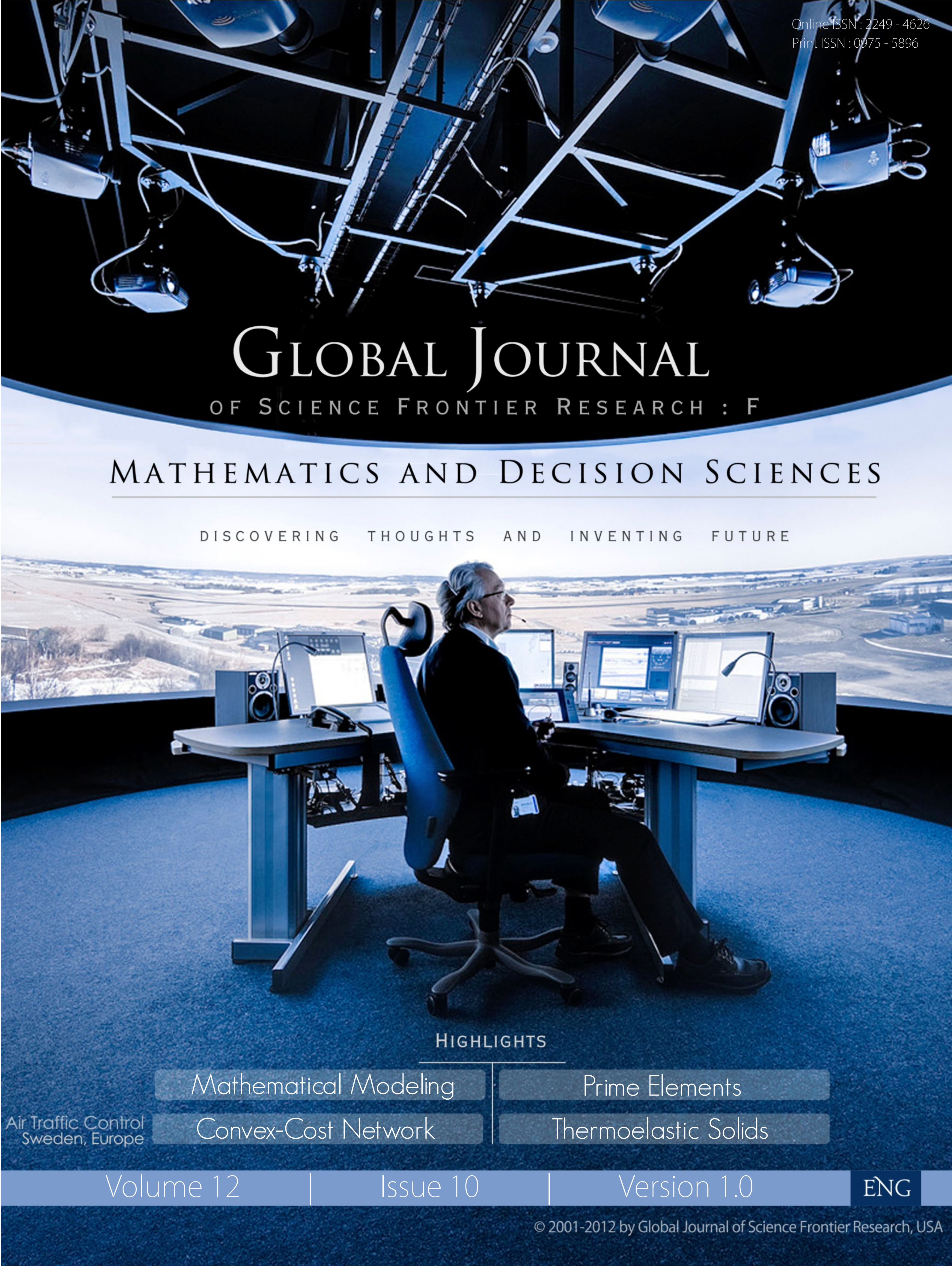
Online ISSN : 2249 - 4626
Print ISSN : 0975 - 5896



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Global Journal of Science Frontier Research: F
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GLOBAL JOURNAL OF SCIENCE FRONTIER RESEARCH: F

MATHEMATICS & DECISION SCIENCES

VOLUME 12 ISSUE 10 (VER. 1.0)

OPEN ASSOCIATION OF RESEARCH SOCIETY

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 10 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Generalization of the Dependent Function in Extenics for Nested Sets with Common Endpoints to 2D-Space, 3D-Space, and Generally to n-D-Space

By Florentin Smarandache
University of New Mexico, USA

Abstract - In this paper we extend Prof. Yang Chunyan and Prof. Cai Wen's dependent function of a point P with respect to two nested sets $X_0 \subset X$, for the case the sets X_0 and X have common ending points, from $1D$ - space to n-D-space. We give several examples in 2D- and 3D-spaces. When computing the dependent function value $k(.)$ of the optimal point O, we take its maximum possible value. Formulas for computing $k(O)$, and the geometrical determination the Critical Zone are also given.

GJSFR-F Classification : MSC 2010: 00A73

GENERALIZATION OF THE DEPENDENT FUNCTION IN EXTENICS FOR NESTED SETSWITH COMMON ENDPOINTS TO 2D-SPACE, 3D-SPACE, AND GENERALLY TO N-D-SPACE

Generalization of the Dependent Function in Extenics for Nested Sets with Common Endpoints to 2D-Space, 3D-Space, and Generally to n-D-Space

Florentin Smarandache

Abstract - In this paper we extend Prof. Yang Chunyan and Prof. Cai Wen's dependent function of a point *P* with respect to two nested sets $X_0 \subset X$, for the case the sets X_0 and X have common ending points, from *1D* - space to *n-D*-space. We give several examples in *2D*- and *3D*-spaces. When computing the dependent function value *k(.)* of the optimal point *O*, we take its maximum possible value.

Formulas for computing *k(O)*, and the geometrical determination the Critical Zone are also given.

I. PRINCIPLE OF DEPENDENT FUNCTION

Principle of Dependent Function of a point $P(x)$ with respect to a nest of two sets $X_0 \subset X$, i.e. *the degree of dependence of point P with respect to the nest of the sets* $X_0 \subset X$, is the following.

The dependent function value, $k(x)$, is computed as follows:

- the extension distance between the point P and the larger set's closest frontier, divided by the extension distance between the frontiers of the two sets {both extension distances are taken on the line/geodesic that passes through the point P and the optimal/attracting point O};
- the dependent function value is positive if point P belongs to the larger set, and negative if point P is outside of the larger set.

II. DEPENDENT FUNCTION FORMULA FOR NESTED SETS HAVING COMMON ENDING POINTS IN 1D-SPACE

For two nested sets $X_0 \subset X$ from the one-dimensional space of real numbers R, with X_0 and X having common endpoints, the **Dependent Function** $K(x)$, which gives the degree of dependence of a point x with respect to this pair of included *1D* -intervals, was defined by Yang Chunyan and Cai Wen in [2] as:

Ref.

2. Yang Chunyan, Cai Wen. Extension Engineering [M]. Beijing: Public Library of Science, 2007.

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$$K(x)=\begin{cases}\dfrac{\rho(x,X)}{\rho(x,X)-\rho(x,X_0)} & \rho(x,X)-\rho(x,X_0)\neq 0, x\in X\\ -\rho(x,X_0)+1 & \rho(x,X)-\rho(x,X_0)=0, x\in X_0\\ -\rho(x,X) & \rho(x,X)-\rho(x,X_0)=0, x\notin X_0, x\in X\\ \dfrac{\rho(x,X)}{\rho(x,X)-\rho(x,\hat{X})} & \rho(x,X)-\rho(x,\hat{X})\neq 0, x\in R-X\\ -\rho(x,\hat{X})-1 & \rho(x,X)-\rho(x,\hat{X})=0, x\in R-X\end{cases} \tag{1}$$

where $X_0 = <a_0, b_0>$, $X = <a, b>$, $\hat{X} = <c, d>$, and $X_0 \subset X \subset \hat{X}$.

III. *N-D*-DEPENDENT FUNCTION FORMULA FOR TWO NESTED SETS HAVING NO COMMON ENDING POINTS

The extension *n-D*-dependent function *k(.)* of a point *P*, which represents the degree of dependence of the point *P* with respect to the nest of the two sets $X_0 \subset X$, is:

$$k(P)=\frac{\rho(P,BiggerSet)}{\rho(P,BiggerSet)-\rho(P,SmallerSet)}=\frac{\rho_{nD}(P,X)}{\rho_{nD}(P,X)-\rho_{nD}(P,X_0)}=\pm\frac{|PP_2|}{|PP_2|-|PP_1|}=\pm\frac{|PP_2|}{|P_1P_2|} \tag{2}$$

In other words, the extension *n-D*-dependent function *k(.)* of a point *P* is the *n-D*-extension distance between the point *P* and the closest frontier of the larger set *X*, divided by the *n-D*extension distance between the frontiers of the two nested sets *X* and X_0; all these n-Dextension distances are taken along the line (or geodesic) *OP*.

IV. *N-D*-DEPENDENT FUNCTION FORMULA FOR TWO NESTED SETS HAVING COMMON ENDING POINTS

We generalize the above formulas (1) and (2) to an ***n-D* Dependent Function** of a point $P(x_1, x_2,\ldots, x_n)$ with respect to the nested sets X_0 and *X* having common endpoints, $X_0 \subset X$, from the universe of discourse *U*, in the *n-D*-space:

$$K_{nD}((x_1,x_2,\ldots,x_n))=\begin{cases}\dfrac{\rho_{nD}((x_1,x_2,\ldots,x_n),X)}{\rho_{nD}((x_1,x_2,\ldots,x_n),X)-\rho_{nD}((x_1,x_2,\ldots,x_n),X_0)} & \rho_{nD}((x_1,x_2,\ldots,x_n),X)-\rho_{nD}((x_1,x_2,\ldots,x_n),X_0)\neq 0,(x_1,x_2,\ldots,x_n)\in U\\ -\rho_{nD}((x_1,x_2,\ldots,x_n),X_0)+1 & \rho_{nD}((x_1,x_2,\ldots,x_n),X)-\rho_{nD}((x_1,x_2,\ldots,x_n),X_0)=0,(x_1,x_2,\ldots,x_n)\in X_0\\ -\rho_{nD}((x_1,x_2,\ldots,x_n),X) & \rho_{nD}((x_1,x_2,\ldots,x_n),X)-\rho_{nD}((x_1,x_2,\ldots,x_n),X_0)=0,(x_1,x_2,\ldots,x_n)\in U-X_0\end{cases} \tag{3}$$

V. EXAMPLE 1 OF NESTED RECTANGLES WITH ONE COMMON SIDE

We have a factory piece whose desired *2D*-dimensions should be *20 cm* × *30 cm*, and acceptable *2D* dimensions *22 cm* × *32 cm*, but the two rectangles have common ending points. We define the extension *2D*-distance, and then we compute the extension *2D*-dependent function. Let's do an extension *2D*-diagram:

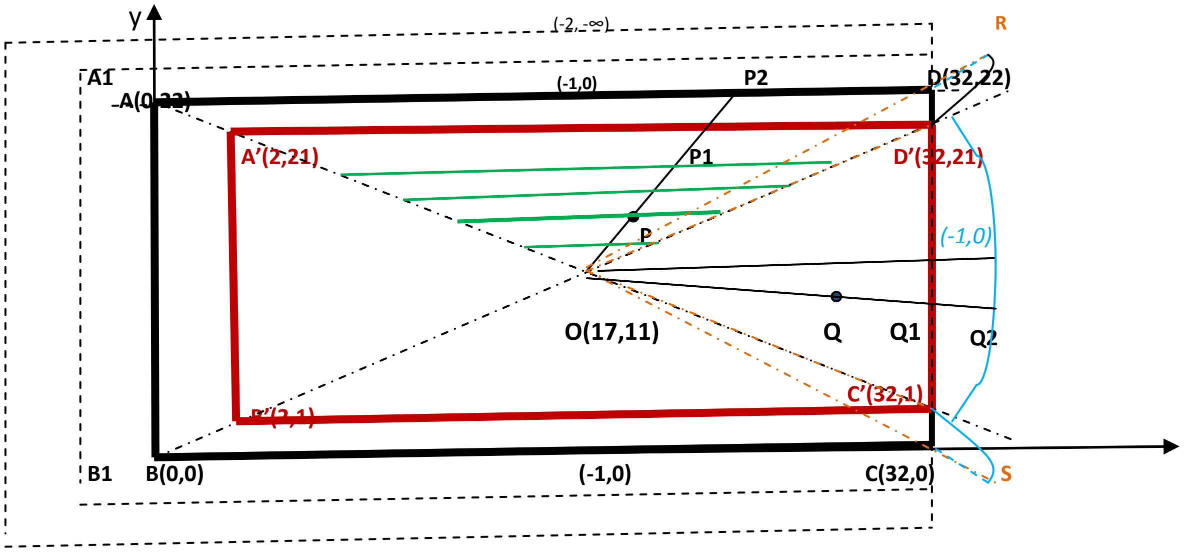

Diagram 1

The Critical Zone in the top, down, and left sides of the Diagram 1 as the same as for the case when the two pink and black rectangles have no common ending points. But on the right-hand side the Critical Zone is delimitated by the a blue curve in the middle and the blue dotted lines in the upper and lower big rectangle's corners.

The dependent function of the points Q, Q_1, Q_2 is respectively:

$$k(Q) = |QQ_1|+1, \text{ and } k(Q_1) = 1 \text{ (if } Q_1 \in A'B'C'D') \text{ or } 0 \text{ (if } Q_1 \notin A'B'C'D'), \text{ and } k(Q_2) = -|Q_2Q_1| = -1, \quad (4)$$

where $|MN|$ means the geometrical distance between the points M and N.

