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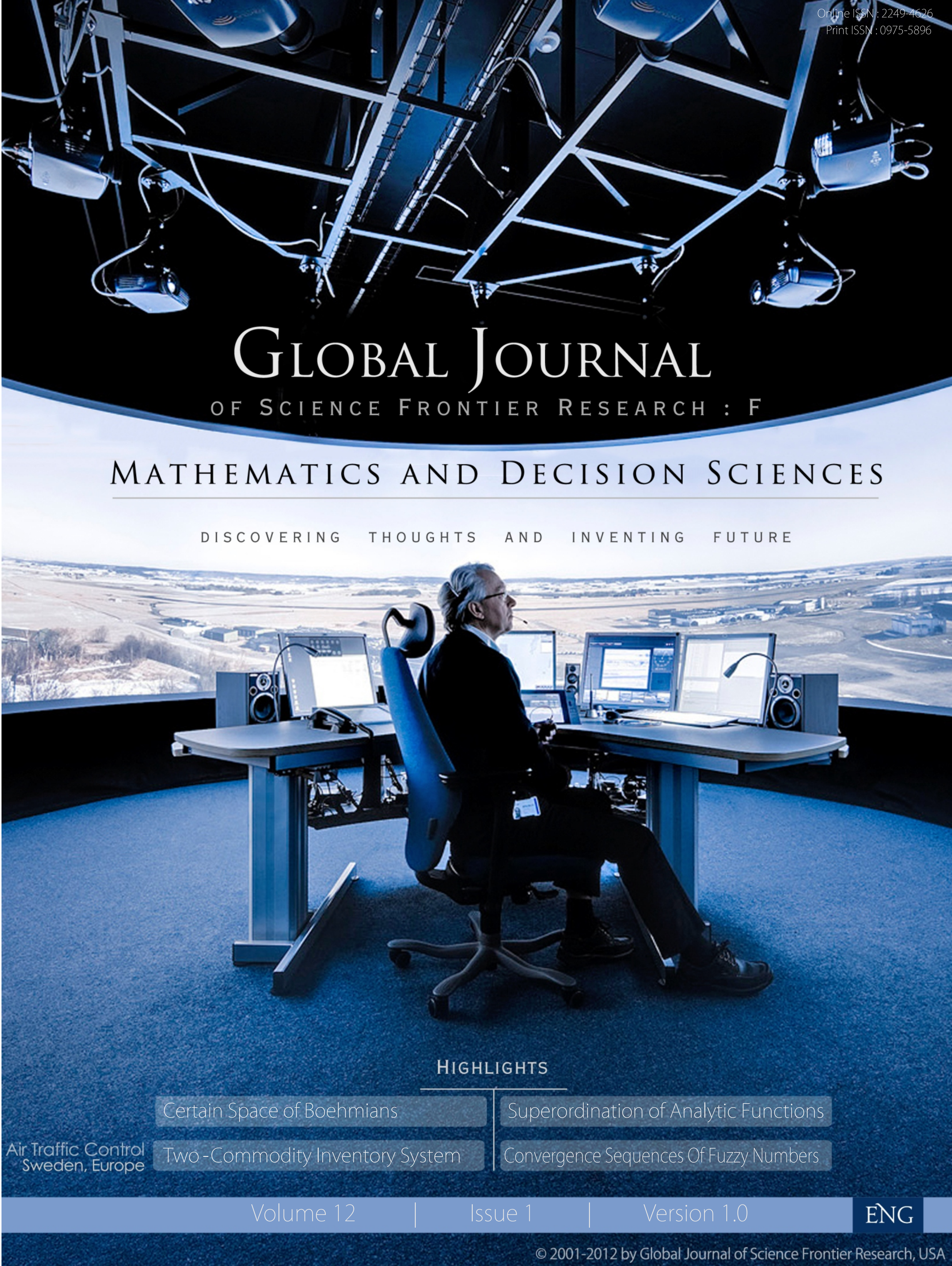

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Highlights

- Certain Space of Boehmians
- Superordination of Analytic Functions
- Two-Commodity Inventory System
- Convergence Sequences Of Fuzzy Numbers

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Note on Elzaki Transform of Distributions and Certain Space of Boehmians

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Abstract - The Elzaki transform transform was discussed in [19] as a motivation of the classical Sumudu transform. In this article, we extend the Elzaki transform to a space of tempered distributions (distributions of slow growth) by known kernel method. Further, we establish two spaces of Boehmians so that the Elzaki transform is well defined. Certain theorems are established in some details.

Keywords and phrases : Generalized function; Elzaki Transform; Sumudu Transform; Tempered Distribution; Boehmian Space.

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NOTE ON ELZAKI TRANSFORM OF DISTRIBUTIONS AND CERTAIN SPACE OF BOEHMIANS

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Note on Elzaki Transform of Distributions and Certain Space of Boehmians

S.K.Q.Al-Omari

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I. Introduction

In order to solve differential equations, several integral transforms were extensively used and applied in theory and application such as the Laplace, Fourier, Mellin, Hankel and Sumudu transforms, to name but a few. In the sequence of these transforms, rescently, Elzaki,T. and Elzaki, S. [17,18,19] introduced a motivation of the Sumudu transform [14 -16] and applied it to the solution of ordinary and partial differential equations as well.

The Elzaki transform over the set functions is defined by

$$A=\left\{f(t):\exists M,\tau_1,\tau_2>0,|f(t)|<Me^{t/\tau_j},t\in(-1)^j\times(0,\infty)\right\} \quad (1)$$

by the formula

$$\tilde{f}(z)=Ef(z)=:\int^{\infty} zf(t)\,e^{\frac{-t}{z}}dt, z\in(-\tau_1,\tau_2). \quad (2)$$

The general properties of Elzaki transforms are found in above citations. In fact there is a relationship between Elzaki transform and some other transforms. In particular, the strong relationship between the Elzaki transform and Laplace transform was already proved in [19] which can be decribed as follows. Let f be a function of exponential order and Lf and Ef be the Laplace and Elzaki transforms of f, respectively, then

$$Ef(z)=zLf\left(\frac{1}{z}\right).$$

and hence

$$Lf\left(\frac{1}{z}\right)=zE\left(\frac{1}{z}\right).$$

The following are needful in the sequel.

(1) If a and b are non-negative real numbers then

$$E(af(t)+bg(t))(z)=aEf(z)+bEg(z).$$

(2) $\lim_{t\to0}f(t)=\lim_{z\to0}Ef(z)=f(0).$

II. Elzaki Transform of Boehmians

The minimal structure necessary for the construction of Boehmians consists of the following: (1) A nonempty set A ; (2) A commutative semigroup $(B,\star)$; (3) An operation $\star:A\times B\to A$ such that for each $x\in A$ and $s_1,s_2,\in B, x\star(s_1\star s_2)=(x\star s_1)\star s_2$; (3) A collection $\Delta\subset B^N$ such that: (a) If $x,y\in A,(s_n)\in\Delta,x\star s_n=y\star s_n$ for all then $x=y,$; (b) If $(s_n),(t_n)\in\Delta$, then $(s_n\star t_n)\in\Delta$.

Elements of Δ are called delta sequences. Consider

$$Q=\{(x_n,s_n):x_n\in A,(s_n)\in\Delta,x_n\star s_m=x_m\star s_n,\forall m,n\in\mathbf{N}\}.$$

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If $(x_n, s_n), (y_n, t_n) \in Q, x_n \star t_m = y_m \star s_n, \forall m, n \in \mathbf{N},$ then we say $(x_n, s_n) \sim (y_n, t_n)$. The relation $\sim$ is an equivalence relation in Q. The space of equivalence clases in Q is denoted by β. Elements of β are called Boehmians. Between A and β there is a canonical embedding expressed as $x \to \frac{x \star s_n}{s_n}$. The operation $\star$ can be extended to $\beta \times A$ by $\frac{x_n}{s_n} \star t = \frac{x_n \star t}{s_n}$. The relationship between the notion of convergence and the product $\star$ is given by:

(1) If $f_n \to f$ as $n \to \infty$ in A and, $\phi \in B$ is any fixed element, then $f_n \star \phi \to f \star \phi$ in $A\,(\text{as } n \to \infty)$; (ii) If $f_n \to f$ as $n \to \infty$ in A and $(\delta_n) \in \Delta$, then $f_n \star \delta_n \to f$ in $A\,(\text{as } n \to \infty)$.

The operation $\star$ is extended to $\beta \times B$ as follows: If $[\frac{f_n}{s_n}] \in \beta$ and $\phi \in B$,then $[\frac{f_n}{s_n}] \star \phi = [\frac{f_n \star \phi}{s_n}]$.
Convergence in β is defined as

(1) : A sequence (h_n) in β is said to be δ convergent to h in β, $h_n \xrightarrow{\delta} h$, if there exists $(s_n) \in \Delta$ such that $(h_n \star s_n), (h \star s_n) \in A, \forall k, n \in \mathbf{N}$, and $(h_n \star s_k) \to (h \star s_k)$ as $n \to \infty$, in A, for every $k \in \mathbf{N}$.

(2) : A sequence (h_n) in β is said to be Δ convergent to h in β, $h_n \xrightarrow{\Delta} h$, if there exists a $(s_n) \in \Delta$ such that $(h_n - h) \star s_n \in A, \forall n \in \mathbf{N}$, and $(h_n - h) \star s_n \to 0$ as $n \to \infty$ in A. For further details, we refer to $[1-8, 10, 11, 13]$.

The convolution product between two functions u and v is given by the integral

$$(u * v)(y) = \int_0^\infty u(y-x)\, v(x)\, dx \qquad (3)$$

or, equivalently,

$$(u * v)(y) = \int u(t)\, \tau_y \tilde{v}(t)\, dt, \qquad (4)$$

where

$$\tilde{v}(t) = v(-t) \text{ and } \tau_y v(t) = v(t-y).$$

Lemma 2.1. $E(u * v)(z) = \frac{1}{z}(Eu)(z)(Ev)(z)$.

Proof See [19, Theo.2-6] :

Denote by S the space of all complex valued functions $s(t)$ that are infinitely smooth and are such that, as $|t| \to \infty$, they and their partial derivatives decrease to zero faster than every power of $\frac{1}{|t|}$.This required behaviour as $|t| \to \infty$ can also be stated in the following alternative way. For t one-dimensional, every function $s(t) \in S$ satisfies the infinite set of inequalities

$$\left|t^m s^{(k)}(t)\right| \le C_{mk}, t \in (0, \infty), \qquad (5)$$

where m and k run through all non negative integers. The elements of S are called testing functions of rapid descents. S is a linear space. The dual space of S is denoted by $\acute{S}$. A distribution $u \in \acute{S}$ is said to be tempered distribution or distribution of slow growth.

Let $\mathbf{R}_+$ be the field of positive real numbers and z be arbitrary but fixed in $\mathbf{R}_+$ then

$$D_t^k\left(ze^{\frac{-t}{z}}\right) = (-1)^k z^{1-k} e^{\frac{-t}{z}}, k = 1, 2, ...$$

Hence for arbitrary but fixedz $z \in \mathbf{R}_+$, we get

$$\left|t^m D_t^k\left(ze^{\frac{-t}{z}}\right)\right| = \left|t^m z^{1-k} e^{\frac{-t}{z}}\right| < \infty, 0 < t < \infty. \qquad (6)$$

By aid of (6) we define the Elzaki transform of $u \in \acute{S}$ by kernel method as

$$Ef(z) = \left\langle u(t), ze^{\frac{-t}{z}} \right\rangle. \qquad (7)$$

$t, z \in \mathbf{R}_+$.

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Denote by D the space of test functions of compact supports on $\mathbf{R}_+$ then

Definition 2.2 Let $u \in \acute{S}$ and $s \in D$ then we define the convolution $u * s$ to be C^∞ function such that

$$(u * v)(y) = \langle u, \tau_y \tilde{v} \rangle , \tag{8}$$

where $\tilde{v}(t) = v(-t)$ and $\tau_y v(t) = v(t-y), t \in R_+$. Equ(8) can also be written as

$$(u * v)(y) = \langle u(t), v(t-y) \rangle \tag{9}$$

Definition 2.3. The convolution of two tempered distributions $u, v \in \acute{S}$ is defined as an element in $\acute{S}$ through

$$\langle u * v, s \rangle = \langle u(y), \langle v(t), \phi(t+y) \rangle \rangle , s \in D. \tag{10}$$

It can be noted that if $u \in \acute{S}, v \in S$ then $u * v \in O_m,$ where O_m is the space of multipliers for $\acute{S}$. In fact $O_m \subset \acute{S}$. This, establishes the following lemma.

Lemma 2.4. If $u \in \acute{S}, s \in D$ then $u * s \in \acute{S}$.

Lemma 2.5. If $u \in \acute{S}, s_1, s_2 \in D$ then

$$(u * s_1) * s_2 = u * (s_1 * s_2) .$$

Proof. Since $D \subset S, u * s_1 \in C^\infty$ and hence $(u * s_1) * s_2 \in C^\infty, u * (s_1 * s_2) \in C^\infty$. Aso, $u \in \acute{S}, s_1 \in D \subset S \subset \acute{S}$, implies

$$u * s_1 \in \acute{S}.$$

We write

$$\begin{aligned} ((u * s_1) * s_2)(y) &= \langle u * s_1, \tau_y \tilde{s}_2 \rangle \\ &= \langle u(t), \langle s_1(x), \tau_y \tilde{s}_2(t+x) \rangle \rangle \\ &= \langle u(t), (s_1 * s_2)(y-t) \rangle \\ &= (u * (s_1 * s_2))(y) . \end{aligned}$$

Hence

$$(u * s_1) * s_2 = u * (s_1 * s_2) .$$

This completes the proof.

Lemma 2.6 If $u_1, u_2 \in \acute{S}, s \in D$ and $\alpha \in R$ then we have $(1)\,(u_1 + u_2) * s = u_1 * s + u_2 * s.;\ (2)\,\alpha(u_1 * s) = (\alpha u_1) * s = u_1 * (\alpha s)$. Let Δ be the collection of all sequences (r_n) from D such that Equ. $(11-13)$ satisfies.

$$\int_{\mathbf{R}_+} r_n(t)\,dt = 1 \tag{11}$$

$$\int_{\mathbf{R}_+} |r_n(t)|\,dt < M, M \in \mathbf{R}_+ \tag{12}$$

$$\operatorname{supp} r_n(t) \to 0 \text{ as } n \to \infty \tag{13}$$

Sequences from Δ are called delta sequences.

Lemma 2.7. If $u_n \to u \; is \; S \; as \; n \to \infty$ then

$$u_n * s \to u * s \text{ as } n \to \infty \text{ in } \acute{S}, s \in D.$$

Lemma 2.8. If $u_n \to u \; in \; \acute{S}$ as $n \to \infty \; then \; u_n * r_n \to u$ as $n \to \infty$ for each $(r_n) \in \Delta$.

The described Boehmian space is denoted by $O\left(\acute{S}, D, \Delta\right)$. Next, we describe another Boehmian space as follows. Let H be the set of all Elzaki transforms of tempered distributions from $\acute{S}$. That is, for each $h \in H,$ there is $u \in \acute{S}$ such that $h = Eu$. Moreover, $h_n \to h$ in H if there are $u_n, u \in \acute{S}$ such that $u_n \to u$ in $\acute{S}$.

Define a mapping $\bullet$ between $h \in H$ and $s \in D$ by

$$(h \bullet s)(z) = h(z) \int e^{\frac{-t}{z}} s(t)\,dt \tag{14}$$

Lemma 2.9. Let $h \in H$ such that $h = Eu, u \in \acute{S}$ and $s \in D$ then

$$E(u * s)(z) = (h \bullet s)(z) .$$

Proof. Using definitions and Leibnitz' rule and change of varibles yields

$$\begin{aligned} E(u*s)(z) &= \int s(t)\,dt \int u(y)\, z e^{-\frac{t+y}{z}}\,dy \\ &= \int u(y)\, z e^{\frac{-y}{z}}\,dy \int e^{\frac{-t}{z}} s(t)\,dt \\ &= h(z) \int e^{\frac{-t}{z}} s(t)\,dt \\ &= (h \bullet s)(z). \end{aligned}$$

Hence the Lemma.

Following lemmas are straightforward. We avoid same details.

Lemma 2.10 If $h \in H, s \in D$ then $h \bullet s \in H$.

Note that if $h \in H$ then $h = Eu$, for some $u \in \acute{S}$. Therefore $h \bullet s = Eu \bullet s = E(u*s)$, by Lemma 2.9. Since $u * s \in \acute{S}$, the lemma follows.

Lemma 2.11. If $h \in H, s \in D$ then $E^{-1}(h \bullet s) = E^{-1} h * \phi$ where E^{-1} is the inverse Elzaki transform

Proof. Let $u \in \acute{S}$ such that $Eu = h$ then

$$E(u*s) = h \bullet s.$$

Hence, employing E^{-1} on both sides yields $E^{-1}(h \bullet s) = u * s = E^{-1} h * s$.

Lemma 2.12. If $h_1, h_2 \in H, s_1, s_2 \in$ then

$$(h_1 + h_2) \bullet s = h_1 \bullet s + h_2 \bullet s; \quad (2)\, h \bullet (s_1 * s_2) = (h \bullet s_1) \bullet s_2.$$

Lemma 2.13. If $h_n \to h$ and $s \in D$ then $h_n \bullet s \to h \bullet s$.

Lemma 2.14. If $h_n \to h$ in H and $(r_n) \in \Delta$ then

$$h_n \bullet r_n \to h \text{ as } n \to \infty.$$

The space $O(H, D, \Delta)$ can therefore be regarded as a Boehmian space.

III. Elzaki Transform of Boehmians

Let $\beta_1 = \left[\frac{u_n}{s_n}\right] \in O\left(\acute{S}, D, \Delta\right)$ then we define the extended Elzaki transform of β_1 as

$$\hat{E}\left[\frac{u_n}{r_n}\right] = \left[\frac{Eu_n}{r_n}\right] \in O(H, D, \Delta), \tag{15}$$

where $(r_n) \in \Delta$.

Theorem 3.1. $\hat{E} : O\left(\acute{S}, D, \Delta\right) \to O(H, D, \Delta)$ is well defined.

Proof: Let $\left[\frac{u_n}{r_n}\right] = \left[\frac{v_n}{\psi_n}\right]$ in $O\left(\acute{S}, D, \Delta\right)$ then

$$u_n * \psi_m = v_m * r_n = v_n * r_m.$$

Employing E on both sides,

$$(Eu_n)(z) \int \psi_m(t)\, e^{\frac{-t}{z}}\,dt = (Ev_n)(z) \int r_m(t)\, e^{\frac{-t}{z}}\,dt.$$

Hence,

$$Eu_n \bullet \psi_m = Ev_n \bullet r_m.$$

That is,

$$\frac{Eu_n}{r_n} \sim \frac{Ev_n}{\psi_n}.$$

Therefore,

$$\left[\frac{Eu_n}{r_n}\right] = \left[\frac{Ev_n}{\psi_n}\right].$$

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This completes the proof of the theorem.

Theorem 3.2. $\hat{E}: O\left(\acute{S}, D, \Delta\right) \to O\left(H, D, \Delta\right)$ is linear

Proof. Let $\left[\frac{u_n}{r_n}\right], \left[\frac{v_n}{\psi_n}\right]$. From definitions and Equ. (15) we get

$$\begin{aligned}\hat{E}\left(\left[\frac{u_n}{r_n}\right]+\left[\frac{v_n}{\psi_n}\right]\right) &= \hat{E}\left(\left[\frac{u_n*\psi_n+v_n*r_n}{r_n*\psi_n}\right]\right)\\ &= \left[\frac{E\left(u_n*\psi_n+v_n*r_n\right)}{r_n*\psi_n}\right]\\ &= \left[\frac{E\left(u_n*\psi_n\right)+E\left(v_n*r_n\right)}{r_n*\psi_n}\right]\\ &= \left[\frac{Eu_n\bullet\psi_n+Ev_n\bullet r_n}{r_n*\psi_n}\right]\\ &= \left[\frac{Eu_n}{r_n}\right]+\left[\frac{Ev_n}{\psi_n}\right].\end{aligned}$$

Hence

$$\hat{E}\left(\left[\frac{u_n}{r_n}\right]+\left[\frac{v_n}{\psi_n}\right]\right)=\hat{E}\left[\frac{u_n}{r_n}\right]+\hat{E}\left[\frac{v_n}{\psi_n}\right].$$

Also, if $\alpha\in\mathbf{R}_+$ then

$$\alpha\hat{E}\left[\frac{u_n}{r_n}\right]=\alpha\left[\frac{Eu_n}{r_n}\right]=\left[\frac{E\left(\alpha u_n\right)}{r_n}\right].$$

Hence

$$\alpha\hat{E}\left[\frac{u_n}{r_n}\right]=\hat{E}\left(\alpha\left[\frac{u_n}{r_n}\right]\right).$$

This completes the proof.

Theorem 3.3. $\hat{E}$ is one-one.

Proof. Let $\beta_1,\beta_2\in O\left(\acute{S}, D, \Delta\right)$ such $\beta_1=\left[\frac{u_n}{r_n}\right]$ and $\beta_2=\left[\frac{v_n}{\psi_n}\right]$.

Assume $E\beta_1=E\beta_2$ then $\left[\frac{Eu_n}{r_n}\right]=\left[\frac{Ev_n}{\psi_n}\right]$. That is,

$$Eu_n\bullet\psi_n=Ev_m\bullet r_n.$$

Using Lemma 2.9,

$$E\left(u_n*\psi_m\right)=E\left(v_m*r_n\right).$$

Therefore

$$u_n*\psi_m=v_m*r_n.$$

Hence

$$\frac{u_n}{r_n}\sim\frac{v_n}{\psi_n}$$

and

$$\left[\frac{u_n}{r_n}\right]\sim\left[\frac{v_n}{\psi_n}\right].$$

This completes the proof of the lemma.

Theorem 3.4. $\hat{E}: O\left(\acute{S}, D, \Delta\right) \to O\left(H, D, \Delta\right)$ is onto.

Proof. Let $\left[\frac{h_n}{r_n}\right]\in O\left(H, D, \Delta\right)$ then

$$h_n=Eu_n,$$

for all n. $\left[\frac{u_n}{r_n}\right]$ is in $O\left(\acute{S}, D, \Delta\right)$ such that

$$\hat{E}\left[\frac{u_n}{r_n}\right] = \left[\frac{Eu_n}{r_n}\right] = \left[\frac{h_n}{r_n}\right].$$

Hence the theorem. Now, we de.ne the inverse $\hat{E}^{-1}$ by the relation

$$\hat{E}^{-1}\left[\frac{h_n}{r_n}\right] = \left[\frac{\hat{E}^{-1}h_n}{r_n}\right], \quad (16)$$

for every $h_n \in O\left(H, D, \Delta\right)$.

Theorem 3.5. $\hat{E}^{-1} : O\left(H, D, \Delta\right) \to O\left(\acute{S}, D, \Delta\right)$ is well defined.

Theorem 3.6. $\hat{E}^{-1} : O\left(H, D, \Delta\right) \to O\left(\acute{S}, D, \Delta\right)$ is linear.

Theorem 3.7. $\hat{E}^{-1} : O\left(H, D, \Delta\right) \to O\left(\acute{S}, D, \Delta\right)$ is an isomorphism.

Proof of Theorem 3.5, 3.6, 3.7, are analogous to that of Theorem 3.1, 3.2, 3.3, and 3.4. Detailed proofs are avoided.

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A Modern Approach to a Unified Field Theory

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Abstract - This paper presents strictly the mathematical derivation of the Einstein field equations from a vanishing divergent field given Lie group *SU(1)* and satisfying cohomology, and a discussion of the implications of these results. Essentially, we obtain the expression $G^{ab} = T^{ab}\, 8\pi$ from given postulates of a Quantum Yang-Mills theory- the gravitational terms reducing to $G^{ab} = T^{ab} 8\pi$ when we have *SU(1),* satisfying ("good") cohomology (i.e. that of the standard formulation of Hodge's conjecture, [3] for a history), and we integrate with respect to the metric solution of the equations in a standard Lorentzian frame. We then discuss the implications of these results, and consider some misconceptions about past approaches to ubiquitous media.

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I. Introduction

It is the purpose of this paper to approach Einstein's field equations from a standard mathematical viewpoint. Historically, the Einstein's equations were given

$$R_{\mu\nu} - g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}\right)T_{g\mu\nu} \qquad (1)$$

With Λ the cosmological constant. Physically, the interpretation of (1) itself did not cause many issues; rather, it was reconciling it with the also experimentally verified conclusions of quantum mechanics that has presented itself as the challenge. A variety of theories and bases for theories have been proposed ([4],[5],[6],[7]) to attempt such a reconciliation, however none have truly proved satisfying, although undeniably very, very interesting.

Eq. (1) is just that; it is an equation- it is general relativity's expression for the curvature of spacetime. Throwing away all concepts of physical interpretation of results, albeit temporarily, and we can show that eq. (1), and its' simpler form

$$G^{ab} = T^{ab}8\pi \qquad (2)$$

May all be given mathematically independent of the physical picture of the results. We assert that eqs. (1) and (2) are, in fact, one of the rare instances in physics where mathematics has left room for physical interpretation, rather than the other way around. What we mean to assert by this is the following:

Theorem 1): Given eqs. (1) and (2) in their above and any equal forms, the physical picture is independent of the correct formulation of the actual theory. We may thus assert that; mathematically, we may develop the physical picture of the theory on a strictly mathematical basis- and then interpret the physical implications.

The above theorem is exactly what this paper seeks to prove; that the implications of (1), (2), and any other forms, are all the bases for a correct physical picture of phenomena in the universe.

II. Getting the Field Equations

Proposition 2.1) Given QYT (Quantum Yang-Mills theory) with curvature $F = da$, gauge group G,

$$L = \frac{1}{4g^2}\int \mathbf{Tr}F \wedge {}^*F \text{ (eq.(3))} \;;\; F = da = aA + A \wedge A$$

,then we have mass gap $\Delta > 0$; *SU(3)* invariance remains given transformations under *SU(3),* under the limit U(<10), there is no chiral symmetry breaking, but under the limit *U(>10)*; there may be, and; lastly under *U(<10);* the Einstein equations derived for the QYT with gauge group G, L as in eq. (3) are exactly solvable.

a) To the mass gap $\Delta > 0$

Method 2.1.1) Let a frame exist in spacetime such that there are no co-ordinates within the frame lying outside of the region $A\left[ds^2\right]$, with $ds^2 = R^3 d\tau^2 + d\tau$. Let a worldline be defined such that a particle with co-ordinates (θ_1, θ_2) and (ϕ_1, ϕ_2) moves along the worldline with metric ds^2 as defined above. Let $\int A\left[ds^2\right] = \int A\left[ds^2\right]$ for an observer at τ_1. τ_1 is stationary on the sphere in the same frame where $\int A\left[ds^2\right] = \int A\left[ds^2\right]$. Let us now define a function $\bar{g}(t)$

$$\bar{g}(t) = \int T^{(vac)} \bar{F}_{\mu\nu} F_{ab} / 2\pi \cos^3\theta \sin^3 d\theta^2 \left(ds^2\right)^2 frac \{m\}\,, \qquad (4)$$

Where

$$\{m\}_{ab} = \{m\}_{AB} = \sum_{n=1}^{n} g_n \left(\lambda^2\right)^n \prod_n^n \left\{\Gamma^{ab}\right\}\Gamma^2. \qquad (5)$$

Lemma 2.1.1.1). $[S2] \times [S3] R^a_{\mu a\nu} = [6] R^a_{\mu a\nu}$;

Proof): Define a term λ such that $\lambda \in SU(n)$. Let λ be isomorphic to the group of quaternions of norm 1. Say that the group of quaternions of norm 1 describes λ on the two-dimensional disc when the disc is compactified from a 5-sphere and the point λ is thus diffeomorphic to the 3-sphere. Integrate out the term $\int d\tau$ with $d\tau = A\left[ds^2\right]$; and we see that we must perform the following operation:

$$\int d\tau = \int \cdots \int [T_a T_b] \otimes \delta_a \delta_b \delta_{ab} d_{ace} d_v \otimes \int \lambda \otimes \int G^{\mu\nu}_{,} \quad (6)$$

With

$$[T_a T_b] = \frac{1}{2n}\delta_{ab} I_n + \frac{1}{2}\sum_{c=1}^{n^2-1}(if_{abc} + d_{abc})T_c, \quad (7)$$

The generators of the *SU(n)* Lie group. Given λ diffeomorphic to the 3-sphere, we see that; after calculating (6), $R^{\mu\nu} = [S2] \times [S3] \times R^a_{\mu a v}$, and, since we have $\int d\tau$ no longer remaining, we see that $R^{\mu\nu} = [S6] \times R^a_{\mu a v}$, or $R^{\mu\nu} = [6] \times R^a_{\mu a v}$. Thus, $[S2] \times [S3] R^a_{\mu a v} = [6] R^a_{\mu a v}$.

Now, define S_n the symmetry group of the *S(n-1)* simplex. The *SU(n)* Lie group is just the traceless antihermitian $n \times n$ matrices with its regular commutator the Lie bracket. Let us now consider the gauge group *U(1)*. Let us now say that; given the conditions of Lemma 2.1.1.1), the following propositions hold true:

Proposition 2.1.1.1): $U(L) = U(1);$

Proposition 2.1.1.2): $L = U\left(\begin{bmatrix} 0 & -1 & -1 \\ 0 & +1 & -1 \\ 0 & -1 & +1 \end{bmatrix}\right) = U(L),$

Proposition 2.1.1.3):

$$\lambda^2 \oplus U = \lambda^2 U\left[((1):(-1,-1,-1,+1)\right]^{\mu a}_{v},$$

Proposition 2.1.1.4): $\lambda^2 U\left[((1):(-1,-1,-1,+1)\right]$ describes the following relationship of the p - form $P^{**}F$ -

$$P^{**}F = AU(L)F^*.$$

Proposition 2.1.1.5): Proposition 2.1.1.4) is equal to

$$\frac{\textbf{cohomology} \uparrow \textbf{class-cohomology} \downarrow \textbf{class}}{\textbf{cohomology} \downarrow \textbf{class} \times \textbf{cohomology} \uparrow \textbf{class}}.$$

Thus, by the conditions of Lemma 2.1.1.1), and the implications of Propositions 2.1.1.1-5); and the fact the these propositions hold given the conditions of Lemma 2.1.1.1), we see that

$$p - forms\ on\ the\ \mathbf{3}\text{-}sphere = 1;$$
$$p - forms\ on\ the\ \mathbf{5}\text{-}sphere = 1.$$

Note that $\frac{T^{ab}}{8\pi} = G^{ab};\ ab = G^{1/2[ab]/8\pi} = T^{1/2[ab]/8\pi}$, and we see that

$$\lambda^2 = \lambda^2\left(\bar{g}(t)\right) = \mathbf{div}\frac{A}{E}\omega t\left(\bar{g}(t)\right), \quad (7)$$

Thus, for $\lambda \in SU(n)$ with the term $\int d\tau$ still present,

$$T^{ab} = G^{ab}/8\pi = \left[\lambda^2\right]^{ab}, \quad (8)$$

And, performing $\int d\tau$,

$$T^{ab} = G^{ab}/8\pi = \left[\lambda^2\right]^{ab} = 8\pi G^{ab}. \quad (9)$$

We see that in (8), the divergence of the terms $\frac{A}{E}\omega t\left(\bar{g}(t)\right)$ vanishes, and in (9) (7) holds true. This suggests that in the limit of general relativity, we have fluid $\nabla \times A$; that this is equivalent to non-linear integral currents isomorphic to the Hodge group isomorphic to the Lie group *SU(1)*, and that without the limit of general relativity, the same holds. The rectifiable currents in the boundary set of the Lie group *SU(1)* in $\int d\tau$ give us non-linear integral currents in the boundary set of the Lie group *SU(1)* without $\int d\tau$ present; and these are equivalent to non-linear integral currents isomorphic to the Hodge group isomorphic to the Lie group *SU(1)*.

This equates to a group action with quartic potential $\left(A \wedge A^2\right)$, and this gives us a mass gap $\Delta > 0$ (see [2]). It is elementary to see that $\Delta > 0$, as if $\Delta = 0$ it implies that all of spacetime is flat.

It is easy to see the next part of our problem, that *SU(3)* invariance remains given transformations under *SU(3)*, as *SU(3)* invariance with $\int d\tau$; in some suitable setting, remains by the ab terms in $\left[\lambda^2\right]^{ab}$, and that certain transformations at a negligible level allow transformations under *SU(3)*.

Similarly, we see that chiral symmetry breaking holds only under the subgroup of non-linear integral currents isomorphic to the Hodge group isomorphic to the Lie group *SU(1)*, as the limit in which quark bare masses vanish is similar to the limit in which $\int d\tau$ remains; thus, the potential vacuum invariance holds only under the subgroup of non-linear integral currents isomorphic in the above way.

In the limit U(<10), the following demonstrate exact solutions to the Einstein equations: [8],[9],[10],[11],[12]. In the limit U(>10), it is impossible to have an exact solution to the Einstein equations, or a solution in general, as the lack of a satisfying boundary state prevents us from even getting to the Einstein equations in the first place.

III. Results

We have established an effective **QYT**, in which the properties of [2] as listed are satisfied. This **QYT** is completely compatible with developments and standard praxis in gauge theory, **QFT**, relativity, and quantum mechanics, being formulated there from. Now, we wish to discuss these results in the context stated at the beginning of the paper- that of an interpretational one.

We have seen the mathematical relationship between excitations in a vacuum and Einstein's equations. What this allows us to assert is nothing new, it is merely a re-phrasing of old ideas- mathematically, the curvature of spacetime is equivalent to a vanishing divergent field in space. Around the turn of the 20th century, it was, in large part the contributions of the special and general theories of relativity that led to the abandonment of the concept of a ubiquitous 'ether', a medium constant throughout all space. It was not, however, the intention of these theories (See []), in fact, it was a necessary misinterpretation- there is nothing that constitutes cause for denying the concept, it is simply unnecessary. In this way, quantum mechanics and relativity progressed forward independently of the ether.

In our arguments, the concept of an ubiquitous ether is neither necessary nor desirable; yet, we choose to overlook it, vaguely keeping it in mind for utilization in situations such as this- we have approached an issue of reconciliation between general relativity and quantum mechanics. Interpretationally, the vanishing divergent field seen in this paper is equivalent to the ether, and we may interpret it as such- but we do not, seeing as the divergent field itself vanishes with $\int d\tau$, and only holds true without $\int d\tau$ - and is negligible (that is to say, essentially irrelevant) in both.

Thus, we wish to conclude this paper, and state that we have hopefully presented a way to reconcile issues between **QM** (Quantum Mechanics) and **GR** (General Relativity).

Clarification: Roughly stated, we are integrating $\int d\tau$ out, thus we are integrating the frame out. Since our arguments are contained in the space of the frame, we find a limit in which our arguments apply only to the contents of the frame, and a limit in which it (our arguments) applies to all space. Our mathematical formulations and the conclusions of other theories as accepted allow for all other conclusions made.

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A Summation Formula of Half Argument Collocated With Contiguous Relation

By Salahuddin, M.P. Chaudhary, Vinesh Kumar
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Abstract - The main aim of the present paper is to compute a summation formula linked with recurrence relation and contiguous relation.

Keywords : *Gauss second summation theorem, Recurrence relation* .

AMS Subject Classification: 33C05, 33C20, 33C45, 33D50, 33D60

A SUMMATION FORMULA OF HALF ARGUMENT COLLOCATED WITH CONTIGUOUS RELATION

Strictly as per the compliance and regulations of :

A Summation Formula of Half Argument Collocated With Contiguous Relation

Salahuddin[α], M.P. Chaudhary[Ω], Vinesh Kumar[β]

Abstract - The main aim of the present paper is to compute a summation formula linked with recurrence relation and contiguous relation.

Keywords : Gauss second summation theorem, Recurrence relation.

I. Introduction

Generalized Gaussian Hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{lll} a_1,a_2,\cdots,a_A & ; & \\ & & z \\ b_1,b_2,\cdots,b_B & ; & \end{array}\right] = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_A)_k z^k}{(b_1)_k(b_2)_k\cdots(b_B)_k k!} \tag{1}$$

Where the parameters $a_1,a_2,\cdots,a_A$ and $b_1,b_2,\cdots,b_B$ are neither zero nor negative integers and $A,\ B$ are non negative integers.

If $A \leq B$, then series ${}_AF_B$ is always convergent for all finite values of z (real or complex).

If $A = B+1$, then series ${}_AF_B$ is convergent when $|z| < 1$.

