816000000}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-369224057546261780368904057615155200000a}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{220652783879915420663070451699261440000a^2}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-580984115157354481964922563704535040 0a^3}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{679890009913045305132937518437944320a^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-18472923622042793122488427319724288a^5 - 2560574556753342210803824950095424a^6}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{132219440896221918244499646073152a^7 + 5904047252981777731527533921776a^8}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-270614639795328768343142065008a^9 - 11480344457959138778418103724a^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{191451864133347179813345532a^{11} + 13328712827913082236490801a^{12}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{734534660604123620033352a^{13} - 5838069209154766013234a^{14}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-1190079466863125809 08a^{15} - 64104578911758209a^{16} + 23731154067653472a^{17}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{293580605335336a^{18} + 706255392612a^{19} - 12674658689a^{20} - 115467528a^{21} - 326954a^{22}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{12a^{23} + a^{24} + 5143686185111071950723658876452864 0000c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-708996461601678798036609182057103360 00ac}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{334185512152892863187096551031832576 00a^2c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-6936716063495713404096607576151162880a^3c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{592162566124636481520902123469422592a^4c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-15076317899149562951312080516300 8a^5c - 2543564167189971442636762196069376a^6c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{455031434172882345094415385771 52a^7c + 5720189897921291255677630092928a^8c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

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$$+\frac{-578504811830318738455333790 72a^{9}c-807940826351028803401607232 0a^{10}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-4265860271563874009498971 2a^{11}c+5457516920998845643189704a^{12}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1050806287178855032593 04a^{13}c-718597316917131560744a^{14}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-42146779055614771736a^{15}c-394633823765699760a^{16}c+1815118515463440a^{17}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{59605259080752a^{18}c+390797915200a^{19}c+260642536a^{20}c-7396664a^{21}c-27912a^{22}c}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-24a^{23}c+538796761361069628539102843790950400 00c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-587018509229252774363463630111375360 00ac^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{222728624072316572945148266796240076 80a^{2}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-364528185023119717318082568719582822 4a^{3}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{213875375121643552409476002283352064a^{4}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{6491596305419248116184393185321984a^{5}c^{2}-964599604570031846728877390420224a^{6}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-56825339716279918232057583768 32a^{7}c^{2}+1885588931111013766114178951680a^{8}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{19394898395592969380873022336a^{9}c^{2}-18515184501145964445642350 88a^{10}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-390833356752419403943651 44a^{11}c^{2}+5804382314429307871829 84a^{12}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{2554691291934126835890 4a^{13}c^{2}+1592728162218785809 36a^{14}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-3904286889491205360a^{15}c^{2}-70854972633732144a^{16}c^{2}-286985878971792a^{17}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{2582951379008a^{18}c^{2}+30021551560a^{19}c^{2}+93509416a^{20}c^{2}-3432a^{21}c^{2}-312a^{22}c^{2}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{322116990889819015724992716676792320 00c^{3}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-285120003898190835953216763263660851 20ac}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{8790460634142328234758745607006846976a^2c^3}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-11244670018635615805370163531237949 44a^3c^3}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{3970871360509153222830217825377484 8a^4c^3+30723118863153802167474286687600 64a^5c^3}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-1868885408991602763371023121290 24a^6c^3-5460277329981356928340767544064a^7c^3}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{292228466284140197571175445760a^8c^3+8719189987830229672642717824a^9c^3}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-16517518768516207317777 2832a^{10}c^3-77705720518467920356106 72a^{11}c^3}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-2287544989254407394512 0a^{12}c^3+240407540665107064849 6a^{13}c^3}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{3532172251156353376 0a^{14}c^3-16711413272084960a^{15}c^3-42753037418931 84a^{16}c^3}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-34756964834592a^{17}c^3-42598876320a^{18}c^3+641040400a^{19}c^3+2658656a^{20}c^3}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{2288a^{21}c^3+12673569486529179630190013064413184000c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-9247136634374056328148193250852732928ac^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{2322904856073730886919387342994735104a^2c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-227997454418110745446600708021592064a^3c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{3169731737784104346723887733391 7696a^4c^4+717153243408099923350630112145920a^5c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-1889442686206212217116514495001 6a^6c^4-125650350376183599338700616012 8a^7c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{2000531209011457849734001036 8a^8c^4+139950634175881689810620880 0a^9c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{110729347570460764424260 8a^{10}c^4-7148299819635615070189 76a^{11}c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

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$$+\frac{-923481585230722397880 0a^{12}c^4+801973687410373878 40a^{13}c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{258204137203815088 0a^{14}c^4+1525843185458592 0a^{15}c^4-78560643024864a^{16}c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-1260904925280a^{17}c^4-4363799440a^{18}c^4+160160a^{19}c^4+16016a^{20}c^4}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{354976755942716816160143743450192281 6c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-2151099368274358759726535915983601664ac^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{439330611840452027347549954963931136a^2c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-32045308509021985102462728518369280a^3c^5-2301170211868213813782305947484 16a^4c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{103351131588547583108228728397824a^5c^5-5261538053120681144191581726 72a^6c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-156554337016017490934865859584a^7c^5-3423703206703360520972497 92a^8c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{120430698898953310510476288a^9c^5+1514876828200500103475136a^{10}c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-31496137302336777598528a^{11}c^5-792727673743617900288a^{12}c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-22140182900939783 68a^{13}c^5+83755863780580992a^{14}c^5+874309907382912a^{15}c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1577740260096a^{16}c^5-16154121984a^{17}c^5-74378304a^{18}c^5-64064a^{19}c^5}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{7435596504404521213026085124607836 16c^6-37538041809895690938958399859274547 2ac^6}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{619349561670265378326023782089359 36a^2c^6-31761156149939730858777191753809 92a^3c^6}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-975708469165201351717557924904 96a^4c^6+989166849354056386744648215244 8a^5c^6}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{116489775146410674839857159168a^6c^6-119748959258709999562607477 76a^7c^6}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-187271227517652896016043008a^8c^6+5743465474229360858844864a^9c^6}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{144554417686389868597440a^{10}c^{6}-230607270695987657472a^{11}c^{6}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-3325122289083820825 6a^{12}c^{6}-27723813920285683 2a^{13}c^{6}+778197251255424a^{14}c^{6}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{20174474960640a^{15}c^{6}+78548870400a^{16}c^{6}-2882880a^{17}c^{6}-320320a^{18}c^{6}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1204191092498182606990068068552540 16c^{7}-5064392020414638474258246775550 7712ac^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{6678206027798100901235978413015040a^{2}c^{7}-2163499340681201284281111784652 80a^{3}c^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-38498027628346074126359336222 72a^{4}c^{7}+634544063521152036454215260160a^{5}c^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{18576717170330425255016964096a^{6}c^{7}-556528190887633754610422784a^{7}c^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-17422709590477274298209280a^{8}c^{7}+1079629872294293078 20416a^{9}c^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{7038701856856382828544a^{10}c^{7}+44211950709134866944a^{11}c^{7}-6810948855387156 48a^{12}c^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-9700638027015936a^{13}c^{7}-23794160332800a^{14}c^{7}+184617438720a^{15}c^{7}+955468800a^{16}c^{7}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{823680a^{17}c^{7}+154358012434546379554154449375068 16c^{8}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-5394264531666420310950929291542528ac^{8}+55952424817090473092065531854 8480a^{2}c^{8}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-87367961367219047542381819330 56a^{3}c^{8}-1237734142501349223302162522112a^{4}c^{8}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{25253673488097375487494881280a^{5}c^{8}+1455699176221267243359055872a^{6}c^{8}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-12031819362886782611902464a^{7}c^{8}-901850710109690716936704a^{8}c^{8}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-3798874296621284474880a^{9}c^{8}+194341463383802701824a^{10}c^{8}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{2337313594001498112a^{11}c^{8}-2395393722129408a^{12}c^{8}-161395768934400a^{13}c^{8}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-718165493760a^{14}c^{8}+26357760a^{15}c^{8}+3294720a^{16}c^{8}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

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$$+\frac{1592821042644515671417734442254336c^9-4603350744464058365140580895293344ac^9}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{367087315762389086729761686814 72a^2c^9-926919215925502603880510259 2a^3c^9}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-7793101113637005056838205440 0a^4c^9+355242949210634290070487040a^5c^9}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{73346374205830358870228992a^6c^9+2519284025922764947 29216a^7c^9}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-2900203725975250579456 0a^8c^9-3239841591535767961 60a^9c^9}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{2604142834119450624a^{10}c^9+568436105733816 32a^{11}c^9+183910257090560a^{12}c^9}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-1148723896320a^{13}c^9-6788587520a^{14}c^9-5857280a^{15}c^9}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1339446292190427418289870586511 36c^{10}-317929622833417804339218816696 32ac^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{188497743267139030149426262835 2a^2c^{10}+2745383098584091296179001753 6a^3c^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-357789901996362625069331251 2a^4c^{10}-2583106018090455876937318 4a^5c^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{250076295565577076563968 0a^6c^{10}+2977082989680448036044 8a^7c^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-554861886520804335616a^8c^{10}-10477914723705741312a^9c^{10}-650438385640243 2a^{10}c^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{731660679766016a^{11}c^{10}+3798320734208a^{12}c^{10}-139403264a^{13}c^{10}-19914752a^{14}c^{10}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{925908461806919975209201670553 6c^{11}-178843397553639610511239885619 2ac^{11}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{748794779667581360168836792 32a^2c^{11}+227009736666830522948203315 2a^3c^{11}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-119779384831109120949092352a^4c^{11}-2108720637666398843895808a^5c^{11}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{56211218206919898071040a^6c^{11}+1154277387520437207040a^7c^{11}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-427129554592953139 2a^8c^{11}-191483317319024640a^9c^{11}-815139396943872a^{10}c^{11}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{4260696014848a^{11}c^{11}+29350723584a^{12}c^{11}+25346048a^{13}c^{11}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{52917007311099119649752193433 6c^{12}-821998176722737460202574970 88ac^{12}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{222631099803892114280048230 4a^{2}c^{12}+112335291284151811064856576a^{3}c^{12}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-28168593142698488931942 40a^{4}c^{12}-83269328357107000442880a^{5}c^{12}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{7056307343458558115 84a^{6}c^{12}+26456679475060703232a^{7}c^{12}+650942342126469 12a^{8}c^{12}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-1995437837352960a^{9}c^{12}-12430943895552a^{10}c^{12}+456228864a^{11}c^{12}+76038144a^{12}c^{12}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{250845676779265338354129960 96c^{13}-308720122620089528224212582 4ac^{13}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{450909156577630027419484 16a^{2}c^{13}+393918956418989949452288 0a^{3}c^{13}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-405752275824158716723 20a^{4}c^{13}-213970657318478807040 0a^{5}c^{13}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-19545904629488025 6a^{6}c^{13}+382648988297920512a^{7}c^{13}+2188013035192320a^{8}c^{13}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-9832316928000a^{9}c^{13}-81208737792a^{10}c^{13}-70189056a^{11}c^{13}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{987125304445771508818640896c^{14}-944500608021729513696133 12ac^{14}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{392250491442329132466176a^{2}c^{14}+102263791723363946987520a^{3}c^{14}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-10888094263833526 2720a^{4}c^{14}-376481293112898355 20a^{5}c^{14}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-187827301329862656a^{6}c^{14}+3333038389985280a^{7}c^{14}+25954876784640a^{8}c^{14}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-952565760a^{9}c^{14}-190513152a^{10}c^{14}+3219759287688953551493529 6c^{15}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-2337876825862944317243392ac^{15}-105712761772330849402 88a^{2}c^{15}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{19786623841714092113 92a^{3}c^{15}+101870176064341278 72a^{4}c^{15}-447086626545205248a^{5}c^{15}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