The dependent function of point P is normally computing:

$$k(P) = \frac{|PP_2|}{|P_1P_2|}. \quad (5)$$

VI. EXAMPLE 2 OF NESTED RECTANGLES WITH TWO COMMON SIDES

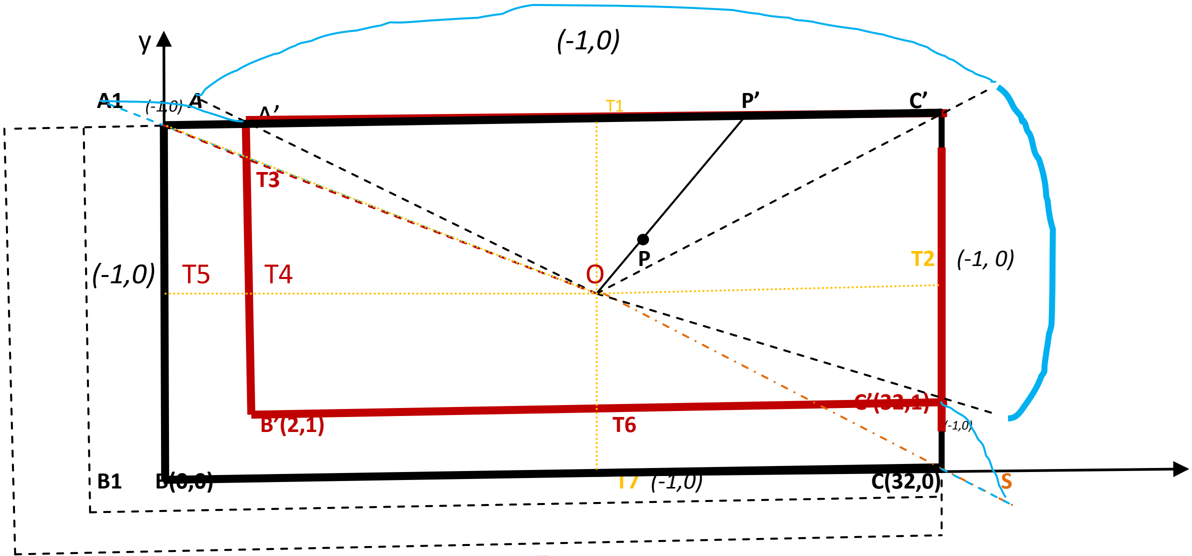

Diagram 2

We observe that the Critical Zone changes dramatically in the places where the common ending points occur, i.e. on the top and respectively left-hand sides. The Critical

Zone is delimitated by blue curves and lines on the top and respectively left-hand sides. Now, the dependent function of point P is different from the Diagram 1:

$$k(P)=|PP'|+1. \tag{6}$$

The dependent function of the optimal point O should be the maximum possible value. Therefore,

$$k(O) = \max\{|OT_1|+1,\ |OT_2|+1,\ |OP'|+1,\ |OC'|+1,\ \frac{|OT_7|}{|T_6T_7|},\ \frac{|OT_5|}{|T_4T_5|},\ \frac{|OA|}{|T_3A|},\ etc.\} \tag{7}$$

VII. EXAMPLE 3 OF NESTED CIRCLES WITH ONE COMMON ENDING POINT

Assume the desirable circular factory piece radius is *6 cm* and acceptable is *8 cm*, but they have a common ending point P'.

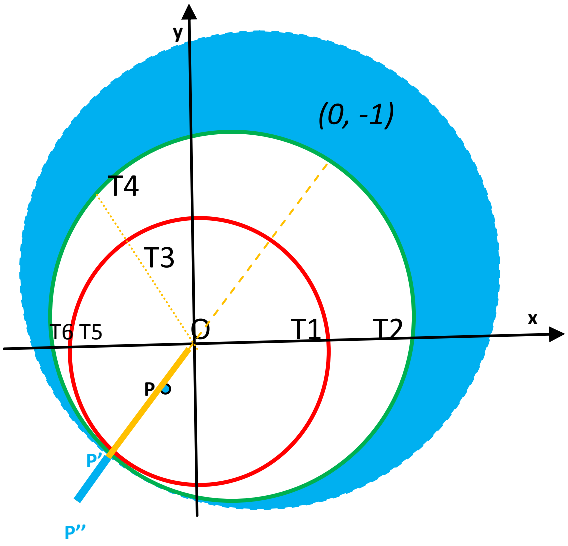

Diagram 3.

The Critical Zone is between the green and blue circles, together with the blue line segment $P''P'$ (this line segment resulted from the fact the P' is a common ending point of the red and green circles).

The dependent function values for the following points are:

$$k(P) = |PP'|+1; \tag{8}$$

$$k(P') = 1 \text{ (if } P' \text{ belongs to the red circle), or } 0 \text{ (if } P' \text{ does not belong to the red circle);} \tag{9}$$

$$k(P'') = |P''P'|; \tag{10}$$

$$k(O) = \max\{|OP'|+1;\ \frac{|OT_4|}{|T_3T_4|}, \tag{11}$$

where T_3 lies arbitrary on the red circle, but $T_3 \neq P'$, and T_4 lies on the green circle but T_4 belongs to the line (or geodesic) OT_3}.

Notes

VIII. EXAMPLE 4 OF NESTED TRIANGLES WITH ONE COMMON BOTTOM SIDE

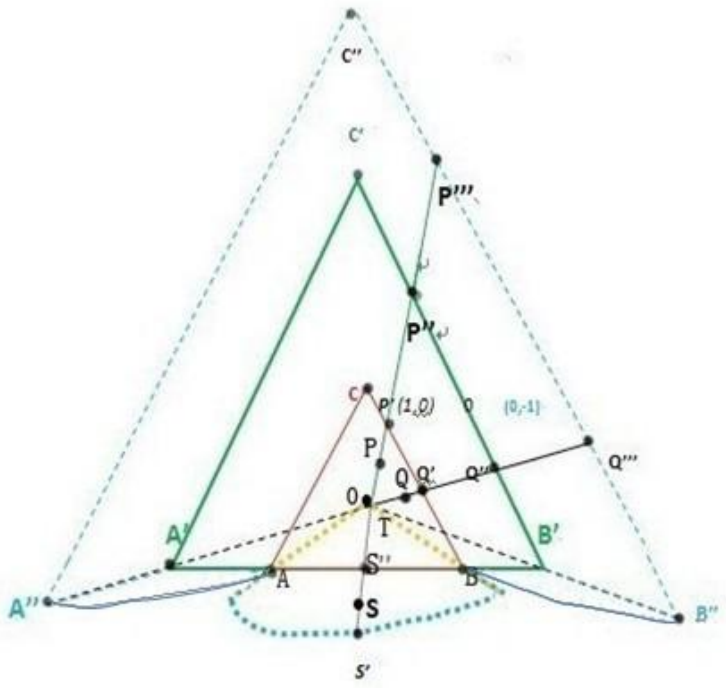

Diagram 4

The Critical Zone is between the green and blue dotted triangle to the left-hand and right-hand sides, while at the bottom side the Critical Zone is delimitated by the blue curve in the middle and the blue small oval triangles *A''AA'* and respectively *B''BB'*.
The dependent function values of the following points are given below:

$$k(P) = \frac{|PP''|}{|P'P''|} > 1; \; k(P')=1; \; k(P'')=0; \; k(P''') = -1. \tag{12}$$

Similarly:

$$k(Q) = \frac{|QQ''|}{|Q'Q''|} > 1; \; k(Q')=1; \; k(Q'')=0; \; k(Q''') = -1. \tag{13}$$

With respect to the bottom common side (where the line segment *AB* lies on line segment *A'B'*) one has:

k(T) = |TS''|+1; k(S'') = 1 (if *S''* belongs to the red triangle *ABC), or 0* (if *S''* does not belong to the red triangle *ABC); k(S) = |SS''|; k(S') = -1.* (14)

$$k(O) = \max\left\{ \max_{S''\in[AB]}(|OS''|+1) \; ; \qquad \max_{\substack{P'\in[AC]\cup[CB],\, P''\in[A'C']\cup[C'B'] \\ P'\in OP'' \\ OPP''\ line/geodesic}}\left(\frac{|OP''|}{|P'P''|}\right) \right\}. \tag{15}$$

Notes

IX. EXAMPLE *5* IN *3D*-SPACE OF TWO PRISMS HAVING A COMMON FACE

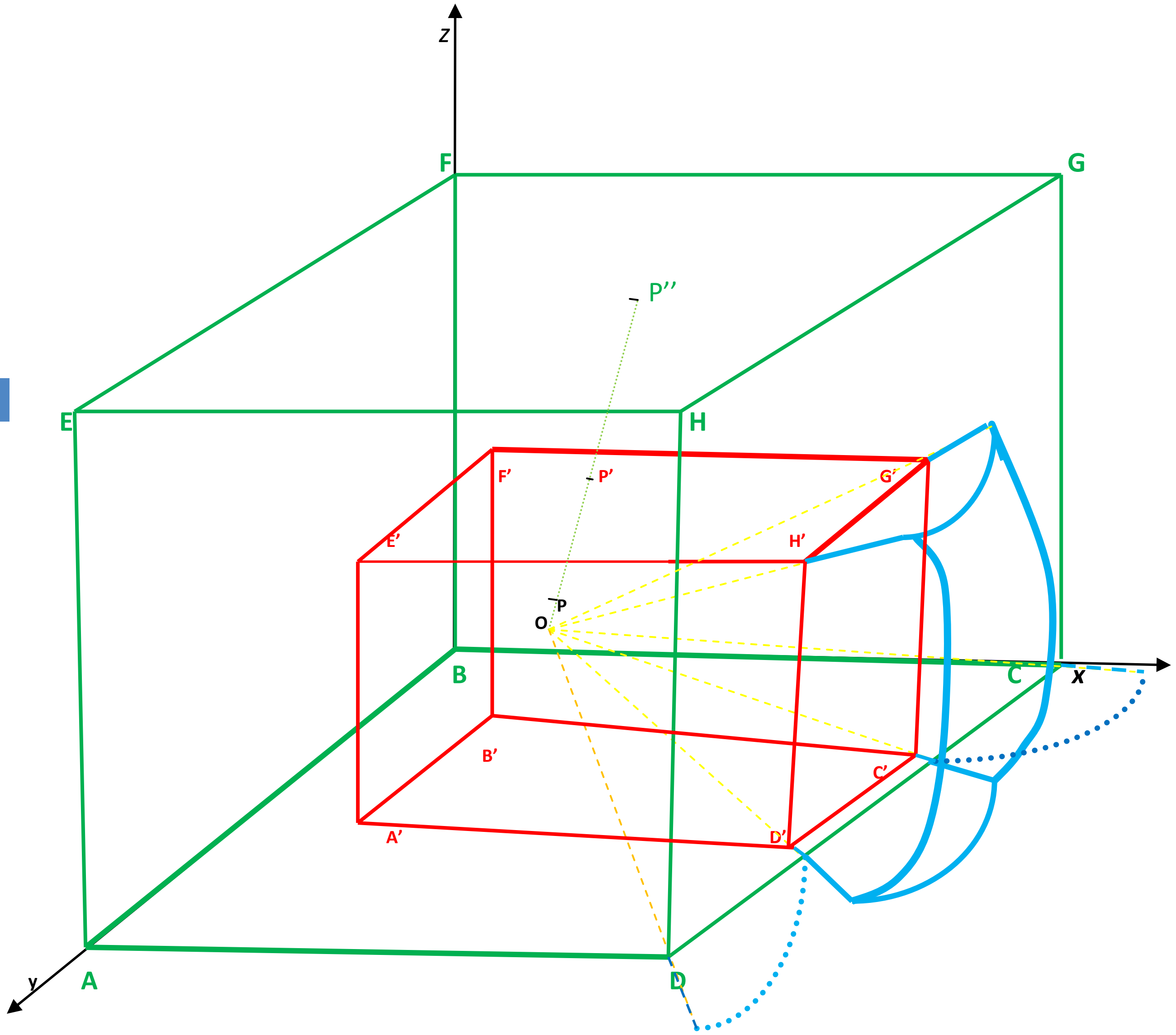

Diagram 5

The Critical Zone (the zone where the extension dependent function takes values between *0* and *-1*) envelopes the larger green prism *ABCDEFGH* at an equal distance from it as the distance between the red prism *A'B'C'D'E'F'G'H'* and the green prism *ABCDEFGH* with respect to the faces *ABCD, ADHE, BCGF, EFGH,* and *ABFE* (because these green faces and their corresponding red faces *A'B'C'D'*, *A'D'H'E'*, *B'C'G'F'*, *E'F'G'H'*, and respectively *A'B'F'E'* have no common points).

But the green face *DCGH* contains the red face *D'C'G'H'*, therefore for all their common points (i.e. all points inside of and on the rectangle *D'C'G'H'*) the extension dependent function has wild values. *D'C'G'H'* entirely lies on *DCGH*. The Critical Zone related to the right-hand green face *DCGH* and the red face *D'C'G'H'* is the solid bounded by the blue continuous and dashed curves on the right-hand side.

Notes

In general, let's consider two *n-D* sets, $S_1 \subset S_2$, that have common ending points (on their frontiers). Let's note by C_E their common ending point zone. Then:
The Dependent Function Formula for computing the value of the Optimal Point O is

$$k(O) = max\ \{ \max_{S''\in C_E}(|OS''|+1)\ ; \quad \max_{\substack{P'\in Fr(S_1-C_E),P''\in Fr(S_2-C_E)\\ P'\in OP''\\ OPP''\ line/geodesic}}\left(\frac{|OP''|}{|P'P''|}\right) \}. \tag{16}$$

We can define the Critical Zone in the sides where there are common ending points as:

$$Z_{C1} = \{P(x)\,|\,P\in U\text{-}S_2,\ 0 < d(P,P'') \leq 1,\ P''\in Fr(S_1)\cap Fr(S_2) \text{ and } P''\in OP\}, \tag{17}$$

where $d(P,P'')$ is the classical geometrical distance between the points P and P''.
And for the sides which have no common ending points, the Critical Zone is:

$$Z_{C2} = \{P(x)\,|\,P\in U\text{-}S_2,\ 0 < d(P,P'') \leq d(P''P'),\ where\ P''\in Fr(S_2)\ and\ P'\in Fr(S_1) \text{ and } P''\in OP\}. \tag{18}$$

Whence, the total Critical Zone is: $Z_C = Z_{C1} \cup Z_{C2}$. (19)

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 10 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Mathematical Modeling of Thin-Layer Drying Behavior of Date Palm

By Hosain Darvishi & Eisa Hazbavi
Islamic Azad University, Tehran, Iran

Abstract - The effect of microwave drying technique on drying kinetics of date palm was investigated. The results showed that the change of moisture ratio with drying time in the power density range from 4 to 9.5 W/g can be successfully described by Page model. Values of drying rate constant (k) were in the range of 0.052–0.142 (1/min) and the effective moisture diffusivities (D_{eff}) of date range palm from 2.72×10^{-6} to 4.73×10^{-6} (m^2/s). The values of k and Deff increased with the increase of power density. The power density dependence of the effective diffusivity coefficient was expressed by an Arrhenius type relationship. Activation energy for the moisture diffusion was determined as 3.908 W/g.

Keywords : Mathematical modeling, moisture diffusivity, activation energy, date palm, microwave drying.

GJSFR-F Classification : MSC 2010: 00A71, 97M10

MATHEMATICAL MODELING OF THIN-LAYER DRYING BEHAVIOR OF DATE PALM

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Mathematical Modeling of Thin-Layer Drying Behavior of Date Palm

Hosain Darvishi [α] & Eisa Hazbavi [σ]

Abstract - The effect of microwave drying technique on drying kinetics of date palm was investigated. The results showed that the change of moisture ratio with drying time in the power density range from 4 to 9.5 W/g can be successfully described by Page model. Values of drying rate constant (k) were in the range of 0.052–0.142 (1/min) and the effective moisture diffusivities (D_{eff}) of date range palm from 2.72×10^{-6} to 4.73×10^{-6} (m^2/s). The values of k and Deff increased with the increase of power density. The power density dependence of the effective diffusivity coefficient was expressed by an Arrhenius type relationship. Activation energy for the moisture diffusion was determined as 3.908 W/g.

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I. INTRODUCTION

Drying is the process of removing the moisture in the product up to certain threshold value by evaporation. In this way, the product can be stored for a long period, since it decreases the water activity of the product, reduces microbiological activity and minimizes physical and chemical changes during storage.

Different drying methods are used in the drying of fruits and vegetables. Air-drying is the most common method in the drying of foodstuffs. The major drawback of air-drying is the longer drying period, low drying rates in the falling rate period, worsening of the taste, colour and nutritional content of the product, higher drying temperature, low energy efficiency and high costs which is not a desirable situation for food industry [1,6,10].

The desire to eliminate this problem, prevent significant quality loss, and achieve fast and effective thermal processing, has resulted in the increase use of other drying heat sources such as microwave and infrared (IR) drying.