Contiguous Relation is defined by [Andrews p.363(9.16), E. D. p.51(10)]

$$(a-b)\ {}_2F_1\left[\begin{array}{ll} a,\ b\ ; & \\ c\quad ; & z\end{array}\right] = a\ {}_2F_1\left[\begin{array}{ll} a+1,\ b\ ; & \\ c\quad ; & z\end{array}\right] - b\ {}_2F_1\left[\begin{array}{ll} a,b+1; & \\ c\quad ; & z\end{array}\right] \tag{2}$$

Gauss Second Summation Theorem is defined by [Prudnikov,491(7.3.7.5)]

$$ {}_2F_1\left[\begin{array}{ccc} a,\ b & ; & \frac{1}{2} \\ \frac{a+b+1}{2} & ; & \end{array}\right] = \frac{\Gamma(\frac{a+b+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})} \tag{3}$$

$$ = \frac{2^{(b-1)}\Gamma(\frac{b}{2})\Gamma(\frac{a+b+1}{2})}{\Gamma(b)\Gamma(\frac{a+1}{2})} \tag{4}$$

In a monograph of Prudnikov et al., a summation theorem is given in the form [Prudnikov, p.491(7.3.7.3)]

$$ {}_2F_1\left[\begin{array}{ccc} a,\ b & ; & \frac{1}{2} \\ \frac{a+b-1}{2} & ; & \end{array}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\ \Gamma(\frac{b+1}{2})} + \frac{2\ \Gamma(\frac{a+b-1}{2})}{\Gamma(a)\ \Gamma(b)}\right] \tag{5}$$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

$$ {}_2F_1\left[\begin{array}{ccc} a,\ b & ; & \frac{1}{2} \\ \frac{a+b-1}{2} & ; & \end{array}\right] = \frac{2^{(b-1)}\ \Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right] \tag{6}$$

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Recurrence Relation is defined by

$$\Gamma(\zeta+1)=\zeta\,\Gamma(\zeta) \tag{7}$$

II. Main Summation Formula

$${}_2F_1\left[\begin{array}{cc} a,\quad b & ; \\ \frac{a+b+44}{2} & ; \end{array}\ \frac{1}{2}\right]=\frac{2^b\ \Gamma(\frac{a+b+44}{2})}{(a-b)\ \Gamma(b)}\times$$

$$\times\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{2097152(255108265612582846464 0000a-4589065620297665740800000a^2)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+\right.\right.$$

$$+\frac{2097152(3618572796858388709376000a^3-1687018700164430403993600a^4)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(526766035608866113191936a^5-117964128844107192729600a^6)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(19769717617137637130240a^7-2550445204778112122880a^8)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(258174206358711894016a^9-20771430124567449600a^{10})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(1338915850793364480a^{11}-69419613644559360a^{12}+2895430910817536a^{13})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(-96782231616000a^{14}+2570993384320a^{15}-53512986240a^{16})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(853247136a^{17}-10054800a^{18}+82460a^{19}-420a^{20}+a^{21})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(2551082656125828464640000b+1803428561869756327526400 0a^2b)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(-5302160797104273044275200a^3b+5607018417653412916101120a^4b)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(-932256615762419968376832a^5b+363791583564959961055232a^6b)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

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$$+\frac{2097152(-37315288811906944204800a^{7}b+7489957442085282054144a^{8}b)}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(-498166507892515897344a^{9}b+58633762795613451264a^{10}b)}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(-2576442421949061120a^{11}b+188756045605906688a^{12}b)}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(-5435341998729216a^{13}b+252735800846976a^{14}b-4577441890560a^{15}b)}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(133017777312a^{16}b-1369444608a^{17}b+23317028a^{18}b-103320a^{19}b)}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(861a^{20}b+4589065620297665740800000b^{2}+18034285618697563275264000ab^{2})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(15762845986109545324216320a^{3}b^{2}-1715582741848198927613952a^{4}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(1948319481506497921024000a^{5}b^{2}-158335044742339255074816a^{6}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(66037165498677645279232a^{7}b^{2}-3734163901229061500928a^{8}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(793246448982601729024a^{9}b^{2}-30755148197026885632a^{10}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(3789344928747961856a^{11}b^{2}-98535301715655168a^{12}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(7467613794096512a^{13}b^{2}-124108229323008a^{14}b^{2}+5886041837312a^{15}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(-56283597552a^{16}b^{2}+1635834564a^{17}b^{2}-6805344a^{18}b^{2}+111930a^{19}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(361857279685838870937600 0b^3+530216079710427304427520 0ab^3)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(1576284598610954532421632 0a^2b^3+4279477751493666085797888a^4b^3)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(-2239173161783538218434 56a^5b^3+253642950373348367663104a^6b^3)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(-114707858235817963683 84a^7b^3+4808408644935979539456a^8b^3)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(-161326575795225544704a^9b^3+3440946647463778252 8a^{10}b^3)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(-806178380777742336a^{11}b^3+99219476929326208a^{12}b^3)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(-1523034274347264a^{13}b^3+114316013559552a^{14}b^3-1027334730240a^{15}b^3)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(47621245308a^{16}b^3-188848296a^{17}b^3+5245786a^{18}b^3)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(168701870016443040399360 0b^4+5607018417653412916101120ab^4)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(1715582741848198927613952a^2b^4+4279477751493666085797888a^3b^4)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(4847374285616319987712 00a^5b^4-138258355866387691683 84a^6b^4)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(151206507902143175372 80a^7b^4-4044254186267493918 72a^8b^4)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(16527480784860139494 4a^9b^4-3363994992766087680a^{10}b^4)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(7005573665596163 84a^{11}b^{4}-9761204824320768a^{12}b^{4})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(1169020433721728a^{13}b^{4}-9795900072000a^{14}b^{4}+709859761520a^{15}b^{4})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(-2675350860a^{16}b^{4}+118030185a^{17}b^{4}+526766035608866113191936b^{5})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(932256615762419968376832ab^{5}+1948319481506497921024000a^{2}b^{5})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(223917316178353821843456a^{3}b^{5}+484737428561631998771200a^{4}b^{5})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(26429876561704912930816a^{6}b^{5}-4378257529043906027 52a^{7}b^{5})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(4546882768824968 73216a^{8}b^{5}-73312482184688947 20a^{9}b^{5})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(287078767335167 8080a^{10}b^{5}-34723823234929920a^{11}b^{5})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(694161314662233 6a^{12}b^{5}-52920909565440a^{13}b^{5}+6064403010960a^{14}b^{5})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(-21402806880a^{15}b^{5}+1471442973a^{16}b^{5}+117964128844107192729600b^{6})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(363791583564959961055232ab^{6}+1583350447423392550748 16a^{2}b^{6})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(25364295037334836766 3104a^{3}b^{6}+13825835586638769168384a^{4}b^{6})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(26429876561704912930816a^{5}b^{6}+747218528298866111488a^{7}b^{6})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(-735110191674495 4368a^{8}b^{6}+7182019754496915328a^{9}b^{6})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(-68357994717492480a^{10}b^{6}+2537349296679142 4a^{11}b^{6})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(-167526567988416a^{12}b^{6}+31784430433616a^{13}b^{6}-102074925120a^{14}b^{6})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(11058116888a^{15}b^{6}+19769717617137637130240b^{7})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(37315288811906944204800ab^{7}+66037165498677645279232a^{2}b^{7})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(11470785823581796368384a^{3}b^{7}+15120650790214317537280a^{4}b^{7})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(4378257529043906027 52a^{5}b^{7}+747218528298866111488a^{6}b^{7})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(11286363877994105472a^{8}b^{7}-647522537970147 84a^{9}b^{7})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(5917878540782208 0a^{10}b^{7}-305570484570624a^{11}b^{7}+10673164324804 8a^{12}b^{7})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(-296017282848a^{13}b^{7}+52860229080a^{14}b^{7}+2550445204778112122880b^{8})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(7489957442085282054144ab^{8}+3734163901229061500928a^{2}b^{8})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(4808408644935979539456a^{3}b^{8}+4044254186267493918 72a^{4}b^{8})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(4546882768824968732 16a^{5}b^{8}+73511019167449543 68a^{6}b^{8})}{\left[\prod_{\heartsuit=0}^{20}\left\{a-b-2\heartsuit\right\}\right]\left[\prod_{\diamondsuit=1}^{21}\left\{a-b+2\diamondsuit\right\}\right]}+$$

$$+\frac{2097152(1128636387799410547 2a^7b^8+89937442713134016a^9b^8)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(-277179402820320a^{10}b^8+236085009732936a^{11}b^8-509323854312a^{12}b^8)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(166509721602a^13b^8+2581742063587118940 16b^9)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(4981665078925158973 44ab^9+7932464489826017290 24a^2b^9)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(1613265757952255447 04a^3b^9+1652748078486013949 44a^4b^9)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(7331248218468894720a^5b^9+718201975449691532 8a^6b^9)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(647522537970147 84a^7b^9+89937442713134016a^8b^9+349750605804600a^{10}b^9)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(-446775310800a^{11}b^9+353697121050a^{12}b^9+2077143012456744960 0b^{10})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(5863376279561345126 4ab^{10}+3075514819702688563 2a^2b^{10})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(3440946647463778252 8a^3b^{10}+3363994992766087680a^4b^{10})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(2870787673351678080a^5b^{10}+683579947174924 80a^6b^{10})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(5917878540782208 0a^7b^{10}+277179402820320a^8b^{10}+349750605804600a^9b^{10})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(513791607420a^{11}b^{10}+1338915850793364480b^{11}+25764424219490611 20ab^{11})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(3789344928747961856a^{2}b^{11}+8061783807777742336a^{3}b^{11})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(7005573665596163 84a^{4}b^{11}+347238232349299 20a^{5}b^{11})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(25373492966791424a^{6}b^{11}+3055704845 70624a^{7}b^{11}+236085009732936a^{8}b^{11})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(446775310800a^{9}b^{11}+513791607420a^{1}0b^{11}+694196136445593 60b^{12})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(1887560456059066 88ab^{12}+985353017156551 68a^{2}b^{12})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(99219476929326208a^{3}b^{12}+9761204824320768a^{4}b^{12})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(6941613146622336a^{5}b^{12}+167526567988416a^{6}b^{12}+106731643248048a^{7}b^{12})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(509323854312a^{8}b^{12}+353697121050a^{9}b^{12}+28954309108175 36b^{13})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(5435341998729216ab^{13}+7467613794096512a^{2}b^{13}+1523034274347264a^{3}b^{13})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(1169020433721728a^{4}b^{13}+52920909565440a^{5}b^{13}+31784430433616a^{6}b^{13})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(296017282848a^{7}b^{13}+166509721602a^{8}b^{13}+96782231616000b^{14})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(252735800846976ab^{14}+124108229323008a^{2}b^{14}+114316013559552a^{3}b^{14})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(9795900072000a^{4}b^{14}+6064403010960a^{5}b^{14}+102074925120a^{6}b^{14})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(52860229080a^7b^{14}+2570993384320b^{15}+4577441890560ab^{15})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(5886041837312a^2b^{15}+1027334730240a^3b^{15}+709859761520a^4b^{15})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(21402806880a^5b^{15}+11058116888a^6b^{15}+53512986240b^{16})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(133017777312ab^{16}+56283597552a^2b^{16}+47621245308a^3b^{16})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(2675350860a^4b^{16}+1471442973a^5b^{16}+853247136b^{17}+1369444608ab^{17})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(1635834564a^2b^{17}+188848296a^3b^{17}+118030185a^4b^{17}+10054800b^{18})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(23317028ab^{18}+6805344a^2b^{18}+5245786a^3b^{18}+82460b^{19}+103320ab^{19})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(111930a^2b^{19}+420b^{20}+861ab^{20}+b^{21})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304b(255108265612582846464 0000+8530455637768840151040 00a)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(5046844677567391413043200a^2+9081136125069704036352 00a^3)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(943293793948861653319680a^4+1048747496714114086993 92a^5)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(43804563947434794549248a^6+320665715712314900480 0a^7)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(69588173900532409958 4a^8+3450217334088441446 4a^9)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(437032804160812953 6a^{10}+147413395681244160a^{11})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(115199737802938 88a^{12}+259529058500096a^{13}+12727480051584a^{14})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(182186263680a^{15}+5499857952a^{16}+44760048a^{17}+771932a^{18}+2660a^{19})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(4194304b(21a^{20}-853045563776884015104000b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(1155916906042116915855 3600ab+1142031698036946716590080a^{2}b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(542395215716520673935 3600a^{3}b+483340488494855570325504a^{4}b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(46209877417962517351628 8a^{5}b+29739154860751054831616a^{6}b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(1187836816369626172620 8a^{7}b+540003762528448172032a^{8}b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(1135112500030868869 12a^{9}b+3598980835570312192a^{10}b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(44265500712079462 4a^{11}b+9538778579925504a^{12}b+720991701034752a^{13}b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(10015737262208a^{14}b+470276092928a^{15}b+3770546960a^{16}b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(106810248a^{17}b+370804a^{18}b+5740a^{19}b+504684467756739141304320 0b^{2})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(-1142031698036946716590080ab^{2}+916213853714368756816281 6a^{2}b^{2})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(397807783079586941632512a^3b^2+1606236897992772302405632a^4b^2)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(79985901310925941112832a^5b^2+70203458129013818785792a^6b^2)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(2770353357030304694272a^7b^2+1044011438827617827840a^8b^2)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(30238110324164627456a^9b^2+6058651455359726080a^{10}b^2)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(122806313632564224a^{11}b^2+14420734130391680a^{12}b^2)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(192246021387392a^{13}b^2+13807642532608a^{14}b^2+108000161152a^{15}b^2)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(4760540668a^{16}b^2+16405740a^{17}b^2+425334a^{18}b^2)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(-908113612506970403635200b^3+5423952157165206739353600ab^3)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(-397807783079586941632512a^2b^3+2387078061942597064065024a^3b^3)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(54165858339340169117696a^4b^3+191084514358987959304192a^5b^3)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(5685631363901324279808a^6b^3+4579807235019583217664a^7b^3)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(113820211057919015936a^8b^3+40163467045632939008a^9b^3)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(7399825222866768896a^{10}b^3+139896504005293568a^{11}b^3)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

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$$+\frac{4194304b(1751777944705152a^{12}b^3+194139922099200a^{13}b^3+1458526810240a^{14}b^3)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(98249075392a^{15}b^3+331233916a^{16}b^3+13489164a^{17}b^3)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(9432937939488616533196800b^4-4833404884948555703255040ab^4)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(1606236897992772302405632a^2b^4-541658583393401691176960a^3b^4)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(264441231300695277830144a^4b^4+34345249747560745943040a^5b^4)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(10753422354294917699584a^6b^4+1976023601015476080640a^7b^4)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(146137881502128465664a^8b^4+22875823167243904000a^9b^4)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(752245508455199616a^{10}b^4+85194173264462080a^{11}b^4)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(1508934143912832a^{12}b^4+106256704310400a^{13}b^4+1101695764400a^{14}b^4)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(3567134480a^{15}b^4+222945905a^16b^4-10487474967141140869939200b^5)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(4620987741796251735162880ab^5-799859013109259411128320a^2b^5)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(19108451435898795930419200a^3b^5-34345249747560745943040a^4b^5)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(1422190991211059102924800a^5b^5+1107187604733634744320a^6b^5)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(3107754377937945415 68a^{7}b^{5}+354277004252063692 8a^{8}b^{5})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(240692436731678131 2a^{9}b^{5}+22967896652821120a^{10}b^{5})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(7015062932620288a^{11}b^{5}+4447469696 6336a^{12}b^{5}+7339626148064a^{13}b^{5})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(22225991760a^{14}b^{5}+2140280688a^{15}b^{5}+4380456394743479454924 8b^{6})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(-2973915486075105483161 6ab^{6}+7020345812901381878579 2a^{2}b^{6})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(-5685631363901324279808a^{3}b^{6}+1075342235429491769958 4a^{4}b^{6})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(-11071876047336347443 2a^{5}b^{6}+39839285035423332659 2a^{6}b^{6})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(18825969484672430 08a^{7}b^{6}+4769925051865733760a^{8}b^{6})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(32753273458249344a^{9}b^{6}+2045180135321523 2a^{10}b^{6}+108494163290240a^{11}b^{6})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(30698633436016a^{12}b^{6}+833611888 48a^{13}b^{6}+12759365640a^{14}b^{6})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(-3206657157123149004800b^{7}+1187836816369626172620 8ab^{7})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(-2770353357030304694272a^{2}b^{7}+4579807235019583217664a^{3}b^{7})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(-1976023601015476080 64a^{4}b^{7}+3107754377937945415 68a^{5}b^{7})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{4194304b(-1882596948467243008a^6b^7+5978379300010996736a^7b^7)}{\left[\prod\limits_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(16738902516445824a^8b^7+38470291913923584a^9b^7+145457559854464a^{10}b^7)}{\left[\prod\limits_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(83501312754496a^{11}b^7+188638464560a^{12}b^7+49336213808a^{13}b^7)}{\left[\prod\limits_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(6958817390053240995 84b^8-5400037625284481720 32ab^8)}{\left[\prod\limits_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(10440114388276178278 40a^2b^8-11382021105791901593 6a^3b^8)}{\left[\prod\limits_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(14613788150212846566 4a^4b^8-35427700425206369 28a^5b^8)}{\left[\prod\limits_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(4769925051865733760a^6b^8-16738902516445824a^7b^8)}{\left[\prod\limits_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(47411663226003648a^8b^8+72173360207520a^9b^8+150988354535304a^{10}b^8)}{\left[\prod\limits_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(241258667832a^{11}b^8+127330963578a^{12}b^8-3450217334088441446 4b^9)}{\left[\prod\limits_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(11351125000308688691 2ab^9-30238110324164627456a^2b^9)}{\left[\prod\limits_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(40163467045632939008a^3b^9-22875823167243904 00a^4b^9)}{\left[\prod\limits_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(240692436731678131 2a^5b^9-327532734582493 44a^6b^9)}{\left[\prod\limits_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(38470291913923584a^7b^9-72173360207520a^8b^9+183709752798000a^9b^9)}{\left[\prod\limits_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(117012581400a^{10}b^{9}+223387655400a^{11}b^{9}+4370328041608129536b^{10})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(-3598980835570312192ab^{10}+6058651455359726080a^{2}b^{10})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(-7399825228667688896a^{3}b^{10}+7522455084551996 16a^{4}b^{10})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(-2296789665282112 0a^{5}b^{10}+2045180135321 5232a^{6}b^{10})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(-1454575598544 64a^{7}b^{10}+1509883545353 04a^{8}b^{10}-117012581400a^{9}b^{10})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(269128937220a^{10}b^{10}-1474133956812441 60b^{11}+442655007120794624ab^{11})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(-1228063136325642 24a^{2}b^{11}+1398965040052935 68a^{3}b^{11})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(-8519417326446208a^{4}b^{11}+7015062932620288a^{5}b^{11})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(-108494163290240a^{6}b^{11}+83501312754496a^{7}b^{11}-241258667832a^{8}b^{11})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(223387655400a^{9}b^{11}+11519973780293888b^{12}-9538778579925504ab^{12})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(14420734130391680a^{2}b^{12}-1751777944705152a^{3}b^{12})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(1508934143912832a^{4}b^{12}-44474696966336a^{5}b^{12}+30698633436016a^{6}b^{12})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(-188638464560a^{7}b^{12}+127330963578a^{8}b^{12}-2595290585000096b^{13})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(720991701034752ab^{13}-192246021387392a^{2}b^{13}+194139922099200a^{3}b^{13})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(-10625670431040a^{4}b^{13}+7339626148064a^{5}b^{13}-83361188848a^{6}b^{13})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(49336213808a^{7}b^{13}+127274800515584b^{14}-100157372622208ab^{14})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(13807642532608a^{2}b^{14}-1458526810240a^{3}b^{14}+1101695764400a^{4}b^{14})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(-22225991760a^{5}b^{14}+12759365640a^{6}b^{14}-182186263680b^{15})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(470276092928ab^{15}-108000161152a^{2}b^{15}+98249075392a^{3}b^{15})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(-3567134480a^{4}b^{15}+2140280688a^{5}b^{15}+5499857952b^{16}-3770546960ab^{16})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(4760540668a^{2}b^{16}-331233916a^{3}b^{16}+222945905a^{4}b^{16}-44760048b^{17})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{4194304b(106810248ab^{17}-16405740a^{2}b^{17}+13489164a^{3}b^{17}+771932b^{18})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\left.\frac{4194304b(-370804ab^{18}+425334a^{2}b^{18}-2660b^{19}+5740ab^{19}+21b^{20})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}\right\}-$$

$$-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{4194304a(25510826561258284646400000-8530455637768840151040000a)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}\right.+$$

$$+\frac{4194304a(5046844677567391413043200a^{2}-908113612506970403635200a^{3})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(94329379394886165331968 0a^{4}-10487474967141140869939 2a^{5})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

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$$+\frac{4194304a(4380456394743479454924 8a^6-320665715712314900480 0a^7)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(695881739005324099584a^8-345021733408844144 64a^9)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(43703280416081295 36a^{10}-147413395681244160a^{11})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(1151997378029388 8a^{12}-259529058500096a^{13}+12727480051584a^{14})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-182186263680a^{15}+5499857952a^{16}-44760048a^{17}+771932a^{18}-2660a^{19})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(21a^{20}+853045563776884015104000b+1155916906042116915855360 0ab)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-11420316980369467165900 80a^2b+542395215716520673935360 0a^3b)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-4833404884948555703255 04a^4b+462098774179625173516288a^5b)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-29739154860751054831616a^6b+1187836816369626172620 8a^7b)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-540003762528448172032a^8b+11351125000308688691 2a^9b)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-3598980835570312192a^{10}b+442655007120794624a^{11}b)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-9538778579925504a^{12}b+720991701034752a^{13}b-10015737262208a^{14}b)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(470276092928a^{15}b-3770546960a^{16}b+106810248a^{17}b-370804a^{18}b)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(5740a^{19}b+504684467756739141304320 0b^{2})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(114203169803694671659008 0ab^{2}+916213853714368756816281 6a^{2}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-3978077830795869416325 12a^{3}b^{2}+16062368979927723024056 32a^{4}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-79985901310925941112832a^{5}b^{2}+70203458129013818785792a^{6}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-2770353357030304694272a^{7}b^{2}+1044011438827617827840a^{8}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-302381103241646274456a^{9}b^{2}+60586514553597260 80a^{10}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-1228063136325642 24a^{11}b^{2}+14420734130391680a^{12}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-192246021387392a^{13}b^{2}+13807642532608a^{14}b^{2}-108000161152a^{15}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(4760540668a^{16}b^{2}-16405740a^{17}b^{2}+425334a^{18}b^{2})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(908113612506970403635200b^{3}+5423952157165206739353600ab^{3})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(397807783079586941632512a^{2}b^{3}+238707806194259706 4065024a^{3}b^{3})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-541658583393401691176 96a^{4}b^{3}+191084514358987959304192a^{5}b^{3})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-56856313639013242798 08a^{6}b^{3}+45798072350195832176 64a^{7}b^{3})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-11382021105791901593 6a^{8}b^{3}+40163467045632939008a^{9}b^{3})}{\left[\prod\limits_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod\limits_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-7399825228667688 96a^{10}b^{3}+1398965040052935 68a^{11}b^{3})}{\left[\prod\limits_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod\limits_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-1751777944705152a^{12}b^{3}+194139922099200a^{13}b^{3}-1458526810240a^{14}b^{3})}{\left[\prod\limits_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod\limits_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(98249075392a^{15}b^{3}-331233916a^{16}b^{3}+13489164a^{17}b^{3})}{\left[\prod\limits_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod\limits_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(943293793948861653319680b^{4}+483340488494855570325504ab^{4})}{\left[\prod\limits_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod\limits_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(1606236897992772302405632a^{2}b^{4}+541658583393401691176 96a^{3}b^{4})}{\left[\prod\limits_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod\limits_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(264441231300695277830144a^{4}b^{4}-34345249747560745943 04a^{5}b^{4})}{\left[\prod\limits_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod\limits_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(10753422354294917699584a^{6}b^{4}-1976023601015476080 64a^{7}b^{4})}{\left[\prod\limits_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod\limits_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(146137881502128465664a^{8}b^{4}-22875823167243904 00a^{9}b^{4})}{\left[\prod\limits_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod\limits_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(752245508455199616a^{10}b^{4}-8519417326446208a^{11}b^{4})}{\left[\prod\limits_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod\limits_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(1508934143912832a^{12}b^{4}-10625670431040a^{13}b^{4}+1101695764400a^{14}b^{4})}{\left[\prod\limits_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod\limits_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-3567134480a^{15}b^{4}+222945905a^{16}b^{4}+104874749671411408699392b^{5})}{\left[\prod\limits_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod\limits_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(462098774179625173516288ab^{5}+799859013109259411128 32a^{2}b^{5})}{\left[\prod\limits_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod\limits_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(1910845143589879593041 92a^{3}b^{5}+343452497475607459430 4a^{4}b^{5})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(1422190991211059102924 8a^{5}b^{5}-110718760473363474432a^{6}b^{5})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(310775437793794541568a^{7}b^{5}-35427700425206369 28a^{8}b^{5})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(2406924367316781312a^{9}b^{5}-22967896652821120a^{10}b^{5})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(7015062932620288a^{11}b^{5}-44474696966336a^{12}b^{5}+7339626148064a^{13}b^{5})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-22225991760a^{14}b^{5}+2140280688a^{15}b^{5}+438045639474347945492 48b^{6})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(297391548607510548316 16ab^{6}+702034581290138187857 92a^{2}b^{6})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(568563136390132427980 8a^{3}b^{6}+107534223542949176995 84a^{4}b^{6})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(110718760473363474432a^{5}b^{6}+398392850354233326592a^{6}b^{6})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-188259694846724300 8a^{7}b^{6}+47699250518657337 60a^{8}b^{6})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-32753273458249344a^{9}b^{6}+2045180135321523 2a^{10}b^{6})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-108494163290240a^{11}b^{6}+30698633436016a^{12}b^{6}-83361188848a^{13}b^{6})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(12759365640a^{14}b^{6}+320665715712314900480 0b^{7})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(11878368163696261726208ab^7+27703533570303046942272a^2b^7)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(45798072350195832176640a^3b^7+19760236010154760806 4a^4b^7)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(3107754377937945415 68a^5b^7+1882596948467243008a^6b^7)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(59783793000109967 36a^7b^7-167389025164445824a^8b^7)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(3847029191392358 4a^9b^7-145457559854464a^{10}b^7+83501312754496a^{11}b^7)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(-188638464560a^{12}b^7+49336213808a^{13}b^7+6958817390053240995 84b^8)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(5400037625284481720 32ab^8+1044011438827617827840a^2b^8)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(1138202110579190159 36a^3b^8+1461378815021284656 64a^4b^8)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(354277004252063692 8a^5b^8+476992505186573376 0a^6b^8)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(167389025164445824a^7b^8+4741166322600364 8a^8b^8-72173360207520a^9b^8)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(150988354535304a^{10}b^8-241258667832a^{11}b^8+127330963578a^{12}b^8)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(34502173340884414464b^9+113511250003086886912ab^9)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(30238110324164627456a^2b^9+40163467045632939008a^3b^9)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(22875823167243904000a^4b^9+24069243673167813120a^5b^9)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(327532734582493440a^6b^9+38470291913923584a^7b^9+72173360207520a^8b^9)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(183709752798000a^9b^9-117012581400a^{10}b^9+223387655400a^{11}b^9)}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(437032804160812953600b^{10}+3598980835570312192ab^{10})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(6058651455359726080a^2b^{10}+7399825228667688960a^3b^{10})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(7522455084551996160a^4b^{10}+2296789665282112000a^5b^{10})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(204518013532152320a^6b^{10}+14545755985446400a^7b^{10}+150988354535304a^8b^{10})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(117012581400a^9b^{10}+269128937220a^{10}b^{10}+147413395681244160b^{11})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(442655007120794624ab^{11}+1228063136325642240a^2b^{11})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(1398965040052935680a^3b^{11}+851941732644620800a^4b^{11})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(70150629326202880a^5b^{11}+10849416329024000a^6b^{11}+83501312754496a^7b^{11})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(241258667832a^8b^{11}+223387655400a^9b^{11}+11519973780293888b^{12})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(95387785799255040ab^{12}+144207341303916800a^2b^{12})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(1751777944705152a^3b^{12}+1508934143912832a^4b^{12}+444746969663336a^5b^{12})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(30698633436016a^6b^{12}+188638464560a^7b^{12}+127330963578a^8b^{12})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(2595290585000096b^{13}+720991701034752ab^{13}+192246021387392a^2b^{13})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(194139922099200a^3b^{13}+10625670431040a^4b^{13}+7339626148064a^5b^{13})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(83361188848a^6b^{13}+49336213808a^7b^{13}+127274800051584b^{14})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(10015737262208ab^{14}+13807642532608a^2b^{14}+1458526810240a^3b^{14})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(1101695764400a^4b^{14}+22225991760a^5b^{14}+12759365640a^6b^{14})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(182186263680b^{15}+470276092928ab^{15}+108000161152a^2b^{15})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(98249075392a^3b^{15}+3567134480a^4b^{15}+2140280688a^5b^{15}+5499857952b^{16})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(3770546960ab^{16}+4760540668a^2b^{16}+331233916a^3b^{16}+222945905a^4b^{16})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(447600048b^{17}+106810248ab^{17}+16405740a^2b^{17}+13489164a^3b^{17})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{4194304a(771932b^{18}+370804ab^{18}+425334a^2b^{18}+2660b^{19}+5740ab^{19}+21b^{20})}{\left[\prod_{\heartsuit=0}^{20}\{a-b-2\heartsuit\}\right]\left[\prod_{\diamondsuit=1}^{21}\{a-b+2\diamondsuit\}\right]}+$$

$$+\frac{2097152(2551082656125828464640000a+4589065620297665740800000a^2)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(3618572796858388709376000a^3+1687018700164430403993600a^4)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(526766035608866113191936a^5+117964128844107192729600a^6)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(19769717617137637130240a^7+2550445204778112122880a^8)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(258174206358711894016a^9+20771430124567449600a^{10})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(1338915850793364480a^{11}+69419613644559360a^{12}+2895430910817536a^{13})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(96782231616000a^{14}+2570993384320a^{15}+53512986240a^{16}+853247136a^{17})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(10054800a^{18}+82460a^{19}+420a^{20}+a^21+2551082656125828464640000b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(18034285618697563275264000a^2b+5302160797104273044275200a^3b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(5607018417653412916101120a^4b+932256615762419968376832a^5b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(363791583564959961055232a^6b+37315288811906944204800a^7b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(7489957442085282054144a^8b+498166507892515897344a^9b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(5863376279561345126 4a^{10}b+2576442421949061120a^{11}b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(188756045605906688a^{12}b+5435341998729216a^{13}b+252735800846976a^{14}b)}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(4577441890560a^{15}b+133017777312a^{16}b+1369444608a^{17}b+23317028a^{18}b)}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(103320a^{19}b+861a^{20}b-4589065620297665740800000b^{2})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(18034285618697563275264000ab^{2}+15762845986109545324216320a^{3}b^{2})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(17155827418481989276139520a^{4}b^{2}+19483194815064979210240000a^{5}b^{2})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(1583350447423392550748160a^{6}b^{2}+660371654986776452792320a^{7}b^{2})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(37341639012290615009280a^{8}b^{2}+7932464489826017290240a^{9}b^{2})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(307551481970268856320a^{10}b^{2}+37893449287479618560a^{11}b^{2})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(98535301715655168a^{12}b^{2}+7467613794096512a^{13}b^{2}+124108229323008a^{14}b^{2})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(5886041837312a^{15}b^{2}+56283597552a^{16}b^{2}+1635834564a^{17}b^{2}+6805344a^{18}b^{2})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(111930a^{19}b^{2}+3618572796858388709376000b^{3})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(-5302160797104273044275200ab^{3}+15762845986109545324216320a^{2}b^{3})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(4279477751493666085797888a^{4}b^{3}+2239173161783538218434560a^{5}b^{3})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(253642950373348367663104a^{6}b^{3}+114707858235817963683840a^{7}b^{3})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(480840864493597953945 6a^{8}b^{3}+161326575795225544704a^{9}b^{3})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(34409466474637782528a^{10}b^{3}+8061783807777742336a^{11}b^{3})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(992194769293262 08a^{12}b^{3}+1523034274347264a^{13}b^{3}+114316013559552a^{14}b^{3})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(1027334730240a^{15}b^{3}+47621245308a^{16}b^{3}+188848296a^{17}b^{3}+5245786a^{18}b^{3})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(-1687018700164430403993600b^{4}+5607018417653412916101120ab^{4})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(-7155827418481989276139 52a^{2}b^{4}+4279477751493666085797888a^{3}b^{4})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(484737428561631998771200a^{5}b^{4}+138258355866387691683 84a^{6}b^{4})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(1512065079021431753728 0a^{7}b^{4}+404425418626749391872a^{8}b^{4})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(165274807848601394944a^{9}b^{4}+3363994992766087680a^{10}b^{4})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(700557366559616384a^{11}b^{4}+97612048243207 68a^{12}b^{4})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(1169020433721728a^{13}b^{4}+97959000 72000a^{14}b^{4}+709859761520a^{15}b^{4})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(2675350860a^{16}b^{4}+118030185a^{17}b5267660356088661131919 36b^{5})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(-932256615762419968376832ab^{5}+1948319481506497921024000a^{2}b^{5})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(-22391731617835382184 3456a^{3}b^{5}+484737428561631998771200a^{4}b^{5})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(264298765617049129308 16a^{6}b^{5}+43782575290439060275 2a^{7}b^{5})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(4546882768824968732 16a^{8}b^{5}+73312482184688947 20a^{9}b^{5})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(287078767335167808 0a^{10}b^{5}+3472382323492992 0a^{11}b^{5})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(6941613146622336a^{12}b^{5}+52920909565440a^{13}b^{5}+6064403010960a^{14}b^{5})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(21402806880a^{15}b^{5}+1471442973a^{16}b^{5}-117964128844107192729600b^{6})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(363791583564959961055232ab^{6}-158335044742339255074816a^{2}b^{6})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(25364295037334836766 3104a^{3}b^{6}-13825835586638769168384a^{4}b^{6})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(26429876561704912930816a^{5}b^{6}+747218528298866111488a^{7}b^{6})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(7351101916744954368a^{8}b^{6}+718201975449691532 8a^{9}b^{6})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(68357994717492480a^{10}b^{6}+2537349296679142 4a^{11}b^{6})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(167526567988416a^{12}b^{6}+31784430433616a^{13}b^{6}+102074925120a^{14}b^{6})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(11058116888a^{15}b^{6}+197697176171376371302 40b^{7})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(-37315288811906944204800ab^{7}+66037165498677645279232a^{2}b^{7})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(-11470785823581796368384a^{3}b^{7}+15120650790214317537280a^{4}b^{7})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(-4378257529043906027 52a^{5}b^{7}+747218528298866111488a^{6}b^{7})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(11286363877994105472a^{8}b^{7}+647522537970147 84a^{9}b^{7})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(5917878540782 2080a^{10}b^{7}+305570484570624a^{11}b^{7}+10673164324 8048a^{12}b^{7})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(296017282848a^{13}b^{7}+52860229080a^{14}b^{7}-2550445204778112122880b^{8})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(74899574420852820541 44ab^{8}-37341639012290615009 28a^{2}b^{8})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(48084086449359795394 56a^{3}b^{8}-40442541862674939187 2a^{4}b^{8})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(45468827688249687321 6a^{5}b^{8}-735110191674495436 8a^{6}b^{8})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(11286363877994105472a^{7}b^{8}+899374427131340 16a^{9}b^{8})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(277179402820320a^{10}b^{8}+236085009732936a^{11}b^{8}+509323854312a^{12}b^{8})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(166509721602a^{13}b^{8}+2581742063587118940 16b^{9}-49816650789251589734 4ab^{9})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(79324644898260172902 4a^{2}b^{9}-16132657579522554470 4a^{3}b^{9})}{\left[\prod_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(1652748078486013949944a^{4}b^{9}-733124821846889472 0a^{5}b^{9})}{\left[\prod\limits_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(71820197544969153 28a^{6}b^{9}-647522537970147 84a^{7}b^{9})}{\left[\prod\limits_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(8993744271313401 6a^{8}b^{9}+349750605804600a^{10}b^{9}+446775310800a^{11}b^{9})}{\left[\prod\limits_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(353697121050a^{12}b^{9}-207714301245674496 00b^{10}+5863376279561345126 4ab^{10})}{\left[\prod\limits_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(-3075514819702688563 2a^{2}b^{10}+3440946647463778252 8a^{3}b^{10})}{\left[\prod\limits_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(-336399499276608768 0a^{4}b^{10}+287078767335167808 0a^{5}b^{10})}{\left[\prod\limits_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(-6835799471749248 0a^{6}b^{10}+5917878540782208 0a^{7}b^{10})}{\left[\prod\limits_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(-277179402820320a^{8}b^{10}+349750605804600a^{9}b^{10}+513791607420a^{11}b^{10})}{\left[\prod\limits_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(1338915850793364480b^{11}-2576442421949061120ab^{11})}{\left[\prod\limits_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(378934492874796185 6a^{2}b^{11}-806178380777742336a^{3}b^{11})}{\left[\prod\limits_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(70055736655961638 4a^{4}b^{11}-34723823234929920a^{5}b^{11})}{\left[\prod\limits_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(2537349296679142 4a^{6}b^{11}-305570484570624a^{7}b^{11}+2360850097329 36a^{8}b^{11})}{\left[\prod\limits_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(-446775310800a^{9}b^{11}+513791607420a^{1}0b^{11}-694196136445593 60b^{12})}{\left[\prod\limits_{\clubsuit=0}^{21}\left\{a-b-2\clubsuit\right\}\right]\left[\prod\limits_{\spadesuit=1}^{20}\left\{a-b+2\spadesuit\right\}\right]}+$$

$$+\frac{2097152(188756045605906688ab^{12}-98535301715655168a^2b^{12})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(99219476929326208a^3b^{12}-9761204824320768a^4b^{12})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(6941613146622336a^5b^{12}-167526567988416a^6b^{12}+106731643248048a^7b^{12})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(-509323854312a^8b^{12}+353697121050a^9b^{12}+2895430910817536b^{13})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(-5435341998729216ab^{13}+7467613794096512a^2b^{13}-1523034274347264a^3b^{13})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(1169020433721728a^4b^{13}-52920909565440a^5b^{13}+31784430433616a^6b^{13})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(-296017282848a^7b^{13}+166509721602a^8b^{13}-967822316160000b^{14})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(2527358008469760ab^{14}-124108229323008a^2b^{14}+114316013559552a^3b^{14})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(-9795900072000a^4b^{14}+6064403010960a^5b^{14}-102074925120a^6b^{14})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(52860229080a^7b^{14}+25709933843200b^{15}-4577441890560ab^{15})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(5886041837312a^2b^{15}-1027334730240a^3b^{15}+709859761520a^4b^{15})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(-21402806880a^5b^{15}+11058116888a^6b^{15}-535129862400b^{16})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(1330177773120ab^{16}-56283597552a^2b^{16}+47621245308a^3b^{16})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(-2675350860a^4b^{16}+1471442973a^5b^{16}+853247136b^{17}-1369444608ab^{17})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(1635834564a^2b^{17}-188848296a^3b^{17}+118030185a^4b^{17}-10054800b^{18})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$+\frac{2097152(23317028ab^{18}-6805344a^2b^{18}+5245786a^3b^{18}+82460b^{19}-103320ab^{19})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}+$$

$$\left.\left.+\frac{2097152(111930a^2b^{19}-420b^{20}+861ab^{20}+b^{21})}{\left[\prod_{\clubsuit=0}^{21}\{a-b-2\clubsuit\}\right]\left[\prod_{\spadesuit=1}^{20}\{a-b+2\spadesuit\}\right]}\right\}\right] \tag{8}$$

III. Derivation of the Summation Formula

Using the same parallel method of Ref.[5],we obtain the main result.