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$$+\frac{-3611715821371392a^6c^{15}+14233364594688a^7c^{15}+146822135808a^8c^{15}+127008768a^9c^{15}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{866877134589278443012096c^{16}-462821133281597094625 28ac^{16}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-53913501854520600166 4a^2c^{16}+28101817396392886272a^3c^{16}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{266925771696439296a^4c^{16}-3333037754941440a^5c^{16}-34606714060800a^6c^{16}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1270087680a^7c^{16}+317521920a^8c^{16}+191265371238247547535 36c^{17}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-7195827723314019696 64ac^{17}-1256649927351808819 2a^2c^{17}+281603400092614656a^3c^{17}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{3578843352268800a^4c^{17}-12558776401920a^5c^{17}-172582502400a^6c^{17}-149422080a^7c^{17}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{3419977550912668303 36c^{18}-853859503669719859 2ac^{18}-18997409357706035 2a^2c^{18}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{1829902732492800a^3c^{18}+28499821199360a^4c^{18}-1045954560a^5c^{18}-348651520a^6c^{18}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{487476043504523673 6c^{19}-7373693574656819 2ac^{19}-1950607118172160a^2c^{19}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{6169186795520a^3c^{19}+127055953920a^4c^{19}+110100480a^5c^{19}+5405068947462553 6c^{20}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{-423766867836928ac^{20}-13199997927424a^2c^{20}+484442112a^3c^{20}+242221056a^4c^{20}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{449048494473216c^{21}-1292583829504ac^{21}-53196357632a^2c^{21}-46137344a^3c^{21}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\frac{2628587094016c^{22}-96468992ac^{22}-96468992a^2c^{22}+9663676416c^{23}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}+$$

$$+\left.\frac{8388608ac^{23}+16777216c^{24}}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a+49}{2})}\right] \quad (8)$$

Derivation of result (8):

Substituting $b=-a-48, z=\frac{1}{2}$ in given result (2),we get

$$(2a+48)\ {}_2F_1\left[\begin{array}{ccc} a & , & -a-48 & ; & \frac{1}{2} \\ & c & & ; & 2 \end{array}\right]$$

$$= a\ {}_2F_1\left[\begin{matrix} a+1\ , & -a-48 & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] + (a+48)\ {}_2F_1\left[\begin{matrix} a\ , & -a-47 & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right]$$

Now applying same parallel method which is used in Ref[6], we can prove the main formula.

Notes

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Orbit - Orbit Resonance of Pluto and Neptune

By M. A. Sharaf & L. A. Alaqal
King Abdul Aziz University, Jeddah, Saudi Arabia

Abstract - In the present paper, an algorithm for the planar restricted circular threebody problem in rotating symbolic system is developed to determine orbit-orbit resonance of Pluto and Neptune.

Keywords : Celestial Mechanics-Planetary close encounters-Pluto- Neptune- solar system dynamics

GJSFR-F Classification : MSC 2010: 70M20

ORBIT -ORBIT RESONANCE OF PLUTO AND NEPTUNE

Strictly as per the compliance and regulations of :

Orbit - Orbit Resonance of Pluto and Neptune

M. A. Sharaf [α] & L. A. Alaqal [σ]

Abstract - In the present paper, an algorithm for the planar restricted circular threebody problem in rotating symbolic system is developed to determine orbit-orbit resonance of Pluto and Neptune.

Keywords : Celestial Mechanics-Planetary close encounters-Pluto- Neptune- solar system dynamics

I. Introduction

Because of Pluto is locked into a 3:2 resonance with Neptune, Pluto completes 2 orbits every 3 orbits of the Sun completed by Neptune. Although, the ratio is not exactly 3:2, sometimes Pluto's period is slightly faster than its average value or slower. It is the possibility to indefinitely close approach if the perihelion and nod of Pluto are unrestricted, where, the radius of perihelion of Pluto is less than the radius of Neptune's orbit. This is the most important case of orbit - orbit resonance in the solar system. It is well known that the orbit of Pluto has a large eccentricity of $e = 0.247$, which brings the planet at a certain moment inside the orbit of Neptune. The two planets are trapped in an orbit-orbit resonance. The period of Pluto is 3:2 times the period of Neptune.

II. Equations of Motion

We consider the planar restricted circular three-body problem in rotating synodic system (e.g .Szebehely 1967) in which the two primaries are the Sun and Neptune while the third body is Pluto. The equations of motion to be solved are

$$\dot{x} = u, \tag{1.1}$$

$$\dot{y} = v, \tag{1.2}$$

$$\dot{u} = -(1-\mu)\frac{x-\mu}{R_1^3} - \mu\frac{x+1-\mu}{R_2^3} + x + 2v, \tag{1.3}$$

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$$\dot{v} = -(1-\mu)\frac{y}{R_1^3} - \mu\frac{y}{R_2^3} + y + 2u, \quad (1.4)$$

with

$$R_1 = \sqrt{(x-\mu)^2 + y^2} \quad (2.1)$$

and

$$R_2 = \sqrt{(x+1-\mu)^2 + y^2}\quad, \quad (2.2)$$

where dot denotes differentiation with respect to the time t, (x, y) are the coordinates of the third body , μ denotes the mass of the smaller primary when the total mass of the primaries has been normalized to unity.

In these equations, the unit of length is the distance between the primaries, the unit of mass is the sum of the masses of the primaries. The unit of time is $1/n$. (n is the mean motion).Normally, n is expressed in a number of radians per second, hence in $1/\sec$. Its inverse $1/n$ is therefore expressed in seconds and may be interpreted as a unit of time.

III. Orbit Determination of Pluto and Neptune

All the numerical values of the following are taken from the reference (Hellings,1994)

a) *Orbital elements*

Neptune:

$$e = 0$$
$$P = 165.62$$
$$a = 30.1584$$

Pluto:

$$e = 0.247$$
$$P = 248.43$$
$$a = 39.5187$$

and $\mu : 0.0000525$

Amplitude of the libration is known to be 38^0.

b) *Initial conditions*

The Initial conditions are

$x_0 = -0.6073955952$; $y_0 = -0.7774968265$; $u_0 = 0.1083342234$; $v_0 = -0.0863997159$.

Notes

c) The results

It should be noted that ,all the computations are performed using *Mathematica* 7. For clear illustrations of our analysis, the results are displayed graphically in the following manner

Notes

Fig.1 : Basic part of the orbit of Pluto relative to the Sun and Neptune

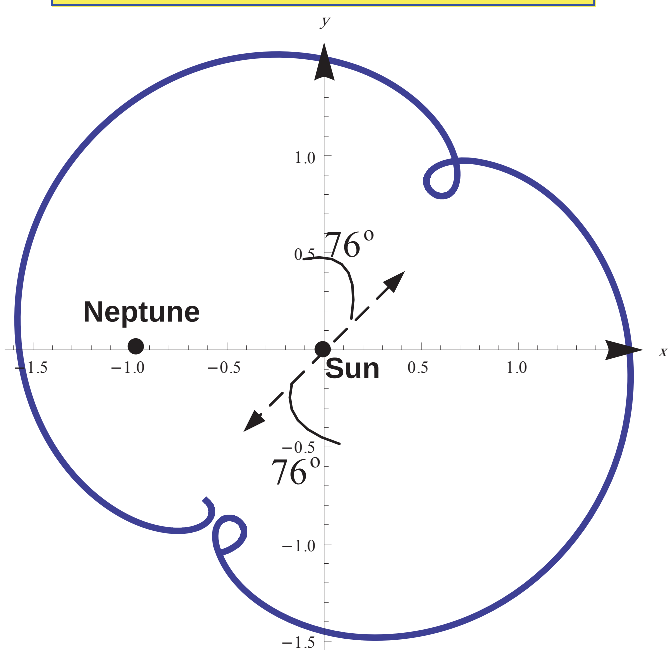


Fig.2 : Plot between u and v for the basic part of the orbit of Pluto relative to the Sun and Neptune

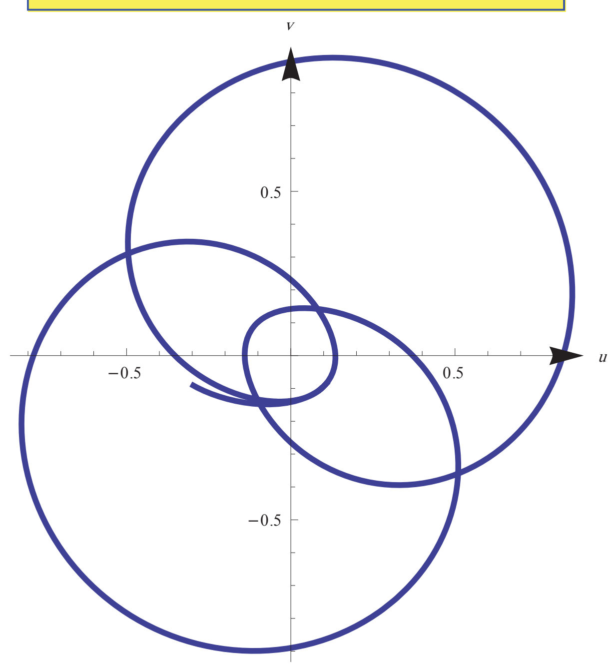


Fig.3 : Plot between x and u
for the basic part of the orbit of Pluto relative
to the Sun and Neptune

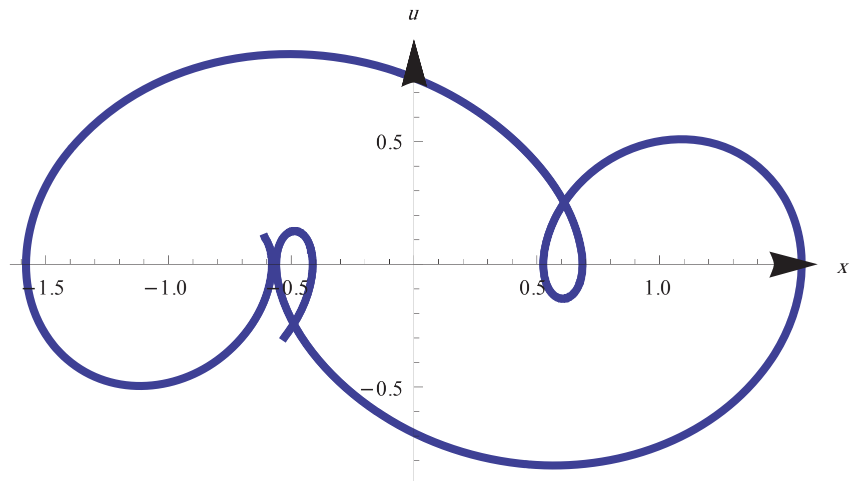
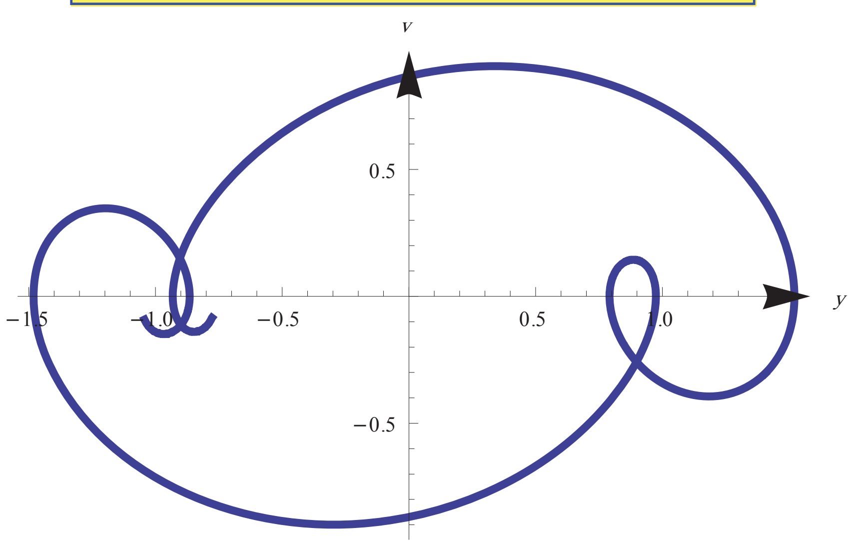