Microwave drying is more rapid, more uniform and more highly energy efficient compared to conventional hot air drying and infrared drying [1,2,9]. In recent years, microwave drying has gained popularity as an alternative drying method for a variety of food products such as fruit, vegetable, snack food and dairy product [1,2,3,4,5,6,7,8,9,10]. The usual means of applying microwaves to a drying process is at the end or should be applied in the falling rate period.

The most relevant aspects of drying technology are the mathematical modeling of the process and the experimental setup. The modeling is basically based on the design of a set of equations to describe the system as accurately as possible.

Ref.

1. Wang, J. and Sheng, K. Far-infrared and microwave drying of peach, LWT, 39 (2006), 247–255.

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No information is available on the ohmic drying behavior of tomato in the open literature. Therefore, the aim of this study was to (i) effect of power density on the drying kinetic of date palm, (ii) compare the measured findings obtained during the drying of date palm with the predicted values obtained with Page thin layer drying semi-empirical model, (iii) to calculate the effective moisture diffusivity and activation energy.

II. Materials and Methods

Date palm, procured from the local market, was used in the present study. They were stored at a temperature of 4 ffi 0.5 °C until the drying process. Before the drying experiments, the samples were taken out of the refrigerator and kernel of samples was separated. To determine the initial moisture content, three 30 g of samples were dried in an oven (Memmert UM-400) at 70 °C for 3 days. The initial moisture content of date palm was calculated 18ffi1.2 % (w.b.) as an average of the results obtained.

A domestic microwave oven (M945, Samsung Electronics Ins) with maximum output of 1000 W at 2450MHz was used for the drying experiments. The dimensions of the microwave cavity were 327×370×207 mm. The oven has a fan for air flow in drying chamber and cooling of magnetron. The moisture from drying chamber was removed with this fan by passing it through the openings on the right side of the oven wall to the outer atmosphere. The microwave dryer was operated by a control terminal which could control both microwave power level and emission time. Experiments were performed at four initial mass of 20, 30, 40 and 50 g at microwave power of 200 W (or power densities (microwave power/mass) of 9.5, 6.5, 5 and 4 W/g). The moisture losses of samples were recorded at 15 s intervals during the drying process by a digital balance (GF-600, A & D, Japan) and an accuracy of ± 0.001 g.

For measuring the weight of the sample during experimentation, the tray with sample was taken out of the drying chamber, weighed on the digital top pan balance and placed back into the chamber. Drying was carried out until the final moisture content reaches to a level less than 7.5% (w.b.).

It has been accepted that the drying characteristics of biological products in the falling rate period can be described by using Fick's diffusion equation. The following assumptions have been made: moisture is initially uniformly distributed throughout the sample, the thermo-physical properties of the material are constant, shrinkage or deformation of the material during drying is negligible, a spherical shape for sample, the resistance to transfer in medium surrounding the sphere is negligible, heat generation inside the moist sample is negligible, and radiation effects are negligible. General equation mass transfer for sphere shape is:

$$\frac{\partial X}{\partial t} = D_{\mathrm{eff}} \left(\frac{\partial^2 X}{\partial r^2} + \frac{2}{r} \frac{\partial X}{\partial r} \right) \tag{1}$$

With the appropriate initial and boundary conditions:

$$X(r,t)\big|_{t=0} = X_0 \tag{2}$$

$$\left. \frac{\partial X(r,t)}{\partial x} \right|_{r=0} = 0 \tag{3}$$

$$X(R,t)\big|_{t>0} = X_e \tag{4}$$

The first boundary condition stipulates that the moisture is initially uniformly distributed throughout the product sample. The second implies that the mass transfer is symmetrical with respect to the centre of the product. The third condition states that the surface moisture content of the samples instantaneously reaches equilibrium with the conditions of the surrounding air. The values of X_e are relatively small. Thus third condition can be simplified $X(R,t)|_{t>0} = 0$.

Following the numerical procedure, assume a solution of the following form in order to separate the variables:

Notes

$$X(r,t) = F(r) \times G(t) \tag{5}$$

where F is function of r only, and G is function of t only. Combining equations (1), (5) and using the initial and boundary conditions

$$X(r,t) = X_0\left(1 + \frac{2R}{\pi}\sum_{n=0}^{\infty}\left(\frac{(-1)^{n+1}}{n}\frac{1}{r}\sin\left(\frac{n\pi r}{R}\right)\exp\left(-\frac{D_{eff}\,n^2\pi^2 t}{R^2}\right)\right)\right) \tag{6}$$

The rate of transfer at time t across the surface of the sphere is:

$$4\pi R^2 N_A(t) = -4\pi R^2 D_{eff}\left(\frac{\partial X}{\partial r}\right)_{r=R} \tag{7}$$

with evaluating $(\partial X/\partial r)$ at $r = R$ form equation (6)

$$4\pi R^2 N_A(t) = 8\pi R^2 X_0 D_{eff}\sum_{n=1}^{\infty}\left(\exp\left(-\frac{D_{eff}\,n^2\pi^2 t}{R^2}\right)\right) \tag{8}$$

The total transfer per unit surface up to time t, N'_A is, where:

$$\frac{N'_A}{4\pi R^2} = \int_0^t N_A(t)dt = X_0\frac{R}{3}\left(1 - \frac{6}{\pi^2}\sum_{n=0}^{\infty}\left(\frac{1}{n^2}\exp\left(-\frac{D_{eff}\,n^2\pi^2 t}{R^2}\right)\right)\right) \tag{9}$$

A material balance on the transfer up to time t is:

$$\frac{4\pi R^3}{3}(X_0 - X) = N'_A \tag{10}$$

Where X is the average moisture throughout the sphere at time t. Combining equations (10) and (9):

$$\frac{X_0 - X}{X_0} = 1 - \frac{6}{\pi^2}\sum_{n=0}^{\infty}\left(\frac{1}{n^2}\exp\left(-\frac{D_{eff}\,n^2\pi^2 t}{R^2}\right)\right) \tag{11}$$

By simplify equation (11):

$$\frac{X}{X_0} = \frac{6}{\pi^2}\sum_{n=0}^{\infty}\left(\frac{1}{n^2}\exp\left(-\frac{D_{eff}\,n^2\pi^2 t}{R^2}\right)\right) \tag{12}$$

The moisture ratio (MR) was calculated using the following equation (13):

$$MR = \frac{X - X_e}{X_0 - X_e} \quad (13)$$

Form third condition; the equation (13) was simplified:

$$MR = \frac{X}{X_0} = \frac{6}{\pi^2}\sum_{n=0}^{\infty}\left(\frac{1}{n^2}\exp\left(-\frac{D_{eff}\, n^2\pi^2 t}{R^2}\right)\right) \quad (14)$$

The diffusion coefficients are typically determined by plotting experimental drying data in terms of ln(MR) versus drying time (t), because the plot gives a straight line with a slope as $\pi^2 D_{eff}/R^2$.

$$\ln(MR) = \ln\left(\frac{6}{\pi^2}\right) - \left(\frac{\pi^2 D_{eff}}{R^2}\right)t \quad (15)$$

The drying rate of date palm was calculated using the following equation:

$$DR = \frac{X_{t+\Delta t} - X_t}{\Delta t} \quad (16)$$

where $X_{t+\Delta t}$ is moisture content at time t+Δt (% d.b.), t is the time (min) and DR is the drying rate (% d.b./min).

Effectively modeling the drying behavior is important for investigation of drying characteristics of bioproduct. In this study, Experimental results of moisture ratio versus drying time was fitted to the semi-theoretical Page model, which are widely used by other workers to describe the kinetics of the drying process. Page's model was defined as follows:

$$MR = \exp(-kt^n) \quad (17)$$

where k is the drying rate constant (1/s) and n is equation constant model.

There are several criteria such as coefficient of determination (R^2) and chi-square (χ^2) are used to determine the quality of the fit. The model is said to be good if R^2 value is high and χ^2 value is low. These parameters are defined as follows:

$$R^2 = 1 - \left(\frac{\sum_{i=1}^{N}\left(MR_{pre,i} - MR_{exp,i}\right)^2}{\sum_{i=1}^{N}\left(MR_{pre,i} - \overline{MR}_{exp}\right)^2}\right) \quad (18)$$

$$\chi^2 = \frac{\sum_{i=1}^{N}\left(MR_{pre,i} - MR_{exp,i}\right)^2}{N - z} \quad (19)$$

where $MR_{pre,i}$ is the ith predicted moisture ratio, $MR_{exp,i}$ is the ith experimental moisture ratio, N is the number of observations and z is the number of constants in drying model.

The dependence of the effective moisture diffusivity on the power density is generally described by the Arrhenius equation:

$$D_{eff} = D_0 \exp\left(-\frac{E_a}{P_d}\right) \quad (20)$$

where E_a is the activation energy (W/g), P_d is the power density (w/g), and D_0 is the pre-exponential factor (m^2/s).

III. Results and Discussion

The variations in moisture content of the date palm as a function of drying time at different temperatures are presented in Fig. 1. It can be seen that the moisture content of the date palm samples decreased with the increase in drying time. Based on these results, the required drying times for the date palm samples to reach a moisture content of 0.28ffi0.2 (% d.b.) in order to obtain safe storage, were found to vary from 150 to 240 s depending on the drying microwave power.

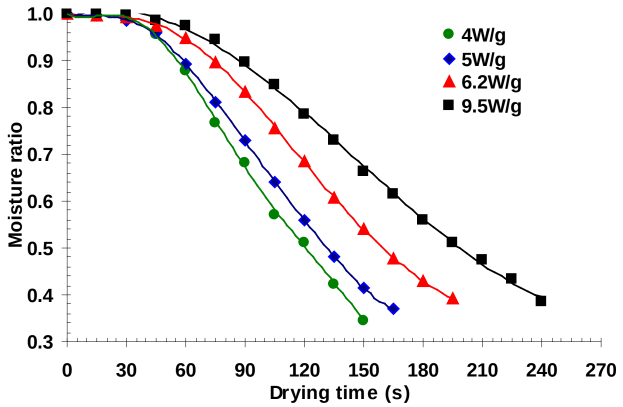

Fig. 1 : Variation of moisture ratio with drying time for the date palm

Validation of the Page model was confirmed by comparing the estimated or predicted moisture ratio at any particular drying condition. The validation of the Page model at different power densities is shown in Fig. 2. The predicted data generally banded around the straight line which showed the suitability of the Page model in describing the microwave drying behavior of the date palm.

Table 1 : Results of statistical analysis on the modeling of moisture content and drying time for the microwave dried date palm

P_d (W/g)	K (1/min)	n	R^2	χ^2
4	0.142	2.221	0.994	0.00038
5	0.126	2.093	0.994	0.00038
6.2	0.078	2.169	0.993	0.00038
9.5	0.052	2.127	0.994	0.00051

The statistical results from page model are summarized in Table 1. The statistical parameter estimations showed that R^2 and χ^2 values were ranged from 0.993 to 0.994 and 0.00038 to 0.0051, respectively. It was determined that the value of the drying rate constant (k) increased with the decrease in the power density.

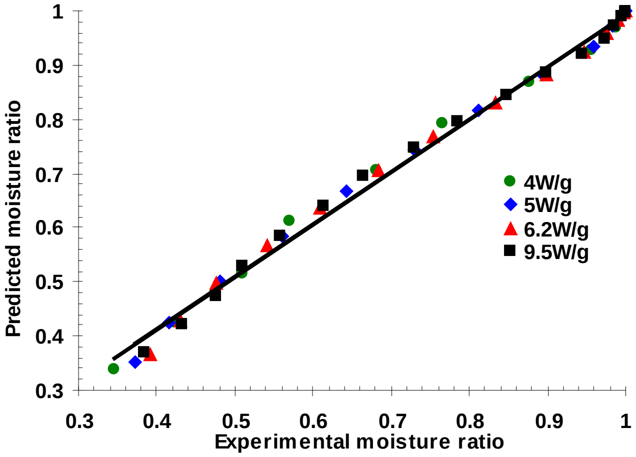

Fig. 2 : Experimental and predicted moisture ratio values for date palm

In order to take into account the effect of power density level on the constants of the Page model, namely, k, n (seen in Table 1), the regression analysis was used to set up the relations between these parameters and the power density level. Thus, the regression equations of these parameters against power density, P_d, (W/g) and the accepted model are as follows:

$$\mathrm{MR} = \exp(-kt^n)$$

where,

$$k = 0.8094 P_d^{-1.2282} \qquad R^2 = 0.961 \tag{21}$$

$$n = -0.0189 P_d^3 + 0.374 P_d^2 - 2.3423 P_d + 6.8142 \quad R^2 = 0.961 \tag{22}$$

The drying rate curves for date palm samples dried at different microwave power densities are given in Fig. 3. In general, two distinct periods are identifiable, namely warming up and falling-rate periods. The initial short period coincides with the warming-up stage which corresponds to sample heating and non-isothermal drying conditions due to the low temperature of samples. The drying rates were more after an initial short period of the process probably due to evaporation and moisture from the surface of the date palm and later decreased with decreasing moisture content, for all the drying conditions once the drying process was governed by moisture diffusion. The accelerated drying rates may be attributed to internal heat generation. The absence of a constant drying rate period may be due to the thin layer of product that did not provide a constant supply of water for an applied period of time. Also, some resistance to water movement may exist due to shrinkage of the product on the surface, which reduces the drying rate considerably. The results indicates that mass transfer within the sample was more rapid during higher power density because more heat was generated within the sample creating a large vapor pressure difference between the centre and the surface of the product due to characteristic microwave volumetric heating. Thus, the power density had a crucial effect on the drying rate.

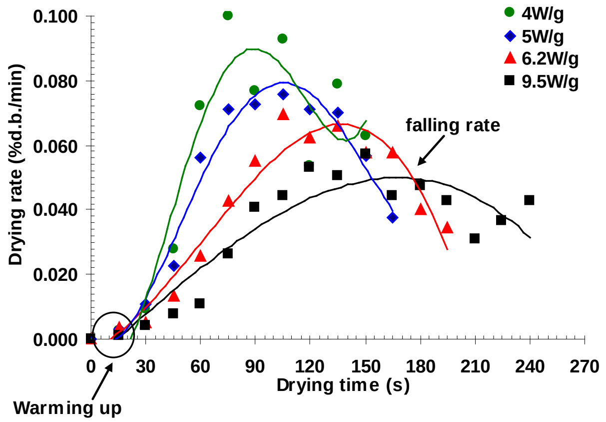

Fig. 3 : Variation of drying rate with drying time for the date palm

The variation in ln (MR) and drying time (t) for different power densities have been plotted in Fig. 4 to obtain the slope S, which can give the effective moisture diffusivity (D_{eff}). The effective diffusivity was calculated using Eq. (15) and is shown in Table. 2. The D_{eff} values of dried samples at power density level of 4–9.5 W/g were varied in the range of 2.72×10^{-6} to 4.73×10^{-6} m^2/s. It can be seen that D_{eff} values increased with increasing power density. When samples were dried at higher power density, increased heating energy would increase the activity of water molecules leading to higher moisture diffusivity.