IV. Open Problem

The result established in this paper having number of terms. It is open problem to all researchers working in mathematics as well as computer science to suggest an technique to reduce the number of terms or provide its general result which is yet to be discovered.

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Some Fixed Point Theorems via W-Distance on Cone Metric Spaces

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Keywords and phrases : *Cone metric space, w-distance, expansive mapping, fixed point.*

GJSFR-F Classification: *MSC 2010: 54H25*

SOME FIXED POINT THEOREMS VIA W-DISTANCE ON CONE METRIC SPACES

Strictly as per the compliance and regulations of :

Some Fixed Point Theorems via w-Distance on Cone Metric Spaces

Sushanta Kumar Mohanta[α], Rima Maitra[Ω]

Abstract - In this paper we present some fixed point theorems with the help of the concept of w- distance on cone metric spaces. Our results generalize and extend several well known results in the existing literature.

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I. Introduction and Preliminaries

In 1996, Kada et.al.[10] introduced the notion of w-distance on a metric space and proved a nonconvex minimization theorem which generalizes Caristi's fixed point theorem and the ϵ-variational principle. Afterwards, Huang and Zhang [8] initiated the notion of cone metric spaces by replacing the set of real numbers with an ordered Banach space . They also proved some fixed point theorems of contractive mappings on complete cone metric spaces with the assumption of normality of a cone. After that series of articles about cone metric spaces started to appear. In this work we extend the idea of w -distance on metric spaces to cone metric spaces and prove some fixed point theorems by considering w -distance on cone metric spaces. Our results generalize some recent results in fixed point theory.

Let E be a real Banach space and P be a subset of E Then P is called a cone if and only if

(i) P is closed; nonempty and $P \neq \{\theta\}$;

(ii) $a, b \in R,\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ (equivalently, $y \geq x$) if and only if $y - x \in P$. $x < y$ (equivalently, $y > x$) will stand for $x \leq y$ and $x \neq y$ while $x \ll y$ will stand for $y - x \in int\, P$, where $int\, P$ denotes the interior of P. For a finite subset A of E, if there exists an element $x \in A$ such that $x \leq a$ for all $a \in A$, we write $x = min\, A$. If there is an element $y \in A$ such that $a \leq y$ for all $a \in A$, we write $y = max\, A$. It is to be noted that $min\, A$, $max\, A$ are exist if the ordering $\leq$ on E is complete. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \ implies \ \| x \| \leq M \| y \| .$$

The least positive number satisfying the above inequality is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if (x_n) is sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\| x_n - x \| \to 0\ (n \to \infty)$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Razapour and Hamlbarani [13] proved that there are no normal cones with normal constants $M < 1$ and for each $k > 1$ there are cones with normal constants $M > k$.

Definition 1.1. [8] Let X be a non empty set. Suppose the mapping $d : X \times X \to E$ satisfies

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(i) $\theta \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 1.2. [8] Let (X,d) be a cone metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is a natural number n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$, then (x_n) is said to be convergent and (x_n) converges to x, and x is the limit of (x_n) We denote this by $\lim_{n\to\infty} x_n = x \ or \ x_n \to x \ (n \to \infty)$.

Definition 1.3. [8] Let (X,d) be a cone metric space, (x_n) be a sequence in X. If for any $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n,m > n_0$, $d(x_n, x_m) \ll c$, then (x_n) is called a Cauchy sequence in X.

Definition 1.4. [8] Let (X,d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

We also note that the relations $int\,P + int\,P \subseteq int\,P$ and $\lambda\, int\,P \subseteq int\,P\,(\lambda > 0)$ hold.

Lemma 1.1. [14] Let (X,d) be a cone metric space and $a,b,c \in X$. Then

(i) If $a \ll b$ and $b \ll c$ then $a \ll c$.

(ii) If $a \leq b$ and $b \ll c$ then $a \ll c$.

Here we present some elementary results of [8].

Let (X,d) be a cone metric space, P a normal cone with normal constant M, $x \in X$ and (x_n) a sequence in X.Then

(i) (x_n) converges to x if and only if $d(x_n, x) \to \theta$ (Lemma 1).

(ii) Limit point of every sequence is unique (Lemma 2).

(iii) Every convergent sequen ce is Cauchy (Lemma 3).

(iv) (x_n) is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ as $n,m \to \infty$ (Lemma 4).

(v) If $x_n \to x$ and $y_n \to y$ then $d(x_n, y_n) \to d(x,y)$ as $n \to \infty$ (Lemma 5).

Proposition 1.1. [9] If E is a real Banach space with cone P and if $a \leq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$ then $a = \theta$.

In the following definition we extend the idea of w-distance on metric spaces to cone metric spaces.

Definition 1.5. Let (X,d) be a cone metric space. Then a function $p : X \times X \to P$ is called a w-distance on X if the following conditions are satisfied:

(i) $p(x,z) \leq p(x,y) + p(y,z)$ for any $x,y,z \in X$;

(ii) for any $x \in X$, $p(x,.) : X \to P$ is lower semicontinuous $i.e.$, if $x \in X$,

$$y_n \to y \ \in X \ then \ p(x,y) \leq \lim_{n\to\infty} \ inf\, p(x, y_n);$$

(iii) for any $\theta \ll \alpha$, there exists $\theta \ll \beta$ such that $p(z,x) \ll \beta$ and $p(z,y) \ll \beta$ imply $d(x,y) \ll \alpha$.

Example 1.1. Let $E = R^2$, $P = \{(x,y) \in E : x,y \geq 0\}$, $X = R$ and $d : X \times X \to E$ defined by $d(x,y)$ $(|x-y|, a|x-y|)$ where $a \geq 0$ is a constant. Then (X,d) is a cone metric space. We define $p : X \times X \to P$ by $p(x,y) = (c,c)$ for every $x,y \in X$, where c is a positive real number. Then p is a w - distance on X.

Proof. (i) and (ii) are obvious. To show (iii), for any $\theta \ll \alpha$, put $\beta = (\frac{c}{2}, \frac{c}{2})$.Then $p(z,x) \ll \beta$ and $p(z,y) \ll \beta$ imply $d(x,y) \ll \alpha$.

Example 1.2. Let (X,d) be a cone metric space, P a normal cone. Then d is a w-distance on X.

Proof. (i) and (ii) are obvious. To show (iii), let $0 \ll \alpha$ be given and put $\beta = \frac{\alpha}{2}$. Then if $d(z,x) \ll \beta$ and $d(z,y) \ll \beta$, we have

$$d(x,y) \leq d(z,x) + d(z,y) \ll \beta + \beta = \alpha.$$

Definition 1.6. Let (X,d) be a cone metric space. A mapping $T : X \to X$ is said to be expansive if there exists a real constant $c > 1$ satisfying $d(T(x),T(y)) \geq c\,d(x,y)$ for all $x,y \in X$.

2. Main Results

In this section we always suppose that E is a real Banach space, P is a non normal cone in E with $int\,P \neq \emptyset$ and $\leq$ is the partial ordering on E with respect to P. Throughout the paper we denote by N the set of all natural numbers.

We start with the following lemma that will be needed in the sequel.

Lemma 2.1. Let (X,d) be a cone metric space and let p be a w-distance on X. Let (x_n) and (y_n) be sequences in X. Let (α_n) and (β_n) be sequences in P converging to θ and let $x,y,z \in X$. Then the following hold:

(i) If $p(x_n,y_n) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in N$, then (y_n) converges to z;

(ii) If $p(x_n,y) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in N$, then $y = z$. In particular, if $p(x,y) = \theta$ and $p(x,z) = \theta$, then $y = z$;

(iii) If $p(x_n,x_m) \leq \alpha_n$ for any $n,m \in N$ with $m > n$, then (x_n) is a Cauchy sequence.

Proof. (i) Let $\theta \ll \alpha$ be given. Then there exists $\theta \ll \beta$ such that $p(u,v) \ll \beta$ and $p(u,z) \ll \beta$ imply $d(v,z) \ll \alpha$. Choose $n_0 \in N$ such that $\alpha_n \ll \beta$ and $\beta_n \ll \beta$ for every $n \geq n_0$. Now, for any $n \geq n_0$, $p(x_n,y_n) \leq \alpha_n \ll \beta$ and $p(x_n,z) \leq \beta_n \ll \beta$ and hence $d(y_n,z) \ll \alpha$. This implies that (y_n) converges to z.

It follows from (i) that (ii) holds.

To prove (iii), let $\theta \ll \alpha$ be given. As in the proof of (i), choose $\theta \ll \beta$ and then $n_0 \in N$. Now for any $n,m \geq n_0+1$, $p(x_{n_0},x_n) \leq \alpha_{n_0} \ll \beta$ and $p(x_{n_0},x_m) \leq \alpha_{n_0} \ll \beta$ and hence $d(x_n,x_m) \ll \alpha$. This implies that (x_n) is a Cauchy sequence.

Theorem 2.1. Let (X,d) be a complete cone metric space with w-distance p and $\leq$ be a complete ordering on E with respect to P. Let T_1,T_2 be mappings from X into itself. Suppose that there exists $r \in [0,1)$ such that

$$max\,\{p(T_1(x),T_2T_1(x)),\; p(T_2(x),T_1T_2(x))\} \leq r\,min\,\{p(x,T_1(x)),p(x,T_2(x))\} \qquad (2.1)$$

for every $x \in X$ and that

$$inf\,\{p(x,y) + min\,\{p(x,T_1(x)),\,p(x,T_2(x))\} : x \in X\} > \theta \qquad (2.2)$$

for every $y \in X$ with y is not a common fixed point of T_1 and T_2. Then there exists $z \in X$ such that $z = T_1(z) = T_2(z)$. Moreover, if $v = T_1(v) = T_2(v)$, then $p(v,v) = \theta$.

Proof. Let u_0 be an arbitrary element of X. A sequence (u_n) in X is defined by

$$\begin{aligned} u_n &= T_1(u_{n-1}), \text{if } n \text{ is odd} \\ &= T_2(u_{n-1}), \text{if } n \text{ is even.} \end{aligned}$$

Then applying condition (2.1), we have for any positive integer n,

$$p(u_n,u_{n+1}) \leq r\,p(u_{n-1},u_n). \qquad (2.3)$$

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By repeated use of (2.3), we obtain

$$p(u_n, u_{n+1}) \leq r^n \, p(u_0, u_1).$$

If $m > n$,then

$$\begin{aligned} p(u_n, u_m) &\leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{m-1}, u_m) \\ &\leq \left[r^n + r^{n+1} + \cdots + r^{m-1}\right] p(u_0, u_1) \\ &\leq \frac{r^n}{1-r} p(u_0, u_1). \end{aligned}$$

Obviously, $\left(\frac{r^n}{1-r} p(u_0, u_1)\right)$ is a sequence in P converging to θ. So, by Lemma 2.1(iii), (u_n) is a Cauchy sequence in X. Since X is complete, (u_n) converges to some point $z \in X$. Let $n \in N$ be fixed. Then since (u_m) converges to z and $p(u_n, .)$ is lower semicontinuous, we have

$$p(u_n, z) \leq \lim_{m\to\infty} inf \, p(u_n, u_m) \leq \frac{r^n}{1-r} p(u_0, u_1).$$

Assume that z is not a common fixed point of T_1 and T_2. Then by hypothesis

$$\begin{aligned} \theta &< inf \, \{p(x, z) + min \, \{p(x, T_1(x)), p(x, T_2(x))\} : x \in X\} \\ &\leq inf \, \{p(u_n, z) + min \, \{p(u_n, T_1(u_n)), p(u_n, T_2(u_n))\} : n \in N\} \\ &\leq inf \, \left\{ \frac{r^n}{1-r} p(u_0, u_1) + p(u_n, u_{n+1}) : n \in N \right\} \\ &\leq inf \, \left\{ \frac{r^n}{1-r} p(u_0, u_1) + r^n \, p(u_0, u_1) : n \in N \right\} \\ &= \theta \end{aligned}$$

which is a contradiction. Therefore, $z = T_1(z) = T_2(z)$.

Suppose that $v = T_1(v) = T_2(v)$ for some $v \in X$. Then

$$\begin{aligned} p(v, v) &= max \, \{p(T_1(v), T_2T_1(v)), p(T_2(v), T_1T_2(v))\} \\ &\leq r \, min \, \{p(v, T_1(v)), p(v, T_2(v))\} \\ &= r \, min \, \{p(v, v), p(v, v)\} \\ &= r \, p(v, v). \end{aligned}$$

By Proposition 1.1, it follows that $p(v, v) = \theta$.

The following Corollary is the generalization of the result [10; Theorem 4] to cone metric spaces.

Corollary 2.1. Let (X , d) be a complete cone metric space, let p be a w-distance on X and let T be a mapping from X into itself. Suppose that there exists $r \in [0, 1)$ such that

$$p(T(x), T^2(x)) \leq r \, p(x, T(x))$$

for every $x \in X$ and that

$$inf \, \{p(x, y) + p(x, T(x)) : x \in X\} > \theta$$

for every $y \in X$ with $y \neq T(y)$. Then there exists $z \in X$ such that $z = T(z)$. Moreover, if $v = T(v)$, then $p(v, v) = \theta$.

Proof. Taking $T_1 = T_2 = T$ in Theorem 2.1, the conclusion of the Corollary follows.

Note: It is worth mentioning that for the cases $T_1 = T_2$ it is suficient to assume that $\leq$ is a partial ordering on

E with respect to P instead of a complete ordering.

Using Corollary 2.1, we obtain the following theorem:

Theorem 2.2. Let (X,d) be a complete cone metric space, let p be a w-distance on X and let $T:X\to X$ be continuous. Suppose that there exists $r\in[0,1)$ such that

$$p(T(x),T^2(x))\leq r\,p(x,T(x))$$

for every $x\in X$ .Then there exists $z\in X$ such that $z=T(z)$.Moreover, if $v=T(v)$, then $p(v,v)=\theta.$

Proof. Assume that there exists $y\in X$ with $y\neq T(y)$ and

$$inf\{p(x,y)+p(x,T(x)):x\in X\}=\theta.$$

Then, there is a sequence (x_n) in X such that

$$\lim_{n\to\infty}\{p(x_n,y)+p(x_n,T(x_n))\}=\theta.$$

So, it must be the case that $p(x_n,y)\to\theta$ and $p(x_n,T(x_n))\to\theta$. By Lemma 2.1(i),$(T(x_n))$converges to y.

Now,

$$\begin{aligned} p(x_n,T^2(x_n)) &\leq p(x_n,T(x_n))+p(T(x_n),T^2(x_n))\\ &\leq p(x_n,T(x_n))+r\,p(x_n,T(x_n))\\ &\longrightarrow \theta. \end{aligned}$$

Again, by Lemma 2.1(i), $(T^2(x_n))$ converges to y . Using continuity of T , we obtain

$$T(y)=T(\lim_n T(x_n))=\lim_n T^2(x_n)=y$$

which is a contradiction.

Hence, if $y\neq T(y)$,then

$$inf\{p(x,y)+p(x,T(x)):x\in X\}>\theta.$$

Now Corollary 2.1 applies to obtain the desired conclusion.

As an application of Corollary 2.1, we obtain the following results [8; Theorem 1; Theorem 3; Theorem 4].

Theorem 2.3. Let (X,d) be a complete cone metric space, P be a normal cone with normal constant M. Suppose the mapping $T:X\to X$ satisfies the contractive condition

$$d(T(x),T(y))\leq k\,d(x,y),\ \ for\ all\ x,y\in X, \qquad (2.4)$$

where $k\in[0,1)$ is a constant. Then T has a unique fixed point in X.

Proof. Since P is normal, we treat d as a w- distance on X.From (2.4), it follows that

$$d(T(x),T^2(x))\leq k\,d(x,T(x))\ \ for\ every\ x\in X.$$

Assume that there exists $y\in X$ with $y\neq T(y)$ and

$$inf\{d(x,y)+d(x,T(x)):x\in X\}=\theta.$$

Then, there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty}\{d(x_n,y)+d(x_n,T(x_n))\}=\theta.$$

So, we have $d(x_n,y)\to\theta$ and $d(x_n,T(x_n))\to\theta$.Then by Lemma 2.1(i), $(T(x_n))$ converges to y.Since P is normal, $d(T(x_n),T(y))\to d(y,T(y))$ as $n\to\infty$.

By using (2.4), we have

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$$d(T(x_n), T(y)) \leq k\, d(x_n, y) \text{ for any } n \in N.$$

Taking limit as $n \to \infty$, it follows that $d(y, T(y)) \leq \theta$ which implies that $-d(y, T(y)) \in P$. Also, $d(y, T(y)) \in P$ and hence $d(y, T(y)) = \theta$. So it must be the case that $y = T(y)$.

This is a contradiction.

Hence, if $y \neq T(y)$, then

$$inf\{d(x, y) + d(x, T(x)) : x \in X\} > \theta.$$

Now Corollary 2.1 applies to obtain a fixed point of T. Clearly a fixed point of T is unique.

Theorem 2.4. Let (X, d) be a complete cone metric space, P a normal cone with normal constant M. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$$d(T(x), T(y)) \leq k\,(d(T(x), x) + d(T(y), y)), \ \ for\ all\ x, y \in X, \tag{2.5}$$

where $k \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X.

Proof. Replacing y by $T(x)$ in (2.5), we have

$$d(T(x), T^2(x)) \leq k\,(d(x, T(x)) + d(T(x), T^2(x)) \text{ for every } x \in X.$$

So, it must be the case that

$$d(T(x), T^2(x)) \leq r\, d(x, T(x)) \text{ for every } x \in X,$$

where $0 \leq r = \frac{k}{1-k} < 1$.

By an argument similar to that used above, we have if $y \neq T(y)$, then

$$inf\{d(x, y) + d(x, T(x)) : x \in X\} > \theta.$$

Applying Corollary 2.1 we have the desired conclusion.

Theorem 2.5. Let (X, d) be a complete cone metric space, P a normal cone with normal constant M. Suppose the mapping $T : X \to X$ satisfies the contractive condition

$$d(T(x), T(y)) \leq k\,(d(T(x), y) + d(T(y), x)), \ \ for\ all\ x, y \in X,$$

where $k \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X.

Proof. The proof obtained by the same techniques as used above.

Theorem 2.6. Let (X, d) be a complete cone metric space with a w-distance p and $\leq$ be a complete ordering on E with respect to P. Let T_1, T_2 be mappings from X onto itself. Suppose that there exists $r > 1$ such that

$$min\{p(T_2T_1(x), T_1(x)),\ p(T_1T_2(x), T_2(x))\} \geq r\, max\{p(T_1(x), x), p(T_2(x), x)\} \tag{2.6}$$

for every $x \in X$ and that

$$inf\ \{p(x, y) + min\{p(T_1(x), x), p(T_2(x), x)\} : x \in X\} > \theta \tag{2.7}$$

for every $y \in X$ with y is not a common fixed point of T_1 and T_2. Then there exists $z \in X$ such that $z = T_1(z) = T_2(z)$. Moreover, if $v = T_1(v) = T_2(v)$, then $p(v, v) = \theta$.

Proof. Let u_0 be an arbitrary element of X. T_1 being onto, there exists an element u_1 satisfying $u_1 \in T_1^{-1}(u_0)$. Since T_2 is also onto, there is an element u_2 such that $u_2 \in T_2^{-1}(u_1)$. Proceeding in a similar way, we can find

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$u_{2n+1} \in T_1^{-1}(u_{2n})$ and $u_{2n+2} \in T_2^{-1}(u_{2n+1})$ for $n = 1, 2, 3, \cdots$.

Therefore, $u_{2n} = T_1(u_{2n+1})$ and $u_{2n+1} = T_2(u_{2n+2})$ for $n = 0, 1, 2, \cdots$.

Using condition (2.6), we have for any positive integer n,

$$p(u_{n-1}, u_n) \geq r\, p(u_n, u_{n+1})$$

which implies that,

$$p(u_n, u_{n+1}) \leq \frac{1}{r} p(u_{n-1}, u_n) \leq \cdots \leq \left(\frac{1}{r}\right)^n p(u_0, u_1). \tag{2.8}$$

Let $\alpha = \frac{1}{r}$, then $0 < \alpha < 1$ since $r > 1$.

Now, (2.8) becomes

$$p(u_n, u_{n+1}) \leq \alpha^n p(u_0, u_1).$$

If $m > n$, then

$$\begin{aligned} p(u_n, u_m) &\leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{m-1}, u_m) \\ &\leq \left[\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}\right] p(u_0, u_1) \\ &\leq \frac{\alpha^n}{1-\alpha} p(u_0, u_1). \end{aligned}$$

But $\left(\frac{\alpha^n}{1-\alpha} p(u_0, u_1)\right)$ is a sequence in P converging to θ. So, by Lemma 2.1(iii), (u_n) is a Cauchy sequence in X . Since X is complete, (u_n) converges to some point $z \in X$. Let $n \in N$ be fixed. Then since (u_m) converges to z and $p(u_n, .)$ is lower semicontinuous, we have

$$p(u_n, z) \leq \lim_{m\to\infty} inf\, p(u_n, u_m) \leq \frac{\alpha^n}{1-\alpha} p(u_0, u_1).$$

Assume that z is not a common fixed point of T_1 and T_2.Then by hypothesis

$$\begin{aligned} \theta &< inf\ \{p(x,z) + min\, \{p(T_1(x), x), p(T_2(x), x)\} : x \in X\} \\ &\leq inf\ \{p(u_n,z) + min\, \{p(T_1(u_n), u_n), p(T_2(u_n), u_n)\} : n \in N\} \\ &\leq inf\ \left\{\frac{\alpha^n}{1-\alpha} p(u_0,u_1) + p(u_{n-1}, u_n) : n \in N\right\} \\ &\leq inf\ \left\{\frac{\alpha^n}{1-\alpha} p(u_0,u_1) + \alpha^{n-1} p(u_0, u_1) : n \in N\right\} \\ &= \theta \end{aligned}$$

which is a contradiction. Therefore, $z = T_1(z) = T_2(z)$.

Suppose that $v = T_1(v) = T_2(v)$ for some $v \in X$.Then

$$\begin{aligned} p(v,v) &= min\ \{p(T_2T_1(v), T_1(v)), p(T_1T_2(v), T_2(v))\} \\ &\geq r\, max\ \{p(T_1(v), v), p(T_2(v), v)\} \\ &= r\, max\ \{p(v,v), p(v,v)\} \\ &= r\, p(v,v). \end{aligned}$$

By Proposition 1.1, we have $p\,(v,v) = \theta$.

Corollary 2.2. Let p be a w-distance on a complete cone metric space (X,d) and let $T : X \to X$ be an onto mapping. Suppose that there exists $r > 1$such that

$$p(T^2(x), T(x)) \geq rp(T(x), x) \tag{2.9}$$

for every $x \in X$ and that

$$inf\{p(x,y) + p(T(x), x) : x \in X\} > \theta \tag{2.10}$$

for every $y \in X$ and that $y \neq T(y)$. Then T has a fixed point in X. Moreover, if $v = T(v)$, then $p(v, v) = \theta$.

Proof. Taking $T_1 = T_2 = T$ in Theorem 2.6, we have the desired result.

The following theorem is the generalization of the result [15; Theorem 3] to cone metric spaces.

Theorem 2.7. Let (X, d) be a complete cone metric space, P a normal cone and T be a mapping of X into itself. If there is a real number r with $r > 1$ satisfying

$$d(T(x), T(y)) \geq r\, min\{d(T(x), x), d(T(y), y), d(x, y)\} \tag{2.11}$$

for any $x, y \in X$, and T is onto continuous, then T has a fixed point.

Proof. Since P is normal, d is a w-distance on X. Replacing y by $T(x)$ in (2.11), we obtain

$$d(T(x), T^2(x)) \geq r\, min\{d(T(x), x), d(T^2(x), T(x)), d(x, T(x))\} \tag{2.12}$$

for all $x \in X$.

We assume that $T(x) \neq T^2(x)$. Otherwise, T has a fixed point.

So, it follows from (2.12) that

$$d(T^2(x), T(x)) \geq rd(T(x), x)$$

for every $x \in X$.

Assume that there exists $y \in X$ with $y \neq T(y)$ and

$$inf\{d(x, y) + d(T(x), x) : x \in X\} = \theta.$$

Then, there exists a sequence (x_n) In X such that

$$\lim_{n\to\infty} \{d(x_n, y) + d(T(x_n), x_n)\} = \theta,$$

which gives that $d(x_n, y) \to \theta$ and $d(x_n, T(x_n)) \to \theta$. By Lemma 2.1(i), $T(x_n))$converges to y. Using continuity of T, we have

$$T(y) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = y,$$

which is a contradiction.

Hence, if $y \neq T(y)$, then

$$inf\{d(x, y) + d(T(x), x) : x \in X\} > \theta.$$

Thus, condition (2.10) is satisfied and Corollary 2.2 applies to obtain a fixed point of T.

Remark 2.1. For an expansive mapping $T : X \to X$, there exists $r > 1$ such that

$$d(T(x), T(y)) \geq r\, d(x, y) \geq r\, min\{d(T(x), x), d(T(y), y), d(x, y)\}$$

for all $x, y \in X$. However, the identity mapping satisfies condition (2.11) but it is not expansive. Thus, the class of mappings that considered in Theorem 2.7 is strictly larger than that of expansive mappings.

Theorem 2.8. Let (X,d) be a complete cone metric space, P a normal cone and the mapping $T:X\to X$ is continuous, onto and satisfies the condition

$$d(T(x),T(y))\geq k\,[d(T(x),x)+d(T(y),y)] \qquad (2.13)$$

for all $x,y\in X,$ where $\frac{1}{2}<k<1$ is a constant. Then T has a fixed point in X.

Proof. Replacing x by $T(x)$ and y by x in (2.13), we have

$$d(T^2(x),T(x))\geq k\,[d(T^2(x),T(x))+d(T(x),x)]$$

which implies that

$$d(T^2(x),T(x))\geq r\,d(T(x),x)\ \ for\ all\ x\in X,$$

where $r=\frac{k}{1-k}>1.$ By the same methods used above, if $y\neq T(y),$then

$$inf\{d(x,y)+d(T(x),x):x\in X\}>\theta,$$

which is condition (2.10) of Corollary 2.2.
Applying Corollary 2.2 we obtain the desired conclusion.
The following is the generalization of Caristi's theorem[2] to cone metric spaces.

Theorem 2.9. Let p be a w-distance in a complete cone metric space $(X,d),$ P a regular cone. Let T be a continuous mapping from X into itself. Suppose that there exists a mapping $Q:X\to P$ such that

$$p(x,T(x))\leq Q(x)-Q(T(x))$$

for all $x\in X$.Then T has a fixed point in X. Moreover, if $v=T(v)$ then $p(v,v)=\theta.$

Proof. Let $u_0\in X$ and let (u_n) be defined as follows:

$$u_n=T(u_{n-1})=T^n(u_0)\ \ for\ n=1,2,3,\cdots.$$

For any positive integer r, we have

$$\begin{aligned} p(u_r,u_{r+1}) &= p(u_r,T(u_r)) \\ &\leq Q(u_r)-Q(T(u_r)) \\ &= Q(u_r)-Q(u_{r+1}). \end{aligned}$$

Therefore,

$$\sum_{r=0}^{n-1}p(u_r,u_{r+1})\leq\sum_{r=0}^{n-1}[Q(u_r)-Q(u_{r+1})]=Q(u_0)-Q(u_n)\leq Q(u_0).$$

Since P is regular, the series $\sum_{r=0}^{\infty}p(u_r,u_{r+1})$ is convergent.

If $m,n\in N,\ m>n,$then

$$\begin{aligned} p(u_n,u_m) &\leq p(u_n,u_{n+1})+p(u_{n+1},u_{n+2})+\cdots+p(u_{m-1},u_m) \\ &= \sum_{r=n}^{m-1}p(u_r,u_{r+1}). \end{aligned} \qquad (2.14)$$

Since the series $\sum_{r=0}^{\infty}p(u_r,u_{r+1})$ is convergent, by applying Lemma 2.1(iii), it follows from (2.14) that (u_n)

is Cauchy. By completeness of (X, d), there exists $v \in X$ such that $\lim_n u_n = v$.

Using continuity of T we have

$$T(v) = \lim_n T(u_n) = \lim_n u_{n+1} = v.$$

So, v is a fixed point of T.

Now,

$$p(v,v) = p(v, T(v)) \leq Q(v) - Q(T(v)) = \theta$$

implies that $-p(v,v) \in P$.

Also, $p(v,v) \in P$. Since $P \cap (-P) = \theta$, we have $p(v,v) = \theta$.

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A Note on Strongly $(\Delta_{(r)})^{\lambda}$ - Summable And $(\Delta_{(r)})^{\lambda}$ - Statistical Convergence Sequences Of Fuzzy Numbers

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Abstract - In this article, we define and study the concepts of strongly $(\Delta_{(r)})^{\lambda}$ - summable and $(\Delta_{(r)})^{\lambda}$- statistical convergence of sequence of fuzzy numbers for several relations among them.

Keywords : Sequence of fuzzy numbers; Difference sequence; Statistical convergence; Summability .

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Strictly as per the compliance and regulations of :

A Note on Strongly $(\Delta_{(r)})^{\lambda}$ - Summable And $(\Delta_{(r)})^{\lambda}$ - Statistical Convergence Sequences Of Fuzzy Numbers

Iqbal H. Jebril

Abstract - In this article, we define and study the concepts of strongly $\left(\Delta_{(r)}\right)^{\lambda}$ - summable and $\left(\Delta_{(r)}\right)^{\lambda}$ - statistical convergence of sequence of fuzzy numbers for several relations among them.

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I. Introduction

The idea of the statistical convergence of sequence was introduced by Fast [4] and Schoenberg [12] independently in order to extend the notion of convergence of sequences. It is also found in Zygmund [16]. Later on it was linked with summability by Fridy and Orhan [5], Maddox [9] and many others. In [11] Nuray and Savaş extended the idea to sequences of fuzzy numbers and discussed the concept of statistically Cauchy sequences of fuzzy numbers. On strongly λ-summability and λ-statistical convergence can be found in [14]. In this article we extend these notions to difference sequences of fuzzy numbers.

Let $C(R^n)$ = $\{A \subset R^n$: A compact and convex$\}$. The space $C(R^n)$ has a liner structure induced by the operations $A+B = \{a + b: a \in A, b \in B\}$ and $\lambda A = \{\lambda a: a \in A\}$ for $A, B \in C(R^n)$ and $\lambda \in R$. The Hausdroff distance between A and B of $C(R^n)$ is defined as:

$$\delta_\infty(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

Let $L(R^n)$ denote the set of all fuzzy numbers. The linear structure of $L(R^n)$ induces addition $X + Y$ and scalar multiplication λX, $\lambda \in R$, in terms of α-level sets, by

$[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha$ and $[\lambda X]^\alpha = \lambda [X]^\alpha$ for each $0 \le \alpha \le 1$, where the α-level set $[X]^\alpha$ = $\{x \in R^n : X(x) \ge \alpha\}$ is a nonempty compact and convex subset of R^n and X is a fuzzy number i.e., a function from R^n to [0, 1] which is normal, fuzzy convex, upper semi-continuous and the closure $X^0 = \{x \in R^n : X(x) > 0\}$ is compact.

Define for each $1 \le q < \infty$

$$d_q(X,Y) = \left(\int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d_\infty\right)^{1/q}$$

And $d_\infty = \sup_{0\le\alpha\le1} \delta_\infty(X^\alpha, Y^\alpha)$. Clearly $d_\infty(X,Y) = \lim_{q\to\infty} d_q(X,Y)$ with $d_q \le d_r$ if $q \le r$. Moreover d_q is a complete, separable and locally compact metric space (see [1]).

Throughout the paper, d will denote d_q with $1 \le q < \infty$.

We now state the following definitions which can be found in [8, 11, 13].

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A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of all positive integers into $L(R)$. The fuzzy number X_k denotes the value of the function at $k \in N$ and is called the k-th term or general term of the sequence.

A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to the fuzzy number X_0, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$d(X_k, X_0) < \varepsilon \text{ for } k > n_0$$

Again $X = (X_k)$ is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$d(X_k, X_l) < \varepsilon \text{ for } k, l > n_0$$

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in N\}$ of fuzzy numbers is bounded.

The natural density of a set K of positive integers is denoted by $\delta(K)$ and defined by

$$\delta(K) = \lim_n \frac{1}{n} \text{card}\{k \le n : k \in K\}$$

A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number X_0 if for every $\varepsilon > 0$, $\lim_n \frac{1}{n} \text{card}\{k \le n : d(X_k, X_0) \ge \varepsilon\} = 0$ and we write st-lim $X_k = X_0$.

Let Z be a real sequence space, then Kizmaz [7] introduced the following difference sequence spaces:

$$Z(\Delta) = \{(x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = \ell_\infty, c, c_0$, where $\Delta x_k = x_k - x_{k+1}$, for all $k \in N$.

II. New Definitions and Main Results

In this section we define some new definitions and investigate the main results of this article.

Let r be a non-negative integer. Let $\lambda = (\lambda_k)$ be a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$. Then the sequence $X = (X_k)$ of fuzzy numbers is said to be strongly $\left(\Delta_{(r)}\right)^{\lambda}$ - summable to a fuzzy number X_0 if

$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k \in I_n} d\left(\Delta_{(r)} X_k, X_0\right) = 0$, where $I_n = [n - \lambda_n + 1, n]$ and $\left(\Delta_{(r)} X_k\right) = \left(X_k - X_{k-r}\right)$ and $\Delta_{(0)} X_k = X_k$ for all $k \in N$. For details about the operator, one can refer to Dutta [2, 3]

In this expansion it is important to note that we take $X_k = \bar{0}$ for non-positive values of k.

If we take $r = 0$, then strongly $\left(\Delta_{(r)}\right)^{\lambda}$- summability reduces to strongly λ- summability. It is clear that strongly λ-summability implies strongly $\left(\Delta_{(r)}\right)^{\lambda}$-summability.

In particular if we take $\lambda_n = n$, for all $n \in N$ then we say $X = (X_k)$ is strongly $\Delta(r)$ - Cesàro summable to X_0.

A sequence $X = (X_k)$ of fuzzy numbers is said to be $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent to a fuzzy number X_0 if for every $\varepsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \text{card}\{k \in I_n : d(\Delta_{(r)} X_k, X_0) \ge \varepsilon\} = 0$$

In particular if we take $\lambda_n = n$, for all $n \in N$, then we say that $X = (X_k)$ is $\Delta_{(r)}$ - statistically convergent to X_0.

Again if we take $\lambda_n = n$, for all $n \in N$, $r = 0$, then $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergence reduces to statistically convergence. Our next aim is to present some relationship between strongly $\left(\Delta_{(r)}\right)^{\lambda}$ - summability and $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent.