Fig.4 : Plot between y and v
for the basic part of the orbit of Pluto relative
to the Sun and Neptune



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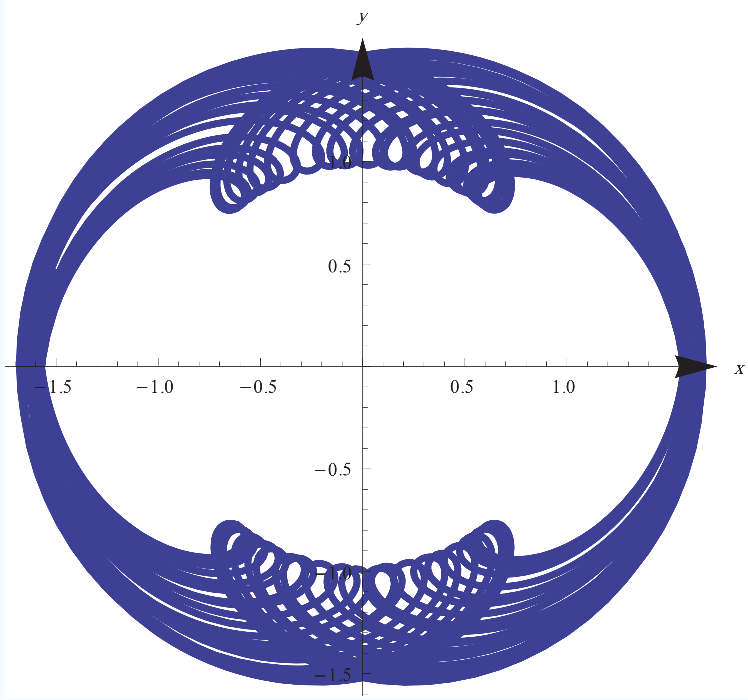


Fig.5 : The basic orbit of Fig.1 oscillates with an amplitude of 38^o

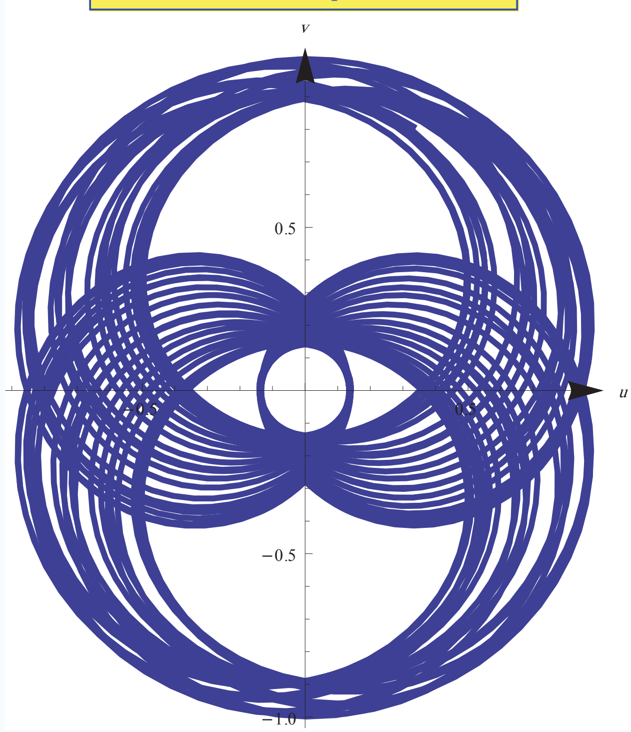


Fig.6 : The basic orbit of Fig.2 oscillates with an amplitude of 38^o

Fig.7 : The basic orbit of Fig.3 oscillates with an amplitude of 38^o

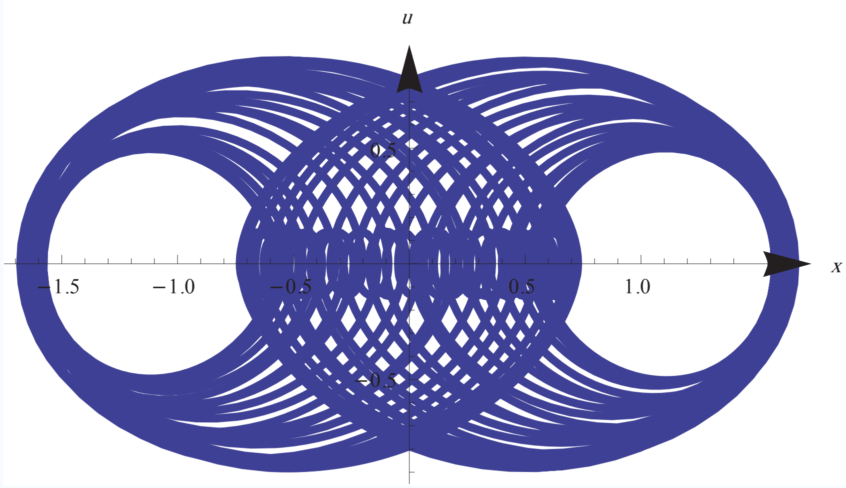
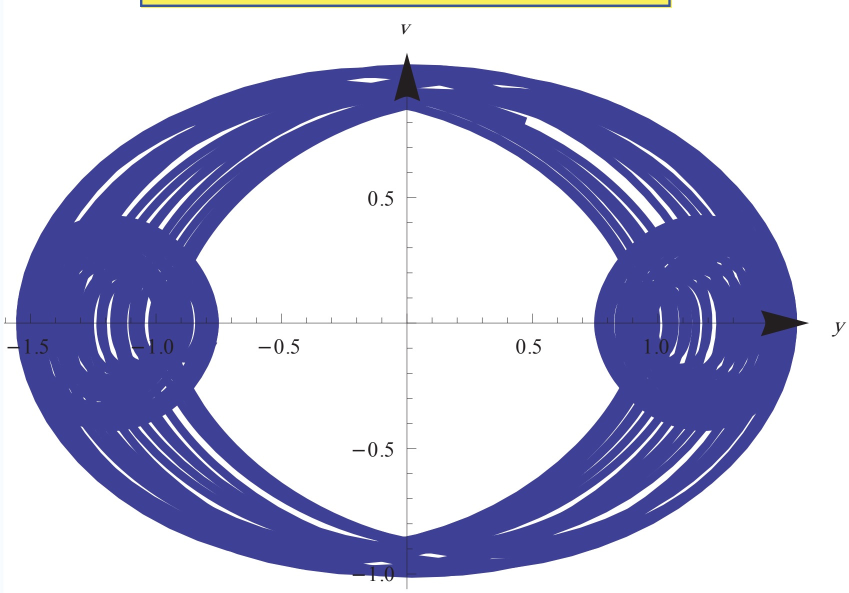


Fig.8 : The basic orbit of Fig.4 oscillates with an amplitude of 38^o



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IV. Conclusion

In this paper, general computational algorithm for the planar restricted circular three-body problem in rotating synodic system is developed in Section 2. This algorithm is applied to determinate orbit-orbit resonance of Pluto and Neptune. Finally the results are illustrate graphically in Section 3 which could be summarized as :

1-Figure 1 shows the basic part of the orbit of Pluto relative to the Sun and Neptune. This part represents two revolutions of Pluto around the Sun.

2- Pluto reaches two times a distance closer to the sun than Neptune.

3-The next two revolutions have the same shape, but the figure is rotated a little bit counter clockwise as the two perihelia approach the y- axis. This phenomenon increases until the whole figure is rotated over 76^0 .

4- A total libration, shown in Fig. 5, illustrates that Pluto will never collide with Neptune since its distance to Neptune is always larger than about 17 A.U.

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New Representations in Terms of q-product Identities for Ramanujan's Results IV

By M.P. Chaudhary, Upendra Kumar Pandit & Ashish Arora
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Abstract - In this paper author has established seven q-product identities, which are presumably new, and not available in the literature.

Keywords : *Theta functions, functions, triple product identities.*

GJSFR-F Classification : *AMS Subject Classifications: Primary 05A17, 05A15; Secondary 11P83*

NEW REPRESENTATIONS IN TERMS OF Q-PRODUCT IDENTITIES FOR RAMANUJANS RESULTS IV

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New Representations in Terms of q-product Identities for Ramanujan's Results IV

M.P. Chaudhary [α], Upendra Kumar Pandit [σ] & Ashish Arora [ρ]

Abstract - In this paper author has established seven q-product identities, which are presumably new, and not available in the literature.

Keywords : Theta functions, functions, triple product identities.

I. Introduction

For $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) \tag{1.1}$$

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{(n-1)}) \tag{1.2}$$

$$(a_1,a_2,a_3,...,a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty...(a_k;q)_\infty \tag{1.3}$$

Ramanujan has defined general theta function, as

$$f(a,b) = \sum_{-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \quad ; \quad |ab| < 1, \tag{1.4}$$

Jacobi's triple product identity [9,p.35] is given, as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{1.5}$$

Special cases of Jacobi's triple products identity are given, as

$$\Phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty \tag{1.6}$$

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$$\Psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.7}$$

$$f(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.8}$$

Equation (1.8) is known as Euler's pentagonal number theorem. Euler's another well known identity is as

$$(q;q^2)_\infty^{-1} = (-q;q)_\infty \tag{1.9}$$

Roger-Ramanujan identities [6, p.578] are given as

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty} \tag{1.10}$$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty} \tag{1.11}$$

Roger-Ramanujan function is given by

$$R(q) = q^{\frac{1}{5}}\frac{H(q)}{G(q)} = q^{\frac{1}{5}}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} \tag{1.12}$$

Throughout this paper we use the following representations

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty = (q^a,q^b,q^c\cdots q^t;q^n)_\infty \tag{1.13}$$

$$(q^a;q^n)_\infty(q^a;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty = (q^a,q^a,q^c\cdots q^t;q^n)_\infty \tag{1.14}$$

Now we can have following q-products identities, as

$$(q^2;q^2)_\infty = \prod_{n=0}^{\infty}(1-q^{2n+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{2(4n)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+1)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+2)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+3)+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{8n+2}) \times \prod_{n=0}^{\infty}(1-q^{8n+4}) \times \prod_{n=0}^{\infty}(1-q^{8n+6}) \times \prod_{n=0}^{\infty}(1-q^{8n+8})$$

$$= (q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty = (q^2,q^4,q^6,q^8;q^8)_\infty \tag{1.15}$$

Ref.