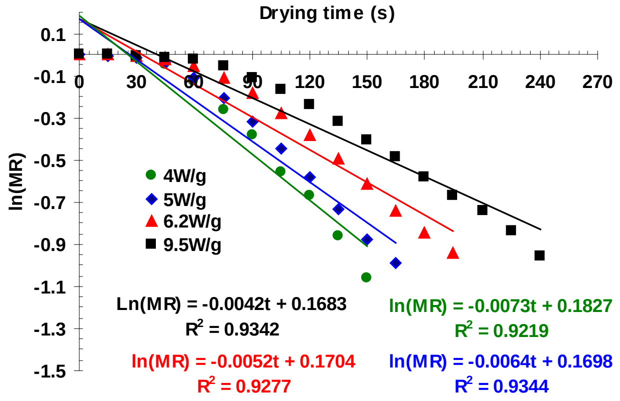

Fig. 4 : Variation in ln (MR) and drying time (in s) for date palm dried at different power densities

Table 2 : Values of effective diffusivity obtained for date palm at different power densities

P_d (W/g)	D_{eff} (m^2/s)
4	2.72×10^{-6}
5	3.37×10^{-6}
6.2	4.15×10^{-6}
9.5	4.73×10^{-6}

The values of effective diffusivity versus $1/P_d$ accurately fit to the exponential model as evident from Fig. 5 with coefficient of determination (R^2) of 0.975. The dependence of the effective diffusivity of date samples on the power density can be represented by the following equation:

$$D_{eff} = 7 \times 10^{-6} \exp\left(-\frac{3.9082}{P_d}\right) \quad R^2 = 0.975 \tag{23}$$

The activation energy for date palm samples was found to be 3.908 W/g.

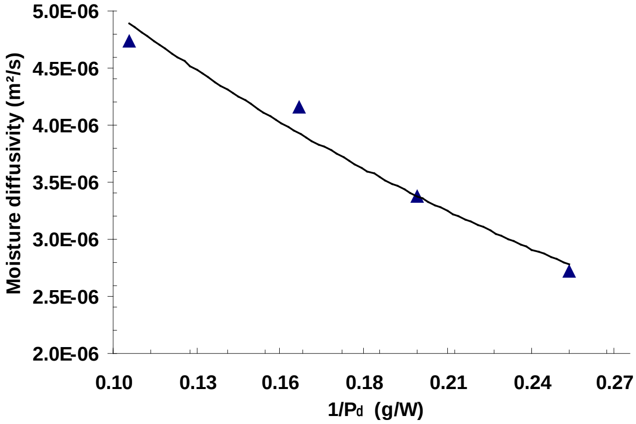

Fig. 5 : Relationship between the values of effective diffusivity and power density

IV. CONCLUSION

The increase in power density significantly reduced the drying time of the date palm. Drying curves date palm did not show a constant rate-drying period under the experimental employed and showed a warming up rate and falling rate-drying periods. Effective diffusivity varied from 2.72×10^{-6} to 4.73×10^{-6} m^2/s and increased with the power density. An Arrhenius relation with an activation energy value of 3.908 W/g expressed effect of power density on the diffusivity.

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Notes

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 10 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Relation Between Weakly Prime Elements and Weakly Prime Sub Modules

By Ayaz Ahmad
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Abstract - In this paper we give the defition of weakly prime element of a module. Therefore we give a new definition of factorization in a module, which is called weakly factorization. So we call a module weakly unique factorization which is unique. We give the relation between weakly prime elements and weakly prime sub modules Then we characterize such weakly unique factorization modules.

Keywords and pharases : Weakly prime element, weakly prime sub module, factorization.

GJSFR-F Classification : MSC 2010: 11A41

RELATION BETWEEN WEAKLY PRIME ELEMENTS AND WEAKLY PRIME SUB MODULES

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Relation Between Weakly Prime Elements and Weakly Prime Sub Modules

Ayaz Ahmad

Abstract - In this paper we give the defition of weakly prime element of a module. Therefore we give a new definition of factorization in a module, which is called weakly factorization. So we call a module weakly unique factorization which is unique. We give the relation between weakly prime elements and weakly prime sub modules Then we characterize such weakly unique factorization modules.

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I. INTRODUCTION

The study of factorization in tortion free modules was begun0 in Nicolas (5). She defined the module M to be factorial if (1) every non zero element of M has a irreducible factorization,(2) every irreducible element of R is prime, and (3)every irreducible element of M is primitive. She showed that if M is factorial then R is a UFD. After this she showed that M is a un0ique factorization module (UFM)if and only if (1)every element of M 0has an irreducible factorization, and(2)if $x= a_1a_2......a_km =b_1b_2......b_tm'$ are two factorization of x € M then $k = a_i\sim b_i$ for all i € { 1,2,........k } and m ~ m'. Later, Lu(3) gives some characterizations of UFM and relations between prime submodules and primitive elements such modules. Further she investigates polynomial modules. There is an another work about factorization of modules, by Anderson and Valdes-Leon(1).They generalize factorization of any modules over a ring with zero divisor, which have non zero tortion elements. They showed that their defition and definition of Nicolas are concides if M is tortion free module and R is and integral domain.

We give a new definition of factorization for modules, named weakly factorization, and give relations between ewakly prime elements and weakly prime submodules. After this investigate the direct sum of modules, the direct product of modules, fractions of modules and polynomial of modules.

Throughout this paper all rings, R are commutative ring with identity 1 and all modules, M are non zero torsion free module which are unitary.

We will give some definitions:

Definition 1. Let M be a tortion free R-module and m be anon zero element of M.

(1) m is irreducible in M if m=am' implies that a € U(R) for every a € R and m' €M

(2) M primitive in M if m | am' implies m | m' for all 0 ≠ a € R and m'€M.

(3) An irreducible element p of R is called prime to the module M if p | am implies p|a in R or p | m in M.

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Definition 2. Let M be an R- module. Then a submodule N of M is called pure sub module if for all a € R we have aM∩ N =An

Definition 3. Anonzero element m of M is called weakly prime (w-prime) if for a,b € R and m' € M , m | abm' implies m | am' or m | bm'.

Definition 4. A submodule N of an R-module M is called weakly prime if abk € N implies ak € N or bk € N for all k € M and a,b € R

Definition 5. Atortion- free module M over a commutative ring with identity R is called a weakly unique factorization module (w-UFM) or w-factorial module if the following two conditions are satisfied:

(w-ufm 1) Each nonzero element x€ M has a w-factorization, $x=a_1a_2\ldots\ldots a_km$, where $a_i{}^{'s}$ are irreducible elements in R (possibly with k=0) and m is a w-prime element in M.

(w-ufm2) if $x=a_1a_2\ldots\ldots{}_ak_m = b_1b_2\ldots\ldots b_tm'$ are two factorization of x, then k= t , $a_i \sim b_i$ and $m \sim m'$ for all i € { 1,2,……,k}.

Definition 6. Let Mbe an R-module and a € R, m € M

(1) An element d € R is called greatest common divisor (gcd) of a and m if the following two condition hold

i) d | a in R and d | m in M , and

ii) if there is an element c € R such that c|a in R and c | m in M then c is a divisor of d.

(2) An element m' € M is called least common multiple (lcm) of ab and m if the following two condition hold

i) a|m' and m| m' in M respetively, and

ii) if there is an element n € M such that a | n and m | n in M then m' is a factor of n

The following propositions are given by (3) without their proof, we will give now their proof

Proposition 1. Let Mbe an R- module then every w-primitive element of M is an irreducible element.

Proof, suppose that m is aprimitive element of M and let m =am' for some a € R, m' € M. Then m | am' and since m is primitive we get m | m'. Since m = am´ implies m´ | m. Therefore m ~ m´, hence m is a irreducible element.

Proposition 2. Let M be an R module, then every primitive element of M is w-prime.

Proof, Assume that m | abm´ for some a,b € R and m´ € M .Then since is primitive. We get m | m´ .Hence m | am´ and m | bm´.

Example 1. Let R be a commutative ring with an identity and M = R| ẍ|, the polynomial ring over Ris an R- module then x € M is w-prime (primitive, irreducible) element.

Example 2. Let R = Z and M = Z | x|. Then the element 2x is a w-prime element but is neither primitive nor irreducible.

Theorem1. Let M be a torsion- free R-module. Then M is a UFM if and only if M is w-UFM.

Proof, The follows from theorem "Let M be a module over a UFD R which satisfie (w-ufm 1) Then M is a w-UFM if and only if every weakly prime element of M is primitive.

With this there m we get that in a w-UFD M, every weakly prime element of M is irreducible element of M. And this gives u that weakly factorial modules and factorial module concides. From this note we obtain the following corollaries.

Corllary 1, Let M be an R –module then two primitive elements m and m^{J} of M are non associates if and only if $Rm' \cap Rm = 0$

Corollary 2. Every vector space is w-UFM.

Theorem 2. Let { M_i / i € I } be a set of modules over a UFD R. Then the following statements are equivalent.

i) $\prod M_i$ is a w-UFM over R,

ii) O M_i is a w-UFM over R ,

iii) Each M_i is a w-UFM over R

Proof. (i) => (ii) => (iii) it is clear

(iii) implies(i) now assume that each M_i is a w-UFM over R for i €

I .Let $M = \prod_i M_i$ and $m = (m_i)_{i\,€\,I}$ € M where $m_i = a_i m_i'$ for some a_i€ R and a w-prime element m_i' of M_i. First we will show that $m = (m_i)_{i\,€\,I}$ € M is a w-prime element in M iff { a_i }$_{i\,€\,I}$ has no gcd. In R. Let m € M be a w- prime element. Assume that d = g.c.d {a_i} And set $a_i = db_i$ for b_i € R. Then $m = dm'$ where $m' = (b_i m_i)_{i\,€\,t'}$. Then m | m =dm´ but m | m´ gives us a contradiction. For the converse assume that { a_i }$_{i\,€\,I}$ has no g.cd. in R. Let m | cbn for c,b € R and $n = (n_i)_{i\,€\,I}$ € M then there exist r € R uch that rm =cbn. So this gives us that for all i € I $rm_i = cbn_i$. Thus for all i € I , $Ra_i m_i' = cbn_i$ Since m_i' is w-prime and { a_i }$_{Ii\,€\,I}$ has no g.c.d. then for all i € I we get $a_i m_i' \mid cn_i$ or $a_i m_i' \mid bn_i$. Hence m | cn or m | bn, so m is w-prime. Now we will show that M is a w-UFM. Let $m = (m = (m_i)_{i\,€\,I}$ € M, since each M_i is a w-UFM over R we have a w-factorization for m_i € M and i € I such that m_i € M_i i € I such that $m_i = a_i m_i'$ where m_i' is w- prime in M_i .If we let d =g.c.d. { a_i } then for $a_i = db_i$ we obtain the equation $m = dm'$ where $m' = (b_i m_i')$. Now by Theorem "Let M be a w-factorization module over a UFD R such that Pm ≠ M for every non unit element p € R .Then the following statements are equivalent:

i) p is prime to M

ii) pM is a weaky prime submodule of M with (pM:M) =(p) "m' is w-prime in M and since R is UFD M satisfies w-UFMI. Now, let p € R be a w-irreducible element such that p | abm in M for some a,b € R and $m = (m_i)_{i\,€\,I}$ € M. Then for all i € I , p |abm_i in M_i is w-UFM if p| ab then p | m_i for all i € I. Consequently p | m and therefore M is w-UFM.

Corollary 3, Every free module over a UFD is w-UFM

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Notes

GLOBAL JOURNAL OF SCIENCE FRONTIER RESEARCH
MATHEMATICS AND DECISION SCIENCES
Volume 12 Issue 10 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

A Solution Procedure for Minimum Convex-Cost Network Flow Problems

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Abstract - This paper presents a procedure to solve Minimum Convex-Cost Network Flow Problems (MC-CNFP). This solution algorithm is constructed on the concepts of Network Simplex Method (NSM) for minimum cost network flow problem, Convex Simplex Method (CSM) of Zangwill, the decomposition of convex simplex method and non-linear transformation problem.

Keywords : *Convex, Convex Simplex Method, directed network, Karush-Kuhn-Tucker conditions, Network Flow, Network Simplex Method, node-arc-incidence matrix.*

GJSFR-F Classification : *MSC 2010: 91B32*

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A Solution Procedure for Minimum Convex-Cost Network Flow Problems

Dewan Ferdous Wahid [α], Farjana Habiba [σ] & Ganesh Chandra Ray [ρ]

Abstract - This paper presents a procedure to solve Minimum Convex-Cost Network Flow Problems (MC-CNFP). This solution algorithm is constructed on the concepts of Network Simplex Method (NSM) for minimum cost network flow problem, Convex Simplex Method (CSM) of Zangwill, the decomposition of convex simplex method and non-linear transformation problem.

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I. INTRODUCTION

The minimum convex-cost network flow problem is a class of minimum cost network flow problems with convex cost function. This problem structure may occur in different practical problems as cost of power losses in electrical networks due to resistance, delay cost of communication networks and congestion costs in city transportation networks etc.

Consider $\mathcal{G}(N, A)$ is a directed network, where $N=\{1,..,m\}$and $A=\{(i, j),..,(s, t)\}\subset N\times N$ are node and arc sets respectively. Let x_{ij} be the flow through the arc$(i\ j)$, and the vector $\mathbf{x}=\{x_{ij}|(i, j\in A)\}$. Then MC-CNFP can be formulated as-

$$\text{minimize} \quad \sum\sum_{(i,j)\in A} c_{ij}(x_{ij})$$

$$\text{subject to} \quad \sum_{\{j|(i,j)\in A\}} x_{ij} - \sum_{\{k|(k,i)\in A\}} x_{ki} = b_i \; ; \; \forall i \in N$$

$$x_{ij} >= 0 \qquad ; \; \forall (i, j) \in A$$

where b_i is the net flow generated at node i and $c_{ij}:\mathbb{R}\rightarrow\mathbb{R}$ are given convex cost functions with continuous first derivative for arcs (i, j).
The above formulation also written as-

$$\text{minimize} \quad C(\mathbf{x}) \tag{1}$$

$$\text{subject to} \quad \mathbf{Ax}=\mathbf{b} \;\; ; \;\; \mathbf{x}\geq 0$$

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where $C(\mathbf{x})$ is convex and the constraints are linear equations. The matrix $\mathbf{A}$ is the *node-arc incidence matrix* with rank $(m-1)$. (Bazaraa, M.S., Jarvis, J.J.& Sherali, H.D., 2005)

This paper represents an optimality condition to minimize the objective function in (1) with subject to linear constraints.

II. Condition for Optimality

We introduce an artificial arc to root node (any other node would do), that lead to the extended constraint matrix $\mathbf{A}_e = (\mathbf{A}, e_m)$ of rank m, where e_m is a unit vector. (Bazaraa, M.S., Jarvis, J.J.& Sherali, H.D., 2005).