Theorem 2.1. *If a sequence $X = (X_k)$ is strongly $\left(\Delta_{(r)}\right)^{\lambda}$ - summable then it is $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent.*

Proof. Suppose $X = (X_k)$ is strongly $\left(\Delta_{(r)}\right)^{\lambda}$ - summable to X_0. Then

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} d\left(\Delta_{(r)} X_k, X_0\right) = 0.$$

Now the result follows from the following inequality:

$$\sum_{k\in I_n} d\left(\Delta_{(r)} X_k, X_0\right) \geq \varepsilon \text{card}\left\{k \in I_n : d(\Delta_{(r)} X_k, X_0) \geq \varepsilon\right\}$$

Theorem 2.2. *If a sequence $X = (X_k)$ is $\Delta_{(r)}$ - bounded and $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent then it is strongly $\left(\Delta_{(r)}\right)^{\lambda}$ - summable.*

Proof. Suppose $X = (X_k)$ is $\Delta_{(r)}$ -bounded and $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent to X_0. Since $X = (X_k)$ is $\Delta_{(r)}$ - bounded, we can find a fuzzy number M such that $d(\Delta_{(r)} X_k, X_0) \leq M$ for all $k \in N$

Again since $X = (X_k)$ is $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent to X_0, for every $\varepsilon > 0$

$$\lim_{n} \frac{1}{\lambda_n} \text{card}\left\{k \in I_n : d(\Delta_{(r)} X_k, X_0) \geq \varepsilon\right\} = 0$$

Now the result follows from the following inequality:

$$\frac{1}{\lambda_n} \sum_{k\in I_n} d\left(\Delta_{(r)} X_k, X_0\right) = \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ d(\Delta_{(r)} X_k, X_0)\geq\varepsilon}} d(\Delta_{(r)} X_k, X_0) + \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ d(\Delta_{(r)} X_k, X_0)<\varepsilon}} d(\Delta_{(r)} X_k, X_0)$$

$$\leq \frac{M}{\lambda_n} \text{card}\left\{k \in I_n : d(\Delta_{(r)} X_k, X_0) \geq \varepsilon\right\} + \varepsilon$$

Corollary 2.3. *If a sequence $X = (X_k)$ is $\Delta_{(r)}$ - bounded and $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent then it is strongly $\Delta_{(r)}$ - Cesàro summable.*

Proof. Proof follows by combining the above Theorem and the following inequality:

$$\frac{1}{n} \sum_{k=1}^{n} d\left(\Delta_{(r)} X_k, X_0\right) = \frac{1}{n} \sum_{k=1}^{n-\lambda_n} d\left(\Delta_{(r)} X_k, X_0\right) + \frac{1}{n} \sum_{k\in I_n} d\left(\Delta_{(r)} X_k, X_0\right)$$

$$\leq \frac{1}{\lambda_n} \sum_{k=1}^{n-\lambda_n} d\left(\Delta_{(r)} X_k, X_0\right) + \frac{1}{\lambda_n} \sum_{k\in I_n} d\left(\Delta_{(r)} X_k, X_0\right)$$

$$\leq \frac{2}{\lambda_n} \sum_{k\in I_n} d\left(\Delta_{(r)} X_k, X_0\right)$$

Theorem 2.4. *If a sequence $X = (X_k)$ is $\Delta_{(r)}$ - statistically convergent and* lim $\inf_n \left(\frac{\lambda_n}{n}\right) > 0$ *then it is $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent.*

Proof. Assume the given conditions. For a given $\varepsilon > 0$, we have

$$\{k \in I_n : d(\Delta_{(r)}X_k, X_0) \geq \varepsilon\} \subset \{k \leq n : d(\Delta_{(r)}X_k, X_0) \geq \varepsilon\}$$

Hence the proof follows from the following inequality:

$$\frac{1}{n}\text{card}\{k \leq n : d(\Delta_{(r)}X_k, X_0) \geq \varepsilon\} \geq \frac{1}{n}\text{card}\{k \in I_n : d(\Delta_{(r)}X_k, X_0) \geq \varepsilon\}$$
$$= \frac{\lambda_n}{n}\frac{1}{\lambda_n}\text{card}\{k \in I_n : d(\Delta_{(r)}X_k, X_0) \geq \varepsilon\}$$

Remark. It is easy to see that if a sequence $X = (X_k)$ is bounded then it is $\Delta_{(r)}$ - bounded. If $X = (X_k)$ is λ -statistically convergent then it is $(\Delta_{(r)})^{\lambda}$ -statistically convergent. Again if $X = (X_k)$ is strongly λ - summable then it is strongly $(\Delta_{(r)})^{\lambda}$ - summable. Therefore we can replace the phrases 'if a sequence $X = (X_k)$ is strongly $(\Delta_{(r)})^{\lambda}$ - summable' by 'if a sequence $X = (X_k)$ is strongly λ - summable', 'if a sequence $X = (X_k)$ is $\Delta_{(r)}$ - bounded and $(\Delta_{(r)})^{\lambda}$ - statistically convergent' by 'if a sequence $X = (X_k)$ is bounded and λ - statistically convergent', 'if a sequence $X = (X_k)$ is $\Delta_{(r)}$ - bounded and $(\Delta_{(r)})^{\lambda}$ - statistically convergent' by 'if a sequence $X = (X_k)$ is bounded and λ - statistically convergent' and 'if a sequence $X = (X_k)$ is $\Delta_{(r)}$ - statistically convergent' by 'if a sequence $X = (X_k)$ is statistically convergent' respectively in Theorem 2.1, Theorem 2.2, Corollary 2.3 and Theorem 2.4.

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Some Characterizations of Space-Like Rectifying Curves in the Minkowski Space-Time

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GJSFR Classification MSC: *14H45; 14H50; 53C40; 53C50*

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Strictly as per the compliance and regulations of :

Some Characterizations of Space-Like Rectifying Curves in the Minkowski Space-Time

Ahmad T. Ali [α], Mehmet Ä Onder [Ω]

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I. Introduction

Lorentzian geometry helps to bridge the gab between modern differential geometry and the mathematical physics of general relativity by giving an invariant treatment of Lorentzian geometry. The fact that relativity theory is expressed in terms of Lorentzian geometry is attractive for geometers, who can penetrate surprising quickly into cosmology (redshift, expanding universe and big bang) and a topic no less interesting geometrically, the gravitation of a single star (perihelion procession, bending of light and black holes) [18]. Despite its long history, the theory of curve is still one of the most important interesting topics in a differential geometry and its is being study by many mathematicians until now, see for example [1, 2, 3, 4, 5, 16, 19, 21, 24].

In the Euclidean space E^3, rectifying curves are introduced by Chen in [7] as space curves whose position vector always lies in its rectifying plane, spanned by the tangent and the binormal vector fields $\vec{T}$ and $\vec{B}$ of the curve. Therefore, the position vector $\vec{\alpha}$ of a rectifying curve satisfies the equation

$$\vec{\alpha}(s) = \lambda(s)\vec{T}(s) + \mu(s)\vec{B}(s),$$

for some differentiable functions λ and μ in arclength function s. The Euclidean rectifying curves are studied in [7, 8]. In particular, it is shown in [8] that there exists a simple relationship between the rectifying curves and the centrodes, which play some important roles in mechanics, kinematics as well as in differential geometry in defining the curves of constant precession. The rectifying curves are also studied in [8] as the extremal curves. In the Minkowski 3-space E_1^3, the rectifying curves are investigated in [10]. The rectifying curves are also studied in [11] as the centrodes and extremal curves. In the Euclidean 4-space E^4, the rectifying curves are investigated in [9].

In analogy with the rectifying curve, the curve whose position vector always lies in its normal plane spanned by the principal normal and the binormal vector fields $\vec{N}$ and $\vec{B}$ of the curve is called normal curve in Euclidean 3-space E^3 and it is well known that normal curves are spherical curves in E^3 [8]. Similar definition and characterizations of space-like, time-like (and also null) and dual time-like normal curves are given in references [10, 11, 17]. The space-like normal curve in Minkowski 4-space E_1^4 is defined in [14] as a curve whose position vector always lies in the orthogonal complement $\vec{T}^{\perp}$ of its tangent vector field $\vec{T}$ which is given by

$$\vec{T}^{\perp} = \{\vec{W} \in E_1^4 \mid g(\vec{W}, \vec{T}) = 0\}.$$

In [6], Camci and others have shown that a space-like curve lies in pseudohyperbolic space H_0^3 iff the following equation holds

$$\vec{\alpha} - m = -(1/k_1)\vec{N} - (1/k_2)(1/k_1)'\vec{B}_1 + (1/k_3)[k_2/k_1 + ((1/k_2)(1/k_1)')']\vec{B}_2,$$

Where m is constant, k_1, k_2 and k_3 are the first, the second and the third curvatures of the curve α, respectively. By using the definition of space-like normal curves in Minkowski 4-space E_1^4 and the last equality, it follows that every space-like curve lying in pseudohy-perbolic space H_0^3 is a normal curve in Minkowski 4-space.

In this paper, in analogy with the Minkowski 3-dimensional case, we define the rectifying curve in the Minkowski 4-space E_1^4 as a curve whose position vector always lies in the orthogonal complement $\vec{N}^{\perp}$ of its principal normal vector field $\vec{N}$ Consequently, $\vec{N}^{\perp}$ is given by

$$\vec{N}^{\perp} = \{\vec{W} \in E_1^4 \mid g(\vec{W}, \vec{N}) = 0\},$$

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Where $g(.,.)$ denotes the standard pseudo scalar product in E_1^4. Hence $\vec{N}^{\perp}$ is a 3-dimensional subspace of E_1^4, spanned by the tangent, the first binormal and the second binormal vector fields $\vec{T}$, $\vec{B}_1$ and $\vec{B}_2$ respectively. Therefore, the position vector with respect to some chosen origin, of a space-like rectifying curve $\vec{\alpha}$ in Minkowski space-time E_1^4, satisfies the equation

$$\vec{\alpha}(s) = \lambda(s)\vec{T}(s) + \mu(s)\vec{B}_1(s) + \nu(s)\vec{B}_2, \tag{1}$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ in arclength function s. Next, characterize space-like rectifying curves in terms of their curvature functions $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ and give the necessary and the sufficient conditions for arbitrary curve in E_1^4 to be a rectifying. Moreover, we obtain an explicit equation of a space-like rectifying curve in E_1^4 and give the relation between rectifying and normal space-like curves in E_1^4.

II. Preliminaries

In this section, we prepare basic notations on Minkowski space-time E_1^4. Let $\vec{\alpha} : I \subset R \to E_1^4$ be arbitrary curve in the Minkowski space - time E_1^4. Recall that the curve $\vec{\alpha}$ is said to be unit speed (or parameterized by arclength function s) if $g(\vec{\alpha}', \vec{\alpha}') = \pm 1$, where $g(.,.)$ denotes the standard pseudo scalar product in E_1^4 given by

$$g(\vec{v}, \vec{w}) = -v_1w_1 + v_2w_2 + v_3w_3 + v_4w_4,$$

for each $\vec{v} = (v_1, v_2, v_3, v_4) \in E_1^4$ and $\vec{w} = (w_1, w_2, w_3, w_4) \in E_1^4$. An arbitrary vector $\vec{v} \in E_1^4$ can have one of three Lorentzian causal characters; it can be space-like if $g(\vec{v}, \vec{v}) > 0$ or $\vec{v} = 0$, time-like if and null (light-like) if $g(\vec{v}, \vec{v}) = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ can locally be space-like, time-like or null (light-like), if all of its velocity vectors $\vec{\alpha}'(s)$ are respectively space-like, time-like or null (light-like). Also recall that the pseudo-norm of an arbitrary vector $\vec{v} \in E_1^4$ is given by $\| \vec{v} \| = \sqrt{| g(\vec{v}, \vec{v}) |}$. Therefore $\vec{v}$ is a unit is a unit vector if $g(\vec{v}, \vec{v}) = \pm 1$. The velocity of the curve $\vec{\alpha}(s)$ is given by $\| \vec{\alpha}'(s) \|$. Next, vectors $\vec{v}, \vec{w}$ in E_1^4 are said to be orthogonal if $g(\vec{v}, \vec{w}) = 0$.

Denote by $\{\vec{T}(s), \vec{N}(s), \vec{B}_1(s), \vec{B}_2(s)\}$ the moving Frenet frame along the curve $\vec{\alpha}(s)$ in the space E_1^4, where $\vec{T}(s)$, $\vec{N}(s)$, $\vec{B}_1(s)$ and $\vec{B}_2(s)$ are the tangent, principal normal, the first binormal and second binormal fields, respectively. For an arbitrary space - like curve $\vec{\alpha}(s)$ with space - like principal normal $\vec{N}$ in the space E_1^4, the following Frenet formula are given in [23, 20, 6, 15, 22, 25]:

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}_1' \\ \vec{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\varepsilon\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & \kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B}_1 \\ \vec{B}_2 \end{bmatrix}, \tag{2}$$

Where

$$g(\vec{B}_1, \vec{B}_1) = -g(\vec{B}_2, \vec{B}_2) = \varepsilon = \pm 1, \quad g(\vec{T}, \vec{T}) = g(\vec{N}, \vec{N}) = 1. \tag{3}$$

Recall the functions $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ are called respectively, the first, the second and the third curvatures of space-like curve $\vec{\alpha}(s)$. Here, ε determines the kind of space - like curve $\alpha(s)$. If $\varepsilon = 1$, then $\alpha(s)$ is a space-like curve with space - like first binormal $\vec{B}_1$ and time - like second binormal B_2. If $\varepsilon = -1$, then $\alpha(s)$ is a space-like curve with time - like first binormal $\vec{B}_1$ and space - like second binormal B_2. If $\kappa_3(s) \neq 0$ for each $s \in I \subset R$, the curve $\vec{\alpha}$ lies fully in E_1^4. Recall that the pseudohyperbolic space $H_0^3(1)$ in E_1^4, centered at the origin, is the hyperquadric defined by

$$H_0^3(1) = \{\vec{X} \in E_1^4 \mid g(\vec{X}, \vec{X}) = -1\}.$$

Recently, Yilmaz et al. [27, 26] defined a vector product in Minkowski spacetime $\mathbf{E}_1^4$ as follows:

Definition 2.1. *Let* $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ *and* $c = (c_1, c_2, c_3, c_4)$ *be vectors in* $\boldsymbol{E}_1^4$. *The vector product in Minkowski space-time* E_1^4 *is defined by the determinant*

$$a \wedge b \wedge c = -\begin{bmatrix} -e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}, \tag{4}$$

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where e_1, e_2, e_3 *and* e_4 *are mutually orthogonal vectors (coordinate direction vectors) satisfying equations*

$$e_1 \wedge e_2 \wedge e_3 = e_4,\ \ e_2 \wedge e_3 \wedge e_4 = e_1,\ \ e_3 \wedge e_4 \wedge e_1 = e_2, e_4 \wedge e_1 \wedge e_2 = -e_3.$$

Lemma 2.2. *Let* $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ *and* $c = (c_1, c_2, c_3, c_4)$ *be vectors in* $\boldsymbol{E}_1^4$. *From the definition of vector product, there is a property in Minkowski space - time* E_1^4 *as the following:*

$$g(a \wedge b \wedge c, a) = g(a \wedge b \wedge c, b) = g(a \wedge b \wedge c, c) = 0. \tag{5}$$

The proof of above lemma is elementary.

III. Some Characterizations of Rectifying Curves In E_1^4

In this section, we firstly characterize the space-like rectifying curves with space-like principal normal in Minkowski space - time in terms of their curvatures. Let $\vec{\alpha} = \vec{\alpha}(s)$ be a unit speed space - like rectifying curve in E_1^4, with non - zero curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ By definition, the position vector of the curve $\vec{\alpha}$ sacrifies the equation (1), for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$. Differentiating the equation (1) with respect to s and using the Frenet equations (2), we obtain

$$\vec{T} = \lambda' \vec{T} + (\lambda\kappa_1 - \varepsilon\mu\kappa_2)\vec{N} + (\mu' + \nu\kappa_3)\vec{B}_1 + (\nu' + \mu\kappa_3)\vec{B}_2. \tag{6}$$

It follows that

$$\begin{aligned} &\lambda' = 1,\\ &\lambda\kappa_1 - \varepsilon\mu\kappa_2 = 0,\\ &\mu' + \nu\kappa_3 = 0,\\ &\nu' + \mu\kappa_3 = 0, \end{aligned} \tag{7}$$

and therefore

$$\begin{aligned} &\lambda = s + c,\\ &\mu = \varepsilon\frac{\kappa_1(s)(s+c)}{\kappa_2},\\ &\nu = -\varepsilon\frac{\kappa_1(s)\kappa_2(s) + (s+c)(\kappa_1'(s)\kappa_2(s) - \kappa_1(s)\kappa_2'(s))}{\kappa_2^2(s)\kappa_3(s)}, \end{aligned} \tag{8}$$

where $c \in R$. In this way the functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are expressed in terms of the curvature functions $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ of the curve $\alpha(s)$. Moreover, by using the last equation in (7) and relation (8) we easily find that the curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ satisfy the equation

$$\frac{\kappa_1(s)\kappa_3(s)(s+c)}{\kappa_2(s)} - \Big[\frac{\kappa_1(s)\kappa_2(s) + (s+c)(\kappa_1'(s)\kappa_2(s) - \kappa_1(s)\kappa_2'(s))}{\kappa_2^2(s)\kappa_3(s)}\Big]' = 0,\ \ c \in R. \tag{9}$$

The condition (9) can be written as:

$$\frac{\kappa_1(s)(s+c)}{\kappa_2(s)} - \frac{1}{\kappa_3(s)}\frac{d}{ds}\Big[\frac{1}{\kappa_3(s)}\frac{d}{ds}\Big(\frac{\kappa_1(s)(s+c)}{\kappa_2(s)}\Big)\Big] = 0. \tag{10}$$

If we change the variable s by the variable t as the following

$$\frac{d}{dt} = \frac{1}{\kappa_3(s)}\frac{d}{ds} \ \ \text{or} \ \ t = \int_0^s \kappa_3(s)ds,$$

the equation (10) takes the following form

$$\frac{\kappa_1(s)(s+c)}{\kappa_2(s)} - \frac{d^2}{dt^2}\Big[\frac{\kappa_1(s)(s+c)}{\kappa_2(s)}\Big] = 0. \tag{11}$$

General solution of this equation is

$$\frac{\kappa_1(s)(s+c)}{\kappa_2(s)} = \varepsilon\Big(A\cosh\textstyle\int_0^s \kappa_3(s)ds + B\sinh\int_0^s \kappa_3(s)ds\Big), \tag{12}$$

where A and B are arbitrary constants. Then from (8) we have

$$\begin{aligned}\lambda(s) &= s + c\\ \mu(s) &= A\cosh\int_0^s \kappa_3(s)ds + B\sinh\int_0^s \kappa_3(s)ds\\ \nu(s) &= -\Big(A\sinh\int_0^s \kappa_3(s)ds + B\cosh\int_0^s \kappa_3(s)ds\Big).\end{aligned} \tag{13}$$

Conversely, assume that the curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ of an arbitrary unit speed space - like curve in E_1^4, satisfy the equation (12). Let us consider the vector $\vec{X} \in E_1^4$ given by

$$\begin{aligned}\vec{X}(s) &= \vec{\alpha}(s) - (s+c)\vec{T}(s) - \Big(A\cosh\int_0^s \kappa_3(s)ds + B\sinh\int_0^s \kappa_3(s)ds\Big)\vec{B}_1(s)\\ &+\Big(A\sinh\int_0^s \kappa_3(s)ds + B\cosh\int_0^s \kappa_3(s)ds\Big)\vec{B}_2(s).\end{aligned} \tag{14}$$

By using the relations (2) and (12), we easily find $\vec{X}'(s) = 0$, which means that $\vec{X}$ is a constant vector. This implies that $\alpha(s)$ is congruent to a rectifying curve. In this way, the following theorem is proved.

Theorem 3.1. *Let $\vec{\alpha}(s)$ be unit speed space - like curve with space - like principal normal in E_1^4 and with non - zero curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$. Then $\vec{\alpha}(s)$ is congruent to a space - like rectifying curve if and only if*

$$\frac{\kappa_1(s)(s+c)}{\kappa_2(s)} = \varepsilon\Big(A\cosh\int_0^s \kappa_3(s)ds + B\sinh\int_0^s \kappa_3(s)ds\Big).$$

In particular, assume that all curvature functions $\kappa_1(s)$, $\kappa_2(s)$, and $\kappa_3(s)$ of space - like rectifying curve, $\vec{\alpha}$ in E_1^4 are constant and different from zero. Then equation (9) easily implies a contradiction. Hence we obtain the following theorem.

Theorem 3.2. *There are no space - like rectifying curves with space - like principal normal lying in E_1^4, with non-zero constant curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$.*

In the next theorem, we give the necessary and the suficient conditions for the space - like curve $\alpha(s)$ in E_1^4 to be a rectifying curve.

Theorem 3.3. *Let $\alpha(s)$ be unit speed space-like rectifying curve with space-like prini-pal normal in E_1^4, with non-zero curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$. Then the following statements hold:*

(i) The distance function $\rho(s) = \| \vec{\alpha}(s) \|$ satisfies $\rho^2(s) = s^2 + c_1 s + c_2,\ c_1 \in R$ and $c_2 \in R_0$.

(ii) The tangential component of the position vector of the space-like rectifying curve is given by $g(\vec{\alpha}(s), \vec{T}(s)) = s + c,\ c \in R$.

(iii) The normal component $\vec{\alpha}^N(s)$ of the position vector of the space - like rectifying curve has constant length and the distance function $\rho(s)$ is non - constant.

(iv) The first binormal component and the second binormal component of the position vector of the space-like rectifying curve are respectively given by

$$\begin{aligned}g(\vec{\alpha}(s), \vec{B}_1(s)) &= \varepsilon\Big(A\cosh\int_0^s \kappa_3(s)ds + B\sinh\int_0^s \kappa_3(s)ds\Big)\\ g(\vec{\alpha}(s), \vec{B}_2(s)) &= \varepsilon\Big(A\sinh\int_0^s \kappa_3(s)ds - B\cosh\int_0^s \kappa_3(s)ds\Big).\end{aligned} \tag{15}$$

Conversely, if $\vec{\alpha}(s)$ is a unit speed curve in E_1^4 with non - zero curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ and one of the statements (i), (ii), (iii) or (iv) holds, then $\vec{\alpha}(s)$ is a space - like rectifying curve.

Proof. Let us first suppose that $\vec{\alpha}(s)$ is a unit speed space - like rectifying curve in E_1^4 with non - zero curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$. The position vector of the curve $\vec{\alpha}(s)$ satisfies the equation (1), where the functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ satisfy relation (13). From relation (1) and (13) we have

$$\begin{aligned}g(\vec{\alpha}, \vec{\alpha}) &= \lambda^2 + \varepsilon\Big(\mu^2(s) - \nu^2(s)\Big),\\ &= (s+c)^2 + \varepsilon(A^2 - B^2).\end{aligned} \tag{16}$$

Therefore, $\rho^2(s) = s^2 + c_1 s + c_2,\ c_1 \in R$ and $c_2 \in R_0$, which proves statement (i).

But using the relations (1) and (8) we easily get $g(\vec{\alpha}(s), \vec{T}(s)) = s + c,\ c \in R$, so the statement (ii) is proved.

Note that the position vector of an arbitrary curve $\vec{\alpha}(s)$ in E_1^4 can be decomposed as $\vec{\alpha}(s) = m(s)\vec{T}(s) + \vec{\alpha}^N(s)$, where $m(s)$ is arbitrary differentiable function and $\vec{\alpha}^N(s)$ is the normal component of the position vector. If $\vec{\alpha}(s)$ is a space-like rectifying curve, relation (1) implies $\vec{\alpha}^N(s) = \mu(s)\vec{B}_1(s) + \nu(s)\vec{B}_2(s)$ and therefore $g(\vec{\alpha}^N(s), \vec{\alpha}^N(s)) = \varepsilon\Big(\mu^2(s) - \nu^2(s)\Big)$.Moreover, by using (13), we find $\| \vec{\alpha}^N(s) \| = \varepsilon(A^2 - B^2) = a$, $a \in R$. By statement (i), $\rho(s)$ is non-constant function, which proves statement (iii).

Finally, using (1), (3) and (13) we easily obtain (15), which proves statement (iv).

Conversely, assume that statement (i) holds. Then $g(\vec{\alpha}(s), \vec{\alpha}(s)) = s^2 + c_1 s + c_2,\ c_1 \in R,\ c_2 \in R$ By differentiating the previous equation two times with respect to s and using (2), we obtain $g(\vec{\alpha}(s), \vec{N}(s)) = 0$, which implies that $\vec{\alpha}$ is a space-like rectifying curve.

If statement (ii) holds, in a similar way it follows that $\vec{\alpha}$ is a space-like rectifying curve.

If statement (iii) holds, let us put $\vec{\alpha}(s) = m(s)\vec{T}(s) + \vec{\alpha}^N(s)$, where $m(s)$ is arbitrary differentiable function. Then

$$g(\vec{\alpha}^N(s), \vec{\alpha}^N(s)) = g(\vec{\alpha}(s), \vec{\alpha}(s)) - 2g(\vec{\alpha}(s), \vec{T}(s))m(s) + m^2(s). \tag{17}$$

Since $g(\vec{\alpha}(s), \vec{T}(s)) = m(s)$, it follows that

$$g(\vec{\alpha}^N(s), \vec{\alpha}^N(s)) = g(\vec{\alpha}(s), \vec{\alpha}(s)) - g(\vec{\alpha}(s), \vec{T}(s))^2, \tag{18}$$

where $g(\vec{\alpha}(s), \vec{\alpha}(s)) = \rho^2(s) \neq$ constant. Defferentiating the previous equation with respect to *s* and using (2), we find

$$\kappa_1(s)\, g(\vec{\alpha}(s), \vec{T}(s))\, g(\vec{\alpha}(s), \vec{N}(s)) = 0. \tag{19}$$

It follows that $g(\vec{\alpha}(s), \vec{N}(s)) = 0$ and hence the space-like curve $\vec{\alpha}$ is a rectifying.

If the statement (iv) holds, by taking the derivative of the equation

$$g(\vec{\alpha}(s), \vec{B}_1(s)) = \varepsilon\Big(A\cosh\textstyle\int_0^s \kappa_3(s)ds + B\sinh\int_0^s \kappa_3(s)ds\Big), \tag{20}$$

with respect to s and using (2), we obtain

$$-\varepsilon\kappa_2(s)g(\vec{\alpha}(s), \vec{N}(s)) + \kappa_3 g(\vec{\alpha}(s), \vec{B}_2(s)) = \varepsilon\kappa_3\Big(A\sinh\textstyle\int_0^s \kappa_3(s)ds + B\cosh\int_0^s \kappa_3(s)ds\Big). \tag{21}$$

By using (15), the last equation becomes $g(\vec{\alpha}(s), \vec{N}(s)) = 0$, which means that $\vec{\alpha}$ is a space-like rectifying curve. This proves the theorem.

In the next theorem, we find the parametric equation of a rectifying curve.

Theorem 3.4. *Let $\alpha : I \subset R \to E_1^4$ be a space-like curve with space-like principal normal in E_1^4 given by $\vec{\alpha}(t) = \rho(t)\ \vec{y}(t)$ where $\rho(t)$ is an arbitrary positive function and $\vec{y}(t)$ is a unit speed space-like curve lying in pseudohyperbolic space $H_0^3(1)$. Then $\vec{\alpha}$ is a space-like rectifying curve if and only if*

$$\rho(t) = \frac{a}{\cosh(t + t_0)},\quad a \in R_0,\quad t_0 \in R. \tag{22}$$

Proof. Let $\vec{\alpha}$ be a curve in E_1^4 given by

$$\vec{\alpha}(t) = \rho(t)\vec{y}(t) \tag{23}$$

where $\rho(t)$ is arbitrary positive function and $\vec{y}(t)$ is a unit speed space-like curve in the pseudohyperbolic space $H_0^3(1)$.. By taking the derivative of the previous equation with respect to t, we get

$$\vec{\alpha}'(t) = \rho'(t)\vec{y}(t) + \rho(t)\vec{y}'(t). \tag{24}$$

Hence the unit tangent vector of $\vec{\alpha}$ is given by

$$\vec{T} = \frac{\rho'(t)}{v(t)}\vec{y}(t) + \frac{\rho(t)}{v(t)}\vec{y}'(t), \tag{25}$$

where $v(t) = \| \vec{\alpha}'(t) \|$ is the speed of $\vec{\alpha}$. Differentiating the equation (25) with respect to t, we find

$$\vec{T}' = \Big(\frac{\rho'}{v}\Big)'\vec{y} + \Big(\frac{2\rho'}{v} - \frac{\rho\rho'(\rho + \rho'')}{v^3}\Big)\vec{y}' + \Big(\frac{\rho}{v}\Big)\vec{y}''. \tag{26}$$

Let $\vec{Y}$ be the unit vector field in E_1^4 satisfying the equations $g(\vec{Y},\vec{y}) = g(\vec{Y},\vec{y}') = 0$. Then $\{\vec{y},\vec{y}',\vec{Y},\vec{y}\wedge\vec{y}' \wedge \vec{Y}\}$ is orthonormal frame in E_1^4. Therefore decomposition of $\vec{y}''$ with respect the frame $\{\vec{y},\vec{y}',\vec{Y},\vec{y}\wedge\vec{y}'\wedge\vec{Y}\}$ reads

$$\begin{aligned}\vec{y}'' = \; & g(\vec{y}'',\vec{y})g(\vec{y},\vec{y})\,\vec{y} + g(\vec{y}'',\vec{y}')\,g(\vec{y}',\vec{y}')\vec{y}' + g(\vec{y}'',\vec{Y})\,g(\vec{Y},\vec{Y})\vec{Y} \\ & + g(\vec{y}'',\vec{y}\wedge\vec{y}'\wedge\vec{Y})\,g(\vec{y}\wedge\vec{y}'\wedge\vec{Y},\vec{y}\wedge\vec{y}'\wedge\vec{Y})\vec{y}\wedge\vec{y}'\wedge\vec{Y}.\end{aligned} \tag{27}$$

Since $g(\vec{y},\vec{y}) = -1$ and $g(\vec{y}',\vec{y}') = 1$, it follows that $g(\vec{y}'',\vec{y}) = -1$ and $g(\vec{y}'',\vec{y}') = 0$, so the equation (27) the equation (27) becomes

$$\vec{y}'' = \vec{y} + g(\vec{y}'',\vec{Y})\,\vec{Y} + g(\vec{y}'',\vec{y}\wedge\vec{y}'\wedge\vec{Y})\,\vec{y}\wedge\vec{y}'\wedge\vec{Y}. \tag{28}$$

Substituting (28) into (26) and applying Frenet formulas for arbitrary speed curves in E_1^4, we find

$$\begin{aligned}\kappa_1 v\vec{N} \; & = \Big[\Big(\frac{\rho'}{v}\Big)' + \frac{\rho}{v}\Big]\vec{y} + \Big(\frac{2\rho'}{v} - \frac{\rho\rho'(\rho+\rho'')}{v^3}\Big)\vec{y}' + \Big(\frac{\rho}{v}\Big)g(\vec{y}'',\vec{Y})\vec{Y} \\ & + \frac{g(\vec{y}'',\vec{y}\wedge\vec{y}'\wedge\vec{Y})}{v}\,\vec{\alpha}\wedge\vec{y}'\wedge\vec{Y}.\end{aligned} \tag{29}$$

Since $g(\vec{y},\vec{y}) = -1$, we have $g(\vec{y},\vec{y}') = 0$ and thus $g(\vec{\alpha},\vec{y}') = 0$. We also have $g(\vec{\alpha},\vec{Y}) = 0$. By definition, $\vec{\alpha}$ is a space - like rectifying curve in E_1^4 if and only if $g(\vec{\alpha},\vec{N}) = 0$. Therefore, after taking the scalar product of (29) with $\vec{\alpha}$, we have $g(\vec{\alpha},\vec{N}) = 0$ if and only if

$$\Big(\frac{\rho'}{v}\Big)' + \frac{\rho}{v} = 0, \tag{30}$$

whose general solutions are $\rho(t) = \dfrac{a}{\cosh(t+t_0)}$ or $\rho(t) = \dfrac{a}{\sinh(t+t_0)}$, $a \in R_0, t \in R$. Since, $g(\vec{T},\vec{T}) = 1$, it follows that $\rho(t) = \dfrac{a}{\cosh(t+t_0)}$ is the only solution. This proves the theorem.

In Theorem 3.4, since $\vec{y}(s)$ is a unit speed space - like curve in the pseudohyperbolic space $H_0^3(1)$, $\vec{y}(s)$ is a space - like normal curve in Minkowski 4- space E_1^4. So, Theorem 3.4 gives the relation between space - like rectifying and space-like normal curves in Minkowski 4-space E_1^4. Then we can give the following corollary.

Corollary 3.5. *In Minkowski 4-space E_1^4, the construction of the space - like rectifying curve with space - like principal normal can be made by using the space - like normal curves.*

Example: Let us consider the curve

$$\vec{\alpha}(s) = \frac{a}{\cosh(s+s_0)}\Big(\sqrt{2}\cosh(s/\sqrt{3}), \sqrt{2}\sinh(s/\sqrt{3}), \sin(s/\sqrt{3}), \cos(s/\sqrt{3})\Big),$$

where $a \in R_0, s_0 \in R$ in E_1^4 This curve has a form $\vec{\alpha}(s) = \rho(s)\vec{y}(s)$ where $\rho(s) = \dfrac{a}{\cosh(s+s_0)}$ and $\vec{y}(s) = \Big(\sqrt{2}\cosh(s/\sqrt{3}), \sqrt{2}\sinh(s/\sqrt{3}), \sin(s/\sqrt{3}), \cos(s/\sqrt{3})\Big)$ Since $g(\vec{y}(s),\vec{y}(s)) = -1$ and $\| \vec{y}'(s) \| = 1$, $\vec{y}(s)$ is a unit speed space-like curve in pseudohyper- bolic space $H_0^3(1)$. According to theorem 3.4, $\vec{\alpha}$ is a space - like rectifying curve lying fully in E_1^4.

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Analysis of inventory system with three stages of deterioration

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Abstract - In this article, we consider a continuous review perishable inventory system with instantaneous replenishment policy. The status of perishable item in inventory is assumed to be in any one of the three stages good, average and damaged. The demand process is assumed to be Poisson, replenishment is instantaneous and the deterioration process is prescribed by certain transition probability matrix. Various stationary system performance measures are obtained. The total system maintenance cost rate is calculated and an optimal value of the S is obtained. The results are illustrated numerically.

Keywords : *Inventory control system; Perishable inventory; Poisson demand; Continuous time Markov chain; Optimization.*

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ANALYSIS OF INVENTORY SYSTEM WITH THREE STAGES OF DETERIORATION

Strictly as per the compliance and regulations of :

Analysis of inventory system with three stages of deterioration

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Abstract - In this article, we consider a continuous review perishable inventory system with instantaneous replenishment policy. The status of perishable item in inventory is assumed to be in any one of the three stages good, average and damaged. The demand process is assumed to be Poisson, replenishment is instantaneous and the deterioration process is prescribed by certain transition probability matrix. Various stationary system performance measures are obtained. The total system maintenance cost rate is calculated and an optimal value of the S is obtained.. The results are illustrated numerically.

Keywords : Inventory control system; Perishable inventory; Poisson demand; Continuous time Markov chain; Optimization.

I. Introduction

The classical inventory theory did not take in to accounts items that have finite lifetime (deteriorating items). However, items stocked in real life situations are subject to perishability due to excessive storage time or because of technology/style of change (obsolescence) occur. Examples for perishable items include certain foods, chemical, medicines, seasonal products and so on. Analysis of inventory systems stocking perishable items has been the theme of many researchers in the last three decades. The often quoted review articles of Nahmias (1982) and Rafat (1991) provide excellent summaries of many of the perishable inventory models. A recent comprehensive review paper focusing on the management of items with finite shelf life is published by Karesman et al. (2009). According to their classification three different kinds of perishable inventory problems have been studied earlier. Continuous review models ; (i) without fixed ordering cost, zero lead time, (ii) without fixed ordering cost, positive lead time, (iii) with fixed ordering cost, zero lead time.

Category (i) problem was studied by Graves (1982), who assumed that items are continuously produced, perish after a deterministic time and that demand follows a compound Poisson process.

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Category (ii) problem was studied first by Pal (1989) who investigated the system with (S-1, S) policy. Nahimias et al. (2004), analyse the same type of problems with emphasise on the performance measures rather than cost optimization.

Category (iii) problem originated by Weiss (1980) is most relevant for our paper. Lian et al. (2005) considered discrete demand for items and time to perish is either fixed or that follows a phase type distribution.

Our model is almost close to this model in terms of deterioration process. We consider an inventory system in which each item has an exponential life time and deterioration process passes through three different stages (good, average and damaged). We indicate the states with numerical indices 1, 2 and 3 respectively. It is also assumed that damaged (completely unusable) items were removed from stock at review time (demand epoch). Demand process is assumed to be Poisson and the replenishment is instantaneous (zero lead time). Fixed ordering cost K and holding cost h are assumed.

II. Problem Formulation

Consider an inventory system which stocks perishable items whose life time is exponentially distributed having three states (sojourn time of each state is exponentially distributed). Demand process follows a Poisson process with unit demand at a time, Identify the three states as good, average and damaged with numerical indicators 1, 2 and 3 respectively. The damaged items have to be removed immediately after the current demand is satisfied at time points (review epochs). The following assumptions are made:

Demand process is assumed to be Poisson with parameter $\lambda > 0$.

The replenishment is assumed to be instantaneous with (0, S) policy, where S is the maximum inventory level. Transition from one state of the inventory item to another in the process of deterioration during demand processing time is a random- phenomena with transition probabilities are given by the matrix $P = (p_{ij}), i, j = 1,2,3.$

III. Model Description and Analysis

Let I (t) and S (t) denote the number of items in stock and the status of the perished items in inventory respectively at time 't'. Then {(I(t), S(t)); t >0 } is a two dimensional stochastic process with state space .

$$E = \begin{cases} (i, j) : i = S,\ S-1,\ S-2,.....,1 \\ j = 1, 2, 3. \end{cases}$$

The embedded Markov chain $\{(I_n , S_n); n \geq 0\}$, where I_n denote the inventory level when the nth demand occurs and S_n, the status (stages) of the perishable item at that time.