6. G.E. Andrews, R. Askey and R. Roy; *Special Functions*, Cambridge University Press, Cambridge, 1999.

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$$(q^4;q^4)_\infty = \prod_{n=0}^{\infty}(1-q^{4n+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{4(3n)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+1)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+2)+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{12n+4}) \times \prod_{n=0}^{\infty}(1-q^{12n+8}) \times \prod_{n=0}^{\infty}(1-q^{12n+12})$$

$$= (q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty = (q^4,q^8,q^{12};q^{12})_\infty \tag{1.16}$$

$$(q^4;q^{12})_\infty = \prod_{n=0}^{\infty}(1-q^{12n+4}) = \prod_{n=0}^{\infty}(1-q^{12(5n)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+1)+4}) \times$$

$$\times \prod_{n=0}^{\infty}(1-q^{12(5n+2)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+3)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+4)+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{60n+4}) \times \prod_{n=0}^{\infty}(1-q^{60n+16}) \times \prod_{n=0}^{\infty}(1-q^{60n+28}) \times \prod_{n=0}^{\infty}(1-q^{60n+40}) \times \prod_{n=0}^{\infty}(1-q^{60n+52})$$

$$= (q^4;q^{60})_\infty(q^{16};q^{60})_\infty(q^{28};q^{60})_\infty(q^{40};q^{60})_\infty(q^{52};q^{60})_\infty = (q^4,q^{16},q^{28},q^{40},q^{52};q^{60})_\infty \tag{1.17}$$

Similarly we can compute following as

$$(q^4;q^{12})_\infty = (q^4;q^{60})_\infty(q^{16};q^{60})_\infty(q^{28};q^{60})_\infty(q^{40};q^{60})_\infty(q^{52};q^{60})_\infty$$
$$= (q^4,q^{16},q^{28},q^{40},q^{52};q^{60})_\infty \tag{1.18}$$

$$(q^6;q^6)_\infty = (q^6;q^{24})_\infty(q^{12};q^{24})_\infty(q^{18};q^{24})_\infty(q^{24};q^{24})_\infty = (q^6,q^{12},q^{18},q^{24};q^{24})_\infty \tag{1.19}$$

$$(q^6;q^{12})_\infty = (q^6;q^{60})_\infty(q^{18};q^{60})_\infty(q^{30};q^{60})_\infty(q^{42};q^{60})_\infty(q^{54};q^{60})_\infty$$
$$= (q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty \tag{1.20}$$

$$(q^8;q^8)_\infty = (q^8;q^{48})_\infty(q^{16};q^{48})_\infty(q^{24};q^{48})_\infty(q^{32};q^{48})_\infty(q^{40};q^{48})_\infty(q^{48};q^{48})_\infty$$
$$= (q^8,q^{16},q^{24},q^{32},q^{40},q^{48};q^{48})_\infty \tag{1.21}$$

$$(q^8;q^{12})_\infty = (q^8;q^{60})_\infty(q^{20};q^{60})_\infty(q^{32};q^{60})_\infty(q^{44};q^{60})_\infty(q^{56};q^{60})_\infty$$
$$= (q^8,q^{20},q^{32},q^{44},q^{56};q^{60})_\infty \tag{1.22}$$

$$(q^8;q^{16})_\infty = (q^8;q^{48})_\infty(q^{24};q^{48})_\infty(q^{40};q^{48})_\infty = (q^8,q^{24},q^{40};q^{48})_\infty \tag{1.23}$$

$$(q^{10};q^{20})_\infty = (q^{10};q^{60})_\infty(q^{30};q^{60})_\infty(q^{50};q^{60})_\infty = (q^{10},q^{30},q^{50};q^{60})_\infty \tag{1.24}$$

$$(q^{12};q^{12})_\infty = (q^{12};q^{60})_\infty(q^{24};q^{60})_\infty(q^{36};q^{60})_\infty(q^{48};q^{60})_\infty(q^{60};q^{60})_\infty$$
$$= (q^{12},q^{24},q^{36},q^{48},q^{60};q^{60})_\infty \tag{1.25}$$

$$(q^{16};q^{16})_\infty = (q^{16};q^{48})_\infty(q^{32};q^{48})_\infty(q^{48};q^{48})_\infty = (q^{16},q^{32},q^{48};q^{48})_\infty \tag{1.26}$$

$$(q^{20};q^{20})_\infty = (q^{20};q^{60})_\infty(q^{40};q^{60})_\infty(q^{60};q^{60})_\infty = (q^{20},q^{40},q^{60};q^{60})_\infty \tag{1.27}$$

The outline of this paper is as follows. In sections 2, we have recorded some well known results, those are useful to the rest of the paper. In section 3, we state and prove seven new q-product identities, which are not available in the literature of special functions.

II. Preliminaries

Let us recall the definition of cubic theta functions $A(q)$,$B(q)$ and $C(q)$ due to Borwein et al.[4], as

$$A(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \tag{2.1}$$

$$B(q) = \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}; \quad \omega = exp\left(\frac{2\pi\imath}{3}\right) \tag{2.2}$$

$$C(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} \tag{2.3}$$

Borwein et al.[4] established the following relations

$$A(q) = A(q^3) + 2qC(q^3) \tag{2.4}$$

$$B(q) = A(q^3) - qC(q^3) \tag{2.5}$$

$$C(q) = \frac{3(q^3;q^3)^3_\infty}{(q;q)_\infty} \tag{2.6}$$

$$A(q)A(q^2) = B(q)B(q^2) + qC(q)C(q^2) \tag{2.7}$$

Entry-2, in Ramanujan's first note book [8, p.230], [10, p.356] is stated as

$$\Psi(q)\Psi(q^3) - \Psi(-q)\Psi(-q^3) = 2q\Phi(q^2)\Psi(q^{12}) \tag{2.8}$$

Entry-4(iv), in the chapter 20 of Ramanujan's second note book [8], [9, p.359] is stated as

$$\Phi(q)\Phi(q^{27}) - \Phi(-q)\Phi(-q^{27}) = 4qf(-q^6)f(-q^{18}) + 4q^7\Psi(q^2)\Psi(q^{54}) \tag{2.9}$$

Entry-9(i), in the chapter 20 of Ramanujan's second note book [8], [9, p.277] is stated as

$$\Psi(q^3)\Psi(q^5) - \Psi(-q^3)\Psi(-q^5) = 2q^3\Psi(q^2)\Psi(q^{30}) \tag{2.10}$$

Ref.

4. J.M. Borwein, P.B. Borwein and F.G.Garvan;*Some cubic modular identities of Ramanujan*, trans.Amer.Math.Soc.,343(1994),35-47.

Entry-9(iii), in the chapter 20 of Ramanujan's second note book [8], [9, p.377] is stated as

$$\Phi(q^3)\Phi(q^5) = \Phi(-q^2)\Phi(-Q^2) + 2q^2\Psi(q)\Psi(Q); \quad where\ Q = q^{15} \tag{2.11}$$

Entry-9(iv), in the chapter 20 of Ramanujan's second note book [8], [9, p.377] is stated as

$$\Psi(q)\Psi(q^{15}) + \Psi(-q)\Psi(-q^{15}) = 2\Psi(q^6)\Psi(q^{10}) \tag{2.12}$$

Entry-25, in Ramanujan's note book [9, p.39] is stated as

$$\Phi(q) + \Phi(-q) = 2\Phi(q^4) \tag{2.13}$$

$$\Phi(q) - \Phi(-q) = 4q\Psi(q^8) \tag{2.14}$$

$$\Phi(q)\Phi(-q) = \Phi(-q^2) \tag{2.15}$$

III. Main Results

We have establish following

$$(q^2, q^4, q^6; q^8)_\infty[(-q; q^2)^2_\infty + (q; q^2)^2_\infty] = 2(-q^4; q^8)^2_\infty \tag{3.1}$$

$$(q^2, q^4, q^6, q^8; q^8)_\infty[(-q; q^2)^2_\infty - (q; q^2)^2_\infty] = 4q\frac{(q^{16}, q^{32}, q^{48}; q^{48})_\infty}{(q^8, q^{24}, q^{40}; q^{48})_\infty} \tag{3.2}$$

$$\frac{(-q; q^2)^2_\infty + (q; q^2)^2_\infty}{(-q; q^2)^2_\infty - (q; q^2)^2_\infty} = \frac{(-q^4; q^8)^2_\infty(q^8, q^8, q^{24}, q^{24}, q^{40}, q^{40}; q^{48})_\infty}{2q} \tag{3.3}$$

$$(-q; q^2)^2_\infty(q; q^2)^2_\infty(q^2; q^2)^2_\infty = (q^2, q^2, q^4; q^4)_\infty \tag{3.4}$$

$$\frac{(-q; q^2)_\infty(-q^3; q^6)_\infty - (q; q^2)_\infty(q^3; q^6)_\infty}{(-q; q^2)_\infty \times (-q^3; q^6)_\infty \times (q; q^2)_\infty \times (q^3; q^6)_\infty} = \frac{2q(-q^2; q^4)^2_\infty(q^4, q^8, q^{16}, q^{20}, q^{24}; q^{24})_\infty}{(q^2, q^4, q^6, q^8; q^8)_\infty(q^6, q^{12}, q^{18}; q^{24})_\infty} \tag{3.5}$$

$$\frac{(-q^3; q^6)_\infty(-q^5; q^{10})_\infty - (q^3; q^6)_\infty(q^5; q^{10})_\infty}{(-q^3; q^6)_\infty \times (-q^5; q^{10})_\infty \times (q^3; q^6)_\infty \times (q^5; q^{10})_\infty} = \frac{(q^4, q^8, q^{12}; q^{12})_\infty}{(q^6, q^{12}, q^{18}, q^{24}; q^{24})_\infty} \times$$

$$\times \frac{2q^3}{(q^2, q^6, q^{10}; q^{12})_\infty(q^{10}, q^{20}, q^{30}, q^{30}, q^{40}, q^{50}; q^{60})_\infty} \tag{3.6}$$

$$\frac{[(q; q^2)_\infty(q^{15}; q^{30})_\infty] + [(-q; q^2)_\infty(-q^{15}; q^{30})_\infty]}{[(q; q^2)_\infty(q^{15}; q^{30})_\infty][(-q; q^2)_\infty(-q^{15}; q^{30})_\infty]} = \frac{(q^{12}, q^{20}, q^{24}, q^{36}, q^{40}, q^{48}, q^{60}, q^{60}; q^{60})_\infty}{(q^{10}, q^{30}, q^{30}, q^{50}, q^{60}; q^{60})_\infty} \times$$

$$\times \frac{2}{(q^2, q^4, q^6, q^8, q^8; q^8)_\infty(q^6, q^{18}, q^{30}, q^{42}, q^{54}; q^{60})_\infty} \tag{3.7}$$

Proof of (3.1): Employing equation (1.6) in equation (2.13), we have

$$(-q; q^2)^2_\infty(q^2; q^2)_\infty + (q; q^2)^2_\infty(q^2; q^2)_\infty = 2(-q^4; q^8)^2_\infty(q^8; q^8)_\infty$$

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$$(q^2;q^2)_\infty[(-q;q^2)_\infty^2+(q;q^2)_\infty^2]=2(-q^4;q^8)_\infty^2(q^8;q^8)_\infty$$
$$(q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty[(-q;q^2)_\infty^2+(q;q^2)_\infty^2]=2(-q^4;q^8)_\infty^2(q^8;q^8)_\infty$$
$$(q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty[(-q;q^2)_\infty^2+(q;q^2)_\infty^2]=2(-q^4;q^8)_\infty^2$$
$$(q^2,q^4,q^6;q^8)_\infty[(-q;q^2)_\infty^2+(q;q^2)_\infty^2]=2(-q^4;q^8)_\infty^2$$

which establish the result (3.1).
Proof of (3.2): Employing equations (1.6) and (1.7) in equation (2.14), we have

$$(-q;q^2)_\infty^2(q^2;q^2)_\infty-(q;q^2)_\infty^2(q^2;q^2)_\infty=\frac{4q(q^{16};q^{16})_\infty}{(q^8;q^{16})_\infty}$$
$$(q^2;q^2)_\infty[(-q;q^2)_\infty^2-(q;q^2)_\infty^2]=4q\frac{(q^{16},q^{32},q^{48};q^{48})_\infty}{(q^8,q^{24},q^{40};q^{48})_\infty}$$
$$(q^2,q^4,q^6,q^8;q^8)_\infty[(-q;q^2)_\infty^2-(q;q^2)_\infty^2]=4q\frac{(q^{16},q^{32},q^{48};q^{48})_\infty}{(q^8,q^{24},q^{40};q^{48})_\infty}$$

which establish the result (3.2).
Proof of (3.3): Dividing equation (3.1) by (3.2), we get equation (3.3).
Proof of (3.4): Employing equation (1.6) in equation (2.15), we have

$$(-q;q^2)_\infty^2(q^2;q^2)_\infty(q;q^2)_\infty^2(q^2;q^2)_\infty=(q^2;q^4)_\infty^2(q^4;q^4)_\infty$$
$$(-q;q^2)_\infty^2(q;q^2)_\infty^2(q^2;q^2)_\infty^2=(q^2;q^4)_\infty^2(q^4;q^4)_\infty$$
$$(-q;q^2)_\infty^2(q;q^2)_\infty^2(q^2;q^2)_\infty^2=(q^2;q^4)_\infty(q^2;q^4)_\infty(q^4;q^4)_\infty$$
$$(-q;q^2)_\infty^2(q;q^2)_\infty^2(q^2;q^2)_\infty^2=(q^2,q^2,q^4;q^4)_\infty$$

which establish the result (3.4).
Proof of (3.5): Employing equations (1.6) and (1.7) in equation (2.8), we get.