Then (1) can be rewrite as-

$$\begin{array}{ll} \text{minimize} & C(\mathbf{x}_e) \\ \text{subject to} & \mathbf{A}_e\mathbf{x}_e = \mathbf{b} \;\;;\;\; \mathbf{x}_e \geq 0 \end{array} \tag{2}$$

where $\mathbf{x}_e$ is $n \times 1$ and $\mathbf{A}_e$ is $m \times n$, and $\mathbf{b}$ is $m \times 1$ matrix, here n is the number of arc including with artificial arc. Now the Lagrangian for (2) can be formulated as-

$$z(\mathbf{x}_e, \boldsymbol{\mu}, \boldsymbol{\lambda}) = C(\mathbf{x}_e) + \boldsymbol{\mu}^T(b - \mathbf{A}_e\mathbf{x}_e) - \boldsymbol{\lambda}\mathbf{x}_e$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are Lagrange multipliers. The optimum value $\bar{\mathbf{x}}$ of (2) should satisfy the Karush-Kuhn-Tucker (KKT) conditions: (Zangwill, 1967)

$$\nabla z = \nabla C(\bar{\mathbf{x}}) - \boldsymbol{\mu}^T\mathbf{A}_e - \boldsymbol{\lambda} = 0 \tag{3}$$

$$\boldsymbol{\lambda}\bar{\mathbf{x}} = 0, \quad \bar{\mathbf{x}} \geq 0, \;\; \boldsymbol{\lambda} \geq 0$$

For each arc flow x_{ij} associated with the arc *(i, j),* we get

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial C(\bar{\mathbf{x}})}{x_{ij}} - \boldsymbol{\mu}^T\mathbf{a}_{ij} - \lambda_{ij} = 0 \tag{4}$$

$$\lambda_{ij}x_{ij} = 0, \;\; x_{ij} \geq 0, \;\; \lambda_{ij} \geq 0$$

where $\boldsymbol{\mu}^{\mathrm{T}} \in \mathbb{R}_m$ and $\mathbf{a}_{ij}$ is column vector associated to x_{ij} (has positive identity at the and negative identity at j-th row position in $\mathbf{A}_e$.).Therefore from (4) we get-

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial C(\bar{\mathbf{x}})}{x_{ij}} - (\mu_i - \mu_j) - \lambda_{ij} = 0 \tag{5}$$

$$\lambda_{ij}x_{ij} = 0, \;\; x_{ij} \geq 0, \;\; \lambda_{ij} \geq 0$$

Therefore (5) can be written as-

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial C(\bar{\mathbf{x}})}{\partial x_{ij}} - (\mu_i - \mu_j) \geq 0$$

and

$$x_{ij}\frac{\partial z}{\partial x_{ij}} = x_{ij}\left[\frac{\partial C(\bar{\mathbf{x}})}{\partial x_{ij}} - (\mu_i - \mu_j)\right] = 0 \tag{6}$$

$$x_{ij} \geq 0$$

Therefore, a point $\bar{\mathbf{x}}$ will minimize the MC-CNFP (2) if it satisfies the optimality conditions (6).

III. SOLUTION PROCEDURE FOR MC-CNFP

Here our goal is to minimize (2) by satisfying the optimality conditions (6). To start a solution procedure first we need an initial basic feasible solution and then we use iterative procedure for moving towards optimal solution.

a) Determination of an Initial Feasible Solution

Since the constraints in (2) are linear, we use inspection of a spanning tree (basis sub-graph) as NSM with linear constraints (Bazaraa, M.S., Jarvis, J.J. & Sherali, H.D.,, 2005). Let $\bar{\mathbf{x}}^o = (\bar{\mathbf{x}}_B^o, \bar{\mathbf{x}}_N^o)$ is a initial feasible solution, where $\bar{\mathbf{x}}_B^o$ and $\bar{\mathbf{x}}_N^o$ are the basic and nonbasic solutions respectively. Next we have to improve this initial feasible solution to an optimal solution.

b) Testing Optimality of a Feasible Solution

Any feasible point of (2) would be optimal solution, if it satisfies the conditions in (6). Let $\bar{\mathbf{x}}^k = (\bar{\mathbf{x}}_B^k, \bar{\mathbf{x}}_N^k)$ be a feasible solution in any k-th iteration and $I_B^k = \{ij : x_{ij}^k \in \bar{\mathbf{x}}_B^k\}$, $I_N^k = \{ij : x_{ij}^k \in \bar{\mathbf{x}}_N^k\}$. We have $x_{ij}^k > 0;\ ij \in I_B^k$, then the complementary slackness condition implies that-

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial C(\bar{\mathbf{x}})}{\partial x_{ij}} - \left(\mu_i - \mu_j\right) = 0; \qquad \forall\, ij \in I_{\mathbf{B}}^k$$

Then compute,

$$\frac{\partial z}{\partial x_{rl}^k} = \min\left\{\frac{\partial z}{\partial x_{ij}^k}\ ;\ ij \in I_N^k\right\}$$

$$x_{st}^k \frac{\partial z}{\partial x_{st}^k} = \max\left\{x_{ij}^k \frac{\partial z}{\partial x_{ij}^k};\ ij \in I_N^k\right\} \tag{7}$$

Now, if $\left|\frac{\partial z}{\partial x_{rl}^k}\right| = x_{st}^k \frac{\partial z}{\partial x_{st}^k} = 0$, then $\bar{\mathbf{x}}$ is optimal. (Hsia, 1973)

Theorem* : If $\left|\frac{\partial z}{\partial x_{rl}^k}\right| = x_{st}^k \frac{\partial z}{\partial x_{st}^k} = 0$ then $\bar{\mathbf{x}}$ is optimal.

Proof : Since $\left|\frac{\partial z}{\partial x_{rl}^k}\right| = x_{st}^k \frac{\partial z}{\partial x_{st}^k} = 0$, we have-

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial C(\bar{\mathbf{x}})}{\partial x_{ij}} - \left(\mu_i - \mu_j\right) = 0, \qquad \text{if}\ \ x_{ij} \geq o \tag{8}$$

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial C(\bar{\mathbf{x}})}{\partial x_{ij}} - \left(\mu_i - \mu_j\right) \geq 0, \qquad \text{if}\ \ x_{ij} = o \tag{9}$$

Here (8) & (9) and the feasibility of $\bar{\mathbf{x}}$ are simply the conditions in (6), which also provides a condition for optimality for (2). (Hsia, 1973)

c) Iterative Procedure for Moving Towards Optional Solution

Any feasible solution which fails to satisfy the optimal condition (*Theorem**), has to improve to optimal solution by changing nonbasic variables to basic. Since the objective function of (2) so we use iterative procedure by Hisa (1975). To improve a feasible solution following cases need to be considered:

Case-1 : If $\left|\frac{\partial z}{\partial x_{rl}^{k}}\right| \geq x_{st}^{k} \frac{\partial z}{\partial x_{st}^{k}}$; increase x_{rl}^{k} by Δ^{k}, where Δ^{k} is compute as-

Let $I_{B_{rl}}^{k} = \{lu,..., ij,.., wr\} =$ {are the indices of the basic flows of the loop contacting the arc *(r,l)* according to the loop direction}.

Then,

$$\Delta^{k} = \min\left\{\left|x_{ij}\right| : ij \in I_{B_{rl}}^{k} \textit{ and } x_{ij} \in \overline{\mathbf{x}}_{B}^{k}\right\} \tag{10}$$

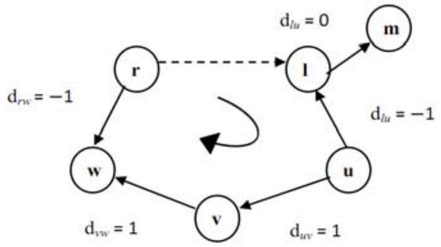


Fig.1 : Direction of the basic loop containing the arc *(r,l).*

Next adjust the flow of the network according to loop direction *(Fig.1)* as follows-

$$y_{ij}^{k} = x_{ij}^{k} \ ; ij \in I_{N}^{k} - \{rl\} \tag{11}$$

$$y_{rl}^{k} = x_{rl}^{k} + \Delta^{k}$$

$$y_{ij}^{k} = x_{ij}^{k} + d_{ij}\Delta^{k}; ij \in I_{B}^{k} \text{ and } d_{ij} = \begin{cases} 1\,; & ij \in I_{B_{rl}}^{k} \\ -1\,; & ij \in I_{B_{rl}}^{k} \\ 0\,; ij \textit{ or } ji \notin I_{B_{rl}}^{k} & \end{cases}$$

By doing so, one of the basic flow say $x_{B_{ij}}^{k}$ may be driven to zero. Let $\overline{\mathbf{y}}^{k}$ be the value of $\overline{\mathbf{x}}^{k}$ after making the necessary adjustment. Since the function is convex, so a better point could be found before reaching $\overline{\mathbf{y}}^{k}$ (Bazaraa, M.S., Sherali, H.D.& Shetty, C.M.,2006).

To check this, find $\overline{\mathbf{x}}^{k+1}$ by using the line search-

$$C(\overline{\mathbf{x}}^{k+1}) = \min\{C(\overline{\mathbf{x}}) : \overline{\mathbf{x}} = \lambda\overline{\mathbf{x}}^{k} + (1-\lambda)\overline{\mathbf{y}}^{k} \ \ \& \ 0 < \lambda < 1\} \tag{12}$$

If, $\overline{\mathbf{x}}^{k+1} \neq \overline{\mathbf{y}}^{k}$ do not change the former basis and go to the next iteration. If $\overline{\mathbf{x}}^{k+1} = \overline{\mathbf{y}}^{k}$ and if a basic flow becomes zero during the adjustment made, change the former basis and go to the next iteration.

Case-2: If $\left|\frac{\partial z}{\partial x_{rl}^{k}}\right| \prec x_{st}^{k} \frac{\partial z}{\partial x_{st}^{k}}$; decrease x_{st}^{k} by Δ^{k}, where Δ^{k} is determined as *Case-1.*

Next adjust the flow of the network as follows-

$$y_{ij}^{k} = x_{ij}^{k} \ ; ij \in I_{N}^{k} - st\}$$

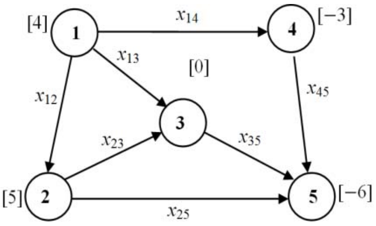
$y_{st}^k = x_{st}^k - \Delta^k$

$y_{ij}^k = x_{ij}^k + d_{ij}\Delta^k;\ ij \in I_B^k$,where d_{ij} can calculate similarly as in (11).

Then we obtain $\bar{\mathbf{y}}^k$. As we decrease x_{st}^k and then either x_{st}^k itself or any basic flow say x_{ij}^k will be driven to zero. Now calculate $\bar{\mathbf{x}}^{k+1}$ from the line search (12). If $\bar{\mathbf{x}}^{k+1} \neq \bar{\mathbf{y}}^k$, do not change the former basis and go to next iteration and if $\bar{\mathbf{x}}^{k+1} = \bar{\mathbf{y}}^k$ change the basis.

Example - 1

Consider the following network flow problem:



$$\text{minimize} \quad z = C(\bar{\mathbf{x}}) = 5x_{12} + 8x_{12}^2 + 4x_{13}^2 + 5x_{14}^2 + x_{23}^2 + x_{25}^2 + 3x_{35} + 7x_{45}^2 + x_{45}$$

with subject to linear constraints:

$$\begin{aligned} x_{12} + x_{13} + x_{14} &= 4 \\ -x_{12} + x_{23} + x_{25} &= 5 \\ -x_{13} - x_{23} + x_{35} &= 0 \\ -x_{14} + x_{45} &= -3 \\ -x_{25} - x_{35} - x_{45} &= -6 \\ x_{ij} \geq 0 \quad ; i, j &= 1,2,\ldots,5 \end{aligned}$$

Now adding an artificial arc at node 5 and let x_5 be the corresponding arc and by using spanning tree method find the initial basic solution.

$$\bar{\mathbf{x}}^0 = (x_{12}^0, x_{13}^0, x_{14}^0, x_{23}^0, x_{25}^0, x_{35}^0, x_{45}^0, x_5^0)^T = (0,1,3,5,0,6,0,0)^T$$

Here, $\bar{\mathbf{x}}_B^0 = (x_{13}^0, x_{14}^0, x_{23}^0, x_{35}^0 x_5^0)^T = (1,3,5,6,0)^T$

and $\bar{\mathbf{x}}_N^o = (x_{12}^0, x_{25}^0, x_{45}^0)^T = (0,0,0)^T$

Therefore $I_B^0 = \{13,14,23,35,5\}$, $I_N^0 = \{12,25,45\}$ and cost at $\bar{\mathbf{x}}^0$ is $C(\bar{\mathbf{x}}^0) = 92$.

Iteration-1: we have,

$$\frac{\partial z}{\partial x_{ij}^o} = \frac{\partial C(\bar{\mathbf{x}}^0)}{\partial x_{ij}^o} - \mu_i + \mu_j = 0, \text{ for each } ij \in I_B^0$$

which gives, $\mu_1 = 11,\ \mu_2 = 13,\ \mu_3 = 3,\ \mu_4 = -19,\ \mu_5 = 0,$

Then calculate the related cost for $x_{ij}^0; ij \in I_N^0$, these are

$$\frac{\partial z}{\partial x_{12}^0} = 7, \quad \frac{\partial z}{\partial x_{25}^0} = -13 \ \textit{and}\ \frac{\partial z}{\partial x_{45}^0} = 20$$

Then compute, $\frac{\partial z}{\partial x_{rl}^{0}} = \min\left\{\frac{\partial z}{\partial x_{ij}^{0}} : ij \in I_N^0\right\} = \frac{\partial z}{\partial x_{25}^{0}} = -13$

and $x_{st}^{0}\frac{\partial z}{\partial x_{st}^{0}} = \max\left\{x_{ij}^{0}\frac{\partial z}{\partial x_{ij}^{0}} : ij \in I_N^0\right\} = 0$

Here $\left|\frac{\partial z}{\partial x_{rl}^{k}}\right| \neq x_{st}^{k}\frac{\partial z}{\partial x_{st}^{k}} = 0$; so go to next step.

Since $\left|\frac{\partial z}{\partial x_{rl}^{k}}\right| \succ x_{st}^{k}\frac{\partial z}{\partial x_{st}^{k}}$; increase x_{25}^{0} by-

$$\Delta^0 = \min\left\{\left|x_{53}^{0}\right|, \left|x_{32}^{0}\right|\right\} = 5$$

Then after necessary adjustment we find

$$\bar{\mathbf{y}}^0 = (0,1,3,0,5,1,0,0)^T .$$

To find the value of $\bar{\mathbf{x}}^1$, we calculate

$$C(\bar{\mathbf{x}}^1) = \min\{C(\bar{\mathbf{x}}) : \bar{\mathbf{x}} = \lambda\bar{\mathbf{x}}^0 + (1-\lambda)\bar{\mathbf{y}}^0 \ \& \ 0 < \lambda < 1\}$$

By solving we get $\lambda = \frac{7}{20}$, and

$$\bar{\mathbf{x}}^1 = \left(0,1,3,\frac{7}{4},\frac{13}{4},\frac{11}{4},0,0\right)$$

and so that $I_B^1 = \{13,14,23,35,5\}$ and $I_N^1 = \{12,25,45\}$.

Since $\bar{\mathbf{y}}^0 \neq \bar{\mathbf{x}}^1$, we do not change the former basic and go to next iteration.

Iteration-2: Similarly we get $\left|\frac{\partial z}{\partial x_{rl}^{k}}\right| = x_{st}^{k}\frac{\partial z}{\partial x_{st}^{k}} = 0$, which is our optimality condition.