The one step transition between stages of perishing is given by the tpm

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ 0 & P_{22} & P_{23} \\ 0 & 0 & 1 \end{bmatrix},$$

$$\textit{where } P_{11} = 1-(P_{12}+P_{13});\ P_{22} = 1-P_{23};\ P_{33} = 1.$$

Thus the two dimensional Markov process {(I(t), S(t)); t >0 } has transitions from one state to another as business cycle advances. From the assumptions we made on the input and output processes (replenishment and demand), it can be shown that the transition probabilities $\left(p_{(i,k)}^{(j,l)}(t)\right)$, of the Markov process has the derivative at time t=0. The intensity of transitions from state (i, k) to (j, l) is defined as $q_{(i,k)}^{(j,l)} = \frac{d}{dt} p_{(i,k)}^{(j.l)}(t) / t = 0.$

Now the infinitesimal generator, $Q = \left(q_{(i,k)}^{(j,l)}\right)$ of the Markov process be defined with intensity of transition defined as follows: System transition takes place from state:

- (i, k) to (i-1, k) with rate $\lambda > 0$ for k=1,2,3.
- (i, k) to (i-1, k+1) with rate $\lambda p_{i(k+1)}$ for k=1,2,3.
- (i, 3) to (S-1, 1) with rate $\lambda > 0$.

Infinitesimal generator (rate matrix) Q can be conveniently expressed as a block partitioned matrix $Q = (Q_{ij})$ where, i, j=1, 2, 3, …S.

$$Q_{ij} = \begin{cases} B & j = i = S \\ A & j = i, i = S-1, S-2,..1 \\ \Lambda_P & j = i-1, i = S, S-1,...2 \\ \Lambda_0 & j = S, i = S-1,...2 \\ \Lambda_1 & i = 1, j = S. \\ 0, \textit{otherwise}. \end{cases}$$

More explicitly,

$$Q = \begin{bmatrix} B & \wedge_P & 0 & 0 & \ldots & 0 \\ \wedge_0 & A & \wedge_P & 0 & \ldots & 0 \\ \wedge_0 & 0 & A & \wedge_P \ldots & & 0 \\ \wedge_0 & 0 & 0 & A & \ldots & \Lambda_p \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ \wedge_1 & 0 & 0 & 0 & \cdots & A \end{bmatrix}$$

Where,

$$B = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ \lambda & 0 & -\lambda \end{bmatrix}$$

$$\wedge_P = \begin{bmatrix} \lambda P_{11} & \lambda P_{12} & \lambda P_{13} \\ 0 & \lambda P_{22} & \lambda P_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = -\lambda I$$

$$\wedge_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}.$$

$$\wedge_1 = \begin{bmatrix} \lambda & 0 & 0 \\ \lambda & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix}$$

Since the state space of the Markov chain is finite, the states of the system are recurrent non-null and aperiodic. So by limiting probability arguments the Markov chain embedded in the process is ergodic. Let the steady state probability distribution of the states of the system (π_j), exists and can be obtained by solving the matrix equation

$$\pi Q = 0.$$

Let $\pi = (\pi_S, \pi_{S-1}, \ldots, \pi_1)$, where each $\pi_j = \left(\pi_j^{(1)}, \pi_j^{(2)}, \pi_j^{(3)}\right)$, with reference to the perished state of the system. Here the inventory means the on hand inventory, (not the position inventory). Now we get a system of matrix equations

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$$\pi_S B + \sum_{i=1}^{S-2} \pi_{S-i} \wedge_0 + \pi_1 \Lambda_1 = 0 \quad and$$

$$\pi_j \Lambda_p + \pi_{j-1} A = 0, j = S, S-1, \ldots, 2.$$

Assuming the initial probability vector $\pi_S = \left(\pi_S^{(1)}, \pi_S^{(2)}, \pi_S^{(3)}\right)$, we are able to get the solution for the above system of equations using recurrence method together with normalizing equation

$$\sum_{j=1}^{3} \left(\sum_{i=1}^{S} \pi_i^{(j)} \right) = 1.$$

The solution in terms of π_S is given by

$$\pi_k = (-1)^{S+k} \pi_S (\Lambda_p)^{S-k} A^{-S+k}, \; 1 \le k \le S-1.$$

And $\pi_S = 1' \left(I + \sum_{j=1}^{S-1} M_j \right)^{-1}$, where $1' = (1,1,1)$ and

$$M_K = (-1)^{S+k} (\Lambda_p)^{S-k} A^{-S+k}.$$

IV. System Performance Measures

The system performance measures of the perishable inventory system we considered can be obtained using the steady state probability vector $\pi_i^{(j)}, i = 1,2,3,\ldots S \; and \; j = 1,2,3.$

a) Mean inventory level

Let $\overline{L}$ denote the mean inventory level of the system in steady state. Then

$$\overline{L} = \sum_{i=1}^{S} \left(\sum_{j=1}^{3} i \pi_i^{(j)} \right).$$

b) Mean Reordering rate

The inventory maintained in the system is of perishable nature with three different states j = 1, 2, 3, and the last state j =3 represent the 'damaged' state of the items in inventory. All items with this condition can be removed from the system immediately after the supply of the demanded item to the customer. Simultaneously the instantaneous replenishment takes place for S items with zero lead time ((0, S) policy assumed). By the above arguments we can establish that the reorder rate 'β' for the system is given by

$$\beta = \left(\sum_{j=1}^{3} \pi_1^{(j)} + \sum_{i=2}^{S} \pi_i^{(3)} \right) \lambda.$$

V. Cost Optimization

The total cost incurred for the proposed system can be obtained with the assumption of proper cost structure as follows.

1. The reorder cost be 'K' per order per unit time.
2. The holding cost is 'h' per item per unit time.

Thus the total cost per unit time [total cost rate] is given by

$$TCU(S) = h\overline{L} + K\beta$$

$$= h \sum_{i=1}^{3} \left(\sum_{j=1}^{3} i \pi_i^j \right) + K \left(\sum_{j=1}^{3} \pi_1^{(j)} + \sum_{i=2}^{S} \pi_i^{(3)} \right) \lambda.$$

VI. Numerical Examples

The convexity of the total cost rate function TCU(S) cannot be proved analytically, due to its complex form. Hence a detailed computational study of the cost function is carried out and try to get the optimal solution S*(optimum ordering quantity) by implementing proper searching algorithms.The criterion used here is the minimization of total expected cost rate. Consider the numerical example with following parameters and transition probability matrix for the deterioration process:

$$P = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0 & 0.7 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}.$$

The system parameters S, λ, and the cost parameters K and h are varied and the corresponding expected total cost rates are obtained using the system performance measures we derived in the previous section IV.

All cost rates together with its optimal values are given in tables-1, 2 and 3. Table-1 shows the expected total costs for fixed parameters K= 100 and h=3, S varied from 4 to 10 and λ varied from 2 to 5. The local convex nature of the total expected cost rate function yields the following optimal pair of system parameters

(λ, S*): {(2, 8), (3, 6), (4, 5), (5, 5)}.

Table-2 shows the expected total costs for fixed parameters λ =2, h =3, S varied from 4 to 10 and K varied from 25 to 100 with step 25. The local convex nature of the expected cost rate function yields the following optimal pair of system parameters:

(K, S*): {(25, 5), (50, 6), (75, 7), (100, 8)}.

Table -1 (K = 100, and h = 3)

Total expected cost rate				
S \ λ	2	3	4	5
4	283.5396106	365.7873845	455.3403856	549.2222358
5	267.0650956	351.9852187	**446.6706657**	**544.6869059**
6	258.4846614	**349.7303686**	449.1308320	549.7077697
7	254.4437136	352.9306701	455.6550818	557.5243486
8	**253.2166870**	358.8100108	463.6914970	566.1660979
9	253.8151260	366.0487049	472.3026071	575.0547297
10	255.6312430	374.0032820	481.1386357	584.0188567

Table -2 (λ= 2, and h = 3)

Total expected cost rate				
S \ K	100	75	50	25
4	283.5396106	218.2797080	153.0198053	87.75990265
5	267.0650956	207.0488217	147.0325478	**87.01627391**
6	258.4846614	201.7384961	**144.9923307**	88.24616535
7	254.4437136	**199.8327852**	145.2218568	90.61092840
8	**253.2166870**	200.0375153	146.8583435	93.67917178
9	253.8151260	201.6113445	149.4075630	97.20378150
10	255.6312430	204.0984323	152.5656215	101.0328108

Table-3 (K =100, and λ = 2)

S \ h	3	4	5	6
4	283.5396106	291.0396106	298.5396106	306.0396106
5	267.0650956	276.0650956	285.0650956	294.0650956
6	258.4846614	268.9846614	279.4846614	**289.9846614**
7	254.4437136	**266.4437136**	**278.4437136**	290.4437136
8	**253.2166870**	266.7166870	280.2166871	293.7166871
9	253.8151260	268.8151260	283.8151260	298.8151260
10	255.6312430	272.1312430	288.6312431	305.1312431

Table-3 shows the expected total costs for fixed parameters, K= 100 and λ = 2, S varied from 4 to 10 and h varied from 3 to 6. The local convex nature of the total expected cost rate function yields the following optimal pair of system parameters:

(h, S*): {(3, 8), (4, 7), (5, 7), (6, 6)}.

The above sensitivity analysis shows that the expected total cost rate is sensitive to both the system parameters S, λ and the cost parameters K and h.

VII. Conclusion

Here we formulated and analyzed a perishable inventory system with (0, S) policy. The different stages of deterioration are considered (good, average, damaged). This model dealing with perishable inventory control system is tractable because of the assumed instantaneous replenishment ((0, S) policy). But in reality there exist a positive lead time. So further work in direction is possible by generalizing the policy to (s, S) type with ordering quantities Q= S - s .This batch ordering policy may increase the complexity of the problem and hence the tractability of the system becomes a question.

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Two-Commodity Inventory System for Base-Stock Policy with Service Facility

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Abstract - This article considers a two-commodity continuous review inventory system at a service facility, wherein an item demanded by a customer is issued after performing service on the item. The service facility is assumed to have a finite waiting hall. The arrival time points of customers form a Poisson process. A customer with probability p and a negative customer with probability $q = (1 - p), (0 \leq p \leq 1)$. An ordinary customer, on arrival, joins the queue and the negative customer does not join the queue and takes away one waiting customer if any. The life time of each item and service time are assumed to have independent exponential distribution. The joint probability distribution of the number of customers in the system and the inventory level is obtained in the steady state. Various system performance measures in the steady state are derived. The results are illustrated numerically.

Keywords : *Inventory system; base stock policy; service facility; negative customer; two-commodity .*

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TWO-COMMODITY INVENTORY SYSTEM FOR BASE-STOCK POLICY WITH SERVICE FACILITY

Strictly as per the compliance and regulations of :

Two-Commodity Inventory System for Base-Stock Policy with Service Facility

Gomathi, D[α]., Jeganathan. K[Ω]., Anbazhagan, N[β]

Abstract - This article considers a two-commodity continuous review inventory system at a service facility, wherein an item demanded by a customer is issued after performing service on the item. The service facility is assumed to have a finite waiting hall. The arrival time points of customers form a Poisson process. A customer with probability p and a negative customer with probability $q=(1-p), (0 \le p \le 1)$. An ordinary customer, on arrival, joins the queue and the negative customer does not join the queue and takes away one waiting customer if any. The life time of each item and service time are assumed to have independent exponential distribution. The joint probability distribution of the number of customers in the system and the inventory level is obtained in the steady state. Various system performance measures in the steady state are derived. The results are illustrated numerically.

Keywords : Inventory system; base stock policy; service facility; negative customer; two-commodity.

I. Introduction

The $(S-1,S)$ or one-to-one policies are usually implemented for inventory systems stocking expensive, slow moving items. Analysis of continuous review perishable inventory systems with positive lead times under $(S-1,S)$ policy have been carried out by Schmidt and Nahmias (1985), Pal (1989) and Kalpakam and Sapna (1995) and (1996). In all these models, whenever the inventory level drops by one unit, either due to a demand or a failure, an order for one item is placed. Kalpakam and Arivarignan (1998) dealt with a $(S-1,S)$ system with renewal demands for non-perishable items. Kalpakam and Shanthi (2000) have considered modified base stock policy and random supply quantity. Sren Gled Johansen (2005) has considered base-stock policies for the lost sales inventory system with Poisson demand and Erlangian lead times.

Krishnamoorthy et al. (1994) considered a two-commodity continuous review inventory system without lead time. In their model, each demand is for one unit of first commodity or one unit of second commodity or one unit of each commodity with prefixed probabilities. Krishnamoorthy and Varghese (1994) considered a two-commodity inventory problem without leadtime and with Markov shift in demand for the type of commodity namely "commodity-1", "commodity-2" or "both commodity". Yadavalli et al. (2006) have considered a two commodity inventory system with Poisson demands. It is further assumed that the demand for the first commodity require the one unit of second commodity in addition to the first commodity with probability p_1. Similarly, the demand for the second commodity require the one unit of first commodity in addition to the second commodity with probability p_2. Yadavalli et al. (2004) have considered a two commodity inventory system with individual and joint ordering policies.

In most of the inventory models considered in the literature, the demanded items are directly issued from the stock, if available. The demands that occur during stock-out period are either not satisfied (lost sales case) or satisfied only on receipt of the ordered items (backlog case). In the later case, it is assumed that either all (full backlogging) or any prefixed number of demands (partial backlogging) that occurred during stock-out period are satisfied. For review of these works see Nahmias (1982), Raafat (1981), Kalpakam and Arivarignan (1990), Elango and Arivarignan (2003) and Liu and Yang (1999).

But in the case of inventories maintained at service facilities, the demanded items are issued to the customers only after some service is performed on it. In this situation the items are issued not at the time of demand but after a random time of service from the epoch of demand. This forces the formation of queues in these models, which in turn necessitates the study of both inventory level and queue length joint distribution. Berman, Kaplan and Shimshank (1993) have considered an inventory management system at a service facility which uses one item of inventory for each service provided. They assumed that both demand and service times are deterministic and constant, as such queues can from only during stock out periods. They determined optimal order quantity that minimizes the total cost rate.

Berman and Kim (1999) analyzed a problem in stochastic environment where customers arrive at service facilities according to a Poisson process and the service times are exponentially distributed with mean inter-arrival time (assumed to be greater than the mean service time) and each service requires one item from

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inventory. The main result of their work is that under both the discounted cost case and the average cost case, the optimal policy of both the finite and infinite time horizon problem is a threshold ordering policy. The optimal policy in Berman and Kim (1999) is derived given that the order quantity is known. A logically related model was studied by He, Jewkes and Buzacott [9], who analyzed a Markovian inventory-production system, where customer demands arrive at a workshop and are processed by a single machine in batch sizes of one. Berman and Sapna (2000) studied extensively an inventory control problem at a service facility that uses one item of inventory distributed service times and zero lead times. They analyzed the system with the restriction that the waiting space is finite. Under a specified cost structure, they derived the optimal ordering quantity that minimizes the long-run expected cost rate.

Elango (2001) has considered a Markovian inventory system with instantaneous supply of orders at a service facility. The service time is assumed to have exponential distribution with parameter depending on the number of waiting customers. Arivarignan et al. (2002) have extended this model to include exponential inventory system in which the size of the space for the waiting customers is infinite. Arivarignan and Sivakumar (2003) have considered an inventory system with arbitrarily distributed demand, exponential service time and exponential lead time. Finally, Sivakumar et al. (2005) considered a two-commodity perishable inventory system under continuous review at a service facility with a finite waiting room.

In this paper we have considered a $(S-1,S)$ policy for two-commodity stochastic inventory system under continuous review at a service facility with a finite waiting room for customers. The customers arriving to the service station are classified as ordinary (positive or regular) and negative customers. The arrival of ordinary customers to the service station increase the queue length by one and the arrival of negative customer to the service station causes one ordinary customer to be removed if anyone is present.

In the real life situation, the sale agencies deal with two different items with high cost like email server and data server, refrigerator and washing machine etc.,. Keeping them in stock for sales purpose is high risk but yield high profit, wherein the waiting customers may be wooed or taken away by new arriving customers from a large population, many companies look for the prospective customers at others' sales centres. This motivates the researcher to consider the negative customer at a service facility for two commodities with $(S-1,S)$ policy.

The remainder of this paper is organized as follows. In Section 2, we present the mathematical model and the notations. Analysis of the model and the steady state solution are given in Section 3. In Section 4, we derive various measures of system performance in steady state. The total expected cost rate is calculated in Section 5. Our numerical study is presented in Section 6. Section 7 has concluding remarks.

II. Problem Formulation

Consider a two commodity stochastic inventory system with service facility in which the items are delivered to the demanding customers. The demand is for single item per customer. The maximum capacity of the i-th commodity is $S_i (i=1,2)$ units and the waiting hall space is M.

The following assumptions are made:

- The arrival times of customers form a Poisson process with parameter λ. The probability that an ordinary customer is p and a negative is $q(=1-p)$.
- The removal rule adopted in this paper is RCE (Removal of a customer at the end), i.e., arrival of a negative customer removes only a customer at the end including the one who is receiving the service at the time of arrival of a negative customer. The arrival of a negative customer has no effect to the empty service station.
- The demands occur either one unit of first commodity or one unit of second commodity or one unit of each commodity and the service time for each demand follows a negative exponential distribution with parameters γ_1, γ_2 and γ_{12} respectively.
- A one-to-one ordering policy is adopted. According to this policy, orders are placed for one unit of i-th commodity, as and when the inventory level of i-th commodity drops due to a demand $(i=1,2)$.
- The lead times of the reorders for the i-th commodity are assumed to be distributed as a negative exponential with parameter μ_i, $i=1,2$.
- The demands that occur during stock-out periods are lost.

A. Notations

$[A]_{ij}$: The element/submatirx at (i,j) th position of A.

0 : Zero matrix.

1 : An identity matrix.

e : A column vector of 1s appropriate dimension.

$$\delta_{ij} = \begin{cases} 1, & if\ i=j \\ 0, & otherwise. \end{cases}$$

$$\bar{\delta}_{ij} = (1-\delta_{ij})$$

$$E_1 = \{0,1,\ldots,S_1\}$$

$$E_2 = \{0,1,\ldots,S_2\}$$

$$E_3 = \{0,1,\ldots,M\}$$

$E = E_1 \times E_2 \times E_3$

$$[G_1]_{mn} = \begin{cases} \gamma_1 + \delta_{k0}\gamma_{12}, & n = m-1, \quad m = 1,2,\ldots,M \\ 0, & otherwise. \end{cases}$$

$$[G_2]_{mn} = \begin{cases} \gamma_2 + \delta_{i0}\gamma_{12}, & n = m-1, \quad m = 1,2,\ldots,M \\ 0, & otherwise. \end{cases}$$

$$[G_{12}]_{mn} = \begin{cases} \gamma_{12}, & n = m, \quad m = 1,2,\ldots,M \\ 0, & otherwise. \end{cases}$$

$$[P_{ik}]_{mn} = \begin{cases} -(p\lambda + (S_1 - i)\mu_1 + (S_2 - k)\mu_2), & n = m, \quad m = 0 \\ -(\lambda + \bar{\delta}_{0i}\gamma_1 + \bar{\delta}_{0k}\gamma_2 + (1-\delta_{0i}\delta_{0k})\gamma_{12} + (S_1 - i)\mu_1 + (S_2 - k)\mu_2), & n = m, \quad m = 1,2,\ldots,M-1 \\ -(q\lambda + \bar{\delta}_{0i}\gamma_1 + \bar{\delta}_{0k}\gamma_2 + (1-\delta_{0i}\delta_{0k})\gamma_{12} + (S_1 - i)\mu_1 + (S_2 - k)\mu_2), & n = m, \quad m = M \\ 0, & otherwise. \end{cases}$$

where $i = 0,1,\ldots,S_1$ and $k = 0,1,\ldots,S_2$

$$[M_{S_1 - i}]_{mn} = \begin{cases} (S_1 - i)\mu_1, & n = m, \quad m = 0,1,\ldots,M \\ 0, & otherwise. \end{cases}$$

$$[U_{S_2 - k}]_{mn} = \begin{cases} (S_2 - k)\mu_2, & n = m, \quad m = 0,1,\ldots,M \\ 0, & otherwise. \end{cases}$$

III. Analysis

Let $L_i(t)$ denote the inventory level of i-th commodity and $X(t)$ denote the number of customers (waiting and being served) in the system, at time t. From the assumptions made on the input and output processes, it may be shown that the triplet

$$(L_1, L_2, X) = \{(L_1(t), L_2(t), X(t),), t \geq 0\},$$

on the state space E, is a Markov process. The infinitesimal generator of this process,

$$A = (a((i,k,m),(j,l,n))), \quad (i,k,m),(j,l,n) \in E$$

can be obtained by using the following arguments:

- The arrival of an ordinary customer makes a transition from (i,k,m) to $(i,k,m+1), i = 0,1,\ldots,S_1$, $k = 0,1,\ldots,S_2$, $m = 0,1,\ldots,M-1$ with intensity of transition $p\lambda$, and the arrival of a negative customer makes a transition from (i,k,m) to $(i,k,m-1)$, $i = 0,1,\ldots,S_1$, $k = 0,1,\ldots,S_2$, $m = 1,2,\ldots,M$ with intensity of transition $q\lambda$, $q = 1-p$. The arrival of a negative customer has no effect on empty service station.

- The service completion involving the first commodity forces one customer to leave the system and a decrease of one item in the inventory level of the first commodity. Thus a transition takes place from (i,k,m) to $(i-1,k,m-1)$, $i = 1,2,\ldots,S_1$, $k = 0,1,\ldots,S_2$, $m = 1,2,\ldots,M$ with intensity γ_1.

- Similarly, a service completion involving the second commodity forces one customer to leave the system and a decrease of one item in the inventory level of the second commodity. Thus a transition takes place from state (i,k,m) to $(i,k-1,m-1)$, $i = 0,1,\ldots,S_1$, $k = 1,2,\ldots,S_2$, $m = 1,2,\ldots,M$ with intensity γ_2.

- A transition from state (i,k,m) to $(i-1,k-1,m-1)$ takes place when a service completion involving both commodity forces one customer to leave the system and a decrease of one item in the inventory level of first and second commodities. The intensity of the transition is γ_{12}, $i = 1,2,\ldots,S_1$, $k = 1,2,\ldots,S_2$, $m = 1,2,\ldots,M$.

- A transition from state (i,k,m) to $(i+1,k,m)$ for $i = 0,1,\ldots,S_1 - 1$, $k = 0,1,\ldots,S_2$, $m = 0,1,\ldots,M$ takes place when a replenishment occurs for the first commodity with intensity μ_1. Similarly, a transition from state (i,k,m) to $(i,k+1,m)$ for $i = 0,1,\ldots,S_1$, $k = 0,1,\ldots,S_2 - 1$, $m = 0,1,\ldots,M$ takes place when a replenishment occurs for the second commodity with intensity μ_2.

- For other transition from (i,k,m) to (j,l,n) except $(i,k,m) \neq (j,l,n)$, the rate is zero.

- Finally, note that

$$a((i,k,m),(i,k,m)) = -\sum_{j}\sum_{l}\sum_{\substack{n \\ (j,l,n)\neq(i,k,m)}} a((i,k,m),(j,l,n)).$$

Hence we have, $a((i,k,m),(j,l,n))$ is given by

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$$
\begin{cases}
p\lambda, & n=m+1, \; m=0,1,\dots,M-1 \\
 & l=k, \; k=0,1,\dots,S_2 \\
 & j=i, \; i=0,1,\dots,S_1 \\[1em]
q\lambda, & n=m-1, \; m=1,2,\dots,M \\
 & l=k, \; k=0,1,\dots,S_2 \\
 & j=i, \; i=0,1,\dots,S_1 \\[1em]
-(p\lambda+(S_1-i)\mu_1+ & n=m, \; m=0 \\
(S_2-k)\mu_2), & l=k, \; k=0,1,\dots,S_2 \\
 & j=i, \; i=0,1,\dots,S_1 \\[1em]
-(\lambda+\overline{\delta}_{0i}\gamma_1+\overline{\delta}_{0k}\gamma_2+ & n=m, \; m=1,2,\dots,M-1 \\
(1-\delta_{0i}\delta_{0k})\gamma_{12}+ & l=k, \; k=0,1,\dots,S_2 \\
(S_1-i)\mu_1+(S_2-k)\mu_2), & j=i, \; i=0,1,\dots,S_1 \\[1em]
-(q\lambda+\overline{\delta}_{0i}\gamma_1+\overline{\delta}_{0k}\gamma_2+ & n=m, \; m=M \\
(1-\delta_{0i}\delta_{0k})\gamma_{12}+ & l=k, \; k=0,1,\dots,S_2 \\
(S_1-i)\mu_1+(S_2-k)\mu_2), & j=i, \; i=0,1,\dots,S_1 \\[1em]
\gamma_1+\delta_{k0}\gamma_{12}, & n=m-1, \; m=1,2,\dots,M \\
 & l=k, \; k=0,1,\dots,S_2 \\
 & j=i-1, \; i=1,2,\dots,S_1
\end{cases}
$$

$$
\begin{cases}
\gamma_2+\delta_{i0}\gamma_{12}, & n=m-1, \; m=1,2,\dots,M \\
 & l=k-1, \; k=1,2,\dots,S_2 \\
 & j=i, \; i=0,1,\dots,S_1 \\[1em]
\gamma_{12}, & n=m-1, \; m=1,2,\dots,M \\
 & l=k-1, \; k=1,2,\dots,S_2 \\
 & j=i-1, \; i=1,2,\dots,S_1 \\[1em]
(S_1-i)\mu_1, & n=m, \; m=0,1,\dots,M \\
 & l=k, \; k=0,1,\dots,S_2 \\
 & j=i+1, \; i=0,1,\dots,S_1-1 \\[1em]
(S_2-i)\mu_2, & n=m, \; m=0,1,\dots,M \\
 & l=k+1, \; k=0,1,\dots,S_2-1 \\
 & j=i, \; i=0,1,\dots,S_1 \\[1em]
0, & \textit{otherwise}.
\end{cases}
$$

More explicitly,

$$
A=\begin{array}{c} S_1 \\ S_1-1 \\ S_1-2 \\ \vdots \\ 1 \\ 0 \end{array}
\begin{pmatrix}
A_{S_1} & B & & & & \\
C_1 & A_{S_1-1} & B & & & \\
 & C_2 & A_{S_1-2} & B & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & C_{S_1-1} & A_1 & B \\
 & & & & C_{S_1} & A_0
\end{pmatrix}
$$

Where

$$
[B]_{kl}=\begin{cases}
G_1, & l=k, \quad k=0,1,\dots,S_2 \\
G_{12}, & l=k-1, \quad k=1,2,\dots,S_2 \\
0, & \textit{otherwise}
\end{cases}
$$

$$
[C_{S_1-i}]_{kl}=\begin{cases}
M_{S_1-i}, & l=k, \quad k=0,1,\dots,S_2 \\
0, & \textit{otherwise}
\end{cases}
$$

$for \quad i=0,1,\dots,S_1$

$$
[A_i]_{kl}=\begin{cases}
G_2, & l=k-1, \quad k=1,2,\dots,S_2 \\
U_{S_2-k}, & l=k+1, \quad k=0,1,\dots,S_2-1 \\
P_{ik}, & l=k, \quad k=0,1,\dots,S_2 \\
0, & \textit{otherwise}
\end{cases}
$$

A Steady State Analysis

It can be seen from the structure of A that the homogeneous Markov process $\{(L_1(t),L_2(t),X(t))t\geq 0\}$ on the finite state space E is irreducible, aperiodic and persistent non-null. Hence the limiting distribution of the Markov process exists.

Let Π, partitioned as $\Pi=(\Pi^{(S_1)},\Pi^{(S_1-1)},\dots,\Pi^{(1)},\Pi^{(0)})$, denote the steady state probability vector of A. That is, Π satisfies

$$\Pi A = 0 \quad and \quad \Pi e = 1 \qquad (1)$$

The components of the vector $\Pi^{(q)}$ $(0 \le q \le S_1)$ are $\Pi^{(q)} = (\pi^{(q,S_2)},\ldots,\pi^{(q,1)},\pi^{(q,0)})$, where for $0 \le l \le S_2$, $\pi^{(q,l)} = (\pi^{(q,l,0)},\pi^{(q,l,1)},\ldots,\pi^{(q,l,M)})$.

From the structure of A, it is seen that the Markov process under study falls into the class of birth and death process in a Markovian environment as discussed by Gaver et al. (1984). Hence using the same argument, we can calculate the limiting probability vectors. For the sake of completeness, we provide the algorithm here.

Algorithm :

1. Determine recursively the matrices

$$F_0 = A_0$$

$$F_i = A_i + B(-F_{i-1}^{-1})C_{S_1-i+1}, \qquad i = 1,2,\ldots S_1$$

2. Compute recursively the vectors $\Pi^{(i)}$ using

$$\Pi^{(i)} = \Pi^{(i+1)}B(-F_i^{-1}), \qquad i = 0,1,\ldots,S_1 - 1$$

3. Solve the system of equations

$$\Pi^{(S_1)}F_{S_1} = 0$$

$$\sum_{i=0}^{S_1}\Pi^{(i)}e = 1.$$

From the system of equations $\Pi^{(S_1)}F_{S_1} = 0$, vector $\Pi^{(S_1)}$ could be determined uniquely, upto a multiplicative constant. This constant is decided by

$$\Pi^{(i)} = \Pi^{(i+1)}B(-F_i^{-1}), i = 0,1,\ldots,S_1 - 1$$

and $$\sum_{i=0}^{S_1}\Pi^{(i)}e = 1.$$

IV. System Performance Measures

In this section, some performance measures of the system are derived under consideration.

a) Mean Inventory Levels

Let η_1 and η_2 be the average inventory level for the first commodity and the second commodity respectively in the steady state. Then we have,

$$\eta_1 = \sum_{i=1}^{S_1} i\left(\sum_{k=0}^{S_2}\sum_{m=0}^{M}\pi^{(i,k,m)}\right)$$

and

$$\eta_2 = \sum_{k=1}^{S_2} k\left(\sum_{i=0}^{S_1}\sum_{m=0}^{M}\pi^{(i,k,m)}\right)$$

b) Mean Reorder Rates

Let η_3 and η_4 denote the mean reorder rate for the first and second commodities respectively. Then we have,

$$\eta_3 = (\gamma_1 + \gamma_{12})\sum_{i=1}^{S_1}\sum_{k=0}^{S_2}\sum_{m=1}^{M}\pi^{(i,k,m)}$$

and

$$\eta_4 = (\gamma_2 + \gamma_{12})\sum_{i=0}^{S_1}\sum_{k=1}^{S_2}\sum_{m=1}^{M}\pi^{(i,k,m)}$$

c) Mean Rate of Arrivals of Negative Customers

Let η_{NC} denote the mean rate of arrivals of negative customers for the system. Then we have,

$$\eta_{NC} = \sum_{i=0}^{S_1}\sum_{k=0}^{S_2} q\lambda\left(\sum_{m=1}^{M}\pi^{(i,k,m)}\right)$$

d) Mean Balking Rate

Let η_B denote the mean balking rate. Then we have,

$$\eta_B = p\lambda\sum_{i=0}^{S_1}\sum_{k=0}^{S_2}\pi^{(i,k,M)}$$

e) Mean Waiting time

Let $\overline{W}$ denote the mean waiting time of the customers. Then, by Little's formula

$$\overline{W} = \frac{\Gamma}{\lambda_a}$$

where, $$\Gamma = \sum_{m=1}^{M} m\left(\sum_{i=0}^{S_1}\sum_{k=0}^{S_2}\pi^{(i,k,m)}\right).$$

where λ_a denotes the expected arrival rate which is given by

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$$\lambda_a = p\lambda\sum_{i=0}^{S_1}\sum_{k=0}^{S_2}\sum_{m=0}^{M-1}\pi^{(i,k,m)}.$$

V. Cost Optimization

In order to compute the total expected cost per unit time, we introduce the following notations:

- c_{h_1} : The inventory holding cost per unit item per unit time for I-commodity.
- c_{h_2} : The inventory holding cost per unit item per unit time for II-commodity.
- c_{s_1} : The setup cost per order for I-commodity.
- c_{s_2} : The setup cost per order for II-commodity.
- c_N : Cost of loss per unit time due to arrival of a negative customer.
- c_w : Waiting time cost of a customer per unit time.
- c_B : Balking cost per customer per unit time.

Then the long-run expected cost rate is given by

$$TC(S_1,S_2,M)=c_{h_1}\eta_1+c_{h_2}\eta_2+c_{s_1}\eta_3+c_{s_2}\eta_4+c_N\eta_{NC}+c_w\overline{W}+c_B\eta_B.$$

Substituting $\eta's$ and $\overline{W}$ into the above equation, we obtain

$$TC(S_1,S_2,M)=c_{h_1}\left(\sum_{i=1}^{S_1} i\left(\sum_{k=0}^{S_2}\sum_{m=0}^{M}\pi^{(i,k,m)}\right)\right)+c_{h_2}\left(\sum_{k=1}^{S_2} k\left(\sum_{i=0}^{S_1}\sum_{m=0}^{M}\pi^{(i,k,m)}\right)\right)$$

$$+c_{s_1}\left((\gamma_1+\gamma_{12})\sum_{i=1}^{S_1}\sum_{k=0}^{S_2}\sum_{m=1}^{M}\pi^{(i,k,m)}\right)+c_{s_2}\left((\gamma_2+\gamma_{12})\sum_{i=0}^{S_1}\sum_{k=1}^{S_2}\sum_{m=1}^{M}\pi^{(i,k,m)}\right)$$

$$+c_N\left(\sum_{i=0}^{S_1}\sum_{k=0}^{S_2} q\lambda\left(\sum_{m=1}^{M}\pi^{(i,k,m)}\right)\right)+c_w\left(\frac{\Gamma}{\lambda_a}\right)+c_B\left(p\lambda\sum_{i=0}^{S_1}\sum_{k=0}^{S_2}\pi^{(i,k,M)}\right)$$

Due to the complex form of the limiting distribution, it is difficult to discuss the properties of the cost fuction analytically. Hence, a detailed computational study of the cost function is carried out in the next section.

VI Numerical Examples

Since we have not shown analytically the convexity of the function $TC(S_1,S_2,M)$ we have explored the behavior of this function by considering it as functions of any two variable by fixing the other one at a constant value.

The *table 1* gives the total expected cost rate for various combinations of S_1 and S_2 when fixed values for other parameters and costs are assumed. They are $M=3$, $\lambda=22$, $\gamma_1=3$, $\gamma_2=5$, $\gamma_{12}=9$, $p=0.7$, $q=0.3$, $\mu_1=1$, $\mu_2=2.1$, $c_{h_1}=6.7$, $c_{h_2}=7$, $c_{s_1}=0.2$, $c_{s_2}=0.5$, $c_N=5$, $c_w=5$, $c_B=0.5$.

Moreover, *Figure 1.* refers the changes of S_1 and S_2 are how to affect the total expected cost rate.

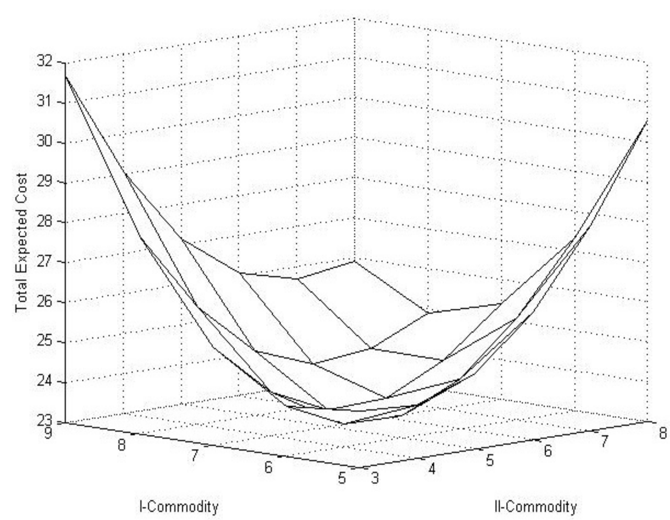

Fig.1. Convexity of the total cost for various combinations of S_1 and S_2.

S_2 / S_1	3	4	5	6	7	8
5	24.3900	<u>24.3208</u>	24.9074	26.1902	28.1159	30.5480
6	**24.2305**	**23.5581**	<u>23.5474</u>	24.2324	25.5464	27.3442
7	25.4291	24.0843	<u>**23.4229**</u>	**23.4859**	**24.1975**	25.3994
8	27.9424	25.9299	24.6207	<u>24.0764</u>	24.2217	**24.8854**
9	31.6487	29.0385	27.1422	26.0501	<u>25.6960</u>	25.8998

Table.1. Total expected cost rate as a function of S_1 and S_2

Let $TC_1(S_1, S_2) = TC(S_1, S_2, 3)$. The values of $TC_1(S_1, S_2)$ are given in the above table. The optimal cost for each S_2 is shown in bold and the optimal cost for each S_1 is underlined. The numerical values shows that $TC_1(S_1, S_2)$ is a convex function in (S_1, S_2) and the (possibly local) optimum occurs at $(S_1, S_2) = (7, 5)$.

The table 2 gives the total expected cost rate for various combinations of S_1 and M. We have assumed constant values for other parameters and costs. Namely, $S_2 = 12$, $\lambda = 11.7$, $\gamma_1 = 3.5$, $\gamma_2 = 2.5$, $\gamma_{12} = 4.5$, $p = 0.7$, $q = 0.3$, $\mu_1 = 0.5$, $\mu_2 = 1.05$, $c_{h_1} = 4$, $c_{h_2} = 3$, $c_{s_1} = 1$, $c_{s_2} = 2$, $c_N = 0.2$, $c_w = 15$, $c_B = 15.5$.