$$\frac{(q^2;q^2)_\infty(q^6;q^6)_\infty}{(q;q^2)_\infty(q^3;q^6)_\infty}-\frac{(q^2;q^2)_\infty(q^6;q^6)_\infty}{(-q;q^2)_\infty(-q^3;q^6)_\infty}=\frac{2q(-q^2;q^4)_\infty^2(q^4;q^4)_\infty(q^{24};q^{24})_\infty}{(q^{12};q^{24})_\infty}$$
$$\frac{(-q;q^2)_\infty(-q^3;q^6)_\infty-(q;q^2)_\infty(q^3;q^6)_\infty}{(-q;q^2)_\infty(-q^3;q^6)_\infty(q;q^2)_\infty(q^3;q^6)_\infty}=\frac{2q(-q^2;q^4)_\infty^2(q^4;q^4)_\infty(q^{24};q^{24})_\infty}{(q^{12};q^{24})_\infty(q^2;q^2)_\infty(q^6;q^6)_\infty}$$
$$\frac{(-q;q^2)_\infty(-q^3;q^6)_\infty-(q;q^2)_\infty(q^3;q^6)_\infty}{(-q;q^2)_\infty(-q^3;q^6)_\infty(q;q^2)_\infty(q^3;q^6)_\infty}=\frac{2q(-q^2;q^4)_\infty^2(q^4,q^8,q^{16},q^{20},q^{24};q^{24})_\infty}{(q^2;q^2)_\infty(q^6,q^{12},q^{18};q^{24})_\infty}$$
$$\frac{(-q;q^2)_\infty(-q^3;q^6)_\infty-(q;q^2)_\infty(q^3;q^6)_\infty}{(-q;q^2)_\infty(-q^3;q^6)_\infty(q;q^2)_\infty(q^3;q^6)_\infty}=\frac{2q(-q^2;q^4)_\infty^2(q^4,q^8,q^{16},q^{20},q^{24};q^{24})_\infty}{(q^2,q^4,q^6,q^8;q^8)_\infty(q^6,q^{12},q^{18};q^{24})_\infty}$$

which establish the result (3.5).
Proof of (3.6): Employing equation (1.7) in equation (2.10), we get.

$$\frac{(q^6;q^6)_\infty(q^{10};q^{10})_\infty}{(q^3;q^6)_\infty(q^5;q^{10})_\infty}-\frac{(q^6;q^6)_\infty(q^{10};q^{10})_\infty}{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty}=\frac{2q^3(q^4;q^4)_\infty(q^{60};q^{60})_\infty}{(q^2;q^4)_\infty(q^{30};q^{60})_\infty}$$

$$\frac{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty-(q^3;q^6)_\infty(q^5;q^{10})_\infty}{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty(q^3;q^6)_\infty(q^5;q^{10})_\infty}=\frac{2q^3(q^4;q^4)_\infty(q^{60};q^{60})_\infty}{(q^2;q^4)_\infty(q^6;q^6)_\infty(q^{10};q^{10})_\infty(q^{30};q^{60})_\infty}$$

$$\frac{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty-(q^3;q^6)_\infty(q^5;q^{10})_\infty}{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty(q^3;q^6)_\infty(q^5;q^{10})_\infty}=\frac{(q^4,q^8,q^{12};q^{12})_\infty}{(q^6,q^{12},q^{18},q^{24};q^{24})_\infty}\times$$

$$\times\frac{2q^3}{(q^2,q^6,q^{10};q^{12})_\infty(q^{10},q^{20},q^{30},q^{30},q^{40},q^{50};q^{60})_\infty}$$

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which establish the result (3.6).

Proof of (3.7): Employing equation (1.7) in equation (2.12), we get.

$$\frac{(q^2;q^2)_\infty(q^{30};q^{30})_\infty}{(q;q^2)_\infty(q^{15};q^{30})_\infty}+\frac{(q^2;q^2)_\infty(q^{30};q^{30})_\infty}{(-q;q^2)_\infty(-q^{15};q^{30})_\infty}=\frac{2(q^{12};q^{12})_\infty(q^{20};q^{20})_\infty}{(q^6;q^{12})_\infty(q^{10};q^{20})_\infty}$$

$$\frac{(-q;q^2)_\infty(-q^{15};q^{30})_\infty+(q;q^2)_\infty(q^{15};q^{30})_\infty}{(-q;q^2)_\infty(-q^{15};q^{30})_\infty(q;q^2)_\infty(q^{15};q^{30})_\infty}=\frac{2(q^{12};q^{12})_\infty(q^{20};q^{20})_\infty}{(q^2;q^2)_\infty(q^6;q^{12})_\infty(q^{10};q^{20})_\infty(q^{30};q^{30})_\infty}$$

$$\frac{(-q;q^2)_\infty(-q^{15};q^{30})_\infty+(q;q^2)_\infty(q^{15};q^{30})_\infty}{(-q;q^2)_\infty(-q^{15};q^{30})_\infty(q;q^2)_\infty(q^{15};q^{30})_\infty}=\frac{2(q^{12},q^{20},q^{24},q^{36},q^{40},q^{48},q^{60},q^{60};q^{60})_\infty}{(q^2;q^2)_\infty(q^6;q^{12})_\infty(q^{10};q^{20})_\infty(q^{30};q^{30})_\infty}$$

$$\frac{(-q;q^2)_\infty(-q^{15};q^{30})_\infty+(q;q^2)_\infty(q^{15};q^{30})_\infty}{(-q;q^2)_\infty(-q^{15};q^{30})_\infty(q;q^2)_\infty(q^{15};q^{30})_\infty}=\frac{(q^{12},q^{20},q^{24},q^{36},q^{40},q^{48},q^{60},q^{60};q^{60})_\infty}{(q^{10},q^{30},q^{30},q^{50},q^{60};q^{60})_\infty}\times$$

$$\times\frac{2}{(q^2,q^4,q^6,q^8,q^8;q^8)_\infty(q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty}$$

which establish the result (3.7).

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Notes

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Note Oncertain Field of Fractions

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Abstract - The set of some real rhotrices of the same dimension D^* was defined in [2] to be an integral domain. An example of a finite field $M[R_3]$ was given in [4] based on this definition also and on the construction of finite fields presented in [3]. It was discovered that the finite sub collection of the elements of $M[R_3]$ as contained in D^* is not closed under rhotrix addition and hence not an integral domain. More generally, D^* is not an integral domain as it is not closed under rhotrix addition. This problem affects the field of fractions constructed in [8]. A solution to this problem is provided in this article and the construction method of such fields is reviewed. This reviewed version gives the generalization of such construction as the n-dimensional rhotrices are used.

Keywords : n-dimensional rhotrix; Quotient rhotrix; Integral domain; Field of fraction.

GJSFR-F Classification : MSC 2010: 83A05

NOTE ONCERTAIN FIELD OF FRACTIONS

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Note Oncertain Field of Fractions

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Abstract -The set of some real rhotrices of the same dimension D^* was defined in [2] to be an integral domain. An example of a finite field $M[R_3]$ was given in [4] based on this definition also and on the construction of finite fields presented in [3]. It was discovered that the finite sub collection of the elements of $M[R_3]$ as contained in D^* is not closed under rhotrix addition and hence not an integral domain. More generally, D^* is not an integral domain as it is not closed under rhotrix addition. This problem affects the field of fractions constructed in [8].A solution to this problem is provided in this article and the construction method of such fields is reviewed. This reviewed version gives the generalization of such construction as the n-dimensional rhotrices are used.

Keywords : *n-dimensional rhotrix; Quotient rhotrix; Integral domain; Field of fraction*

I. Introduction

The idea of classifying the set of all rhotrices of dimension 3 as abstract structures was presented in [1] and [2].In [2] the set of some 3-dimensional real rhotrices

$$D^* = \langle (R - ZD), +, \circ \rangle \tag{1.1}$$

was defined to be an integral domain under rhotrix addition and multiplication, where R is the set of all real rhotrices of dimension 3 as defined in [6] by

$$R = \left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a, b, c, d, e \in \Re \right\},$$

$$ZD = \left\{ \left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle : a, b, d, e, 0 \in \Re \text{ and at least one of } a, b, d, e \neq 0 \right\}.$$

Recall that an integral domain is a commutative ring with out zero divisors. However, D^* is not even a ring because the additive closure is not there. This can be seen as follows: Let $R, Q \in D^*$ such that $h(R) = c$ and $h(Q) = -c$. If $R + Q = S$ then $h(S) = 0$ and at least one of $a, b, d, e \neq 0$ which implies that $S \notin D^*$.

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Ref.

1. Mohammed (2007a). Enrichment exercises through extension to rhotrices. Int. J. Math. Educ. Sci. Technol.38(2007b), 131-136.

A method of constructing finite fields through the use of rhotrices was presented in [3]. The cardinalities of such fields were also provided. In [4] an example of a particular finite field $M[R_3]$ whose cardinality is $|M[R_3]| = 3^5 = 243$ was given. This field gave the clear picture of the problem in D^*.

In [1] a hearty rhotrix of dimension 3 was defined to be a rhotrix S having all its entries(except the heart) as zero. Thus, an integer hearty rhotrix of dimension 3 is of the form $\left\langle \begin{array}{ccc} & 0 & \\ 0 & c & 0 \\ & 0 & \end{array} \right\rangle$, $c \neq 0 \in \mathbb{Z}$. Therefore we may define the set of all integer hearty rhotrices of the same dimension as

$$H = \left\{ \left\langle \begin{array}{ccc} & 0 & \\ 0 & c & 0 \\ & 0 & \end{array} \right\rangle : c \neq 0 \in \mathbb{Z} \right\} \quad (1.2)$$

For any two base rhotrices R, S the quotient $\frac{R}{S}$ is defined in [5] to be the quotient rhotrix as follows:

$$\frac{R}{S} = R \circ S^{-1} \text{, provided } h(S) \neq 0 . \quad (1.3)$$

The concept of n-dimensional heart-oriented rhotrix multiplicationwas introduced in [7]. A rhotrix R_n of dimension n have $|R_n|$ entries where $|R_n| = \frac{1}{2}(n^2+1)$ as indicated in [6]. Thus any given rhotrix R_n with entries $a_1, a_2, \ldots, a_{\frac{1}{2}(n^2+1)}$, is generally represented as

$$R_n = \left\langle \begin{array}{ccccccccc} & & & & a_1 & & & & \\ & & & a_2 & a_3 & a_4 & & & \\ & & a_5 & a_6 & a_7 & a_8 & a_9 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{\frac{n^2-2n+5}{4}} & \cdots & \cdots & \cdots & a_{\frac{1}{4}(n^2+3)} & \cdots & \cdots & \cdots & a_{\frac{n^2+2n+1}{4}} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{\frac{1}{2}(n^2+1)-8} & a_{\frac{1}{2}(n^2+1)-7} & a_{\frac{1}{2}(n^2+1)-6} & a_{\frac{1}{2}(n^2+1)-5} & a_{\frac{1}{2}(n^2+1)-4} & & \\ & & & a_{\frac{1}{2}(n^2+1)-3} & a_{\frac{1}{2}(n^2+1)-2} & a_{\frac{1}{2}(n^2+1)-1} & & & \\ & & & & a_{\frac{1}{2}(n^2+1)} & & & & \end{array} \right\rangle$$

The method of constructing field of fractions through the use of base rhotrices based on definitions (1.1) and (1.3) was presented in [8]. To take care of the problem that affects this construction as pointed out earlier; this article aims at reviewing and generalizing the construction method using definitions (1.2) and (1.3).