Hence $\bar{\mathbf{x}}^1 = \left(0,1,3,\frac{7}{4},\frac{13}{4},\frac{11}{4},0,0\right)$ is the optimal solution, which minimizes the cost function.

d) Optimality Condition during Line Search Problem

In line search problem, we find a optimal solution by solving

$$C(\bar{\mathbf{x}}^{k+1}) = \min\{C(\bar{\mathbf{x}}) : \bar{\mathbf{x}} = \lambda\bar{\mathbf{x}}^k + (1-\lambda)\bar{\mathbf{y}}^k \ \& \ 0 < \lambda < 1\}$$

where $\bar{\mathbf{x}}^{k+1} = \lambda\bar{\mathbf{x}}^k + (1-\lambda)\bar{\mathbf{y}}^k$.

However, from practical experience for some problem we see that when $\lambda=1$, then $\bar{\mathbf{x}}^k = \bar{\mathbf{x}}^{k+1}$; i.e. this line search problem indicates that there is no other better optimal point except $\bar{\mathbf{x}}^k$. Again, if consider next iteration then the feasible solution will not change and the problem circulate here without satisfying the optimal condition (*Theorem**). But this feasible solution makes the cost function least compared to feasible solutions in previous iterations.

Condition: For every k-th iteration $(k \geq 1)$, if $\bar{\mathbf{x}}^k = \bar{\mathbf{x}}^{k+1}$ and $\lambda = 1$, then $\bar{\mathbf{x}}^k$ is optimal solution.

Example - 2

Here we consider the same network and constraints as in *Example-1* with the objective function-

$$\min\ z = C\,(\bar{\mathbf{x}}) = 8x_{12}^2 + 4x_{13}^2 + 5x_{14}^2 + x_{23}^2 + x_{25}^2 + 3x_{35}7x_{45}$$

So the initial feasible solution is

$$\overline{\mathbf{x}}^{0} = (x_{12}^0, x_{13}^0, x_{14}^0, x_{23}^0, x_{25}^0, x_{35}^0, x_{45}^0, x_5^0)^T = (0,1,3,5,0,6,0,0)^T$$

Here, $$\overline{\mathbf{x}}^{0}_{B} = (x_{13}^0, x_{14}^0, x_{23}^0, x_{35}^0 x_5^0)^T = (1,3,5,6,0)^T$$

and $$\overline{\mathbf{x}}^{o}_{N} = (x_{12}^0, x_{25}^0, x_{45}^0)^T = (0,0,0)^T$$

Therefore $I_B^0 = \{13,14,23,35,5\}$, $I_N^0 = \{12,25,45\}$

Iteration-1: Similarly (in Example-1), we get after adjustment $\mathbf{y} = (0,1,3,0,5,1,0,0)^T$. Then in line search,

$$\lambda = \frac{7}{20} \text{ and } \overline{\mathbf{x}}^{1} = \left(0,1,3,\frac{7}{4},\frac{13}{4},\frac{11}{4},0,0\right).$$

So that, $I_B^1 = \{13,14,23,35,5\}$, $I_N^1 = \{12,25,45\}$.

Since $\overline{\mathbf{y}}^{0} \neq \overline{\mathbf{x}}^{1}$, we do not change the former basic and go to next iteration.

Correspondingly, in ***Iteration-3***, we get-

$$\overline{\mathbf{y}}^{2} = (\frac{9}{52},\frac{43}{52},3,0,\frac{269}{52}\frac{43}{52},0,0)^T$$

In line search,

$$\lambda = \frac{191}{200}, \text{ and } \overline{\mathbf{x}}^{3} = (\frac{9}{52},\frac{43}{52},3,\frac{191}{104},\frac{347}{104}\frac{277}{104},0,0)^T$$

So that, $I_B^3 = \{13,14,23,35,5\}$, $I_N^3 = \{12,25,45\}$.

Since $\overline{\mathbf{y}}^{2} \neq \overline{\mathbf{x}}^{3}$, we do not change the former basic and go to next iteration.

In the same way, in ***Iteration-4***, we find

$$\overline{\mathbf{y}}^{3} = (1,0,3,\frac{277}{104},\frac{374}{104}\frac{277}{104},0,0)^T$$

During the line search we get $\lambda = \frac{1109}{1118} \approx 1$ and this gives

$$\overline{\mathbf{x}}^{4} = (\frac{9}{52},\frac{43}{52},3,\frac{191}{104},\frac{347}{104}\frac{277}{104},0,0)^T.$$

Therefore, we find $\overline{\mathbf{x}}^{3} = \overline{\mathbf{x}}^{4}$ and $\lambda = 1$.

Hence accordingly ***Condition**** $\overline{\mathbf{x}}^{3}$ is the optimal solution, which minimizes the objective function.

IV. CONCLUSION

In this paper we propose solution procedure for MC-CNFP and set a numerical example. From empirical, we give another optimal condition when $\lambda = 1$ and set an example. Yet, the *Condition** comes here from practical experience not from mathematical logic, so the research stills open.

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Notes

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 10 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

A New Efficient Approach to Analytical Solutions of Parabolic Equations

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Abstract - In this work the new algorithm entitled Reconstruction of Variational Iteration Method (RVIM) is used as an efficient technique in finding the approximate solutions of the linear and nonlinear equations using only few terms of its own iteration. The algorithm can help overcome the difficulty arising in calculating nonlinear intricate terms. Reconstruction of Variational Iteration Method (RVIM) is independent of any small parameters. Besides, it provides us with a simple way to ensure the convergence of series solution, so that we can always get accurate enough approximations. All of these verify the great potential and validity of the RVIM technique in comparison with those of Variational Iteration Method (VIM) for strongly nonlinear problems in science and engineering.

Keywords : Reconstruction of Variational Iteration Method, Analytical solution, parabolic partial differential equations, Partial Differential Equation, Two Space Variables.

GJSFR-F Classification : MSC 2010: 35K25

A NEW EFFICIENT APPROACH TO ANALYTICAL SOLUTIONS OF PARABOLIC EQUATIONS

A New Efficient Approach to Analytical Solutions of Parabolic Equations

Esmail Hesameddini [α] & Habibolla Latifizadeh [σ]

Abstract - In this work the new algorithm entitled Reconstruction of Variational Iteration Method (RVIM) is used as an efficient technique in finding the approximate solutions of the linear and nonlinear equations using only few terms of its own iteration. The algorithm can help overcome the difficulty arising in calculating nonlinear intricate terms. Reconstruction of Variational Iteration Method (RVIM) is independent of any small parameters. Besides, it provides us with a simple way to ensure the convergence of series solution, so that we can always get accurate enough approximations. All of these verify the great potential and validity of the RVIM technique in comparison with those of Variational Iteration Method (VIM) for strongly nonlinear problems in science and engineering.

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I. INTRODUCTION

It is very difficult to solve nonlinear problems, either numerically or theoretically, and even more difficult to establish a real model for nonlinear problems. Much assumption has to be made artificially or unnecessarily to make the practical engineering problems solvable, leading to loss of most important information. we propose a new kind of analytical method for nonlinear problems called the reconstruction of variational iteration method, in which is pointing out of similarities and differences with VIM and Homotopy Perturbation Method (HPM), not requiring small parameter in an equation as the perturbation techniques conduct and don't use the Lagrange multiplier. It has been shown the capability of this method to solve effectively, easily, and accurately, a large class of nonlinear problems with approximations converging rapidly to accurate solutions.

The method that has been used gives rapidly convergent successive approximations. As stated before, we aim to obtain analytical solutions to problems. We also aim to confirm that the reconstruction of variational iteration method is powerful, efficient, and promising in handling scientific and engineering problems. The RVIM technique is independent of any small parameters in general. In addition, it provides us with a simple way to ensure the convergence of series solution; therefore we can always get accurate enough approximations. The RVIM technique has been successfully applied to many nonlinear problems in science and engineering. All of these facts verifying the great potential and validity of the RVIM technique for strongly nonlinear problems in science and engineering.

II. DESCRIPTION OF THE NEW METHOD

To clarify the basic ideas of our proposed method in [1], we consider the following differential equation the same as VIM based on Lagrange multiplier [2-5]:

$$Lu(x_1,\cdots,x_k)+Nu(x_1,\cdots,x_k)=f(x_1,\cdots,x_k), \tag{1}$$

Ref.

1. E. Hesameddini, H. Latifizadeh, Reconstruction of Variational Iteration Algorithms using the Laplace Transform, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009)1377-1382.

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Subject to:

$$Lu(x_1,\cdots,x_k) = \sum_{i=0}^{k} L_{x_i}u(x_i), \tag{2}$$

where L is a linear operator, N is a nonlinear operator and $f(x_1,\cdots,x_k)$ is an inhomogeneous term. we can rewrite equation (1) down a correction functional as follows:

$$L_{x_j}u(x_j) = \underbrace{f(x_1,\cdots,x_{kn}) - Nu(x_1,\cdots,x_k) - \sum_{\substack{i=0\\ i\neq j}}^{k} L_{x_i}u(x_i)}_{h((x_1,\cdots,x_k),u(x_1,\cdots,x_k))}. \tag{3}$$

Therefore:

$$L_{x_j}u(x_j) = h\big((x_1,\cdots,x_k),u(x_1,\cdots,x_k)\big), \tag{4}$$

with artificial initial conditions being zero regarding the independent variable x_j .

By taking Laplace transform to both sides of the equation (4) in the usual way and using the artificial initial conditions, the result is as follows:

$$P(s).U(x_1,\cdots,x_{i-1},s,x_{i+1},x_k) = H\big((x_1,\cdots,x_{i-1},s,x_{i+1},x_k),u\big), \tag{5}$$

where P(s) is a polynomial with the degree of the highest derivative in the equation (5), (the same as the highest order of the linear operator L_{x_j}). The following relations are possible;

$$\mathcal{L}[h] = H, \tag{6-a}$$

$$B(s) = \frac{1}{P(s)}, \tag{6-b}$$

$$\mathcal{L}[b(x_i)] = B(s), \tag{6-c}$$

where in the equation (6-a) the function $H\big((x_1,\cdots,x_{i-1},s,x_{i+1},x_k),u\big)$ and

, $h\big((x_1,\cdots,x_{i-1},x_i,x_{i+1},x_k),u\big)$ have been abbreviated as H, h, respectively.

Hence, the equation (5) can be rewritten as:

$$U(x_1,\cdots,x_{i-1},s,x_{i+1},x_k) = H\big((x_1,\cdots,x_{i-1},s,x_{i+1},x_k),u\big).B(s). \tag{7}$$

Now, by applying the inverse Laplace transform to both sides of the equation (7) and using the (6-a) - (6-c), we obtain:

$$u(x_1,\cdots,x_{i-1},x_i,x_{i+1},x_k) = \int_0^{x_i} h\big((x_1,\cdots,x_{i-1},\tau,x_{i+1},x_k),u\big).b(x_i-\tau)d\tau. \tag{8}$$

Now, we must impose the actual initial conditions to obtain solution of the equation (1). Thus, we have the following iteration formulation:

$$u_{n+1}(x_1,\cdots,x_{i-1},x_i,x_{i+1},x_k) = u_0(x_1,\cdots,x_{i-1},x_i,x_{i+1},x_k) + \tag{9}$$

$$+\int_0^{x_i}\{h\big((x_1,\cdots,x_{i-1},\tau,x_{i+1},x_k),u_n\big).b(x_i-\tau)\}d\tau,$$

where u_0 is an initial solution with or without unknown parameters. Assuming u_0 is the solution of, Lu, with initial/boundary conditions of the main problem. In case of no unknown parameters, u_0 should satisfy initial/ boundary conditions. When some unknown parameters are involved in u_0, the unknown parameters can be identified by initial/boundary conditions after few iterations, this technology is very effective in dealing with boundary problems. It is worth mentioning that, in fact, the Lagrange multiplier in the He's variational iteration method is $\lambda(\tau) = b(x_i - \tau)$ as shown in [1]. The initial values are usually used for selecting the zeroth approximation u_0. When u_0 is determined, then several approximations $u_n \ n > 0$, follow immediately. Consequently, the exact solution may be obtained by using

$$u(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, x_k) = lim_{n\to} \ u_n(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, x_k). \tag{10}$$

In what follows, we will apply the RVIM method to homogeneous/non-homogeneous parabolic partial differential equations to illustrate the strength of the method and to establish exact solutions for these problems.

In an algorithmic form, the new presented method can be expressed and implemented the solutions as follows:

Algorithm. Let k be the iteration index, set an appropriate value for the tolerance (Tol.) for numerical purposes.

Step 0: Choose an appropriate u_0 so that $L_{x_j}(u_0) = 0$,

Step 1: Set $k = 0$.

Step 2: Use the calculated values of u_n to compute u_{k+1} from Eq. (9).

Step 3: Define $u_n := u_{k+1}$.

Step 4: If $\mathrm{Max}|u_k - u_{k-1}| < Tol$ stop, otherwise continue.

Step 5: Define $u_{k+1} := u_k$.

Step 6: Set $n = n + 1$, and return to step **2**.

III. APPLICATION OF THE PROPOSED METHOD FOR THE PARABOLIC EQUATIONS

Example1. Consider the following one dimensional, variable coefficient of the fourth-order parabolic partial differential equations [6, 7, and 8]

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4} = 0, \qquad \frac{1}{2} < x < 1, t > 0. \tag{11}$$

Subject to the initial conditions:

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = 1 + \frac{x^5}{120}, \tag{12}$$

and the boundary conditions:

$$u\left(\frac{1}{2}, t\right) = \left(1 + \frac{(0.5)^5}{120}\right)\sin(t), \qquad u(1,t) = \left(\frac{121}{120}\right)\sin(t), \tag{13}$$

$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2}, t\right) = \frac{1}{6}\left(\frac{1}{2}\right)^3 \sin(t), \qquad \frac{\partial^2 u}{\partial x^2}(1,t) = \frac{1}{6}\sin(t).$$

By selecting auxiliary linear operator, the Eq. (11) is rewritten as:

$$L_t u(x,t) = \frac{\partial^2 u}{\partial t^2} = \overbrace{-\left(\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4}\right)}^{h(x,\xi,u)}. \tag{14}$$

Now by applying the Laplace transformation with respect to independent variable t on both sides of Eq. (11), using the artificial initial condition [1], solution of the governing equation (11) has aforementioned in Eq. (8),

$$u(x,t) = \int_0^t (t-\xi)\, h(x,\xi,u)\,d\xi. \tag{15}$$

Therefore, using the Eqs. (8), (9) one can obtain the following RVIM's iteration formula in the t-direction

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t (t-\xi)\, h(x,\xi,u_n)\,d\xi. \tag{16}$$

where, $h(x,\xi,u_n)$ is indicated as:

$$h(x,\xi,u_n) = -\left(\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 u_n}{\partial x^4}(x,\xi)\right). \tag{17}$$

Now let's, start with an arbitrary initial approximation $u_0(x,t) = \left(1+\frac{x^5}{120}\right)t$ that satisfies the initial condition. Using the RVIM iteration formula (16), we can have the following successive approximation:

$$u_1(x,t) = \left(1+\frac{x^5}{120}\right)\left(t-\frac{t^3}{3!}\right), \tag{18}$$

$$u_2(x,t) = \left(1+\frac{x^5}{120}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}\right), \tag{19}$$

$$u_3(x,t) = \left(1+\frac{x^5}{120}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}\right). \tag{20}$$

$$\vdots$$

Whereas the RVIM method admits the use of

$$u = lim_{n\to}\ \ u_n.$$

Which gives the exact solution:

$$u(x,t) = \left(1+\frac{x^5}{120}\right)\sin(t). \tag{21}$$

Obtained upon using the Taylor expansion of $\sin(t)$.