S_1 / M	8	9	10	11	12
4	23.52099	21.82993	20.95269	<u>20.94535</u>	21.81381
5	23.13975	21.64211	<u>20.87712</u>	**20.90479**	**21.74334**
6	22.92739	21.55864	<u>**20.87555**</u>	20.94015	21.77652
7	**22.77410**	**21.53674**	<u>20.91023</u>	21.00445	21.84678
8	22.82032	21.54629	<u>20.95764</u>	21.07334	21.92291

Table.2. Total expected cost rate as a function of S_1 and M

Let $TC_2(S_1, M) = TC(S_1, 12, M)$. The values of $TC_2(S_1, M)$ are given in the above table. The optimal cost for each S_1 is shown in bold and the optimal cost for each M is underlined. The numerical values shows that $TC_2(S_1, M)$ is a convex function in (M, S_1) and the (possibly local) optimum occurs at $(M, S_1) = (6, 10)$.

The table 3 gives the total expected cost rate for various combinations of S_2 and M. We have assumed constant values for other parameters and costs. Namely, $S_1 = 6$, $\lambda = 6$, $\gamma_1 = 3.5$, $\gamma_2 = 2.5$, $\gamma_{12} = 4.5$, $p = 0.7$, $q = 0.3$, $\mu_1 = 0.5$, $\mu_2 = 1.05$, $c_{h_1} = 4$, $c_{h_2} = 5$, $c_{s_1} = 1$, $c_{s_2} = 1.5$, $c_N = 0.6$, $c_w = 15$, $c_B = 16.5$.

M / S_2	3	4	5	6	7
2	17.97380	17.43625	<u>17.25528</u>	17.26311	17.36521
3	16.99332	16.66086	<u>16.57168</u>	16.59243	16.65523
4	**16.72198**	**16.52336**	<u>**16.48977**</u>	**16.51536**	**16.55491**
5	17.05421	16.91763	<u>16.90427</u>	16.92590	16.95070
6	17.81789	17.70835	<u>17.69908</u>	17.71507	17.73156

Table.3. Total expected cost rate as a function of S_2 and M

Let $TC_3(S_2, M) = TC(6, S_2, M)$. The values of $TC_3(S_2, M)$ are given in the above table. The optimal cost for each M is shown in bold and the optimal cost for each S_2 is underlined. The numerical values shows that $TC_3(S_2, M)$ is a convex function in (S_2, M) and the (possibly local) optimum occurs at $(S_2, M) = (4, 5)$.

In table 4 the effect of service rates γ_1 and γ_2 on the optimal values (S_1, S_2) and the corresponding total expected cost rate are studied by fixing the parameters and costs $M = 3$, $\lambda = 22$, $\gamma_{12} = 8$, $p = 0.7$, $q = 0.3$, $\mu_1 = 1$, $\mu_2 = 2.1$, $c_{h_1} = 6.7$,

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$c_{h_2} = 7$, $c_{s_1} = 1.2$, $c_{s_2} = 1.5$, $c_N = 5$, $c_w = 5$, $c_B = 0.5$. We observed that the total expected cost rate increase when γ_1 and γ_2 increases.

γ_1 \ γ_2	2.69		3.69		4.69		5.69		6.69	
2.49	7	5	7	5	7	5	7	5	7	6
	5.7310		5.7891		5.9164		25.9582		26.0155	
3.49	8	5	7	5	7	5	7	5	7	5
	25.8860		25.9483		25.9452		26.0038		26.1124	
4.49	8	5	8	5	7	4	7	5	7	5
	25.8614		25.9641		26.0933		26.1189		26.1309	
5.49	8	5	8	5	8	5	7	4	7	4
	25.9715		25.9940		26.0538		26.1418		26.2255	
6.49	8	4	8	4	8	5	8	4	7	4
	25.9017		26.0111		26.1216		26.1513		26.2118	

Table.4. Effect of service rates γ_1 and γ_2 on optimal values

Table 5 illustrates the impact of service rates γ_1 and γ_{12} on the optimal values (S_1, S_2) and the corresponding total expected cost rate when $M = 3$, $\lambda = 20$, $\gamma_2 = 4.5$, $p = 0.7$, $q = 0.3$, $\mu_1 = 1$, $\mu_2 = 2.1$, $c_{h_1} = 7$, $c_{h_2} = 8$, $c_{s_1} = 2$, $c_{s_2} = 1.8$, $c_N = 5$, $c_w = 5$, $c_B = 0.5$. We observed that the total expected cost rate increase when γ_1 and γ_{12} increases.

γ_1 \ γ_{12}	4.5		5.0		5.5		6.0		6.5	
2.5	6	5	6	5	6	5	6	5	6	4
	26.2447		26.2717		26.2962		26.3564		26.4714	
3.0	6	5	6	5	6	5	6	4	6	4
	26.2100		26.2579		26.3601		26.3747		26.3912	
3.5	6	5	6	4	6	4	6	4	6	4
	26.2636		26.2850		26.2896		26.3265		26.3884	
4.0	6	4	6	4	6	4	6	4	7	5
	26.1900		26.2173		26.2733		26.3512		26.4093	
4.5	6	4	6	4	6	4	7	5	7	4
	26.1592		26.2329		26.3255		26.4041		26.4237	

Table.5. Effect of service rates γ_1 and γ_{12} on optimal values

Table 6 illustrates the impact of service rates γ_2 and γ_{12} on the optimal values (S_1, S_2) and the corresponding total expected cost rate when $M = 3$, $\lambda = 19$, $\gamma_1 = 1.5$, $p = 0.7$, $q = 0.3$, $\mu_1 = 1$, $\mu_2 = 2.1$, $c_{h_1} = 6.7$, $c_{h_2} = 7$, $c_{s_1} = 1.2$, $c_{s_2} = 1.5$, $c_N = 5$, $c_w = 5$, $c_B = 0.5$.

We observed that the total expected cost rate increase when γ_2 and γ_{12} increases.

γ_{12} \ γ_2	1.5		2.5		3.5		4.5		5.5	
2.5	5	4	5	5	4	4	4	4	4	5
	23.8494		24.0158		24.0678		24.2092		24.4204	
3.5	5	4	5	4	5	5	5	5	5	5
	24.0907		24.4758		24.6172		24.8916		25.2715	
4.5	5	4	5	4	5	4	5	5	5	5
	24.0095		24.1568		24.4138		24.4772		24.6397	
5.5	6	5	6	5	6	5	5	4	5	5
	23.9139		23.9874		24.1867		24.4688		24.5837	
6.5	6	4	6	5	6	5	6	5	6	5
	24.9960		25.4432		25.7337		26.0965		26.4013	

Table.6. Effect of service rates γ_{12} and γ_2 on optimal values

Table 7 illustrates the impact of replenishments rates μ_1 and μ_2 on the optimal values (S_1, S_2) and the corresponding total expected cost rate when $M=3$, $\lambda=19$, $\gamma_1=3.5$, $\gamma_2=2.5$, $\gamma_{12}=3.5$, $p=0.7$, $q=0.3$, $c_{h_1}=6.7$, $c_{h_2}=7$, $c_{s_1}=1.2$, $c_{s_2}=1.5$, $c_N=5$, $c_w=5$, $c_B=0.5$. We observed that the total expected cost rate decrease when μ_1 and μ_2 increases.

μ_1 \ μ_2	1.7		1.8		1.9		2.0		2.1	
0.6	9	7	9	6	9	6	9	6	9	6
	27.6291		27.6314		27.4658		27.3444		27.2590	
0.7	8	6	8	6	8	6	8	6	8	6
	26.8882		26.7375		26.6289		26.5547		26.5083	
0.8	7	6	7	6	7	6	7	6	7	6
	26.1883		26.1155		26.0735		26.0563		26.0587	
0.9	7	6	7	6	7	6	7	6	7	6
	25.8493		25.7166		25.6196		25.5519		25.5079	
1.0	6	6	6	6	6	6	6	5	6	5
	25.4296		25.3894		25.3737		25.3516		25.2388	

Table.7. Effect of service rates μ_1 and μ_2 on optimal values

In table 8 the impact of holding costs c_{s_1} and c_{s_2} on the optimal values (S_1, S_2) and the corresponding total expected cost rate are studied by fixing the parameters and costs $M=3$, $\lambda=22$, $\gamma_1=1.5$, $\gamma_2=2.5$, $\gamma_{12}=8$, $p=0.7$, $q=0.3$, $\mu_1=1$, $\mu_2=2.1$, $c_{h_1}=6.7$, $c_{h_2}=7$, $c_N=5$, $c_w=5$, $c_B=0.5$. We observed that the total expected cost rate increase when c_{s_1} and c_{s_2} increases.

c_{s_2} \ c_{s_1}	1.2		1.3		1.4		1.5		1.6	
0.7	7	5	7	5	7	5	7	5	7	5
	25.0526		25.2241		25.3957		25.5672		25.7388	
0.8	7	5	7	5	7	5	7	5	7	5
	25.0652		25.2368		25.4083		25.5799		25.7514	
0.9	7	5	7	5	7	5	7	5	7	5
	25.0779		25.2494		25.4210		25.5925		25.7641	
1.0	7	5	7	5	7	5	7	5	7	5
	25.0905		25.2621		25.4336		25.6052		25.7767	
1.1	7	5	7	5	7	5	7	5	7	5
	25.1032		25.2747		25.4463		25.6178		25.7894	

Table.8. Effect of setup costs c_{s_1} and c_{s_2} on optimal values

In table 9 the impact of holding costs c_{h_1} and c_{h_2} on the optimal values (S_1, S_2) and the corresponding total expected cost rate are studied by fixing the parameters and costs $M=3$, $\lambda=22$, $\gamma_1=1.5$, $\gamma_2=2.5$, $\gamma_{12}=8$, $p=0.7$, $q=0.3$, $\mu_1=1$, $\mu_2=2.1$, $c_{s_1}=1.2$, $c_{s_2}=1.5$, $c_N=5$, $c_w=5$, $c_B=0.5$. We observed that the total expected cost rate increase when c_{h_1} and c_{h_2} increases.

c_{h_2} / c_{h_1}	4		5		6		7		8	
3.5	9	7	9	7	9	7	9	7	9	7
	16.5145		16.7850		17.0556		17.3261		17.5967	
4.5	8	6	8	6	8	6	8	6	8	6
	19.3092		19.6377		19.9662		20.2946		20.6231	
5.5	8	7	8	6	8	6	8	6	8	6
	21.8339		22.2329		22.5614		22.8898		23.2183	
6.5	7	6	7	5	7	5	7	5	7	5
	23.8701		24.3651		24.8469		25.2070		25.5670	
7.5	7	6	7	6	7	6	7	5	7	5
	25.8514		26.3464		26.8414		27.3245		27.6846	

Table.9. Effect of holding costs c_{h_1} and c_{h_2} on optimal values

VII. Conclusion

In this paper, we discussed $(S-1, S)$ policy for two-commodity stochastic inventory system under continuous review at a service facility with finite waiting hall. The customers arriving to the service station are classifed as ordinary (positive or regular) and negative customers. Demands occuring during stock out periods are lost. The limiting distribution is obtained by using the algorithm of Gaver (1984). Various system performance measures are derived in the steady state. The results are illustrated with numerically. The model discussed here is useful in studying a service facility for two commodity inventory system which are slow moving items and the high holding cost.

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Some Contraction on G-Banach Space

By Ramakant Bhardwaj

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Abstract - In this paper we prove some results of fixed point theorems in G- Banach space.Our result are version of some known results in ordinary Banach Spaces.

Keywords : Fixed point, Common Fixed point, G-Banach space ,Continuous Mapping, Weakly Compatible Mappings.

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Strictly as per the compliance and regulations of :

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I. Introduction & Prilimnaries

This is well known that, Banach contraction principle is the fundamental contraction principle for proving fixed point results. The concept of G- Banach space is introduce by [11], which is a probable modification of the ordinary Banach Space. In this section some properties about G- Banach space are recall. In section 2, fixed point and common fixed point theorems for four weakly compatible maps in G- Banach space are proved .

In what follows, $\mathbb{N}$ *be the set of natural numbers and* $\mathbb{R}^+$ *be the set of all positive real numbers.*

Let binary operation $\nabla : R^+ \times R^+ \to R^+$ *satisfies the following conditions:*

i. ∇ *is associative and commutative,*

ii. ∇ *is continuous.*

Five typical examples are as follows:

for each $a, b \in R^+$

I. $a \nabla b = max\{a, b\}$

II. $a \nabla b = a + b$

III. $a \nabla b = a.b$

IV. $a \nabla b = a.b + a + b$

V. $a \nabla b = \frac{ab}{max\{a, b, 1\}}$

Definition: 1.1 [11]

The binary operation ∇ *is said to satisfy* α *– property if there exists a positive real number* α*, such that* $a \nabla b \leq max\{a, b\}$ *for every* $a, b \in R^+$

Example: 1.2

If we define $a \nabla b = a + b$*, for each* $a, b \in R^+$*, then for* $\alpha \geq 2$*, we have*

$$a \nabla b \leq \alpha\ max\{a, b\}$$

If we define $a \nabla b = \frac{ab}{max\{a, b, 1\}}$*, for each* $a, b \in R^+$*, then for* $\alpha \geq 1$*, we have*

$$a \nabla b \leq \alpha\ max\{a, b\}$$

Definition:- 1.3[11]

Let X be a nonempty set , A Generalized Normed Space on X, is a function $\|.\|_g : X \times X \to R^+$*, that satisfies the following conditions for each* $x, y, z \in X$.

(1) $\|x - y\|_g > 0$

(2) $\|x - y\|_g = 0$ *if and only if* $x = y$

(3) $\|x - y\|_g = \|y - x\|_g$

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(4) $\|\alpha x\|_g = |\alpha| \|x\|_g$ *for any scalar* α.

(5) $\| x-y\|_g \le \|x-z\|_g \ \nabla\ \|z-x\|_g$

The pair $(X, \|.\|_g)$ *is called Generalized Normed Space, or simply* G*- Normed Space.*

Definition: - 1.4[11]

A sequence $\{x_n\}$ *in* X *is said converges to* x*, if* $\|x_n - x\|_g \to 0$ *, as* $n \to \infty$*. That is for each* $\epsilon > 0$ *there exists* $n_0 \in N$ *such that, for every* $n \ge n_0$ *implies that,* $\|x_n - x\|_g < \epsilon$*.*

Definition:- 1.5 [11]

A sequence $\{x_n\}$ *is said to be Cauchy sequence if for every* $\epsilon > 0$*, there exists* $n_0 \in N$ *such that* $\|x_m - x_n\|_g < \epsilon$ *for each* $m, n \ge n_0$*.* G*- Normed space is said to be* G*- Banach space if every Cauchy sequence is converges in it.*

Definition: 1.6[11]

Let $(X, \|.\|_g)$ *be a* G*- normed space . for* $r > 0$ *we define*

$$B_g(x,r) = \{ y \in X: \|x-y\|_g < r \}$$

Let X *be a* G*- Normed space and* A *be a subset of* X*. Then for every* $x \in A$*, there exists* $r > 0$*, such that* $B_g(x,r) \subset A$*, then the subset* A *is called open subset of* X*. a subset* A *of* X *is said to be closed if the complement of* A *is open in* X*.*

Definition: 1.7[11]

A subset A *of* X *is said to be* G*- bounded if there exists* $r > 0$ *such that*

$$\|x-y\|_g < r \ for\ all\ x, y \in A.$$

Some example of $\|.\|_g$ *are as follows:*

a) let X *be a nonempty set then we define* $\|x-y\|_g = \|x-y\|$ *for every* $x, y \in X$*, Where* $a\ \nabla\ b = a + b$ *for* $a, b \in R^+$ *and* $\|.\|$ *is ordinary normed space on* X*.*

b) let X *be a non empty set. We define,*

$$\|x-y\|_g = \begin{cases} 0 \ , & x = y \\ 1 \ , otherwise \end{cases}$$

For each $x, y \in X$*, where* $a\ \nabla\ b = max\{a, b\}$ *for* $a, b \in R^+$*.*

Lemma: 1.9[11]

Let $(X, \|.\|_g)$ *be a* G*- Normed space such that,* ∇*, setisfy* $\alpha - property$ *with* $\alpha > 0$*.*
if sequence $\{x_n\}$ *in* X *is converges to* x*, then* x*, is unique.*

Lemma: 1.10[11]

Let $(X, \|.\|_g)$ *be a* G*- Normed space such that,* ∇*, setisfy* $\alpha - property$ *with* $\alpha > 0$*.*
if sequence $\{x_n\}$ *in* X *is converges to* x*, then* $\{x_n\}$ *is Cauchy sequence.*

Definition: 1.11[11]

Let A *and* S *be mappings from a* G*- Banach space* X *into itself. Then the mappings are said to be weakly compatible if they are commute at their coincidence point, that is* $Ax = Sx$ *implies that,* $ASx = SAx$*.*

II. Main Results

Theorem 2.1:- *Let* X *be a complete* G*- Banach space such that* ∇ *satisfy* $\alpha - property$ *with* $\alpha \le 1$*. If* T *be a mapping from* X *into it, satisfying the following condition;*

$$\|Tx - Ty\|_g \le k_1 \left(\frac{\|x - Tx\|_g \|x - Ty\|_g}{\|x-y\|_g} \nabla \frac{\|y - Tx\|_g \|y - Ty\|_g}{\|x-y\|_g} \right)$$

$$+ k_2 \left(\frac{\|x - Tx\|_g \|y - Ty\|_g}{\|x-y\|_g} \nabla \frac{\|x - Ty\|_g \|y - Tx\|_g}{\|x-y\|_g} \right)$$

$$+ k_3 \{ \|x - Tx\|_g \nabla \|y - Ty\|_g \nabla \|x - Ty\|_g \nabla \|y - Tx\|_g \nabla \|x - y\|_g \} \qquad 2.1.1$$

For non negative k_1, k_2, k_3 *such that* $0 < k_1 + k_2 + k_3 < 1$ *. Then* T *has unique fixed point in* X*.*

Proof *:- let* x_0 *be arbitrary point in* X *, then we choose a point* x_1 *in* X *such that* $x_1 = Tx_0$*. In general we have a sequence* $\{x_n\}$ *in* X *such that,* $x_{n+1} = Tx_n$

Now $\|x_{n+1} - x_{n+2}\|_g = \|Tx_n - Tx_{n+1}\|_g$

From 2.1.1 we have,

$$\|Tx_n - Tx_{n+1}\|_g \leq k_1\left(\frac{\|x_n - Tx_n\|_g\|x_n - Tx_{n+1}\|_g}{\|x_n - x_{n+1}\|_g} \nabla \frac{\|x_{n+1} - Tx_n\|_g\|x_{n+1} - Tx_{n+1}\|_g}{\|x_n - x_{n+1}\|_g}\right)$$

$$+ k_2\left(\frac{\|x_n - Tx_n\|_g\|x_{n+1} - Tx_{n+1}\|_g}{\|x_n - x_{n+1}\|_g} \nabla \frac{\|x_n - Tx_{n+1}\|_g\|x_{n+1} - Tx_n\|_g}{\|x_n - x_{n+1}\|_g}\right)$$

$$+ k_3\begin{Bmatrix}\|x_n - Tx_n\|_g \nabla \|x_{n+1} - Tx_{n+1}\|_g \nabla \|x_n - Tx_{n+1}\|_g \\ \nabla \|x_{n+1} - Tx_n\|_g \nabla \|x_n - x_{n+1}\|_g\end{Bmatrix}$$

$$\|Tx_n - Tx_{n+1}\|_g \leq k_1\left(\frac{\|x_n - x_{n+1}\|_g\|x_n - x_{n+2}\|_g}{\|x_n - x_{n+1}\|_g} \nabla \frac{\|x_{n+1} - x_{n+1}\|_g\|x_{n+1} - x_{n+2}\|_g}{\|x_n - x_{n+1}\|_g}\right)$$

$$+ k_2\left(\frac{\|x_n - x_{n+1}\|_g\|x_{n+1} - x_{n+2}\|_g}{\|x_n - x_{n+1}\|_g} \nabla \frac{\|x_n - x_{n+2}\|_g\|x_{n+1} - x_n\|_g}{\|x_n - x_{n+1}\|_g}\right)$$

$$+ k_3\begin{Bmatrix}\|x_n - x_{n+1}\|_g \nabla \|x_{n+1} - x_{n+2}\|_g \nabla \|x_n - x_{n+2}\|_g \\ \nabla \|x_{n+1} - x_{n+1}\|_g \nabla \|x_n - x_{n+1}\|_g\end{Bmatrix}$$

$$\|x_{n+1} - x_{n+2}\|_g \leq k_1 max(\|x_n - x_{n+1}\|_g, \|x_{n+1} - x_{n+2}\|_g)$$
$$+ k_2 max(\|x_n - x_{n+1}\|_g, \|x_{n+1} - x_{n+2}\|_g)$$
$$+ k_3 max\{\|x_n - x_{n+1}\|_g, \|x_{n+1} - x_{n+2}\|_g\}$$

If we take max, $\|x_{n+1} - x_{n+2}\|_g$*, then we have,*

$$\|x_{n+1} - x_{n+2}\|_g \leq (k_1 + k_2 + k_3)\|x_{n+1} - x_{n+2}\|_g$$

Which contradiction the hypothesis, so we have

$$\|x_{n+1} - x_{n+2}\|_g \leq (k_1 + k_2 + k_3)\|x_n - x_{n+1}\|_g$$

Similarly we can find,

$$\|x_{n+1} - x_n\|_g \leq (k_1 + k_2 + k_3)\|x_{n-1} - x_n\|_g$$

In this way, we can write,

$$\|x_{n+1} - x_n\|_g \leq (k_1 + k_2 + k_3)^n \|x_{n-1} - x_n\|_g$$

$$\lim_{n\to\infty}\|x_{n+1} - x_n\|_g \to 0$$

which implies, $\{x_n\}$ *is a Cauchy sequence,. Which converges to* $'u'$ *in* X*.*

Now on taking, $x = x_n$ *and* $y = u$ *in* [2.1.1]*, then we get*

$$\|Tx_n - Tu\|_g \leq k_1\left(\frac{\|x_n - Tx_n\|_g\|x_n - Tu\|_g}{\|x_n - u\|_g} \nabla \frac{\|u - Tx_n\|_g\|u - Tu\|_g}{\|x_n - u\|_g}\right)$$

$$+ k_2\left(\frac{\|x_n - Tx_n\|_g\|u - Tu\|_g}{\|x_n - u\|_g} \nabla \frac{\|x_n - Tu\|_g\|u - Tx_n\|_g}{\|x_n - u\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|x_n-Tx_n\|_g\nabla\|u-Tu\|_g\nabla\|x_n-Tu\|_g\\ \nabla\|u-Tx_n\|_g\nabla\|x_n-u\|_g\end{Bmatrix}$$

as $n\to\ \infty$, *we get,* $Tu\ =\ u$.

$i.e.\ \ u, is\ \ a\ fixed\ point\ of\ T\ \ in\ X.$

Uniqueness

Let us assume that, 'w' is another fixed point of T, different from 'u' in X. then, $u\ \neq w$

$$\|Tw-Tu\|_g\le k_1\left(\frac{\|w-Tw\|_g\|w-Tu\|_g}{\|w-u\|_g}\nabla\frac{\|u-Tw\|_g\|u-Tu\|_g}{\|w-u\|_g}\right)$$

$$+k_2\left(\frac{\|w-Tw\|_g\|u-Tu\|_g}{\|w-u\|_g}\nabla\frac{\|w-Tu\|_g\|u-Tw\|_g}{\|w-u\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|w-Tw\|_g\nabla\|u-Tu\|_g\nabla\|w-Tu\|_g\\ \nabla\|u-Tw\|_g\nabla\|w-u\|_g\end{Bmatrix}$$

$$\|u-w\|\ \le\ \ (k_1+k_2+k_3)\,\|u-w\|_g$$

This is a contradiction .so 'u' is unique fixed point of T, in X.

Theorem 2.2:-

Let X be a complete G- Banach space such that ∇ *satisfy* $\alpha-property$ *with* $\alpha\le1$. *If S, T be compatible mapping from X into itself, satisfying the following condition;*

$$\|Sx-Ty\|_g\le k_1\left(\frac{\|x-Sx\|_g\|x-Ty\|_g}{\|x-y\|_g}\nabla\frac{\|y-Sx\|_g\|y-Ty\|_g}{\|x-y\|_g}\right)$$

$$+k_2\left(\frac{\|x-Sx\|_g\|y-Ty\|_g}{\|x-y\|_g}\nabla\frac{\|x-Ty\|_g\|y-Sx\|_g}{\|x-y\|_g}\right)$$

$$+k_3\{\|x-Sx\|_g\nabla\|y-Ty\|_g\nabla\|x-Ty\|_g\nabla\|y-Sx\|_g\nabla\|x-y\|_g\}\qquad 2.2.1$$

for non negative k_1,k_2,k_3 *such that* $0<\ k_1+k_2+k_3<1$. *Then S , T have unique common fixed point in X.*

Proof :-

Let x_0 *be arbitrary point in X , then we choose a point* x_1 *in X such that* $x_1=Tx_0$. *In general we have a sequence* $\{x_n\}$ *in X such that,* $x_{n+1}=\ \ Sx_n,\ x_{n+2}=\ \ Tx_{n+1}$

Now

$$\|x_{n+1}-x_{n+2}\|_g\ \ =\ \ \|Sx_n-Tx_{n+1}\|_g$$

From 2.2.1 *we have,*

$$\|Sx_n-Tx_{n+1}\|_g\le k_1\left(\frac{\|x_n-Sx_n\|_g\|x_n-Tx_{n+1}\|_g}{\|x_n-x_{n+1}\|_g}\nabla\frac{\|x_{n+1}-Sx_n\|_g\|x_{n+1}-Tx_{n+1}\|_g}{\|x_n-x_{n+1}\|_g}\right)$$

$$+k_2\left(\frac{\|x_n-Sx_n\|_g\|x_{n+1}-Tx_{n+1}\|_g}{\|x_n-x_{n+1}\|_g}\nabla\frac{\|x_n-Tx_{n+1}\|_g\|x_{n+1}-Sx_n\|_g}{\|x_n-x_{n+1}\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|x_n-Sx_n\|_g\nabla\|x_{n+1}-Tx_{n+1}\|_g\nabla\|x_n-Tx_{n+1}\|_g\\ \nabla\|x_{n+1}-Sx_n\|_g\nabla\|x_n-x_{n+1}\|_g\end{Bmatrix}$$

$$\|x_{n+1}-x_{n+2}\|_g\le k_1\left(\frac{\|x_n-x_{n+1}\|_g\|x_n-x_{n+2}\|_g}{\|x_n-x_{n+1}\|_g}\nabla\frac{\|x_{n+1}-x_{n+1}\|_g\|x_{n+1}-x_{n+2}\|_g}{\|x_n-x_{n+1}\|_g}\right)$$

$$+k_2\left(\frac{\|x_n-x_{n+1}\|_g\|x_{n+1}-x_{n+2}\|_g}{\|x_n-x_{n+1}\|_g}\nabla\frac{\|x_n-x_{n+2}\|_g\|x_{n+1}-x_n\|_g}{\|x_n-x_{n+1}\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|x_n-x_{n+1}\|_g\nabla\|x_{n+1}-x_{n+2}\|_g\nabla\|x_n-x_{n+2}\|_g\\ \nabla\|x_{n+1}-x_{n+1}\|_g\nabla\|x_n-x_{n+1}\|_g\end{Bmatrix}$$

$$\|x_{n+1}-x_{n+2}\|_g \le k_1 max(\|x_n - x_{n+1}\|_g, 0)$$
$$+ k_2 max(\|x_n - x_{n+1}\|_g, \|x_{n+1} - x_{n+2}\|_g)$$
$$+ k_3 max\{\|x_n - x_{n+1}\|_g, \|x_{n+1} - x_{n+2}\|_g\}$$

If we take max, $\|x_{n+1}-x_{n+2}\|_g$, *then we have,*

$$\|x_{n+1}-x_{n+2}\|_g \le (k_1+k_2+k_3)\|x_{n+1}-x_{n+2}\|_g$$

Which contradiction the hypothesis, so we have

$$\|x_{n+1}-x_{n+2}\|_g \le (k_1+k_2+k_3)\|x_n-x_{n+1}\|_g$$

Similarly we can find,

$$\|x_{n+1}-x_n\|_g \le (k_1+k_2+k_3)\|x_{n-1}-x_n\|_g$$

In this way, we can write,

$$\|x_{n+1}-x_n\|_g \le (k_1+k_2+k_3)^n\|x_{n-1}-x_n\|_g$$

$as \ n \to \infty$

$\lim_{n\to\infty}\|x_{n+1}-x_n\|_g \to 0$

Which implies, $\{x_n\}$ *is a Cauchy sequence,. Which converges to* $'u'$ *in X.*

Since S *and* T *are compatible mapping, which implies,*

$u = \lim_{n\to\infty} STx_n = \mathrm{S}\lim_{n\to\infty} Tx_n = Su$, *also*

$u = \lim_{n\to\infty} STx_n = \mathrm{T}\lim_{n\to\infty} Sx_n = Tu.$

i.e, 'u' is common fixed point of S and T, in X.

Uniqueness,

Let us assume that, 'w' is another fixed point of T, different from 'u' in X. then, $u \neq w$

$$\|Sw-Tu\|_g \le k_1\left(\frac{\|w-Sw\|_g\|w-Tu\|_g}{\|w-u\|_g}\nabla\frac{\|u-Sw\|_g\|u-Tu\|_g}{\|w-u\|_g}\right)$$

$$+ k_2\left(\frac{\|w-Sw\|_g\|u-Tu\|_g}{\|w-u\|_g}\nabla\frac{\|w-Tu\|_g\|u-Sw\|_g}{\|w-u\|_g}\right)$$

$$+ k_3\left\{\begin{matrix}\|w-Sw\|_g\nabla\|u-Tu\|_g\nabla\|w-Tu\|_g \\ \nabla\|u-Sw\|_g\nabla\|w-u\|_g\end{matrix}\right\}$$

$$\|u-w\| \le (k_1+k_2+k_3)\|u-w\|_g$$

This is a contradiction .So 'u' is unique common fixed point of S, and T, in X.

Theorem: 2.3 *Let* X *be a* G*- Banach space , such that* ∇ *satisfy property with* $\alpha - \alpha \le 1$. *If* A, B, S *and* T *be mapping from* X *into itself satisfying the following condition:*

i. $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, *and* $T(X)$ *or* $S(X)$ *is a closed subset of X.*
ii. *The pair* (A,S)*and* (B,T) *are weakly compatible,*
iii. *For all* $x, y \in X$,

$$\|Ax-By\|_g \le k_1\left(\frac{\|Sx-Ax\|_g\|Sx-By\|_g}{\|Sx-Ty\|_g}\nabla\frac{\|Ty-Ax\|_g\|Ty-By\|_g}{\|Sx-Ty\|_g}\right)$$

$$+ k_2\left(\frac{\|Sx-Ax\|_g\|Ty-By\|_g}{\|Sx-Ty\|_g}\nabla\frac{\|Sx-By\|_g\|Ty-Ax\|_g}{\|Sx-Ty\|_g}\right)$$

$$+ k_3\{\|Sx-Ax\|_g\nabla\|Ty-By\|_g\nabla\|Sx-By\|_g\nabla\|Ty-Ax\|_g\nabla\|Sx-Ty\|_g\}$$

Where $k_1, k_2, k_3 > 0$ *and* $0 < k_1 + k_2 + k_3 < 1$. *Then* A, B, S, *and* T *have a unique Common fixed point in* X.

Proof :-

Let x_0 *be an arbitrary point in X. then by (i), we choose a point* x_1 *in* X *such that,* $y_0 = Ax_0 = Tx_1$ *and* $y_1 = Bx_1 = Sx_2$. *In general, there exists a sequence* $\{y_n\}$ *such that,* $y_{2n} = Ax_{2n} = Tx_{2n+1}$ *and* $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ *for* $n = 1,2,3, \dots\dots$

We claim that the sequence $\{y_n\}$ *is Cauchy sequence.*

By (iii), we have,

$$\|y_{2n} - y_{2n+1}\|_g = \|Ax_{2n}, Bx_{2n+1}\|_g$$

$$\|Ax_{2n} - Bx_{2n+1}\|_g \leq k_1\left(\frac{\|Sx_{2n}-Ax_{2n}\|_g\|Sx_{2n}-Bx_{2n+1}\|_g}{\|Sx_{2n}-Tx_{2n+1}\|_g}\nabla\frac{\|Tx_{2n+1}-Ax_{2n}\|_g\|Tx_{2n+1}-Bx_{2n+1}\|_g}{\|Sx_{2n}-Tx_{2n+1}\|_g}\right)$$

$$+k_2\left(\frac{\|Sx_{2n}-Ax_{2n}\|_g\|Tx_{2n+1}-Bx_{2n+1}\|_g}{\|Sx_{2n}-Tx_{2n+1}\|_g}\nabla\frac{\|Sx_{2n}-Bx_{2n+1}\|_g\|Tx_{2n+1}-Ax_{2n}\|_g}{\|Sx_{2n}-Tx_{2n+1}\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|Sx_{2n}-Ax_{2n}\|_g\nabla\|Tx_{2n+1}-Bx_{2n+1}\|_g\nabla\|Sx_{2n}-Bx_{2n+1}\|_g\\ \nabla\|Tx_{2n+1}-Ax_{2n}\|_g\nabla\|Sx_{2n}-Tx_{2n+1}\|_g\end{Bmatrix}$$

$$\|y_{2n} - y_{2n+1}\|_g \leq k_1\left(\frac{\|y_{2n-1}-y_{2n}\|_g\|y_{2n}-y_{2n+1}\|_g}{\|y_{2n-1}-y_{2n}\|_g}\nabla\frac{\|y_{2n}-y_{2n}\|_g\|y_{2n}-y_{2n+1}\|_g}{\|y_{2n-1}-y_{2n}\|_g}\right)$$

$$+k_2\left(\frac{\|y_{2n-1}-y_{2n}\|_g\|y_{2n}-y_{2n+1}\|_g}{\|y_{2n-1}-y_{2n}\|_g}\nabla\frac{\|y_{2n-1}-y_{2n+1}\|_g\|y_{2n}-y_{2n}\|_g}{\|y_{2n-1}-y_{2n}\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|y_{2n-1}-y_{2n}\|_g\nabla\|y_{2n}-y_{2n+1}\|_g\nabla\|y_{2n-1}-y_{2n+1}\|_g\\ \nabla\|y_{2n}-y_{2n}\|_g\nabla\|y_{2n-1}-y_{2n}\|_g\end{Bmatrix}$$

$$\|y_{2n} - y_{2n+1}\|_g \leq k_1 max\left(\frac{\|y_{2n-1}-y_{2n}\|_g\|y_{2n}-y_{2n+1}\|_g}{\|y_{2n-1}-y_{2n}\|_g}\nabla\frac{\|y_{2n}-y_{2n}\|_g\|y_{2n}-y_{2n+1}\|_g}{\|y_{2n-1}-y_{2n}\|_g}\right)$$

$$+k_2\left(\frac{\|y_{2n-1}-y_{2n}\|_g\|y_{2n}-y_{2n+1}\|_g}{\|y_{2n-1}-y_{2n}\|_g}\nabla\frac{\|y_{2n-1}-y_{2n+1}\|_g\|y_{2n}-y_{2n}\|_g}{\|y_{2n-1}-y_{2n}\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|y_{2n-1}-y_{2n}\|_g\nabla\|y_{2n}-y_{2n+1}\|_g\nabla\|y_{2n-1}-y_{2n+1}\|_g\\ \nabla\|y_{2n}-y_{2n}\|_g\nabla\|y_{2n-1}-y_{2n}\|_g\end{Bmatrix}$$

$$\|y_{2n} - y_{2n+1}\|_g \leq (k_1 + k_2 + k_3)\|y_{2n-1} - y_{2n}\|_g$$

That is by induction we can show that

$$\|y_{2n} - y_{2n+1}\|_g \leq (k_1 + k_2 + k_3)^n \|y_0 - y_1\|_g$$

As $n \to \infty$, $\|y_{2n} - y_{2n+1}\|_g \to 0$

For any integer $m \geq n$

It follows that, the sequence $\{y_n\}$ *is a Cauchy sequence which converges to* $y \in X$.

This implies that

$lim_{n\to\infty} y_n = lim_{n\to\infty} Ax_{2n} = lim_{n\to\infty} Bx_{2n+1} = lim_{n\to\infty} Sx_{2n+2} = lim_{n\to\infty} Tx_{2n+1} = y$

Now let us assume that, $T(X)$ *is closed subset of X, then there exists* $v \in X$ *such that*

$Tv = y$. *We now prove that* $Bv = y$. *by (iii), we get*

$$\|Ax_{2n} - Bv\|_g \le k_1\left(\frac{\|Sx_{2n}-Ax_{2n}\|_g\|Sx_{2n}-Bv\|_g}{\|Sx_{2n}-Tv\|_g}\nabla\frac{\|Tv-Ax_{2n}\|_g\|Tv-Bv\|_g}{\|Sx_{2n}-Tv\|_g}\right)$$

$$+k_2\left(\frac{\|Sx_{2n}-Ax_{2n}\|_g\|Tv-Bv\|_g}{\|Sx_{2n}-Tv\|_g}\nabla\frac{\|Sx_{2n}-Bv\|_g\|Tv-Ax_{2n}\|_g}{\|Sx_{2n}-Tv\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|Sx_{2n}-Ax_{2n}\|_g\nabla\|Tv-Bv\|_g\nabla\|Sx_{2n}-Bv\|_g\\ \nabla\|Tv-Ax_{2n}\|_g\nabla\|Sx_{2n}-Tv\|_g\end{Bmatrix}$$

$$as\ n\to\infty,\quad \|y-Bv\|_g \le (k_1+k_2+k_3)\ \|y-Bv\|_g$$

Which contradiction, it follows that $Bv = y = Tv$. *Since* B *and* T *are weakly compatible mappings, then we have* $BTv = TBv$ *which implies* $By = Ty$.