Ref.

3. S.M. Tudunkaya and S.O. Makanjuola (2010). Rhotrices and the construction of finite fields. Bulletin of Pure and Applied Sciences, Vol. 29 E Issue 2, pp225-229

II. A Particular Field of Fraction of an Integral Domain

Theorem 2.1

Let H_n be the set of all n-dimensional integer hearty rhotrices. If $H_n^* = H_n \cup \{O_n\}$ then H_n^* is an integral domain, where O_n is an n-dimensional zero rhotrix.

Notes

Proof

It suffices to show that H_n^* is isomorphic to an integral domain $\mathbb{Z}$. That is $H_n^* \cong \mathbb{Z}$.

Define a mapping $\tau : \mathbb{Z} \to H_n^*$ by $\tau(c) = C_n$.

For homomorphism, let $c, d \in \mathbb{Z}$, then

(i) $\tau(c+d) = C_n + D_n = \tau(c) + \tau(d)$ (ii) $\tau(cd) = C_n \circ D_n = \tau(c)\tau(d)$

Therefore τ is a homomorphism

Since $\forall \tau(c) \in H_n^*$ there exists $c \in \mathbb{Z}$ such that $\tau(c) = C_n$ then τ is onto.

Now let $\tau(c), \tau(d) \in H_n^*$ such that $\tau(c) = \tau(d)$.

$\tau(c) = \tau(d) \Rightarrow C_n = D_n \Rightarrow c = d$.

Thus τ is one to one.

Hence $H_n^* \cong \mathbb{Z}$.

Definition 2.1

Let H_n^* and H_n be as in Theorem 2.1 above. Then a relation ~ on H_n^* x H_n defined by cross multiplication as $(C_{n1}, D_{n1}) \sim (C_{n2}, D_{n2})$ if $C_{n1} \circ D_{n2} = C_{n2} \circ D_{n1}$, $\ni C_{n1}, C_{n2} \in H_n^*$; $D_{n1}, D_{n2} \in H_n$.

Proposition 2.1

The relation ~ as defined in Definition (2.1) is an equivalence relation.

Proof

Reflexivity and Symmetry of the relation are obvious.

For transitivity, let $C_{n1} \circ D_{n2} = C_{n2} \circ D_{n1}$ and $C_{n2} \circ D_{n3} = C_{n3} \circ D_{n2}$. To show that $C_{n1} \circ D_{n3} = C_{n3} \circ D_{n1}$ we have

$$(C_{n1} \circ D_{n3}) \circ D_{n2} = (C_{n1} \circ D_{n2}) \circ D_{n3}$$
$$= (C_{n2} \circ D_{n1}) \circ D_{n3}$$
$$= (C_{n2} \circ D_{n3}) \circ D_{n1}$$
$$= (C_{n3} \circ D_{n2}) \circ D_{n1}$$
$$= (C_{n3} \circ D_{n1}) \circ D_{n2}$$
$$\therefore (C_{n1} \circ D_{n3}) = (C_{n3} \circ D_{n1}) \text{ by cancellation law.}$$

We denote by $\frac{C_n}{D_n}$ the equivalence class of (C_n, D_n) in H_n^*x H_n and define $H_n^*[H_n^{\;1}]$ to be the set of all the equivalenceclasses $\frac{C_n}{D_n}$, where $C_n \in H_n^*$ and $D_n \in H_n$.

For all $\frac{C_{n1}}{D_{n1}}, \frac{C_{n2}}{D_{n2}} \in H_n^*[H_n^{\ 1}]$ we define addition and multiplication on $H_n^*[H_n^{\ 1}]$ as follows: $\frac{C_{n1}}{D_{n1}} +' \frac{C_{n2}}{D_{n2}} = \frac{C_{n1} \circ D_{n2} + C_{n2} \circ D_{n1}}{D_{n1} \circ D_{n2}}$ and $\frac{C_{n1}}{D_{n1}} \circ' \frac{C_{n2}}{D_{n2}} = \frac{C_{n1} \circ C_{n2}}{D_{n1} \circ D_{n2}}$.

Proposition2.2

The operations $(+')$, $(\circ')$ as defined above are well-defined.

Proof

Suppose $\frac{C'_{n1}}{D'_{n1}} = \frac{C_{n1}}{D_{n1}}$ and $\frac{C'_{n2}}{D'_{n2}} = \frac{C_{n2}}{D_{n2}}$; then $C'_{n1} \circ D_{n1} = D'_{n1} \circ C_{n1}$ and $C'_{n2} \circ D_{n2} = D'_{n2} \circ C_{n2}$, so that $(C'_{n1} \circ D'_{n2} + C'_{n2} \circ D'_{n1}) D_{n1} D_{n2} = C'_{n1} \circ D_{n1} \circ D'_{n2} \circ D_{n2} + C'_{n2} \circ D_{n2} \circ D'_{n1} \circ D_{n1}$

$= C_{n1} \circ D'_{n1} \circ D'_{n2} \circ D_{n2} + C_{n2} \circ D'_{n2} \circ D'_{n1} \circ D_{n1}$

$= (C_{n1} \circ D_{n2} + C_{n2} \circ D_{n1}) D'_{n1} \circ D'_{n2}$

implying that $\frac{C'_{n1}}{D'_{n1}} +' \frac{C'_{n2}}{D'_{n2}} = \frac{C_{n1}}{D_{n1}} +' \frac{C_{n2}}{D_{n2}}$.

Similarly $(C'_{n1} \circ C'_{n2}) D_{n1} \circ D_{n2} = (C_{n1} \circ C_{n2}) D'_{n1} \circ D'_{n2}$ implies that $\frac{C'_{n1}}{D'_{n1}} \circ' \frac{C'_{n2}}{D'_{n2}} = \frac{C_{n1} \circ C_{n2}}{D_{n1} \circ D_{n2}}$.

By definition (1.3) the equivalence class $\frac{C_n}{D_n} = C_n \circ D_n^{-1}$ since $D_n \neq 0_n \in H_n$. Therefore, for all $D_n \in H_n$, $0_n \in H_n^*$, $\frac{0_n}{D_n} = 0_n \circ D_n^{-1} = 0_n = 0_n \circ I_n = \frac{0_n}{I_n}$. Thus $\frac{0_n}{I_n} = \frac{0_n}{D_n}$ is the additive identity and $-\frac{C_n}{D_n} = \frac{-C_n}{D_n}$ is the additive inverse. Similarly, $\frac{I_n}{I_n} = \frac{D_n}{D_n}$ is the multiplicative identity.

Theorem 2.2

With the above definitions and the definitions of the operations $(+')$ and $(\circ')$, the set of the equivalence classes $H_n^*[H_n^{\ 1}]$ is a commutative ring.

Proof

One should check that the properties of a ring are fulfilled. But the proof follows from the fact that addition and multiplication are the regular addition and multiplication of fractions.

Proposition 2.3

The function $\psi : H_n^* \to H_n^*[H_n^{\ 1}]$ defined by $\psi(C_n) = \frac{C_n}{I_n}$ is a ring homomorphism whose kernel is $\{C_n \in H_n^* : C_n \circ' D_n = 0 \text{ for some } D_n \in H_n^*[H_n^{\ 1}]\}$.

Notes

Proof

Let $C_{n1}, C_{n2} \in H_n^*$, then

$$\psi(C_{n1}+C_{n2}) = \frac{C_{n1}+C_{n2}}{I_n} = (C_{n1}+C_{n2})\circ I_n = C_{n1}\circ I_n + C_{n2}\circ I_n = \frac{C_{n1}}{I_n}+\frac{C_{n2}}{I_n} = \psi(C_{n1})+\psi(C_{n2})$$

$$\psi(C_{n1}\circ' C_{n2}) = \frac{C_{n1}\circ' C_{n2}}{I_n} = (C_{n1}\circ' C_{n2})\circ I_n = C_{n1}\circ I_n \circ' C_{n2}\circ I_n = \frac{C_{n1}}{I_n}\circ'\frac{C_{n2}}{I_n} = \psi(C_{n1})\circ'\psi(C_{n2})$$

$$\psi(I_n) = \frac{I_n}{I_n}.$$

Now $C_n \in \ker\psi$ if and only if $\frac{C_n}{I_n} = \frac{0_n}{I_n}$, if and only if $C_n \circ I_n = 0_n \circ I_n = 0_n$, which imply that $\ker\psi = \{0_n\}$.

Recall from [1] that, the set $M = \{nI : n \in \mathbb{Z}\}$ where I is the unity element of the commutative ring of 3-dimensionalrhotrices R is a subring and submonoid of R under multiplication $(\circ)$. Thus the set $M_n = \{nI_n : n \in \mathbb{Z}\}$ is a subring and submonoid of the commutative ring R_n^* of n-dimensional rhotrices. Therefore any submonoid, H_n of R_n^* with property that for all $Q_n \neq 0 \in R_n^*$ and $S_n \in H_n$, $Q_n \circ S_n \neq 0$ can serve in the above construction for the generalization of proposition 2.3 as stated in the following proposition.

Proposition 2.4

$R_n^*[H_n{}^1]$ as constructed above is a ring, and there is a homomorphism $\psi : R_n^* \to R_n^*[H_n{}^1]$ given by $\psi(Q_n) = \frac{Q_n}{I_n}$.

Proof

The proof follows from propositions 2.2 and 2.3.

As defined in [1], a diagonal rhotrix of dimension 3 is a rhotrix whose two non-diagonal entries are all zero. Let D be the set of all n-dimensional diagonal rhotrices then it is easy for someone to verify that $W = D \cup \{O_n, I_n\}$, where O_n is the n-dimensional additive identity; I_n is the n-dimensional multiplicative identity is a group and is normal in R_n^* under multiplication.

Proposition 2.5

Let R_n^* be a commutative ring of n-dimensional rhotrices, and let H_n be a submonoid of R_n^* such that $Q_n \circ S_n \neq 0_n$ for every $Q_n \neq 0_n \in R_n^*$ and $S_n \in H_n$. Then every ideal of $R_n^*[H_n{}^1]$ has the form $W[H_n{}^1]$, for suitable W normal in R_n^*.

Proof

Since all the elements of $W[H_n{}^1]$ are also elements of $R_n^*[H_n{}^1]$ and $I_n \in H_n$ then obviously $W[H_n{}^1]$ is an additive subgroup of $R_n^*[H_n{}^1]$.

Ref.

1. Mohammed (2007a). Enrichment exercises through extension to rhotrices. Int. J. Math. Educ. Sci. Technol.38(2007b), 131-136.