Example 2. Consider the following parabolic equation [6, 7, and 8]

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u}{\partial x^4} = 0, \qquad 0 < x < 1, t > 0, \tag{22}$$

subject to the initial conditions:

$$u(x,0) = x - \sin(x), \qquad \frac{\partial u}{\partial t}(x,0) = -(x-\sin(x)), \tag{23}$$

Ref.

1. E. Hesameddini, H. Latifizadeh, Reconstruction of Variational Iteration Algorithms using the Laplace Transform, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009)1377-1382.

and the boundary conditions:

$$u(0,t)=0,\qquad u(1,t)=\mathrm{e}^{-\mathrm{t}}(1-\sin(1)),\tag{24}$$

$$\frac{\partial^2 u}{\partial x^2}(0,t)=0,\qquad \frac{\partial^2 u}{\partial x^2}(1,t)=\mathrm{e}^{-\mathrm{t}}\sin(1).$$

Ref.

1. E. Hesameddini, H. Latifizadeh, Reconstruction of Variational Iteration Algorithms using the Laplace Transform, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009)1377-1382.

As in the previous example, to implement the RVIM technique, first of all we need to choose the auxiliary linear operator as:

$$L_t u(x,t)=\frac{\partial^2 u}{\partial t^2}=\overbrace{-\left(\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u}{\partial x^4}\right)}^{h(x,\xi,u)}.$$

Accordingly, after taking Laplace transform using the artificial initial condition as in [1], on both side of the Eq. (22), the following RVIM iteration formula in the *t*-direction can be obtained as:

$$u_{n+1}(x,t)=u_0(x,t)-\int_0^t(t-\xi)\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u_n}{\partial x^4}(x,\xi)d\xi.\tag{25}$$

By the RVIM's recurrent formula in the Eq. (25), the terms of the sequence $\{u_n\}$ are constructed as follows, after choosing its initial approximate solution as $u_0(x,t)=(x-\sin(x))(1-t)$.

$$u_1(x,t)=(x-\sin(x))\left(1-t+\frac{t^2}{2!}\right),\tag{26}$$

$$u_2(x,t)=(x-\sin(x))\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}\right),\tag{27}$$

$$u_3(x,t)=(x-\sin(x))\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}\right).\tag{28}$$

The next terms of $\{\mathrm{u_n}\}$ can be determined in a similar way and the n-th approximation of u can be constructed as:

$$\mathrm{u_n}=(\mathrm{x}-\sin(\mathrm{x}))\sum_{\mathrm{i}=0}^{\mathrm{n}}\frac{\mathrm{t^i}}{\mathrm{i}!}\,.\tag{29}$$

And since

$$\lim_{\mathrm{n}\to}\ \mathrm{u_n}=(\mathrm{x}-\sin(\mathrm{x}))\mathrm{e}^{-\mathrm{t}}.\tag{30}$$

Therefore, approximate solution that is obtained by RVIM procedure converges to the exact solution.

Example3. Now, we solve the following one dimensional non-homogeneous fourth-order equation [6, 7, and 8]**:**

$$\frac{\partial^2 u}{\partial t^2}+(x+1)\frac{\partial^4 u}{\partial x^4}=\left(x^4+x^3-\frac{6}{7!}x^7\right)\cos(t),\qquad 0<x<1, t>0.\tag{31}$$

Subject to the initial conditions:

$$u(x,0)=\frac{6}{7!}x^7,\qquad \frac{\partial u}{\partial t}(x,0)=0,\tag{32}$$

and the boundary conditions:

$$u(0,t)=0,\qquad u(1,t)=\frac{6}{7!}\cos(\mathrm{t}),\tag{33}$$

$$\frac{\partial^2 u}{\partial x^2}(0,t)=0,\qquad \frac{\partial^2 u}{\partial x^2}(1,t)=\frac{1}{20}\cos(\mathrm{t}).$$

Applying RVIM to this equation with the given initial and boundary conditions, according to (3) and (4), we can have:

$$L_t u(x,t)=\frac{\partial^2 u}{\partial t^2}=\overbrace{\left(x^4+x^3-\frac{6}{7!}x^7\right)\cos(t)-(x+1)\frac{\partial^4 u}{\partial x^4}}^{h(x,\xi,u)}. \tag{34}$$

Now, by applying the Laplace Transform with respect to independent variable t on both sides of Eq. (31) using the artificial initial condition [1], therefore, using the Eqs. (8) and (9), one obtain the following RVIM's iteration formula in the *t*-direction:

$$u_{n+1}(x,t)=u_0(x,t)+\int_0^t(t-\xi)\,h(x,\xi,u_n)d\xi. \tag{35}$$

Where, $h(x,\xi,u_n)$ is indicated as:

$$h(x,\xi,u_n)=\left(x^4+x^3-\frac{6}{7!}x^7\right)\cos(t)-(x+1)\frac{\partial^4 u_n}{\partial x^4}(x,\xi). \tag{36}$$

With the aid of initial approximation $u_0(x,t)=\frac{6}{7!}x^7$ and using the RVIM iteration, we can obtain directly the rest of the other components as follows:

$$u_1(x,t)=x^3+x^4-\frac{t^2}{2!}(x^4+x^3)-x^3\cos(t)-x^4\cos(t)+\frac{1}{7!}x^7\cos(t), \tag{37}$$

$$u_2(x,t)=24+24x-12t^2(x+1)-24\cos(t)+\frac{1}{7!}x^7\cos(t)+t^4(1+x), \tag{38}$$

$$u_3(x,t)=\frac{1}{7!}x^7\cos(t). \tag{39}$$

It can be easily shown that $u_n=\frac{1}{7!}x^7\cos(t)$, $n=4,5,\ldots$; thus, the solution of Eq. (31) is obtained which reads:

$$\lim_{n\to}\ u_n=\frac{1}{7!}x^7\cos(t). \tag{40}$$

That is the exact solution.

Example4. Consider the fourth-order parabolic equation in two space variables [9]

$$\frac{\partial^2 u}{\partial t^2}+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4}=0 \qquad \frac{1}{2}<x<1, t>0. \tag{41}$$

Subject to the initial conditions:

$$u(x,y,0)=0, \qquad \frac{\partial u}{\partial t}(x,y,0)=2+\frac{1}{6!}x^6+\frac{y^6}{6!}, \tag{42}$$

and the boundary conditions:

$$u\left(\frac{1}{2},y,t\right)=\left(2+\frac{(0.5)^6}{6!}+\frac{y^6}{6!}\right)\sin(t), \qquad u(1,y,t)=\left(2+\frac{1}{6!}+\frac{y^6}{6!}\right)\sin(t), \tag{43}$$

$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2},y,t\right)=\left(\frac{(0.5)^4}{4!}\right)\sin(t), \qquad \frac{\partial^2 u}{\partial x^2}(1,y,t)=\frac{1}{4!}\sin(t),$$

$$\frac{\partial^2 u}{\partial x^2}\left(x,\frac{1}{2},t\right)=\left(\frac{(0.5)^4}{4!}\right)\sin(t), \qquad \frac{\partial^2 u}{\partial x^2}(x,1,t)=\frac{1}{4!}\sin(t).$$

Ref.

1. E. Hesameddini, H. Latifizadeh, Reconstruction of Variational Iteration Algorithms using the Laplace Transform, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009)1377-1382.

To implement the RVIM method on this differential equation with the given initial and boundary conditions, according to (3) and (4), the auxiliary linear operator is selected as:

$$L_t u(x,t) = \frac{\partial^2 u}{\partial t^2} = \overbrace{-\left(2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4} + 2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4}\right)}^{h(x,\xi,u)}. \tag{44}$$

Therefore, as in the previous examples the RVIM iterative formula can be expressed as:

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t (t-\xi)\, h(x,\xi,u_n)d\xi. \tag{45}$$

So that $h(x,\xi,u_n)$, is indicated as:

$$h(x,\xi,u_n) = -\left(2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4}(x,\xi) + 2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4}(x,\xi)\right). \tag{46}$$

We start with an initial approximation $\mathrm{u}_0(\mathrm{x},\mathrm{t}) = \mathrm{t}\left(2+\frac{\mathrm{x}^6}{6!}+\frac{\mathrm{y}^6}{6!}\right)$; by the iteration formula (45), one can obtain the first few components as follows:

$$u_1(x,t) = \left(t-\frac{t^3}{3!}\right)\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right), \tag{47}$$

$$u_2(x,t) = \left(t-\frac{t^3}{3!}+\frac{t^5}{5!}\right)\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right), \tag{48}$$

$$u_3(x,t) = \left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}\right)\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right). \tag{49}$$

The rest of the other components are obtained using the iteration formula (45) and then the solution of $u(x,t)$ in closed form is given by:

$$u(x,y,t) = \lim_{n\to}\ u_n(x,y,t) = \left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\sin(t). \tag{50}$$

That follows immediately upon using the Taylor expansion for $\sin(t)$.

IV. DISCUSSION

There are two main goals pursued in this work. The first was employing the powerful Reconstruction of the Variational Iteration to investigate fourth-order parabolic differential equations and the second one was showing the power of this method and its significant features. The two goals are achieved.

It is obvious that the method gives rapidly convergent successive approximations without any restrictive assumptions or transformation that may change the physical behavior of the problem. Reconstruction of the variational iteration gives several successive approximations through using the RVIM's iteration relation. Moreover, the RVIM reduces the size of calculations and the method is direct and straightforward. The RVIM uses the initial values for selecting the zeroth approximation, and boundary conditions, when given for bounded domains, and can be used for justification only.

For nonlinear equations that arise frequently in expressing nonlinear phenomena, RVIM facilitates the computations and gives the solution rapidly. As for concrete problems where an exact solution does not exist, a few approximations can be used for numerical purposes So by these advantages the RVIM method seems to be reliable and promising.

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Notes

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 10 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

A New Class of Harmonic Univalent Functions Defined by an Integral Operator

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Abstract - We define and investigate a new class of harmonic univalent functions defined by Salagean integral operator. We obtain coefficient inequalities and distortion bounds for the functions in this class.

Keywords : Integral operator, harmonic univalent functions, distortion inequalities.

GJSFR-F Classification : MSC 2010: 30C45, 30C50, 31A05

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I. INTRODUCTION

A continuous complex valued function $f = u + iv$ defined in a complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|, z \in D$.(See Clunie and Sheil-Small[2]).

Denote by $\mathcal{H}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense preserving in the unit disc $U = \{z : |z| < 1\}$ so that $f = h + \overline{g}$ is normalized by $f(0) = h(0) = f'_z(0) - 1 = 0$.

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U. We let:

$$A_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1}z^{n+1} + ..., z \in U\}, \quad with \quad A_1 = A.$$

We let $\mathcal{H}[a, n]$ denote the class of analytic functions in $U$of the form

$$f(z) = a + a_n z^n + a_{n+1}z^{n+1} + ..., z \in U.$$

The integral operator I^n is defined in [4] by:

$$(i) \quad I^0 f(z) = f(z);$$

$$(ii) \quad I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt;$$

$$(iii) \quad I^n f(z) = I(I^{n-1}f(z)), \quad n \in \mathbb{N} - \{0\}, \quad f \in A.$$

Ref.

[2] J. Clunie, T. Scheil- Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math., **9**(1984), 3-25.

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Ahuja and Jahangiri [1] defined the class $H(n)$, $n \in \mathbb{N}$, consisting of all univalent harmonic functions $f = h + \overline{g}$ that are sense preserving in U and h and g are of the form:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{1.1}$$

For $f = h + \overline{g}$ given by (1.1) the integral operator I^n is defined as:

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)}, z \in U, \tag{1.2}$$

where

$$I^n h(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k$$

and

$$I^n g(z) = \sum_{k=1}^{\infty} k^{-n} b_k z^k.$$

For fixed positive integers n and for $0 \leq \alpha < 1, \beta \geq 0$ we let $H(n, \alpha, \beta)$ denote the class of univalent harmonic functions of the form (1.1) that satisfy the condition:

$$Re\left\{\frac{I^n f(z)}{I^{n+1} f(z)}\right\} > \beta\left|\frac{I^n f(z)}{I^{n+1} f(z)} - 1\right| + \alpha. \tag{1.3}$$

The subclass $H^-(n, \alpha, \beta)$ consists of functions $f_n = h + \overline{g_n}$ in $H(n, \alpha, \beta)$ so that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.4}$$

II. THE MAIN RESULTS

In the first theorem, we introduce a sufficient coefficient bound for harmonic functions in $H(n, \alpha, \beta)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1.1). If*

$$\sum_{k=1}^{\infty} \{ \quad (n, \alpha, \beta)|a_k| + \theta(n, \alpha, \beta)|b_k|\} \leq 2, \tag{2.1}$$

where

$$(n, \alpha, \beta) = \frac{k^{-n}(1 + \beta) - (\beta + \alpha)k^{-(n+1)}}{1 - \alpha},$$

Ref.

[1]O.P. Ahuja, J.M. Jahangiri, *Multivalent harmonic starlike functions,* Ann. Univ. Marie Curie-Sklodowska Sect. A, LV 1(2001), 1-13.

and
$$\theta(n,\alpha,\beta)=\frac{k^{-n}(1+\beta)+(\beta+\alpha)k^{-(n+1)}}{1-\alpha},$$

$a_1=1,\quad 0\le\alpha<1,\quad \beta\ge 0,\quad n\in\mathbb{N},$ *then* $f\in H(n,\alpha,\beta)$.