Now we prove that, $By = y$, *for this by using (iii), we get*

$$\|Ax_{2n} - By\|_g \le k_1\left(\frac{\|Sx_{2n}-Ax_{2n}\|_g\|Sx_{2n}-By\|_g}{\|Sx_{2n}-Ty\|_g}\nabla\frac{\|Ty-Ax_{2n}\|_g\|Ty-By\|_g}{\|Sx_{2n}-Ty\|_g}\right)$$

$$+k_2\left(\frac{\|Sx_{2n}-Ax_{2n}\|_g\|Ty-By\|_g}{\|Sx_{2n}-Ty\|_g}\nabla\frac{\|Sx_{2n}-By\|_g\|Ty-Ax_{2n}\|_g}{\|Sx_{2n}-Ty\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|Sx_{2n}-Ax_{2n}\|_g\nabla\|Ty-By\|_g\nabla\|Sx_{2n}-By\|_g\\ \nabla\|Ty-Ax_{2n}\|_g\nabla\|Sx_{2n}-Ty\|_g\end{Bmatrix}$$

$$as\ n\to\infty,\quad \|y-By\|_g \le (k_1+k_2+k_3)\ \|y-By\|_g$$

Which contradiction. Thus $By = y = Ty$

Since $B(X)\subseteq S(X)$, *there exists* $w\in X$. *such that* $Sw = y$. *we show that,* $Aw = y$. *from (iii)*

$$\|Aw - By\|_g \le k_1\left(\frac{\|Sw-Aw\|_g\|Sw-By\|_g}{\|Sw-Ty\|_g}\nabla\frac{\|Ty-Aw\|_g\|Ty-By\|_g}{\|Sw-Ty\|_g}\right)$$

$$+k_2\left(\frac{\|Sw-Aw\|_g\|Ty-By\|_g}{\|Sw-Ty\|_g}\nabla\frac{\|Sw-By\|_g\|Ty-Aw\|_g}{\|Sw-Ty\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|Sw-Aw\|_g\nabla\|Ty-By\|_g\nabla\|Sw-By\|_g\\ \nabla\|Ty-Aw\|_g\nabla\|Sw-Ty\|_g\end{Bmatrix}$$

$$\|Aw-By\|_g \le (k_1+k_2+k_3)\ \|Aw-By\|_g$$

$$\|Aw-y\|_g \le (k_1+k_2+k_3)\ \|Aw-y\|_g$$

Which contradiction, so that $Aw = y = Sw$ *Since* A *and* S *are weakly compatible, then* $ASw = SAw$ *and so* $Ay = Sy$.

Now we show that, $Ay = y$, *from (iii),*

$$\|Ay - By\|_g \le k_1\left(\frac{\|Sy-Ay\|_g\|Sy-By\|_g}{\|Sy-Ty\|_g}\nabla\frac{\|Ty-Ay\|_g\|Ty-By\|_g}{\|Sy-Ty\|_g}\right)$$

$$+k_2\left(\frac{\|Sy-Ay\|_g\|Ty-By\|_g}{\|Sy-Ty\|_g}\nabla\frac{\|Sy-By\|_g\|Ty-Ay\|_g}{\|Sy-Ty\|_g}\right)$$

$$+k_3\begin{Bmatrix}\|Sy-Ay\|_g\nabla\|Ty-By\|_g\nabla\|Sy-By\|_g\\ \nabla\|Ty-Ay\|_g\nabla\|Sy-Ty\|_g\end{Bmatrix}$$

$$\|Ay-y\|_g \le (k_1+k_2+k_3)\ \|Ay-y\|_g$$

Which contradiction,

Thus $Ay = y$ *and therefore* $Ay = Sy = By = Ty = y$.

y is a common fixed point of $A, B, S, T,$ in X.

The proof is similar when we assume that, $S(X)$ is a closed subset of X

Uniqueness:-

Let us assume that x is another fixed point of A, B, S, T different from y in X.

Then from (iii), we have

$$\|Ax - By\|_g \le k_1 \left(\frac{\|Sx-Ax\|_g \|Sx-By\|_g}{\|Sx-Ty\|_g} \nabla \frac{\|Ty-Ax\|_g \|Ty-By\|_g}{\|Sx-Ty\|_g} \right)$$

$$+ k_2 \left(\frac{\|Sx-Ax\|_g \|Ty-By\|_g}{\|Sx-Ty\|_g} \nabla \frac{\|Sx-By\|_g \|Ty-Ax\|_g}{\|Sx-Ty\|_g} \right)$$

$$+ k_3 \left\{ \begin{matrix} \|Sx-Ax\|_g \nabla \|Ty-By\|_g \nabla \|Sx-By\|_g \\ \nabla \|Ty-Ax\|_g \nabla \|Sx-Ty\|_g \end{matrix} \right\}$$

$$\|x-y\|_g \le (k_1+k_2+k_3)\ \|x-y\|_g$$

This is contradiction. Thus $x = y$. This completes the proof of the theorem.

Remark:-

1. If we take $S = T$ in theorem- 2.2 then we get theorem 2.1
2. If we take $S = T = I$ in theorem- 2.3 then we get theorem 2.2
3. If we take $A = B$ and $S = T = I$ in theorem - 2.3 then we get theorem 2.1

Corollary 2.4 : Let X be a complete G- Banach space such that ∇ satisfy $\alpha - property$ with $\alpha \le 1$. If T be a mapping from X into it, satisfying the following condition;

$$\|T^r x - T^s y\|_g \le k_1 \left(\frac{\|x-T^r x\|_g \|x-T^s y\|_g}{\|x-y\|_g} \nabla \frac{\|y-T^r x\|_g \|y-T^s y\|_g}{\|x-y\|_g} \right)$$

$$+ k_2 \left(\frac{\|x-T^r x\|_g \|y-T^s y\|_g}{\|x-y\|_g} \nabla \frac{\|x-T^s y\|_g \|y-T^r x\|_g}{\|x-y\|_g} \right)$$

$$+ k_3 \{ \|x-T^r x\|_g \nabla \|y-T^s y\|_g \nabla \|x-T^s y\|_g \nabla \|y-T^r x\|_g \nabla \|x-y\|_g \} \qquad 2.4.1$$

For non negative k_1, k_2, k_3 such that $0 < k_1 + k_2 + k_3 < 1$, and $r, s \in N(set\ of\ natural\ number)$. Then T has unique fixed point in X.

Proof : This can be proved easily by theorem – 2.1, on taking $r = s = 1$.

Corollary 2.5: Let X be a complete G- Banach space such that ∇ satisfy $\alpha - property$ with $\alpha \le 1$. If S, T be compatible mapping from X into itself, satisfying the following condition;

$$\|S^r x - T^u y\|_g \le k_1 \left(\frac{\|x-S^r x\|_g \|x-T^u y\|_g}{\|x-y\|_g} \nabla \frac{\|y-S^r x\|_g \|y-T^u y\|_g}{\|x-y\|_g} \right)$$

$$+ k_2 \left(\frac{\|x-S^r x\|_g \|y-T^u y\|_g}{\|x-y\|_g} \nabla \frac{\|x-T^u y\|_g \|y-S^r x\|_g}{\|x-y\|_g} \right)$$

$$+ k_3 \{ \|x-S^r x\|_g \nabla \|y-T^u y\|_g \nabla \|x-T^u y\|_g \nabla \|y-S^r x\|_g \nabla \|x-y\|_g \} \qquad 2.2.1$$

For non negative k_1, k_2, k_3 such that $0 < k_1 + k_2 + k_3 < 1$. and $r, u \in N$ (set of natural number) Then S, T have unique common fixed point in X.

Proof : This can be proved easily by theorem – 2.2, on taking $r = u = 1$.

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Differential Subordination and Superordination of Analytic Functions Defined By Cho - Kwon - Srivastava Operator

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Abstract - Differential subordination and superordination results are obtained for analytic functions in the open unit disk which are associated with Cho-Kwon-Srivastava operator. These results are obtained by investigating appropriate classes of admissible functions. Some of the result established in this paper would provide extensions of those given in earlier works.

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GJSFR-F Classification: FOR Code: 010207

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Strictly as per the compliance and regulations of :

Differential Subordination and Superordination of Analytic Functions Defined By Cho-Kwon-Srivastava Operator

Jamal M. Shenan

Abstract - Differential subordination and superordination results are obtained for analytic functions in the open unit disk which are associated with Cho-Kwon-Srivastava operator. These results are obtained by investigating appropriate classes of admissible functions. Some of the result established in this paper would provide extensions of those given in earlier works.

Keywords And Phrases : Analytic functions, integral operator, hadmard product, differential subordination, super - ordination.

I. Introduction

Let $H(U)$ be the class of functions analytic in $U=\{z : z\in C \text{ and } |z|<1\}$ and $H[a,n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z)=a+a_n z^n+a_{n+1}z^{n+1}+\ldots$, With $H_0\equiv H[0,1]$ and $H\equiv H[1,1]$. Let $A(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}z^{k+p} \quad \left(p\in\mathbb{N}=\{1,2,3,\ldots\};z\in U\right), \tag{1.1}$$

and let $A(1)=A$. Let f and F be members of $H(U)$. The function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $w(z)$ analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$, such that $f(z)=F(w(z))$. In such a case we write $f(z)\prec F(z)$. In particular, if F is univalent, then $f(z)\prec F(z)$ if and only if $f(0)=F(0)$ and $f(U)\subset F(U)$ (see [1] and [2]).

For two functions $f(z)$ given by (1.1) and

$$g(z)=z^p+\sum_{k=1}^{\infty}b_{k+p}z^{k+p}, \tag{1.2}$$

The hadmard product (or convolution) of f and g is defined by

$$(f*g)(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}b_{k+p}z^{k+p}=(g*f)(z).$$

Saitoh [8] introduce a linear operator:

$$L_p(a,c):A_p\to A_p$$

defined by

$$L_p(a,c)=\phi_p(a,c;z)*f(z) \quad (z\in U), \tag{1.3}$$

where

$$\phi_p(a,c;z)=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}z^{p+k},$$

and $(a)_k$ is the Pochammer symbol. In 2004, Cho, Kwon and Srivastava [4] introduced the linear operator $L_p^{\lambda}(a,c):A_p\to A_p$ analogous to $L_p(a,c)$ defined by

$$L_p^{\lambda}(a,c)f(z)=\phi_p^{\lambda}(a,c;z)*f(z) \quad \left(z\in U;a,c\in R\setminus Z_0^-;\lambda>-p\right), \tag{1.4}$$

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where $\phi_p^{\lambda}(a,c;z)$ is the function defined in terms of the Hadamard product (or convolution) by the following condition :

$$\phi_p(a,c;z)*\phi_p^{\lambda}(a,c;z)=\frac{z^p}{(1-z)^{\lambda+p}} \tag{1.5}$$

We can easily find from (1.4) and (1.5) and for the function $f(z)\in A_p$ that

$$L_p^{\lambda}(a,c)f(z)=z^p+\sum_{k=1}^{\infty}\frac{(\lambda+p)_k(c)_k}{k!(a)_k}a_{k+p}z^{k+p} \quad . \tag{1.6}$$

It is easily verified from (1.6) that

$$z\left(L_p^{\lambda}(a+1,c)f\right)'(z)=aL_p^{\lambda}(a,c)f(z)-(a-p)L_p^{\lambda}(a+1,c)f(z) \quad , \tag{1.7}$$

and

$$z\left(L_p^{\lambda}(a,c)f\right)'(z)=(\lambda+p)L_p^{\lambda+1}(a,c)f(z)-\lambda L_p^{\lambda}(a,c)f(z). \tag{1.8}$$

To prove our results, we need the following definitions and lemmas.

Denote by Q the set of all functions $q(z)$ that are analytic and injective on $\overline{U}/E(q)$ where

$$E(q)=\{\zeta\in\partial U:\lim_{z\to\zeta}q(z)=\infty\},$$

and are such that $q'(\zeta)\neq$ for $\zeta\in\partial U/E(q)$. Further let the subclass of Q for which $q(0)=a$ be denoted by $Q(a)$, $Q(0)\equiv Q_0$ and $Q(1)\equiv Q_1$.

Definition 1 ([6]). let Ω be a set in C, $q\in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega,q]$ consist of those functions $\psi:C^3\times U\to C$ that satisfy the admissibility condition:

$$\psi(r,s,t;z)\notin\Omega$$

whenever

$$r=q(\zeta),\ s=k\zeta q'(\zeta),\ R\left\{\frac{t}{s}+1\right\}\geq kR\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

where $z\in U$, $\zeta\in\partial U/E(q)$ and $k\geq n$ We write $_1[\Omega,q]$ as $\Psi[\Omega,q]$.

Definition 2 ([7]). let Ω be a set in C, $q(z)\in H[a,n]$ with $q'(z)\neq 0$ The class of admissible functions $\Psi_n'[\Omega,q]$ consist of those functions $\psi:C^3\times\overline{U}\to C$ that satisfy the admissibility condition

$$\psi(r,s,t;\zeta)\notin\Omega$$

whenever

$$r=q(z),\ s=\frac{zq'(z)}{m},\ R\left\{\frac{t}{s}+1\right\}\geq\frac{1}{m}R\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

where $z\in U$, $\zeta\in\partial U$ and $m\geq n\geq 1$. In particular , we write $\Psi_1'[\Omega,q]$ as $\Psi'[\Omega,q]$.

Lemma 1 ([6]). Let $\psi\in\Psi_n[\Omega,q]$ with $q(0)=a$. If the analytic function $p(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots$ satisfies

$$\psi(p(z),zp'(z),z^2p''(z);z)\in\Omega,$$

then

$$p(z)\prec q(z).$$

Lemma 2 ([7]). Let $\psi\in\Psi_n'[\Omega,q]$ with $q(0)=a$. If $p(z)\in Q(a)$ and $\psi(p(z),zp'(z),z^2p''(z);z)$ is univalent in U then

$$\Omega\subset\{\psi(p(z),zp'(z),z^2p''(z);z):z\in U\},$$

implies

$$q(z) \prec p(z).$$

In the present investigation, the differential subordination result of Miller and Mocanu [6,7] is extended for analytic functions in the open unit disk , which are associated with Cho - Kwon - Srivastava operator $L_p^\lambda(a,c)$ $\left(a,c \in R \setminus Z_0^-; \lambda > -p\right)$, and we obtain certain other related results . A simililar problem for analytic functions was Srivastava [5], Aouf and Seoudy [3], Aghalary et al. [1], Ali et al. [2]. Additionally, the corresponding differential superordination problem is investigated, and several sandwichtype result are obtained.

II. Subordination Results Involving the Cho-Kwon–Srivastava Operator

$$I_p^\lambda(a,c) f(z).$$

Definition 3. Let Ω be a set in C and $q(z) \in Q_0 \cap H[0,p]$. The class of admissible functions $\Phi_I[\Omega, q]$ consist of those functions $\phi : C^3 \times U \to C$: that satisfy the admissibility condition

$$\phi(u,v,w;z) \notin \Omega$$

whenever

$$u = q(\zeta),\ v = \frac{k\zeta q'(\zeta) + (a-p)q(\zeta)}{a},$$

$$R\left\{\frac{a(a-1)w - (a-p)(a-p-1)u}{av - (a-p)u} - 2(a-p) + 1\right\} \geq kR\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

where $z \in U$, $\zeta \in \partial U / E(q)$, $p \in N$, $a \in R \setminus Z_0^-$ and $k \geq p$.

Theorem 1. Let $\phi \in \Phi_I[\Omega, q]$. If $f(z) \in A(p)$ satisfies

$$\left\{\phi\left(I_p^\lambda(a+1,c)f(z), I_p^\lambda(a,c)f(z), I_p^\lambda(a-1,c)f(z); z\right) : z \in U\right\} \subset \Omega, \quad (2.1)$$

then

$$I_p^\lambda(a+1,c)f(z) \prec q(z)$$

$$\left(z \in U;\ a,c \in R \setminus Z_0^-; \lambda > -p; p \in N\right).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) = I_p^\lambda(a+1,c)f(z) \qquad \left(z \in U;\ a,c \in R \setminus Z_0^-; \lambda > -p; p \in N\right). \quad (2.2)$$

In view of the relation (1.7) from (2.2), we get

$$I_p^\lambda(a,c)f(z) = \frac{zp'(z) + (a-p)p(z)}{a}. \quad (2.3)$$

Further computation show that

$$I_p^\lambda(a+1,c)f(z) = \frac{\left[z^2 p''(z) + 2(a-p)zp'(z) + (a-p)(a-p-1)p(z)\right]}{a(a-1)}. \quad (2.4)$$

Define the transformation from C^3 to C by

$$u = r,\ v = \frac{s + (a-p)r}{a},\quad w = \frac{t + 2(a-p)s + (a-p)(a-p-1)r}{a(a-1)}$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, \frac{s + (a-p)r}{a}, \frac{t + 2(a-p)s + (a-p)(a-p-1)r}{a(a-1)}; z\right). \quad (2.5)$$

The proof shall make use of Lemma 1. Using equation (2.2) , (2.3) and (2.4), then from (2.5) , we obtain

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(I_p^\lambda(a+1,c)f(z), I_p^\lambda(a,c)f(z), I_p^\lambda(a-1,c)f(z); z\right). \quad (2.6)$$

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Hence (2.1) becomes

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega .$$

The proof is completed if it can be shown that the admissibility condition for is equivalent to the admissibility condition for as given in Definition 1.

Note that

$$\left\{\frac{t}{s}+1\right\}=\left\{\frac{a(a-1)w-(a-p)(a-p-1)u}{av-(a-p)u}-2(\lambda+p)+1\right\},$$

and hence $\psi \in \Psi_p[\Omega, q]$. By Lemma 1,

$$p(z) \prec q(z) \text{ or } I_p^{\lambda}(a+1,c)f(z) \prec q(z).$$

If $\Omega \neq C$ is a simply connected domain , then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_I[h(U), q]$ is written as $\Phi_I[h,q]$.

The following result is an immediate consequence of Theorem 1.

Theorem 2. Let $\phi \in \Phi_I[h,q]$, If $f(z) \in A(p)$ satisfies

$$\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right) \prec h(z), \tag{2.7}$$

then

$$I_p^{\lambda}(a+1,c)f(z) \prec q(z),$$

where $\left(p \in N; a,c \in R \setminus Z_0^-; \lambda > -p; z \in U\right)$.

Our next result is an extension of Theorem1 to the case where the behavior of $q(z)$, on ∂U is not known.

Corollary 1. Let $\Omega \subset C$ and let $q(z)$, be univalent in U, $q(0)=0$. Let $\phi \in \Phi_I[\Omega, q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If $f(z) \in A(p)$ and

$$\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right) \in \Omega, \tag{2.8}$$

then

$$I_p^{\lambda}(a+1,c)f(z) \prec q(z)$$

$$\left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right).$$

Proof. Theorem 1 yields $I_p^{\lambda}(a+1,c)f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$.

If $q(z) = Mz, M > 0$, and in view of Definition 1, The class of admissible functions $\Phi_I[\Omega, q]$, denoted by $\Phi_I[\Omega, M]$ is described below .

Definition 4. let Ω be a set in C and $M > 0$. The class of admissible functions $\Phi_I[\Omega, M]$ consist of those functions $\phi: C^3 \times U \to C$ such that

$$\phi\left(Me^{i\theta}, \frac{k+(a-p)}{a}Me^{i\theta}, \frac{L+\left[2(a-p-1)k+(a-p-1)(a-p-2)\right]Me^{i\theta}}{a(a-1)}; z\right) \notin \Omega \tag{2.9}$$

whenever ,$z \in U, \theta \in R, R\left\{Le^{-i\theta}\right\} \geq (k-1)k\mathrm{M}$ for all real θ, $p \in \mathrm{N}$, $a \in R \setminus Z_0^-$ and $k \geq p$.

Corollary 2. Let $\phi \in \Phi_I[\Omega, M]$. If $f(z) \in A(p)$ satisfies

$$\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right) \in \Omega,$$

then

$$\left|I_p^{\lambda}(a+1,c)f(z)\right| < M . \qquad \left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$$

In the special case $\Omega = q(U) = \{w : |w| < M\}$, the class $\Phi_I[\Omega, M]$ is simply denoted by $\Phi_I[M]$, then the corollary (2.2) takes the following form .

Corollary 3. Let $\phi \in \Phi_I[M]$. If $f(z) \in A(p)$ satisfies

$$\left|\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right)\right| < M \quad ,$$

then

$$\left|I_p^{\lambda}(a+1,c)f(z)\right| < M \ . \qquad \left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$$

Now , we introduce a new class of admissible functions $\Phi_{I,1}[\Omega, q]$.

Definition 5. Let Ω be a set in C, $q \in Q_0 \cap H_0$. The class of admissible functions $\Phi_{I,1}[\Omega, q]$ consists of those functions $\phi : C^3 \times U \to C$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \ v = \frac{k\zeta q'(\zeta) + (a-1)q(\zeta)}{a},$$

$$R\left\{\frac{(a-1)\left[aw - (a-2)u\right]}{av - (a-1)u} - 2(a-p) + 3\right\} \geq kR\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

where $z \in U$, $\zeta \in \partial U / E(q)$, $p \in N$, $a \in R \setminus Z_0^-$ and $k \geq p$.

Theorem 3. Let $\phi \in \Phi_{I,1}[\Omega, q$. If $f(z) \in A(p)$ satisfies

$$\left\{\phi\left(\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda}(a-1,c)f(z)}{z^{p-1}}; z\right) : z \in U\right\} \subset \Omega \quad , \qquad (2.10)$$

then

$$\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}} \prec q(z) \ .$$

$\left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$.

Proof . Define the analytic function $p(z)$ in U by

$$p(z) = \frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}} \qquad (2.11)$$

In the view of relation (1.7) and from (2.11) we get ,

$$\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}} = \frac{zp'(z) + (a-1)p(z)}{a} \quad . \qquad (2.12)$$

Further computation show that

$$\frac{I_p^{\lambda}(a-1,c)f(z)}{z^{p-1}} = \frac{\left[z^2 p''(z) + 2(a-1)zp'(z) + (a-1)(a-2)p(z)\right]}{a(a-1)} \quad . \qquad (2.13)$$

Define the transformation from C^3 to C by

$$u = r, \ v = \frac{s + (a-1)r}{a} \quad , \ w = \frac{t + 2(a-1)s + (a-1)(a-2)r}{a(a-1)} \quad . \qquad (2.14)$$

Let

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$$\psi(r,s,t;z)=\phi(u,v,w;z)$$
$$=\phi\left(r,\frac{s+(a-1)r}{a},\frac{t+2(a-1)s+(a-1)(a-2)r}{a(a-1)};z\right) \quad (2.15)$$

The proof shall make use of Lemma 1. Using equation (2.11), (2.12) and (2.13), from (2.15), we obtain

$$\psi(p(z),zp'(z),z^2p''(z);z)=\phi\left(\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}},\frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}},\frac{I_p^{\lambda}(a-1,c)f(z)}{z^{p-1}};z\right) \quad . (2.16)$$

Hence (2.10) becomes

$$\psi(p(z),zp'(z),z^2p''(z);z)\in\Omega .$$

The proof is completed if it can be shown that the admissibility condition for $\phi\in\Phi_{I,1}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.

Note that

$$\left\{\frac{t}{s}+1\right\}=\left\{\frac{(a-1)\left[aw-(a-2)u\right]}{av-(a-1)u}-2(a-p)+3\right\},$$

and hence $\psi\in\Psi_p[\Omega,q]$. By Lemma 1 , $p(z)\prec q(z)$ or

$$\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}\prec q(z).$$

If $\Omega\neq C$ is a simply connected domain , then $\Omega=h(U)$, for some conformal mapping $h(z)$ of U onto Ω In this case the class $\Phi_{I,1}[h(U),q]$ is written as $\Phi_{I,1}[h,q]$.

The following result is an immediate consequence of Theorem 3.

Theorem 4. Let $\phi\in\Phi_{I,1}[\Omega,q]$, If $f(z)\in A(p)$ satisfies

$$\phi\left(\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}},\frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}},\frac{I_p^{\lambda}(a-1,c)f(z)}{z^{p-1}};z\right)\prec h(z), \quad (2.17)$$

then

$$\frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}\prec q(z).$$
$$\left(p\in N;\lambda>-p;a,c\in R\setminus Z_0^-;z\in U\right).$$

If $q(z)=Mz\,,M>0$, The class of admissible functions $\Phi_{I,1}[\Omega,q]$, denoted by $\Phi_{I,1}[\Omega,M]$, is described below .

Definition 6. let Ω be a set in C and $M>0$. The class of admissible functions $\Phi_{I,1}[\Omega,q]$, consists of those functions $\phi:C^3\times U\to C$ such that

$$\phi\left(Me^{i\theta},\frac{k+a-1}{a}Me^{i\theta},\frac{L+(a-1)\{2k+(a-2)\}Me^{i\theta}}{a(a-1)};z\right)\notin\Omega , \quad (2.18)$$

whenever

$$z\in U,\theta\in R,R\left\{Le^{-i\theta}\right\}\geq(k-1)k\mathrm{M}\ \text{ for all real } \theta,\ p\in\mathrm{N} \text{ and } a\in R\setminus Z_0^-,k\geq p.$$

Corollary 4. Let $\phi \in \Phi_{I,1}[\Omega, M]$. If $f(z) \in A(p)$ satisfies

$$\phi\left(\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a-1,c)f(z)}{z^{p-1}}; z\right) \in \Omega ,$$

then

$$\left|\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}\right| < M \ .\left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$$

In the special case $\Omega = q(U) = \{w : |w| < M\}$, the class $\Phi_{I,1}[\Omega, M]$ is simply denoted by $\Phi_{I,1}[M]$, then the previous Corollary 4 takes the following form .

Corollary 5. Let $\phi \in \Phi_{I,1}[M]$. If $f(z) \in A(p)$ satisfies

$$\left|\phi\left(\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a-1,c)f(z)}{z^{p-1}}; z\right)\right| < M ,$$

then

$$\left|\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}\right| < M \ .\left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$$

Next , we introduce a new class of admissible functions $\Phi_{I,2}[\Omega, q]$.

Definition7. Let Ω be a set in C, $q(z) \in Q_1 \cap H$. The class of admissible functions $\Phi_{I,2}[\Omega, q]$ consists of those functions $\phi : C^3 \times U \to C$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta),\ v = \frac{1}{a-1}\left\{-1 + aq(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)}\right\} ,$$

$$R\left\{\frac{\{(a-2)w - (a-1)v + 1\}}{(a-1)v - au + 1} - 2au + (a-1)v - 1\right\} \geq kR\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

where $z \in U$, $\zeta \in \partial U / E(q)$, $p \in N$, $a \in R \setminus Z_0^-$ and $k \geq p$.

Theorem 5. Let $\phi \in \Phi_{I,2}[\Omega, q]$ and $I_p^\lambda(a+1,c)f(z) \neq 0$. If $f(z) \in A(p)$ satisfies

$$\left\{\phi\left(\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)}, \frac{I_p^\lambda(a-2,c)f(z)}{I_p^\lambda(a-1,c)f(z)}; z\right) : z \in U\right\} \subset \Omega \quad , \qquad (2.19)$$

then

$$\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)} \prec q(z) .$$

$$\left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right).$$

Proof . Define the analytic function $p(z)$ in U by

$$p(z) = \frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)} . \qquad (2.20)$$

Using (2.20) , we get

$$\frac{zp'(z)}{p(z)}=\frac{z\,(I_p^{\lambda}(a,c)f(z))'}{(I_p^{\lambda}(a,c)f(z))}-\frac{z\,(I_p^{\lambda}(a+1,c)f(z))'}{(I_p^{\lambda}(a+1,c)f(z))}\quad . \tag{2.21}$$

By making use of the relation (1.7) in (2.21) , we get

$$\frac{I_p^{\lambda}(a-1,c)f(z)}{I_p^{\lambda}(a,c)f(z)}=\frac{1}{(a-1)}\left\{-1++ap(z)+\frac{zp'(z)}{p(z)}\right\} \tag{2.22}$$

Further computation show that

$$\frac{I_p^{\lambda}(a-2,c)f(z)}{I_p^{\lambda}(a-1,c)f(z)}=\frac{1}{(a-1)}$$

$$\left[-2+\frac{zp'(z)}{p(z)}+ap(z)+\frac{\frac{zp'(z)}{p(z)}+\frac{z^2p''(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+azp'(z)}{\frac{zp'(z)}{p(z)}+ap(z)-1}\right]\quad . \tag{2.23}$$

Define the transformation from C^3 to C by

$$u=r\,,v=\frac{1}{(a-1)}\left\{-1+ar+\frac{s}{r}\right\},\quad w=\frac{1}{(a-1)}\left\{-2+\frac{s}{r}+ar+\frac{\frac{t}{r}+\frac{s}{r}-\left(\frac{s}{r}\right)^2+as}{\frac{s}{r}+ar-1}\right\}\quad . \tag{2.24}$$

Let

$$\psi(r,s,t;z)=\phi(u,v,w;z)$$

$$=\phi\left(r,\frac{1}{(a-1)}\left\{-1+ar+\frac{s}{r}\right\},\frac{1}{(a-1)}\left\{-2+\frac{s}{r}+ar+\frac{\frac{t}{r}+\frac{s}{r}-\left(\frac{s}{r}\right)^2+as}{\frac{s}{r}+ar-1}\right\};z\right) \tag{2.25}$$

The proof shall make use of Lemma 1. Using equation (2.20), (2.22) and (2.23), then (2.25), we obtain

$$\psi(p(z),zp'(z),z^2p''(z);z)=\phi\left(\frac{I_p^{\lambda}(a,c)f(z)}{I_p^{\lambda}(a+1,c)f(z)},\frac{I_p^{\lambda}(a-1,c)f(z)}{I_p^{\lambda}(a,c)f(z)},\frac{I_p^{\lambda}(a-2,c)f(z)}{I_p^{\lambda}(a-1,c)f(z)};z\right) \tag{2.26}$$

Hence (2.19) becomes

$$\psi(p(z),zp'(z),z^2p''(z);z)\in\Omega\,.$$

The proof is completed if it can be shown that the admissibility condition for $\phi\in\Phi_{I,2}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition1.

Note that

$$\left\{\frac{t}{s}+1\right\}=\left\{\frac{v\,\{(a-2)w-(a-1)v+1\}}{(a-1)v-au+1}-2\,au+(\,a-1)\,v+1\right\},$$

and hence $\psi \in \Psi_p[\Omega, q]$. By Lemma 1, $p(z) \prec q(z)$ or $\frac{I_p^{\lambda}(a,c)f(z)}{I_p^{\lambda}(a+1,c)f(z)} \prec q(z)$.

If $\Omega \neq C$ is a simply connected domain , then $\Omega = h(U)$, for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_{I,2}[h(U), q]$ is written as $\Phi_{I,2}[h, q]$.

The following result is an immediate consequence of Theorem 5.

Theorem 6. Let $\phi \in \Phi_{I,2}[h, q]$ and $I_p^{\lambda}(a+1,c)f(z) \neq 0$. If $f(z) \in A(p)$ satisfies

$$\phi\left(\frac{I_p^{\lambda}(a,c)f(z)}{I_p^{\lambda}(a+1,c)f(z)}, \frac{I_p^{\lambda}(a-1,c)f(z)}{I_p^{\lambda}(a,c)f(z)}, \frac{I_p^{\lambda}(a-2,c)f(z)}{I_p^{\lambda}(a-1,c)f(z)}; z\right) \prec h \tag{2.27}$$

then

$$\frac{I_p^{\lambda}(a,c)f(z)}{I_p^{\lambda}(a+1,c)f(z)} \prec q(z).$$

$$\left(p \in N; \lambda > -p; a, c \in R \setminus Z_0^-; z \in U\right).$$

If $q(z) = Mz$, $M > 0$, The class of admissible functions $\Phi_{I,2}[\Omega, q]$, denoted by $\Phi_{I,2}[\Omega, M]$, is described below .

Definition 8. let Ω be a set in C and $M > 0$. The class of admissible functions $\Phi_{I,2}[\Omega, M]$ consist of those functions $\phi: C^3 \times U \to C$ such that

$$\phi\left(Me^{i\theta}, \frac{1}{(a-1)}\left(k-1+aMe^{i\theta}\right), \frac{1}{(a-1)}\left\{k-2+aMe^{i\theta} + \frac{Le^{-i\theta}+kM+akM^2-k^2M}{M(k-1)+aM^2e^{i\theta}}\right\}; z\right) \notin \Omega \qquad , (2.28)$$

whenever

$z \in U, \theta \in R, \Re\left\{Le^{-i\theta}\right\} \geq (k-1)k\mathrm{M}$ for all real θ, $p \in \mathrm{N}$; $a \in R \setminus Z_0^-$ and $k \geq p$.

Corollary 6. Let $\phi \in \Phi_{I,2}[\Omega, M]$ and $I_p^{\lambda}(a+1,c)f(z) \neq 0$. If $f(z) \in A(p)$ satisfies

$$\phi\left(\frac{I_p^{\lambda}(a,c)f(z)}{I_p^{\lambda}(a+1,c)f(z)}, \frac{I_p^{\lambda}(a-1,c)f(z)}{I_p^{\lambda}(a,c)f(z)}, \frac{I_p^{\lambda}(a-2,c)f(z)}{I_p^{\lambda}(a-1,c)f(z)}; z\right) \in \Omega,$$

then

$$\left|\frac{I_p^{\lambda}(a,c)f(z)}{I_p^{\lambda}(a+1,c)f(z)}\right| < M, \qquad \left(p \in N; \lambda > -p; a, c \in R \setminus Z_0^-; z \in U\right)$$

In the special case $\Omega = q(U) = \{w : |w| < M\}$, the class $\Phi_{I,2}[\Omega, M]$ is simply denoted by $\Phi_{I,2}[M]$, then Corollary 6 takes the following form .

Corollary 7. Let $\phi \in \Phi_{I,2}[M]$. If $f(z) \in A(p)$ satisfies

$$\left|\phi\left(\frac{I_p^{\lambda}(a,c)f(z)}{I_p^{\lambda}(a+1,c)f(z)}, \frac{I_p^{\lambda}(a-1,c)f(z)}{I_p^{\lambda}(a,c)f(z)}, \frac{I_p^{\lambda}(a-2,c)f(z)}{I_p^{\lambda}(a-1,c)f(z)}; z\right)\right| < M,$$

then

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$$\left|\frac{I_p^{\lambda}(a,c)f(z)}{I_p^{\lambda}(a+1,c)f(z)}\right| < M, \quad \left(p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U\right)$$

III. Superordination of the Cho-Kwon–Srivastava Operator $I_p^{\lambda}(a,c)f(z)$

The dual problem of the differential subordination, that is, differential superordination of the operator $I_p^{\lambda}(a,c)f(z)$ is investigated in this section . For this purpose the class of the admissible functions is given in the following definition.

Definition 9. let Ω be a set in $q(z) \in H[0,p]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_I[\Omega,q]$ consists of those functions $\phi : C^3 \times \overline{U} \to C$ that satisfy the admissibility condition

$$\phi(u,v,w;z) \notin \Omega$$

whenever

$$u = q(z), \; v = \frac{zq'(z) + m(a-p)q(z)}{ma},$$

$$R\left\{\frac{a(a-1)w - (a-p)(a-p-1)u}{av - (a-p)u} - 2(a-p) + 1\right\} \leq \frac{1}{m} R\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

where $z \in U$, $\zeta \in \partial U$, $a \in R \setminus Z_0^-$, $z \in U$ and $m \geq p$.

Theorem 7. Let $\phi \in \Phi'_I[\Omega,q]$. If $f(z) \in A(p)$, $I_p^{\lambda}(a,c)f(z) \in Q_0$ and

$$\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right)$$

is univalent in U, then

$$\Omega \subset \left\{\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right) : z \in U\right\} \tag{3.1}$$

implies

$$q(z) \prec I_p^{\lambda}(a+1,c)f(z).$$

$$\left(z \in U;\, a,c \in R \setminus Z_0^-; \lambda > -p; p \in N\right).$$

Proof . From (2.6) and (3.1),we have

$$\Omega \subset \{\psi(p(z), zp'(z), z^2 p''(z); z) : z \in U\}.$$

From (2.5), we see that the admissibility condition for $\phi \in \Phi'_I[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi'_p[\Omega,q]$, and by

Lemma 2, $q(z) \prec p(z)$ or $q(z) \prec I_p^{\lambda}(a+1,c)f(z)$.

If $\Omega \neq C$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω In this case the class $\Phi'_I[h(U),q]$ is written as $\Phi'_I[h,q]$.

The following result is an immediate consequence of Theorem 7.

Theorem 8. Let $h(z)$ be analytic on U and $\phi \in \Phi'_I[h,q]$. If $f(z) \in A(p)$, $I_p^{\lambda}(a+1,c)f(z) \in Q_0$ and

$$\phi\left(I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z\right)$$

is univalent in U,

then

$$h(z) \prec \phi\left(I_p^\lambda(a+1,c)f(z), I_p^\lambda(a,c)f(z), I_p^\lambda(a-1,c)f(z); z\right), \tag{3.2}$$

implies

$$q(z) \prec I_p^\lambda(a+1,c)f(z).$$

Now, we introduce a new class of admissible functions $\Phi'_{I,1}[\Omega, q]$.

Definition 9. Let Ω be a set in C, $q(z) \in H_0$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{I,1}[\Omega, q]$ consists of those functions $\phi: C^3 \times \overline{U} \to C$ that satisfy the admissibility condition :

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z),\ v = \frac{zq'(z) + m(a-1)q(z)}{ma},$$

$$R\left\{\frac{a(a-1)w - (a-2)u}{av - (a-1)u} - 2(a-p) + 3\right\} \leq \frac{1}{m} R\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

where $z \in U$, $\zeta \in \partial U$ and $m \geq p$.