For all $\frac{C_n}{D_n} \in R_n^*[H_n{}^1]$; $\frac{W_n}{D_n'} \in W[H_n{}^1]$; $\frac{C_n}{D_n} \circ' \frac{W_n}{D_n'} = \frac{C_n \circ W_n}{D_n \circ D_n'} \in W[H_n{}^1]$ since $C_n \circ W_n \in W, D_n \circ D_n' \in H_n$.

W is normal in R_n^* implies that $\frac{W_n}{D_n'} \circ \frac{C_n}{D_n} \in W[H_n{}^1]$.

Proposition 2.6

$H_n^*[H_n{}^1]$ is an integral domain.

Proof

Suppose $\frac{C_{n1}}{D_{n1}} \circ' \frac{C_{n2}}{D_{n2}} = 0_n \in H_n^*[H_n{}^1]$, that is $\frac{C_{n1} \circ C_{n2}}{D_{n1} \circ D_{n2}} = \frac{0_n}{I_n}$

$\Rightarrow (C_{n1} \circ C_{n2}, D_{n1} \circ D_{n2}) \tilde{} (0_n, I_n)$ and $C_{n1} \circ C_{n2} \circ D_n = 0$ for some $D_n \in H_n$.

$C_{n1} \circ C_{n2} \circ D_n = 0_n \in H_n^*$, which is an integral domain, and $D_n \neq 0_n$, thus $C_{n1} \circ C_{n2} = 0_n$.

So either C_{n1} or C_{n2} is 0_n and consequently either $\frac{C_{n1}}{D_{n1}}$ or $\frac{C_{n2}}{D_{n2}}$ is 0_n.

Theorem 2.3

The set $H_n^*[H_n{}^1]$ of all equivalence classes $\frac{C_n}{D_n}$ is a field.

Proof

From Theorem 2.2, $H_n^*[H_n{}^1]$ is a commutative ring with unity $\frac{I_n}{I_n}$. So we just need to show that every non zero element of $H_n^*[H_n{}^1]$ has multiplicative inverse.

Suppose $\frac{C_n}{D_n} \neq \frac{0_n}{I_n}$, then $C_n \neq 0_n$, so $C_n \in H_n$ which implies that $\frac{C_n}{D_n} \in H_n^*[H_n{}^1]$.

Clearly, $\frac{C_n}{D_n} \circ' \frac{D_n}{C_n} = \frac{C_n \circ D_n}{D_n \circ C_n} = \frac{I_n}{I_n}$. Thus $\frac{D_n}{C_n}$ is the multiplicative inverse of $\frac{C_n}{D_n}$.

III. Conclusion

In this short note, amendment concerning some definitions in [2] and [8] with their generalizations were provided. The steps observed in the construction of field of fractions illustrated in [8] were also amended respectively.

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Bianchi Type-VI_0 Dark Energy Cosmological Models in General Relativity

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Abstract - Bianchi type-VI_0 cosmological models of the universe filled with dark energy with constant and time-dependent equation of state parameters are investigated in general relativity. We obtain exact solutions of Einstein's field equations using the condition that the shear scalar is proportional to the expansion scalar, which represent singular and non-singular cosmological models of the universe. The physical behavior of the models are discussed. We conclude that the universe models do not approach isotropy through the evolution of the universe.

Keywords : *Bianchi type-VI. Dark energy. Cosmological models.*

GJSFR-F Classification : *MSC 2010: 83A05*

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3. Perlmutter, S., et al.: Astrophys. J. **483**, 565 (1997)

Bianchi Type-VI$_0$ Dark Energy Cosmological Models in General Relativity

Priyanka, S. Chandel, M.K. Singh & Shri Ram

Abstract - Bianchi type-VI$_0$ cosmological models of the universe filled with dark energy with constant and time-dependent equation of state parameters are investigated in general relativity. We obtain exact solutions of Einstein's field equations using the condition that the shear scalar is proportional to the expansion scalar, which represent singular and non-singular cosmological models of the universe. The physical behavior of the models are discussed. We conclude that the universe models do not approach isotropy through the evolution of the universe.

Keywords : Bianchi type-VI. Dark energy. Cosmological models.

I. Introduction

Recent observations on expansion history of the universe indicate that the universe is currently experiencing a phase of accelerated expansion. This was first observed from high red shift supernova Ia (Reiss et al. [1-2], Perlmutter et al. [3], Astier et al. [4], Spergel et al.[5] etc.) and confirmed later by cross checks from the cosmic microwave background radiation (Bennett et al. [6], Abazajian et al.[7-9], Hawkins et al. [10] etc.). The current accelerating expansion of the universe attributed to the fact that our universe is dominated by an unknown dark energy DE an exotic energy with negative pressure.

The simplest dark energy candidate is the vacuum energy density which is mathematically equivalent to the cosmological constant Λ. As per Copeland et al. [11] "fine tuning "and the cosmic "coincidence "are the two well known difficulties of the cosmological constant problems. There are several alternative theories for the dynamical DE scenario which have been proposed by scientists to interpret the accelerating universe. Wang and Tegmark [12] have shown that the universe is actually undergoing an acceleration with repulsive gravity of some strange energy-form i.e. DE at work. Dark energy is a mysterious substance with negative pressure and accounts for nearly 70% of total matter-energy of universe, but has no clear explanation. Karami et al. [13] introduced a polytropic gas model of DE as an alternative model to explain the accelerated expansion of the universe. Gupta and Pradhan [14] proposed a new candidate known as cosmological nuclear-energy as a possible candidate for the dark energy.

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Bianchi types I-IX cosmological models are important in the sense that these are homogeneous and anisotropic, from which the process of isotropization of the universe is studied through the passage of time. Moreover, from the theoretical point of view anisotropic universe have a greater generality than FRW isotropic models. The simplicity of the field equations made Bianchi space-times useful in constructing models of spatially homogeneous and anisotropic cosmologies. Considerable works have been done in obtaining various Bianchi type cosmological models and their inhomogeneous generalization. Bianchi type-VI$_0$ space-time is of special interest in anisotropic cosmology. Barrow [15] pointed out that Bianchi type-VI$_0$ models of the universe give a better explanation of some of the cosmological problems like primordial helium abundance and they also isotropize in a special sense. Looking to the importance of Bianchi type-VI$_0$ universes, many authors [16-20] have studied it in different context. Shri Ram[21, 22] has presented Bianchi type-VI$_0$ cosmological models filled with dust and perfect fluid in modified Brans-Dicke theory respectively.

Adhav et al. [23] studied Bianchi type-VI$_0$ cosmological models with anisotropic dark energy. Abdussattar and Prajapati [24] obtained a class of bouncing non-singular FRW models by constraining the deceleration parameter (DP) in the presence of an interacting dark energy represented by a time-varying cosmological constant. They have also discussed the role of deceleration parameter and interacting dark energy in singularity avoidance. Bisabr [25] has shown that an accelerating expansion is possible in a spatially flat universe for large values of the Brans-Dicke parameter consistent with the local gravity experiments. Yadav and Saha [26] studied DE models with variable equation of state (EoS) parameter. Recently, Saha and Yadav [27] presented a general relativistic cosmological model with time-dependent DP in LRS Bianchitype-II space-time which can be described by isotopic and variable EoS parameter. In this paper, We present general relativistic cosmological models with constant and time-dependent DP in Bianchi type-VI$_0$ space-time which can be described by isotropic constant and variable EoS parameters. This paper is organized as follows: We present the metric and field equations in Sect.2. In Sect.3, we obtain the solutions of the field equations representing Bianchi type-VI$_0$ cosmological models with perfect fluid by imposing the condition that the shear scalar is proportional to expansion scalar. We also discuss the physical behaviors of the cosmological models with dark energy. Concluding remarks are given in Sect.4.

II. The Metric and Field Equations

We consider the spatially homogeneous and anisotropic Bianchi type-VI$_0$ space-time in the form

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)e^{-2mx}dy^2 + C^2(t)e^{2mx}dz^2 \quad (1)$$

where A, B and C are functions of the cosmic time t and m is a constant

The Einstein's field equations, in natural limits $(8\pi G = c = 1)$are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -T_{\mu\nu} \quad (2)$$

Ref.

15. Barrow, J.D.: Mon. Not. Astron. Soc. **211**, 221(1984)

Notes

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar curvature and $T_{\mu\nu}$ is the energy-momentum tensor of matter. For a perfect fluid distribution, the tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \tag{3}$$

where ρ is the energy density of the cosmic matter p is the isotropic pressure and u^μ is the four-velocity vector. In comoving coordinate system $u^\mu = (0, 0, 0, 1)$, the Einstein's field equation (2) together with (3), for the metric (1), yield

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + \frac{m^2}{A^2} = -\omega\rho, \tag{4}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{m^2}{A^2} = -\omega\rho, \tag{5}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{m^2}{A^2} = -\omega\rho, \tag{6}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{m^2}{A^2} = \rho, \tag{7}$$

$$(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}) = 0 \tag{8}$$

where ω is the EoS parameter given by

$$p = \omega\rho \tag{9}$$

and a dot denotes ordinary differentiation with respect to t.

The average scalar factor a and volume scalar V are given by

$$a^3 = V = ABC. \tag{10}$$

The generalized mean Hubble parameter H is defined by

$$H = \frac{\dot{a}}{a} = \frac{1}{3}(H_1 + H_2 + H_3) \tag{11}$$

where the directional Hubble parameters H_1, H_2 and H_3 are given by

$$H_1 = \frac{\dot{A}}{A}, H_2 = \frac{\dot{B}}{B}, H_3 = \frac{\dot{C}}{C}. \tag{12}$$

The expansion scalar θ and shear scalar σ are given by

$$\theta = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}, \tag{13}$$

$$\sigma^2 = \frac{1}{2}[\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}] - \frac{1}{6}\theta^2. \tag{14}$$

The deceleration parameter q is defined by

$$q = -1 + \frac{d}{dt}(H). \tag{15}$$

The sign of q indicates whether the model inflates or not. A positive sign of q corresponds to the standard decelerating model whereas the negative sign of q indicates inflation. The recent observations of SN Ia (Reiss et al.[1], Perlmutter et al.[3]) reveal that the present universe is accelerating and the value of DP lies somewhere in the range $-1 < q < 0$.

III. Solution of Field Equations

Equation (8), on integration, gives

$$B = C \tag{16}$$

where the constant of integration is absorbed in B or C. Using (16), equations (4) – (7) reduce to

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{m^2}{A^2} = -\omega\rho, \tag{17}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{m^2}{A^2} = -\omega\rho, \tag{18}$$

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} - \frac{m^2}{A^2} = \rho. \tag{19}$$

These are three equations connecting four unknown functions A, B, ρ and ω. In order to solve the above equations we use the physical condition that expansions scalar is proportional to shear scalar, which in our case leads to

$$A = B^n \tag{20}$$

where n is a constant. Roy and Banerjee [28], Bali and Singh [29] have proposed this condition to find exact solutions of cosmological models.

Here we use the procedure of Saha and Yadav [27] to find exact solutions of (17) – (19) combining (10) and (20), we obtain

$$A = V^{\frac{n}{n+1}}, \qquad B = V^{\frac{1}{n+1}}. \tag{21}$$

Subtraction of (18) from (17) gives

$$\frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} + \frac{\dot{B}^2}{B^2} - \frac{\dot{A}\dot{B}}{AB} + \frac{2m^2}{A^2} = 0. \tag{22}$$

Substituting (21) into (22), we obtain

$$\ddot{V} = \frac{2m^2(n+2)}{n-1} V^{\frac{2-n}{n+2}}. \tag{23}$$

The first integral of (23) is

$$\int \frac{dV}{V^{\frac{4}{n+2}+C}} = \frac{m(n+2)t}{\sqrt{n-1}} \tag{24}$$

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3. Perlmutter, S., et al.: Astrophys. J. **483**, 565 (1997)

where C is an arbitrary constant. Clearly (24) imposes some restriction on the choice of n namely, $n > 1$. It is not possible to solve equation (24) in general. So, in order to solve the problem completely, we have to choose either C or n in such a way that (24) be integrable. Therefore we consider the following cases.