Now, we will give the dual result of Theorem 3 for differential superordination.

Theorem 9. Let $\phi \in \Phi'_{I,1}[\Omega, q]$. If $f(z) \in A(p)$, $\dfrac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}} \in Q_0$ and

$$\phi\left(\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a-1,c)f(z)}{z^{p-1}}; z\right)$$

is univalent in U, then

$$\Omega \subset \left\{\phi\left(\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a-1,c)f(z)}{z^{p-1}}; z\right) : z \in U\right\} \tag{3.3}$$

implies

$$q(z) \prec \frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}.$$

$$\left(z \in U;\ a, c \in R \setminus Z_0^-;\ \lambda > -p;\ p \in N\right).$$

Proof. From (2.16) and (3.3), we have

$$\Omega \subset \{\psi(p(z), zp'(z), z^2 p''(z); z) : z \in U\}.$$

From (2.12), we see that the admissibility condition for $\phi \in \Phi'_{I,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi'[\Omega, q]$ and by

Lemma 2, $q(z) \prec p(z$ or $q(z) \prec \dfrac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}$.

If $\Omega \neq C$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω

In this case the class $\Phi'_{I,1}[h(U), q]$ is written as $\Phi'_{I,1}[h, q]$.

The following result is an immediate consequence of Theorem 9.

Theorem 10. Let $q(z)\in H_0, h(z)$ be univalent in U and $\phi\in\Phi'_{I,1}[\Omega,q]$. If $f(z)\in A(p)$, $\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}\in Q_0$ and

$$\phi\left(\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}},\frac{I_p^\lambda(a,c)f(z)}{z^{p-1}},\frac{I_p^\lambda(a-1,c)f(z)}{z^{p-1}};z\right)$$

is univalent in U, then

$$h(z)\prec\phi\left(\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}},\frac{I_p^\lambda(a,c)f(z)}{z^{p-1}},\frac{I_p^\lambda(a-1,c)f(z)}{z^{p-1}};z\right) \tag{3.4}$$

implies

$$q(z)\prec\frac{I_p^\lambda(a+1,c)f(z)}{z^{p-1}}.$$

$$\left(z\in U;\,a,c\in R\setminus Z_0^-;\lambda>-p;p\in N\right)$$

Finally, we introduce down a new class of admissible functions $\Phi'_{I,2}[\Omega,q]$.

Definition 10. let Ω be a set in C, $q(z)\neq 0$, $zq'(z)\neq 0$ and $q(z)\in H$. The class of admissible functions $\Phi'_{I,2}[\Omega,q]$ consists of those functions $\phi: C^3\times\overline{U}\to C$ that satisfy the admissibility condition

$$\phi(u,v,w;z)\notin\Omega$$

whenever

$$u=q(z),\; v=\frac{1}{a-1}\left\{-1+aq(z)+\frac{zq'(z)}{mq(z)}\right\},$$

$$R\left\{\frac{\{(a-2)w-(a-1)v+1\}v}{(a-1)v-au+1}+(a-1)u-2av+1\right\}\le\frac{1}{m}R\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

where $z\in U$, $\zeta\in\partial U$ and $m\ge 1$.

Now , we will give the dual result of theorem 5.

Theorem 11. Let $\phi\in\Phi'_{I,2}[\Omega,q$. If $f(z)\in A(p)$, $\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}\in Q_1$ and

$$\phi\left(\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)},\frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)},\frac{I_p^\lambda(a-2,c)f(z)}{I_p^\lambda(a-1,c)f(z)};z\right)$$

is univalent in U, then

$$\Omega\subset\phi\left(\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)},\frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)},\frac{I_p^\lambda(a-2,c)f(z)}{I_p^\lambda(a-1,c)f(z)};z\right)\quad, \tag{3.5}$$

implies

$$q(z)\prec\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}.$$

$\left(z \in U; a,c \in R \setminus Z_0^-; \lambda > -p; p \in N\right).$

Proof. From (2.26) and (3.5) , we have

$$\Omega \subset \{\psi(p(z), zp'(z), z^2 p''(z); z) : z \in U\}.$$

From (2.24) , we see that the admissibility condition for $\phi \in \Phi'_{I,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi'[\Omega, q]$, and by

lemma 2, $q(z) \prec p(z)$ or $q(z) \prec \dfrac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}$.

If $\Omega \neq C$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω In this case the class $\Phi'_{I,2}[h(U), q]$ is written as $\Phi'_{I,2}[h, q]$.

The following result is an immediate consequence of Theorem 11.

Theorem 12. Let $h(z)$ be analytic in U and $\phi \in \Phi'_{I,2}[h,q]$. If $f(z) \in A(p)$, $\dfrac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)} \in Q_1$ and

$$\phi\left(\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)}, \frac{I_p^\lambda(a-2,c)f(z)}{I_p^\lambda(a-1,c)f(z)}; z\right)$$

is univalent in U , then

$$h(z) \prec \phi\left(\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)}, \frac{I_p^\lambda(a-2,c)f(z)}{I_p^\lambda(a-1,c)f(z)}; z\right), \qquad (3.6)$$

implies

$$q(z) \prec \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a,c)f(z)}.$$

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Regression Analysis of Child Mortality and Per Capital Income

By Adeleke R.A. And Halid O.Y.
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Abstract - Higher income may be a precondition for healthy environment and better health services. There is considerable evidence and academic debate regarding relationships between per capital income and various health indicators including child mortality. In this paper, we proposed a two variable reciprocal regression model to establish the relationship between child mortality and per capital income. The method of ordinary least squares and some statistical inference were employed to analyse critically and ascertain the relationships between the two variables. From the analysis, it was discovered by the test of significance of regression, that there exist a relationship between the child mortality and per capital income at 5percent level of significance.

Keywords : *Per capital Income, Mortality, Reciprocal regression model ordinary least squares technique, statistical Inference.*

GJSFR-F Classification: *FOR Code: 140303*

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Regression Analysis of Child Mortality and Per Capital Income

Adeleke R.A.[α] And Halid O.Y.[Ω]

Abstract - Higher income may be a precondition for healthy environment and better health services. There is considerable evidence and academic debate regarding relationships between per capital income and various health indicators including child mortality. In this paper, we proposed a two-variable reciprocal regression model to establish the relationship between child mortality and per capital income. The method of ordinary least squares and some statistical inference were employed to analyse critically and ascertain the relationships between the two variables. From the analysis, it was discovered by the test of significance of regression, that there exist a relationship between the child mortality and per capital income at 5percent level of significance.

Keywords : Per capital Income, Mortality, Reciprocal regression model ordinary least squares technique, statistical Inference.

I. Introduction

Child mortality is defined as the number of deaths of children under the age of five in a given year per one thousand children in this age group. Income is defined as the money that is received as the results of normal business activities of an individual.

The determinants of child mortality change in less developed countries are not easy to unravel. Improvements in health technology and education play an important role, but effects of these factors are difficult to identify. The variables tend to be collinear with each other and with many other aspects of development, making their isolation difficult.

Identifying the impact of factors which are directly associated with health, is worthwhile for purposes of policy formulation but it may not be critical for a description of child mortality changes in the process of development. Behind these specific variables, the overall economic status of individuals is likely to dominate health changes through nutrition and other aspects of consumption because economic status is a close correlate and determinant of many of the more specific variables noted above.

Higher income may be a precondition for healthier environment and better health services. Thus, for general empirical analysis, it is quite reasonable to propose a sequence of causation which goes from income to child mortality via a number of intermediate variables. This is what this paper attempts to do.

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II. Materials and Methods

The data used in this paper are in respect of child mortality and per capital income for 64 countries for 2005. The data is a secondary. The data collected was analyzed using the following technique:

a) Reciprocal Regression Model

We use the two-variable reciprocal regression model of the form

$$y_i = \beta_1 + \beta_2\left(\frac{1}{x_i}\right) + \varepsilon_i \quad , i = 1, 2, \ldots, n \qquad (1)$$

This model is non-linear in the variable x_i because it enters inversely or reciprocally but linear in parameters β_1 and β_2 which are the intercept and the slope respectively.

The model is therefore a linear regression model and has the feature that as x_i increases indefinitely, the term $\beta_2\left(\frac{1}{x_i}\right)$ approaches zero and y_i approaches the limiting or asymptotic value β_1

b) Parameter Estimation

The parameters of $y_i = \beta_1 + \beta_2\left(\frac{1}{x_i}\right) + \varepsilon_i$ in (1) above can be estimated using the least square method so that,

$$\beta_2 = \frac{n\sum \frac{y_i}{x_i} - \sum\left(\frac{1}{x_i}\right)\sum y_i}{n\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^2 - \left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^2} \qquad (2)$$

And;

$$\beta_1 = \bar{y} - \beta_2\left(\frac{\sum \frac{1}{x}}{n}\right) \qquad (3)$$

c) Test of Significance of Linear Regression

This is used to test the significance of the linear relationship between child mortality y_i and per capital income x_i.

That is, testing the significance of the parameters. The null and alternative hypotheses are of the form

$H_0 : \beta_2 = 0$ and $H_1 : \beta_2 \neq 0$

$H_0 : \beta_2 = 0$ and $H_1 : \beta_2 \neq 0$ respectively.

This is done using the analysis of variance (ANOVA) table below

Source of variation	Sum of Squares	d.f.	Mean Squares	$F_{calculated}$
Due to Regression	$\beta_2 S_{xy}$	1	$\beta_2 S_{xy}$	MSR / MSE
Due to residuals	$S_{yy} - \beta_2 S_{xy}$	n-2	$S_{yy} - \beta_2 S_{xy}$ / n-2	
Total	S_{yy}	n-1		

Where d.f. depicts the degrees of freedom and the critical value of F (called the tabulated value) is given by $F_\alpha(1, n-2)$ and α is the level of significance. We reject the null hypothesis if $F_{calculated} > F_\alpha(1, n-2)$ and conclude that there exists no significant relationship between the variables.

III. Results and Discussions

a) Reciprocal regression model and its Parameter Estimates

By using the reciprocal model (1) of y on x, we obtain

$\beta_2 = 1.92$ and $\beta_1 = 141.5$, therefore,

$$y_i = 141.5 \; 1.92 \left(\frac{1}{x_i}\right)$$

$E(\varepsilon_i) = 0$ by the ordinary least square assumption. This shows that an increase in per capital income would cause a decrease in child mortality.

b) Test of Significance of Regression

In testing the statistical significance of regression, it is necessary to test the relationship between the concerned variables *x* and y. This is carried out and presented in the following ANOVA table.

Source of variation	Sum of Squares	d.f.	Mean Squares		$F_{0.05}(1, 62)$
Regression	36.66	1	36.66	0.008	3.84
Error	278236.34	62	4487.68		
Total	278273	63			

$F_{calculated} < F_\alpha(1, n-2)$ at five percent level of significance (0.008<3.84)

We accept the null hypothesis and conclude that per capital income affects child mortality. That is, there exists a relationship between the two variables.

IV. Conclusion

Based on the analysis carried out, we arrived at the following conclusions:

(i) An increase in per capital income would cause a decrease in child mortality .This we found from the regression analysis.

(ii) The test of hypothesis from the Analysis of Variance (ANOVA) shows that there is a significant relationship between child mortality and per capital income at 5percent level of significance.

V. Recommendation

Based on the above conclusions, the following recommendations were made:

(i) Government should provide free health care services so that low income earners would be able to access them.

(ii) Funds should also be made available by private organizations or firms in form of social development scheme to health sectors to improve medical facilities and personnel.

(iii) UNICEF and other organizations such as World Bank, UNDP should aim the provision of portable water supplies to rural populace of undeveloped and developing countries.

(iv) New findings and development in medicine which include introduction of vaccines for certain diseases should be encouraged to improve the health of infants and children especially in developing countries.

(v) Health extension programmes such as immunization should be extended from national to grassroot level. This will ensure infant survival rate.

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Fourth-Order Four Point Sturn-Liouville Boundary Value Problem With Non homogeneous Conditions

By Djibibe Moussa Zakari, Tcharie Kokou
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Abstract - In this paper, the sucient conditions are given for the existence and uniqueness of solutions of the following nonlinear Sturn-Liouville boundary value problem with non homogeneous four point boundary conditions.

Keywords and phrases : Fourth-order, Four points, Sturn-Liouville, Boundary value problem, Nonhomogeneous, Cone, Concave, Fixed point, Green's function, Dontinuous dependence, Positive solution.

Mathematics Subject Classification: 34B10

FOURTH-ORDER FOUR POINT STURN-LIOUVILLE BOUNDARY VALUE PROBLEM WITH NON HOMOGENEOUS CONDITIONS

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Fourth-Order Four Point Sturn-Liouville Boundary Value Problem With Non homogeneous Conditions

Djibibe Moussa Zakari[α], Tcharie Kokou[Ω]

Abstract - In this paper, the sucient conditions are given for the existence and uniqueness of solutions of the following nonlinear Sturn-Liouville boundary value problem with non homogeneous four point boundary conditions :

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1 \\ u'(0) - h_0 u(0) = \alpha_0 \\ u'(1) - h_1 u(1) = \alpha_1 \\ a_1 u^{(3)}(t_1) - b_1 u''(t_1) = \lambda_1 \\ a_2 u^{(3)}(t_2) + b_2 u''(t_2) = \lambda_2 \end{cases}$$

where $0 \leq t_1 \leq t_2 \leq 1$ and $\lambda_1 \text{ et } \lambda_2$ are nonegative parameters.
The dependence continue of the solution on the parameters λ_1 and λ_2 is also investigated.

Keywords and phrases : Fourth-order, Four points, Sturn-Liouville, Boundary value problem, Nonhomogeneous, Cone, Concave, Fixed point, Green's function, Dontinuous dependence, Positive solution.

I. Introduction

Multi-point boundary value problems for ordinary differential equations arise in variety of areas of applied biologics, chemics, mathematics and physics have been studied. For details, see for exemple, [1], [4] -[11] and referencs therein.

In particular, in a recent article [4], Sun and Wang studied a four-point boundary value problem of the form

$$\begin{aligned} &u^{(4)}(t) = f(t, u(t)), \ 0 < t < 1 \\ &\alpha u(0) - \beta u'(0) = \gamma u(1) + \delta u'(1) = 0 \\ &a u''(\xi_1) - b u'''(\xi_1) = -\lambda, \ c u''(\xi_1) + d u'''(\xi_1) = -\mu \end{aligned}$$

In [2], Kong and Kong, investigated following multi-point boundary value problem :

$$\begin{aligned} &u''(t) + a(t) f(u) = 0 \\ &u(0) = \sum_{i=1}^{m} a_i u(t_i) + \lambda, \ u(1) = \sum_{i=1}^{m} b_i u(t_i) + \mu \end{aligned}$$

where λ and μ are nonegative parameter. They derived some conditions for the above boundary value problems to have a unique solution and then studied the dependence of this solution on the parameters λ and μ.

In another paper [5], Ricardo and Luis :

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$$\begin{aligned}
&u^{(4)}(t) = f(t, u(t), u''(t)), \ 0 < t < 2\pi \\
&u(0) = u(2\pi), \ u'(0) = u'(2\pi) \\
&u''(0) = u''(2\pi), \ u'''(0) = u'''(2\pi) \\
&u(0) = u(\pi) = u''(0) = u''(\pi) = 0
\end{aligned}$$

In the present paper, being inspired by [2] and [5], we investigated the fourth - order differential equation

$$u^{(4)}(t) = f(t, u(t), u''(t)), \ 0 < t < 1 \tag{1.1}$$

and the four-point nonhomogeneous Sturn-Liouville boundary conditions

$$u'(0) - h_0 u(0) = \alpha_0 \tag{1.2}$$
$$u'(1) - h_1 u(1) = \alpha_1 \tag{1.3}$$
$$a_1 u^{(3)}(t_1) - b_1 u''(t_1) = \lambda_1 \tag{1.4}$$
$$a_2 u^{(3)}(t_2) + b_2 u''(t_2) = \lambda_2 \tag{1.5}$$

where $0 \leq t_1 \leq t_2 \leq 1$ and λ_1 and λ_2 are nonnegative parameters.

We investigated the existence, uniqueness and parameter dependence continuous solution of the problem (1.1) -(1.5).

We will suppose the following conditions are satisfied :

Conditions 1.1

$\alpha_0, \ \alpha_1, \ h_0, \ h_1, \ a_1, \ b_1$ are nonegative constants and $a_2, \ b_2$ negative constants such that

$$a = -h_0 + h_1 + h_0 h_1 > 0 \ \textit{and} \ b = b_1 b_2 (t_1 - t_2) - a_1 b_2 - a_2 b_1 > 0.$$

Conditions 1.2

$$h_1 \alpha_0 - h_0 \alpha_1 > 0, \quad \alpha_0 - \alpha_1 - h_1 > 0, \quad a_1 - b_1 t_1 > 0, \quad -b_2 t_2 - a_2 > 0.$$

Conditions 1.3

- $f : [0, +\infty[\times [0, +\infty[\times \mathbb{R} \longrightarrow [0, +\infty[$ is continuous and monotone increasing in u and u''.
- There exists $0 \leq r < 1$, such that : $k^r f(t, u(t), u'') \leq f(t, ku(t), ku''(t))$ for all $t \in (0,1)$, and $k \in (0,1)$.

II. Preliminaries and Some Basic Lemmas

Definition 1

Let $\mathbb{E}$ be a reel Banach space with a norm $||.||_{\mathbb{E}}$ *and* K a nonempty closed convex set of $\mathbb{E}$.

1. K is said to be cone if $\alpha K \subseteq \mathbb{E}$ for all $\alpha \geq 0$ and $K \cap (-K) = \{0_{\mathbb{E}}\}$.
2. Every cone K in $\mathbb{E}$ definies a partial ordering in $\mathbb{E}$ by $x \leq y \Longleftrightarrow x - y \in K$.
3. A cone K is said to be normal if there existe $\lambda > 0$ such that

$$0 \leq x \leq y \Longrightarrow ||x||_{\mathbb{E}} \leq \lambda ||y||_{\mathbb{E}}.$$

4. A cone K is said to be solid if the interior $\overset{\circ}{K}$ of K is nonempty.
5. An operator $A : \overset{\circ}{K} \longrightarrow \overset{\circ}{K}$ is called r-concave if

$$k^r A(u) \leq A(ku) \quad \text{for all} \quad 0 \leq k \leq 1, \ u \in \overset{\circ}{K},$$

where K *is a solid cone and* $0 \le r < 1$.

Lemma 2.1

Let E be u Banach space , K be a normal solid cone in E, $0 \le r < 1$ and $A : \mathring{K} \longrightarrow \mathring{K}$ is a r_- concave increasing operateur. Then A has a unique fixed point in $\mathring{K}$.

proof

The proof of this lemma 2.1 is the same as [1]

Lemma 2.2

Suppose $a \neq 0$ and $b \neq 0$. If $g(t) \in C([0, 1])$ and $g(t) \ge 0$ on $[0, 1]$, the nonhomogeneous boundary value problem :

$$\begin{aligned} & u^{(4)}(t) = g(t), \ 0 < t < 1 \\ & u'(0) - h_0 u(0) = \alpha_0, \quad u'(1) - h_1 u(1) = \alpha_1 \\ & a_1 u^{(3)}(t_1) - b_1 u''(t_1) = \lambda_1, \quad a_2 u^{(3)}(t_2) - b_2 u''(t_2) = \lambda_2 \end{aligned}$$

has a unique solution

$$u(t) = \int_0^1 K_1(t,x) \int_{t_1}^{t_2} K_2(x,y) g(y)\, dy\, dx + h(t) + \lambda_1 \varphi_1(t) + \lambda_2 \varphi_2(t), \quad 0 \le t \le 1$$

where

$$K_1(t,x) = \begin{cases} \dfrac{(1+h_0 x)(1+h_1(1-t))}{a}, & 0 \le x \le t \le 1 \\[2ex] \dfrac{(1+h_0 t)(1+h_1(1-x))}{a}, & 0 \le t \le x \le 1 \end{cases}$$

$$K_2(x,y) = \begin{cases} \dfrac{(b_1(y-t_1)+a_1)(b_2(x-t_2)-a_2)}{b}, & y \le x, \ t_1 \le y \le t_2 \\[2ex] \dfrac{(b_1(x-t_1)+a_1)(b_2(y-t_2)-a_2)}{b}, & x \le y, \ t_1 \le x \le t_2 \end{cases}$$

$$h(t) = \frac{(h_1\alpha_0 - h_0\alpha_1)t}{a} + \frac{\alpha_0 - \alpha_1 - h_1}{a}, \quad 0 \le t \le 1$$

$$\varphi_1(t) = \frac{1}{b} \int_0^1 (b_2(x-t_2) - a_2) K_1(t,x)\, dx, \quad 0 \le t \le 1$$

$$\varphi_2(t) = \frac{1}{b} \int_0^1 (b_1(x-t_1) + a_1) K_1(t,x)\, dx, \quad 0 \le t \le 1$$

Proof

Putting $u''(t) = w(t), \ 0 \le t \le 1$.

By vertue of boundaries conditions (1.2)-(1.5), we obtain two following Sturn-Liouville boundary value problems :

$$(P_1) \ : \ \begin{cases} u''(t) = w(t), \ 0 < t < 1 \\ u'(0) - h_0 u(0) = \alpha_0 \\ u'(1) - h_1 u(1) = \alpha_1 \end{cases} \quad \text{and} \quad (P_2) \ : \ \begin{cases} w''(t) = g(t), \ 0 < t < 1 \\ a_1 w'(t_1) - b_1 w(t_1) = \lambda_1 \\ a_2 w'(t_2) - b_2 w(t_2) = \lambda_2 \end{cases}$$

The Green's fonctions for Sturn-Liouville problems $(P_1$ and (P_2) are respectively K_1 and K_2.

Then the solutions of the boundary value problems (P_1) and (P_2) are :

$$u(t) = -\int_0^1 K_1(t,x) w(x)\, dx + \frac{(h_1\alpha_0 + h_0\alpha_1)t}{a} + \frac{\alpha_1 - (h_1+1)\alpha_0}{a} \tag{2.1}$$

$$w(t) = -\int_{t_1}^{t_2} K_2(t,y) g(y)\, dy + \frac{(b_2(x-t_2) - a_2)\lambda_1}{b} + \frac{(b_1(x-t_1) + a_1)\lambda_2}{b} \tag{2.2}$$

Substituting (2.2) into (2.1), we get

$$u(t) = \int_0^1 K_1(t,x) \int_{t_1}^{t_2} K_2(x,y) g(y)\, dy dx + h(t) + \lambda_1 \varphi_1(t) + \lambda_2 \varphi_2(t), \quad 0 \le t \le 1.$$

This completes the proof of Lemma 2.2. ⊡

Lemma 2.3

Let conditions 1.1 and 1.2 be fulfilled. Then

1. $K_1(t,x) > 0$ and $K_2(t,y) > 0$ for $t, \ x \in [0, \ 1]$ and $y \in [t_1, \ t_2]$.
2. $h(t) > 0, \ \varphi_1(t) > 0$ and $\varphi_2(t) > 0$ for $t \in [0, \ 1]$.

III. Main Results

Throughout this article, for $k = 0, \ 1, \cdots,$ we denote by $C^k[0, \ 1]$ the Banach space of all kth continuously differentiable functions $u(t)$ on $[0, \ 1]$ with the norm $\|u\| = \max\limits_{t\in[0, \ 1]} \{|u(t)|, \ |u'(t)|, \ \cdots |u^k(t)|\}$ and let $E = C^2[0, \ 1]$. We denote by $L[0, \ 1]$ the Banach space of all integrable functions $u(t)$ on $[0, \ 1]$ with the norm

$$\|u\|_{L[0, \ 1]} = \int_0^1 |u(x)|\, dx.$$

Theorem 3.1 (Existence)

Let conditions (1.1), (1.2) and (1.3) be fulfilled. Then the nonhomogeneous Sturn-Liouville boundary value problem has a unique positive solution $u_{\lambda_1,\lambda_2}(t)$ for all $\lambda_1 > 0$ and $\lambda_2 > 0$.

Proof

Let $K = \{u \in \mathbb{E} \ : \ u(t) \ge 0, \ 0 \le t \le 1\}$. Then K is a normal solid cone in $\mathbb{E}$ and his interior is defined by $\overset{\circ}{K} = \{u \in \mathbb{E} \ : \ u(t) > 0, \ 0 \le t \le 1\}$.

The rest of the proof is based on the following proposition

Proposition 3.1

Let $A_{\lambda_1,\lambda_2} \ : \overset{\circ}{K} \longrightarrow \overset{\circ}{K}$ an operator define for any $\lambda_1 > 0$ and $\lambda_2 > 0$ by :

$$A_{\lambda_1,\lambda_2}(u(t)) = \int_0^1 K_1(t,x)\int_{t_1}^{t_2} K_2(x,y)f(y,u(y),u''(y))\,dydx + h(t) + \lambda_1\varphi_1(t) + \lambda_2\varphi_2(t). \quad (3.1)$$

Then A_{λ_1,λ_2} is $r-$concave with $0 \le r < 1$.

Proof of proposition 3.1

Let $k \in [0,\ 1]$ and $u \in \mathring{K}$, it follows from (3.1)

$$A_{\lambda_1,\lambda_2}(ku(t)) = \int_0^1 K_1(t,x)\int_{t_1}^{t_2} K_2(x,y)f(y,ku(y),ku''(y))\,dydx + h(t) + \lambda_1\varphi_1(t) + \lambda_2\varphi_2(t). \quad (3.2)$$

By vertue the conditions (1.3), we obtain

$$A_{\lambda_1,\lambda_2}(ku(t)) \ge k^r \int_0^1 K_1(t,x)\int_{t_1}^{t_2} K_2(x,y)f(y,u(y),u''(y))\,dydx + h(t) + \lambda_1\varphi_1(t) + \lambda_2\varphi_2(t). \quad (3.3)$$

Therefore, by inequality $h(t) + \lambda_1\varphi_1(t) + \lambda_2\varphi_2(t) \ge k^r\{h(t) + \lambda_1\varphi_1(t) + \lambda_2\varphi_2(t)\}$ and (3.3), we obtain

$$A_{\lambda_1,\lambda_2}(ku(t)) \ge k^r \left\{\int_0^1 K_1(t,x)\int_{t_1}^{t_2} K_2(x,y)f(y,u(y),u''(y))\,dydx + h(t) + \lambda_1\varphi_1(t) + \lambda_2\varphi_2(t)\right\}. \quad (3.4)$$

From (3.4), We conclude that

$$A_{\lambda_1,\lambda_2}(u(t)) \le k^r A_{\lambda_1,\lambda_2}(ku(t)). \quad (3.5)$$

Rest of proof of theorem (3.1)

It follows from lemma 2.1 and proposition 3.1 that A_{λ_1,λ_2} has a unique fixed point $u_{\lambda_1,\lambda_2} \in \mathring{K}$, which is the unique positive solution of the boundary value problem (1.1)-(1.5). This completes the proof. ⊡

Lemma 3.1

Under the conditions of Theorem (3.1) . The solution u_{λ_1,λ_2} of the boundary value problem (1.1) - (1.5) satisfies the following propertie :

$$\lim_{(\lambda_1,\lambda_2)\to(+\infty,+\infty)} \|u_{\lambda_1,\lambda_2}(t)\|_{\mathbb{E}} = +\infty$$

Proof

By virtue of the lemma 2.3, and the definition of $u_{\lambda_1,\lambda_2}(t)$:

$$\begin{aligned} u_{\lambda_1,\lambda_2}(t) &= A_{\lambda_1,\lambda_2}(u_{\lambda_1,\lambda_2}(t)) \\ &= \int_0^1 K_1(t,x)\int_{t_1}^{t_2} K_2(x,y)f(y,u_{\lambda_1,\lambda_2}(y),u''_{\lambda_1,\lambda_2}(y))\,dydx + h(t) + \lambda_1\varphi_1(t) + \lambda_2\varphi_2(t), \end{aligned}$$

we have

$$\lambda_1\varphi_1(t) + \lambda_2\varphi_2(t) \le \|u_{\lambda_1,\lambda_2}(t)\|_{\mathbb{E}}.$$

It is clear that $\lim_{(\lambda_1,\lambda_2)\to(+\infty,+\infty)} [\lambda_1\varphi_1(t) + \lambda_2\varphi_2(t)] = +\infty.$

From this last limit, we conclude :

$$\lim_{(\lambda_1,\lambda_2)\to(+\infty,+\infty)} \|u_{\lambda_1,\lambda_2}(t)\|_{\mathbb{E}} = +\infty.$$

The proof of lemma 3.1 is complete. ⊡

Theorem 3.2 (Continuous dependence)

Under the conditions of previous theorem . The solution u_{λ_1,λ_2} of the boundary value problem (1.1) - (1.5) , $u_{\lambda_1,\lambda_2}(t)$ is continuous in λ_1 and λ_2.

Proof

Let $(\lambda_1^0,\lambda_2^0)$ and $(\lambda_1^1,\lambda_2^1)$, such that $(0,0) < (\lambda_1^0,\lambda_2^0) < (\lambda_1^1,\lambda_2^1)(0 < \lambda_1^0 < \lambda_1^1$ and $0 < \lambda_2^0 < \lambda_2^1)$.

Put $\overline{n} = \left\{ n > 0 \; ; \; u_{\lambda_1^0\lambda_2^0}(t) \le n u_{\lambda_1^1\lambda_2^1}(t), \; t \in [0,1] \right\}$.

We assert that $\overline{n} \le 1$.

Thus, we obtain $u_{\lambda_1^0\lambda_2^0}(t) \le n u_{\lambda_1^1\lambda_2^1}(t)$, for $t \in [0,1]$.

Since $A_{\lambda_1\lambda_2}$ is strictly increasing in λ_1 and λ_2, we have

$$u_{\lambda_1^0\lambda_2^0}(t) = A_{\lambda_1^0\lambda_2^0}(u_{\lambda_1^0\lambda_2^0}(t)) \le A_{\lambda_1^0\lambda_2^0}(u_{\lambda_1^1\lambda_2^1}(t)) < A_{\lambda_1^1\lambda_2^1}(u_{\lambda_1^1\lambda_2^1}(t)) = u_{\lambda_1^1\lambda_2^1}(t)$$
$$u_{\lambda_1^0\lambda_2^0}(t) < u_{\lambda_1^1\lambda_2^1}(t), \text{ for } t \in [0, \; 1]$$

It is easy to see that $u_{\lambda_1\lambda_2}(t)$ is also strictly increasing in λ_1 and λ_2.

For any $(\lambda_1^0, \; \lambda_2^0) > (0,0)$,we suppose $(\lambda_1,\lambda_2) \to (\lambda_1^0,\lambda_2^0)$, with $(\lambda_1^0,\lambda_2^0) < (\lambda_1,\lambda_2)$.

We have easily, $u_{\lambda_1^0\lambda_2^0}(t) < u_{\lambda_1\lambda_2}(t), \; t \in [0,1]$.

Put $\overline{m} = \left\{ m > 0, \; u_{\lambda_1\lambda_2}(t) \le m u_{\lambda_1^0\lambda_2^0}(t), \; t \in [0,1] \right\}$

Then $\overline{m} \ge 1$, and $u_{\lambda_1^0\lambda_2^0}(t) \le \dfrac{1}{m} u_{\lambda_1\lambda_2}(t)$ for $t \in [0,1]$.

Set $\Omega_{\lambda_1\lambda_2} = \min \left(\dfrac{\lambda_1}{\lambda_1^0}, \dfrac{\lambda_2}{\lambda_2^0} \right)$.

That implies $\Omega_{\lambda_1\lambda_2} \ge 1$, and

$$u_{\lambda_1^0\lambda_2^0}(t) = A_{\lambda_1^0\lambda_2^0}(u_{\lambda_1^0\lambda_2^0}(t)) \ge A_{\lambda_1^0\lambda_2^0}\left(\frac{1}{\overline{m}} u_{\lambda_1\lambda_2}(t)\right) \tag{3.6}$$

$$A_{\lambda_1^0\lambda_2^0}\left(\frac{1}{\overline{m}} u_{\lambda_1\lambda_2}(t)\right) > \frac{1}{\Omega_{\lambda_1\lambda_2}} A_{\lambda_1\lambda_2}\left(\frac{1}{\overline{m}}(u_{\lambda_1\lambda_2}(t))\right) \tag{3.7}$$

$$A_{\lambda_1\lambda_2}\left(\frac{1}{m} u_{\lambda_1\lambda_2}(t)\right) \ge \frac{1}{\overline{m}^r} A_{\lambda_1\lambda_2}\left(u_{\lambda_1\lambda_2}(t)\right) = \frac{1}{\overline{m}^r} u_{\lambda_1\lambda_2}(t), \quad r,t \in [0,1] \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we can easily obtain :

$$u_{\lambda_1\lambda_2}(t) < \overline{m}^r \Omega_{\lambda_1\lambda_2} u_{\lambda_1^0\lambda_2^0}(t), \quad t \in [0,1]. \tag{3.9}$$

Combining (3.9) and the definition of $\overline{m}$, it follows that

$$\overline{m} \le \Omega_{\lambda_1\lambda_2}^{\frac{1}{1-r}}, \; 0 \le r \le 1.$$

And so

$$u_{\lambda_1\lambda_2}(t) \le \overline{m} u_{\lambda_1^0\lambda_2^0}(t) \le (\Omega_{\lambda_1\lambda_2}^{\frac{1}{1-r}} u_{\lambda_1^0\lambda_2^0}(t), \; 0 \le r \le 1, \; 0 \le t \le 1. \tag{3.10}$$

By virty of (3.10), we can write

$$\|u_{\lambda_1\lambda_2}(t) - u_{\lambda_1^0\lambda_2^0}(t)\| \leq (\Omega_{\lambda_1\lambda_2}^{\frac{1}{1-r}} - 1)\|u_{\lambda_1^0\lambda_2^0}(t)\|, \quad 0 \leq t \leq 1. \tag{3.11}$$

From (3.11) and the fact that $\lim\limits_{(\lambda_1,\lambda_2)\to(\lambda_1^0,\lambda_2^0)} \Omega_{\lambda_1\lambda_2} = 1,$ it follows

$$\lim_{(\lambda_1,\lambda_2)\to(\lambda_1^0,\lambda_2^0)} \|u_{\lambda_1\lambda_2}(t) - u_{\lambda_1^0\lambda_2^0}(t)\| = 0. \tag{3.12}$$

Thus , finaly, $u_{\lambda_1,\lambda_2}(t)$ is continuous in λ_1 and λ_2. This complete the proof of theorem 3.2. ⊡

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3. Think Like Evaluators: If you are in a confusion or getting demotivated that your paper will be accepted by evaluators or not, then think and try to evaluate your paper like an Evaluator. Try to understand that what an evaluator wants in your research paper and automatically you will have your answer.

4. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

5. Ask your Guides: If you are having any difficulty in your research, then do not hesitate to share your difficulty to your guide (if you have any). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work then ask the supervisor to help you with the alternative. He might also provide you the list of essential readings.

6. Use of computer is recommended: As you are doing research in the field of Computer Science, then this point is quite obvious.

7. Use right software: Always use good quality software packages. If you are not capable to judge good software then you can lose quality of your paper unknowingly. There are various software programs available to help you, which you can get through Internet.

8. Use the Internet for help: An excellent start for your paper can be by using the Google. It is an excellent search engine, where you can have your doubts resolved. You may also read some answers for the frequent question how to write my research paper or find model research paper. From the internet library you can download books. If you have all required books make important reading selecting and analyzing the specified information. Then put together research paper sketch out.

9. Use and get big pictures: Always use encyclopedias, Wikipedia to get pictures so that you can go into the depth.

10. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right! It is a good habit, which helps to not to lose your continuity. You should always use bookmarks while searching on Internet also, which will make your search easier.

11. Revise what you wrote: When you write anything, always read it, summarize it and then finalize it.

12. Make all efforts: Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

13. Have backups: When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

14. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several and unnecessary diagrams will degrade the quality of your paper by creating "hotchpotch." So always, try to make and include those diagrams, which are made by your own to improve readability and understandability of your paper.

15. Use of direct quotes: When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

16. Use proper verb tense: Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

17. Never use online paper: If you are getting any paper on Internet, then never use it as your research paper because it might be possible that evaluator has already seen it or maybe it is outdated version.

18. **Pick a good study spot:** To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

19. Know what you know: Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

20. Use good quality grammar: Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

21. Arrangement of information: Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

23. Multitasking in research is not good: Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

24. Never copy others' work: Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

25. Take proper rest and food: No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

26. Go for seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

27. Refresh your mind after intervals: Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

29. **Think technically:** Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

30. Think and then print: When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

31. Adding unnecessary information: Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

32. Never oversimplify everything: To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

33. Report concluded results: Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

34. After conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page

- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.

Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.
- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.

- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

Administration Rules Listed Before Submitting Your Research Paper to Global Journals Inc. (US)

Please carefully note down following rules and regulation before submitting your Research Paper to Global Journals Inc. (US):

Segment Draft and Final Research Paper: You have to strictly follow the template of research paper. If it is not done your paper may get rejected.

- The **major constraint** is that you must independently make all content, tables, graphs, and facts that are offered in the paper. You must write each part of the paper wholly on your own. The Peer-reviewers need to identify your own perceptive of the concepts in your own terms. NEVER extract straight from any foundation, and never rephrase someone else's analysis.
- Do not give permission to anyone else to "PROOFREAD" your manuscript.
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- To guard yourself and others from possible illegal use please do not permit anyone right to use to your paper and files.

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Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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