Notes

Case 3.1 When C=0

In this case the solution of (24) is

$$V = \left(\frac{mn}{\sqrt{n-1}}\right)^{\frac{n+2}{n}} (t + k_1) \tag{25}$$

where k_1 is an arbitrary constant. From (21) and (25) we obtain the scale factor as

$$A = \frac{mn}{\sqrt{n-1}}(t + k), \tag{26}$$

$$B = (\frac{mn}{\sqrt{n-1}})^{\frac{1}{n}}(t + k)^{\frac{1}{n}}. \tag{27}$$

With these scale factors, the metric (1) can be written in form

$$ds^2 = -dT^2 + (\frac{mn}{\sqrt{n-1}})^2 dx^2 + (\frac{mn}{\sqrt{n-1}})^{\frac{2}{n}} T^{\frac{2}{n}} (e^{-2mx} dy^2 + e^{2mx} dz^2) \tag{28}$$

where T=t+k.

The expressions for the energy density ρ and the EOS ω for the model (28) are obtained as

$$\rho = \frac{1+n}{n^2 T^2}, \tag{29}$$

$$\omega = \frac{n-2}{n+1}. \tag{30}$$

The other physical and kinematical parameters are given by

$$nH_1 = H_2 = H_3 = \frac{1}{T}, \tag{31}$$

$$\theta = 3H = \frac{n+2}{nT}, \tag{32}$$

$$\sigma = \frac{1}{\sqrt{3}} \frac{n-1}{nT}, \tag{33}$$

$$q = -\frac{2}{n+2}. \tag{34}$$

The deceleration parameter q is always negative. The EoS parameter is positive when $n > 2$ and is negative if $1 < n < 2$. Thus, the metric (28) represents as ever power-law accelerated expansion universe filled with a perfect fluid. If $1 < n < 2$, $\omega < 0$, we obtain DE cosmological model of Bianchi type-VI$_0$.

The spatial volume V is zero and all physical parameters diverge at $T = 0$. Therefore, the model has a point-type singularity at $T = 0$. For $0 < T < \infty$, the spatial volume is an increasing function of time. The physical parameters are monotonically decreasing function of time and ultimately tend to zero for large T. The anisotropy in the model is maintained throughout the passage of time. For the physical reality of the model we will have to choose n, greater than 1,in such a way that $\left|\frac{n-2}{n+2}\right| \leq 1$. It deserves mention that we are unable to find n for which $\omega = \pm 1$

Notes

Case 3.2 When C$\neq 0$

When $C \neq o$ equation (24) is not integrable for general values of n. However, for $n = 2$, it becomes

$$\int \frac{dV}{\sqrt{V + C}} = 4mt \tag{35}$$

which, after integration, yields

$$V = 4m^2t^2 + 2\beta t + \gamma \tag{36}$$

where β and γ are arbitrary constants. The constant C is absorbed in γ. From (21) and (36), we obtain the scale factors as

$$A = (4m^2t^2 + 2\beta t + \gamma)^{\frac{1}{2}}, \tag{37}$$

$$B = (4m^2t^2 + 2\beta t + \gamma)^{\frac{1}{4}}. \tag{38}$$

Therefore, the metric (1) of our solutions can be written in the form

$$ds^2 = -dt^2 + (4m^2t^2 + 2\beta t + \gamma)dx^2 + (4m^2t^2 + 2\beta t + \gamma)^{\frac{1}{2}}(e^{-2mx}dy^2 + e^{2mx}dz^2) \tag{39}$$

The expressions for (H_1, H_2, H_3), H, ρ, θ and σ are obtained as

$$H_1 = \frac{4m^2t + \beta}{4m^2t^2 + 2\beta t + \gamma}, \tag{40}$$

$$H_2 = H_3 = \frac{1}{2}(\frac{4m^2t + \beta}{4m^2t^2 + 2\beta t + \gamma}), \tag{41}$$

$$H = \frac{2}{3}(\frac{4m^2t + \beta}{4m^2t^2 + 2\beta t + \gamma}), \tag{42}$$

$$\theta = 2(\frac{4m^2t + \beta}{4m^2t^2 + 2\beta t + \gamma}), \tag{43}$$

Notes

$$\sigma = \frac{1}{\sqrt{3}}\left(\frac{4m^2t+\beta}{4m^2t^2+2\beta t+\gamma}\right). \tag{44}$$

The energy density, DP and ω are obtained as

$$\rho = \frac{(8m^2t+2\beta)^2-(4m^2\gamma-\beta^2)}{4(4m^2t^2+2\beta t+\gamma)^2}, \tag{45}$$

$$\omega = -\frac{5(4m^2\gamma-\beta^2)}{(8m^2t+2\beta)^2-(4m^2\gamma-\beta^2)}, \tag{46}$$

$$q = -\frac{2m^2(4m^2t^2+2\beta t+\gamma)}{(4m^2t+\beta)^2}. \tag{47}$$

The value of DP is always negative since V is never negative. The EoS parameter ω is negative if $\gamma > \frac{\beta^2}{4m^2}$. If this condition holds, the model (39) corresponds to a Bianchi type-VI$_0$ energy cosmological model with variable q and ω.

If $\gamma > \frac{\beta^2}{4m^2}$, the model (39) has no finite singularity. The physical and kinematical parameters are all decreasing function of time and ultimately tend to zero for large time. The model essentially gives an empty space-time for large time. The anisotropy in the model never dies out.

IV. Conclusion

In this paper, we have presented exact solutions of Einstein's field equations for a Bianchi-type VI$_0$ space-time filled with perfect fluid satisfying the barotropic equation of state under the assumption that the expansion scalar is proportional to shear scalar. Under some specific choice of problem parameters, the present consideration yields singular and non-singular models of the accelerated expansion universe filled with perfect fluid and dark energy. Models with negative EoS parameter ω may be attributed to the current accelerated expansion of universe. The physical and kinematical parameters are all decreasing function of time and ultimately tend to zero for large time. The universe models do not approach to isotropy. The models presented in this paper can be potential tools to describe the present universe as well as the early universe.

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Search engines for most searches, use Boolean searching, which is somewhat different from Internet searches. The Boolean search uses "operators," words (and, or, not, and near) that enable you to expand or narrow your affords. Tips for research paper while preparing research paper are very helpful guideline of research paper.

Choice of key words is first tool of tips to write research paper. Research paper writing is an art.A few tips for deciding as strategically as possible about keyword search:

- One should start brainstorming lists of possible keywords before even begin searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in research paper?" Then consider synonyms for the important words.
- It may take the discovery of only one relevant paper to let steer in the right keyword direction because in most databases, the keywords under which a research paper is abstracted are listed with the paper.
- One should avoid outdated words.

Keywords are the key that opens a door to research work sources. Keyword searching is an art in which researcher's skills are bound to improve with experience and time.

Numerical Methods: Numerical methods used should be clear and, where appropriate, supported by references.

Acknowledgements: Please make these as concise as possible.

References

References follow the Harvard scheme of referencing. References in the text should cite the authors' names followed by the time of their publication, unless there are three or more authors when simply the first author's name is quoted followed by et al. unpublished work has to only be cited where necessary, and only in the text. Copies of references in press in other journals have to be supplied with submitted typescripts. It is necessary that all citations and references be carefully checked before submission, as mistakes or omissions will cause delays.

References to information on the World Wide Web can be given, but only if the information is available without charge to readers on an official site. Wikipedia and Similar websites are not allowed where anyone can change the information. Authors will be asked to make available electronic copies of the cited information for inclusion on the Global Journals Inc. (US) homepage at the judgment of the Editorial Board.

The Editorial Board and Global Journals Inc. (US) recommend that, citation of online-published papers and other material should be done via a DOI (digital object identifier). If an author cites anything, which does not have a DOI, they run the risk of the cited material not being noticeable.

The Editorial Board and Global Journals Inc. (US) recommend the use of a tool such as Reference Manager for reference management and formatting.

Tables, Figures and Figure Legends

Tables: Tables should be few in number, cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g. Table 4, a self-explanatory caption and be on a separate sheet. Vertical lines should not be used.

Figures: Figures are supposed to be submitted as separate files. Always take in a citation in the text for each figure using Arabic numbers, e.g. Fig. 4. Artwork must be submitted online in electronic form by e-mailing them.

Preparation of Electronic Figures for Publication

Even though low quality images are sufficient for review purposes, print publication requires high quality images to prevent the final product being blurred or fuzzy. Submit (or e-mail) EPS (line art) or TIFF (halftone/photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Do not use pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings) in relation to the imitation size. Please give the data for figures in black and white or submit a Color Work Agreement Form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution (at final image size) ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs) : >350 dpi; figures containing both halftone and line images: >650 dpi.

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Figure Legends: Self-explanatory legends of all figures should be incorporated separately under the heading 'Legends to Figures'. In the full-text online edition of the journal, figure legends may possibly be truncated in abbreviated links to the full screen version. Therefore, the first 100 characters of any legend should notify the reader, about the key aspects of the figure.

6. AFTER ACCEPTANCE

Upon approval of a paper for publication, the manuscript will be forwarded to the dean, who is responsible for the publication of the Global Journals Inc. (US).

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the search? Will I be able to find all information in this field area? If the answer of these types of questions will be "Yes" then you can choose that topic. In most of the cases, you may have to conduct the surveys and have to visit several places because this field is related to Computer Science and Information Technology. Also, you may have to do a lot of work to find all rise and falls regarding the various data of that subject. Sometimes, detailed information plays a vital role, instead of short information.

2. Evaluators are human: First thing to remember that evaluators are also human being. They are not only meant for rejecting a paper. They are here to evaluate your paper. So, present your Best.

3. Think Like Evaluators: If you are in a confusion or getting demotivated that your paper will be accepted by evaluators or not, then think and try to evaluate your paper like an Evaluator. Try to understand that what an evaluator wants in your research paper and automatically you will have your answer.

4. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

5. Ask your Guides: If you are having any difficulty in your research, then do not hesitate to share your difficulty to your guide (if you have any). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work then ask the supervisor to help you with the alternative. He might also provide you the list of essential readings.

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9. Use and get big pictures: Always use encyclopedias, Wikipedia to get pictures so that you can go into the depth.

10. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right! It is a good habit, which helps to not to lose your continuity. You should always use bookmarks while searching on Internet also, which will make your search easier.

11. Revise what you wrote: When you write anything, always read it, summarize it and then finalize it.

12. Make all efforts: Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

13. Have backups: When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

14. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several and unnecessary diagrams will degrade the quality of your paper by creating "hotchpotch." So always, try to make and include those diagrams, which are made by your own to improve readability and understandability of your paper.

15. Use of direct quotes: When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

16. Use proper verb tense: Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

17. Never use online paper: If you are getting any paper on Internet, then never use it as your research paper because it might be possible that evaluator has already seen it or maybe it is outdated version.

18. **Pick a good study spot:** To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

19. Know what you know: Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

20. Use good quality grammar: Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

21. Arrangement of information: Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

23. Multitasking in research is not good: Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

24. Never copy others' work: Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

25. Take proper rest and food: No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

26. Go for seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

27. Refresh your mind after intervals: Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

29. **Think technically:** Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

30. Think and then print: When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

31. Adding unnecessary information: Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

32. Never oversimplify everything: To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

33. Report concluded results: Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

34. After conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page

- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.

Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.
- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

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This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.

- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

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Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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