Proof. According to (1.2) and(1.3) we only need to show that

$$Re\Big(\frac{I^n f(z)-\alpha I^{n+1}f(z)-\beta e^{i\theta}|I^n f(z)-I^{n+1}f(z)|}{I^{n+1}f(z)}\Big)\ge 0.$$

The case $r=0$ is obvious. For $0<r<1$ it follows that

$$Re\Big(\frac{I^n f(z)-\alpha I^{n+1}f(z)-\beta e^{i\theta}|I^n f(z)-I^{n+1}f(z)|}{I^{n+1}f(z)}\Big)=$$

$$=Re\Big\{\frac{(1-\alpha)z+\sum_{k=2}^{\infty}a_k z^k[\gamma^n-\alpha\gamma^{n+1}]}{z+\sum_{k=2}^{\infty}\gamma^{n+1}a_k z^k+(-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}\,\overline{z^k}}+$$

$$+\frac{(-1)^n\sum_{k=1}^{\infty}\overline{b_k}\,\overline{z^k}[\gamma^n+\alpha\gamma^{n+1}]}{z+\sum_{k=2}^{\infty}\gamma^{n+1}a_k z^k+(-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}\,\overline{z^k}}-$$

$$-\frac{\beta e^{i\theta}|\sum_{k=2}^{\infty}a_k z^k[\gamma^n-\gamma^{n+1}]+(-1)^n\sum_{k=1}^{\infty}\overline{b_k}\,\overline{z^k}[\gamma^n+\gamma^{n+1}]|}{z+\sum_{k=2}^{\infty}\gamma^{n+1}a_k z^k+(-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}\,\overline{z^k}}\Big\}=$$

$$=Re\Big\{\frac{1-\alpha+\sum_{k=2}^{\infty}a_k z^{k-1}[\gamma^n-\alpha\gamma^{n+1}]}{1+\sum_{k=2}^{\infty}\gamma^{n+1}a_k z^{k-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}\,\overline{z^k}z^{-1}}+$$

$$\frac{(-1)^n\sum_{k=1}^{\infty}\overline{b_k}\,\overline{z^k}z^{-1}[\gamma^n+\alpha\gamma^{n+1}]}{1+\sum_{k=2}^{\infty}\gamma^{n+1}a_k z^{k-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}\,\overline{z^k}z^{-1}}-$$

$$\frac{\beta e^{i\theta}z^{-1}|\sum_{k=2}^{\infty}[\gamma^n-\gamma^{n+1}]a_k z^k+(-1)^n\sum_{k=1}^{\infty}[\gamma^n+\gamma^{n+1}]\overline{b_k}\,\overline{z^k}|}{1+\sum_{k=2}^{\infty}\gamma^{n+1}a_k z^{k-1}+(-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}\,\overline{z^k}z^{-1}}\Big\}=$$

$$= Re\frac{(1-\alpha)+A(z)}{1+B(z)}, \quad where \quad \gamma=\frac{1}{k}.$$

For $z=re^{i\theta}$ we have

$$A(re^{i\theta})=\sum_{k=2}^{\infty}(\gamma^n-\alpha\gamma^{n+1})a_kr^{k-1}e^{(k-1)\theta i}+$$

$$+(-1)^n\sum_{k=1}^{\infty}(\gamma^n+\gamma^{n+1}\alpha)\overline{b_k}r^{k-1}e^{-(k+1)\theta i}-\beta\mathcal{D}(n+1,n,\alpha),$$

where

$$\mathcal{D}(n+1,n,\alpha)=|\sum_{k=2}^{\infty}(\gamma^n-\gamma^{n+1})a_kr^{k-1}e^{-ki\theta}+(-1)^n\sum_{k=1}^{\infty}(\gamma^n+\gamma^{n+1})\overline{b_k}r^{k-1}e^{-ki\theta}|,$$

and

$$B(re^{i\theta})=\sum_{k=2}^{\infty}\gamma^{n+1}a_kr^{k-1}e^{(k-1)\theta i}+(-1)^{n+1}\sum_{k=1}^{\infty}\gamma^{n+1}\overline{b_k}r^{k-1}e^{-(k+1)\theta i}.$$

Setting $\frac{1-\alpha+A(z)}{1+B(z)}=(1-\alpha)\frac{1+w(z)}{1-w(z)}$.

The proof will be complete if we can show that $|w(z)|\leq r<1$. This is the case since, by the condition (2.1), we can write:

$$|w(z)|=|\frac{A(z)-(1-\alpha)B(z)}{A(z)+(1-\alpha)B(z)+2(1-\alpha)}|\leq$$

$$\leq\frac{\sum_{k=1}^{\infty}[(1+\beta)(\gamma^n-\gamma^{n+1})|a_k|+(1+\beta)(\gamma^n+\gamma^{n+1})|b_k|]r^{k-1}}{4(1-\alpha)-\sum_{k=1}^{\infty}\{[\gamma^n(1+\beta)-\delta\gamma^{n+1}]|a_k|+[\gamma^n(1+\beta)+\delta\gamma^{n+1}]|b_k|\}r^{k-1}}$$

$$<\frac{\sum_{k=1}^{\infty}(1+\beta)(\gamma^n-\gamma^{n+1})|a_k|+(\gamma^n+\gamma^{n+1})(1+\beta)|b_k|}{4(1-\alpha)-\sum_{k=1}^{\infty}\{[\gamma^n(1+\beta)-\delta\gamma^{n+1}]|a_k|+[\gamma^n(1+\beta)+\delta\gamma^{n+1}]|b_k|\}}\leq 1,$$

where $\delta=\beta+2\alpha-1$.

The harmonic univalent functions

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1}{(n,\alpha,\beta)}x_kz^k+\sum_{k=1}^{\infty}\frac{1}{\theta(n,\alpha,\beta)}\overline{y_kz^k},$$

Notes

where $n\in\mathbb{N}, 0\le\alpha<1, \beta\ge 0$ and $\sum\limits_{k=2}^{\infty}|x_k|+\sum\limits_{k=1}^{\infty}|y_k|=1$, show that the coefficient bound given by (2.1) is sharp.

In the following theorem it is show that the condition (2.1)is also necessary for the function $f_n=h+\overline{g_n}$, where h and g_n are of the form (1.4).

Theorem 2.2. *Let $f_n=h+\overline{g_n}$ be given by (1.4). Then $f_n\in H^{-}(n,\alpha,\beta)$ if and only if*

$$\sum_{k=1}^{\infty}[\ \ (n,\alpha,\beta)a_k+\theta(n,\alpha,\beta)b_k]\le 2, \qquad (2.2)$$

$$a_1=1, 0\le\alpha<1, n\in\mathbb{N}.$$

Proof. Since $H^{-}(n,\alpha,\beta)\subset H(n,\alpha,\beta)$, we only need to prove the "only if" part of the theorem. For functions f_n of the form (1.4), we note that the condition

$$Re\{\frac{I^n f(z)}{I^{n+1}f(z)}\}>\beta|\frac{I^n f(z)}{I^{n+1}f(z)}-1|+\alpha$$

is equivalent to

$$Re\{\frac{(1-\alpha)z-\sum\limits_{k=2}^{\infty}(\gamma^n-\alpha\gamma^{n+1})a_k z^k}{z-\sum\limits_{k=2}^{\infty}\gamma^{n+1}a_k z^k+(-1)^{2n}\sum\limits_{k=1}^{\infty}\gamma^{n+1}b_k\overline{z^k}}+$$

$$+\frac{(-1)^{2n-1}\sum\limits_{k=1}^{\infty}(\gamma^n+\gamma^{n+1}\alpha)b_k\overline{z^k}}{z-\sum\limits_{k=2}^{\infty}\gamma^{n+1}a_k z^k+(-1)^{2n}\sum\limits_{k=1}^{\infty}\gamma^{n+1}b_k\overline{z^k}}-$$

$$-\frac{\beta e^{i\theta}|-\sum\limits_{k=2}^{\infty}(\gamma^n+\gamma^{n+1})a_k z^k+(-1)^{2n-1}\sum\limits_{k=1}^{\infty}(\gamma^n-\gamma^{n+1})\overline{b_k}\overline{z^k}|}{z-\sum\limits_{k=2}^{\infty}\gamma^{n+1}a_k z^k+(-1)^{2n+1}\sum\limits_{k=1}^{\infty}\gamma^{n+1}b_k\overline{z^k}}\}\ge 0, \qquad (2.3)$$

where $\gamma=\frac{1}{k}$.

The above required condition (2.3) must hold for all values of $z\in U$. Upon choosing the values of z on the positive real axis where $0\le z=r<1$ and using $Re(-e^{i\theta})\ge-|e^{i\theta}|=-1$ we must have

$$\frac{(1-\alpha)-\sum_{k=2}^{\infty}[\gamma^{n}(1+\beta)-(\alpha+\beta)\gamma^{n+1}]a_k r^{k-1}}{1-\sum_{k=2}^{\infty}\gamma^{n+1}a_k r^{k-1}+\sum_{k=1}^{\infty}\gamma^{n+1}b_k r^{k-1}}- \tag{2.4}$$

$$-\frac{\sum_{k=1}^{\infty}[\gamma^{n}(1+\beta)+\gamma^{n+1}(\beta+\alpha)]b_k r^{k-1}}{1-\sum_{k=2}^{\infty}\gamma^{n+1}a_k r^{k-1}+\sum_{k=1}^{\infty}\gamma^{n+1}b_k r^{k-1}}\geq 0.$$

If the condition (2.3) does not hold, then the expression in (2.4) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which this quotient in (2.4) is negative. This contradicts the required condition for $f_n \in H^{-}(n,\alpha,\beta)$ and so the proof is complete.

The following theorem gives the distortion bounds for functions in $H^{-}(n,\alpha,\beta)$ which yields a covering results for this class.

Theorem 2.3. *Let $f_n \in H^{-}(n,\alpha,\beta)$. Then for $|z| = r < 1$ we have*

$$|f_n(z)| \leq (1+b_1)r + [\theta(n,\alpha,\beta)-\omega(n,\alpha,\beta)b_1]r^{n+2}$$

and

$$|f_n(z)| \geq (1-b_1)r - \{\phi(n,\alpha,\beta)-\omega(n,\alpha,\beta)b_1\}r^{n+2},$$

where

$$\phi(n,\alpha,\beta)=\frac{1-\alpha}{(1/2)^{n}(1+\beta)-(1/2)^{n+1}(\alpha+\beta)},$$

$$\omega(n,\alpha,\beta)=\frac{(1+\beta)+(\alpha+\beta)}{(1/2)^{n}(1+\beta)-(1/2)^{n+1}(\alpha+\beta)}.$$

Proof. We prove the right side inequality for $|f_n|$. The proof for the left hand inequality can be done using similar arguments. Let $f_n \in H^{-}(n,\alpha,\beta)$. Taking the absolute value of f_n then by Theorem 2.2, we can obtain :

$$|f_n(z)| = |z-\sum_{k=2}^{\infty}a_k z^k + (-1)^{n-1}\sum_{k=1}^{\infty}b_k\overline{z^k}| \leq$$

$$\leq r+\sum_{k=2}^{\infty}a_k r^k+\sum_{k=1}^{\infty}b_k r^k = r+b_1 r+\sum_{k=2}^{\infty}(a_k+b_k)r^k \leq$$

$$\leq r+b_1 r+\sum_{k=2}^{\infty}(a_k+b_k)r^2 =$$

Notes

$$= (1+b_1)r + \phi(n,\alpha,\beta)\sum_{k=2}^{\infty}\frac{1}{\phi(n,\alpha,\beta)}(a_k+b_k)r^2 \leq$$

$$\leq (1+b_1)r + \phi(n,\alpha,\beta)r^{n+2}\sum_{k=2}^{\infty}[\quad(n,\alpha,\beta)a_k + \theta(n,\alpha,\beta)b_k] \leq$$

$$\leq (1+b_1)r + [\phi(n,\alpha,\beta) - \omega(n,\alpha,\beta)b_1]r^{n+2}.$$

Notes

The following covering result follows from the left hand inequality in Theorem 2.3.

Corollary 2.4. *Let* $f_n \in H^-(n,\alpha,\beta)$. *Then for* $|z| = r < 1$ *we have* $\{w : |w| < 1 - b_1 - [\phi(n,\alpha,\beta) - \omega(n,\alpha,\eta)b_1] \subset f_n(U)\}$.

Next we determine the extreme points of closed convex hulls of $H^-(n,\alpha,\beta)$, denoted by clco$H^-(n,\alpha,\beta)$.

Theorem 2.5. *Let* f_n *be given by (1.4). Then* $f_n \in H^-(n,\alpha,\beta)$ *if and only if*

$$f_n(z) = \sum_{k=1}^{\infty}[x_k h_k(z) + y_k g_{n_k}(z)],$$

where $h(z) = z$,

$$h_k(z) = z - \frac{1-\alpha}{k^{-n}(1+\beta) - (\beta+\alpha)k^{-(n+1)}}z^k, k = 2,3,...$$

and

$$g_{n_k}(z) = z + (-1)^{n-1}\frac{1-\alpha}{k^{-n}(1+\beta) + (\beta+\alpha)k^{-(n+1)}}\overline{z}^k, k = 1,2,3,...$$

$$x_k \geq 0, y_k \geq 0, \sum_{k=1}^{\infty}(x_k+y_k) = 1.$$

In particular, the extreme points of $H^-(n,\alpha,\beta)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. For functions f_n of the form (2.1) we have:

$$f_n(z) = \sum_{k=2}^{\infty}[x_k h_k(z) + y_k g_{n_k}(z)] =$$

$$= \sum_{k=1}^{\infty}(x_k+y_k)z - \sum_{k=2}^{\infty}\frac{1-\alpha}{k^{-n}(1+\beta) - (\beta+\alpha)k^{-(n+1)}}x_k z^k +$$

$$+(-1)^{n-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{k^{-n}(1+\beta) + (\beta+\alpha)k^{-(n+1)}}y_k \overline{z}^k.$$

Then

$$\sum_{k=2}^{\infty} x_k \frac{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+1)}}{1-\alpha} \cdot \frac{(1-\alpha)}{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+1)}}+$$

$$\sum_{k=1}^{\infty} y_k \frac{k^{-n}(1+\beta)+(\beta+\alpha)k^{-(n+1)}}{1-\alpha} \frac{1-\alpha}{k^{-n}(1+\beta)+(\beta+\alpha)k^{-(n+1)}}$$

$$=\sum_{k=2}^{\infty} x_k+\sum_{k=1}^{\infty} y_k=1-x_1 \leq 1$$

and so $f_n(z) \in H^-(n,\alpha,\beta)$.
Conversely, suppose $f_n(z) \in H^-(n,\alpha,\beta)$. Letting

$$x_1=1-\sum_{k=2}^{\infty} x_k-\sum_{k=1}^{\infty} y_k$$

$$x_k=\frac{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+1)}}{1-\alpha} \cdot a_k, k=2,3,...$$

and

$$y_k=\frac{k^{-n}(1+\beta)+(\beta+\alpha)k^{-(n+1)}}{1-\alpha} \cdot b_k, k=1,2,3,...$$

we obtain the required representation, since

$$f_n(z)=z-\sum_{k=2}^{\infty} a_k z^k+(-1)^{n-1}\sum_{k=1}^{\infty} b_k \overline{z}^k=$$

$$=z-\sum_{k=2}^{\infty} \frac{1-\alpha}{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+1)}} x_k z^k+$$

$$+(-1)^{n-1}\sum_{k=1}^{\infty} \frac{1-\alpha}{k^{-n}(1+\beta)+(\beta+\alpha)k^{-(n+1)}} y_k \overline{z}^k=$$

$$=z-\sum_{k=2}^{\infty}[z-h_k(z)]x_k-\sum_{k=1}^{\infty}[z-g_{n_k}(z)]y_k=$$

$$=[1-\sum_{k=2}^{\infty} x_k-\sum_{k=1}^{\infty} y_k]z+\sum_{k=2}^{\infty} x_k h_k(z)+\sum_{k=1}^{\infty} y_k g_{n_k}(z)=$$

$$=\sum_{k=1}^{\infty}[x_k h_k(z)+y_k g_{n_k}(z)].$$

Notes

Now we show that $H^{-}(n,\alpha,\beta)$ is closed under convex combination of its members.

Theorem 2.6. *The family* $H^{-}(n,\alpha,\beta)$ *is closed under convex combination.*

Proof. For $i=1,2,...$ suppose that $f_n^i \in H^{-}(n,\alpha,\beta)$, where

$$f_n^i(z)=z+\sum_{k=2}^{\infty} a_k^i z^k+(-1)^{n-1}\sum_{k=1}^{\infty} b_k^i \overline{z}^k,$$

then by Theorem 2.2,

$$\sum_{k=1}^{\infty}\frac{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+1)}}{1-\alpha}a_k^i+\sum_{k=1}^{\infty}\frac{k^{-n}(1+\beta)+(\beta+\alpha)k^{-(n+1)}}{1-\alpha}b_k^i\leq 2, \tag{2.5}$$

for $\sum_{i=1}^{\infty} t_i=1, 0\leq t_i\leq 1$, the convex combination of f_n^i may be written as

$$\sum_{i=1}^{\infty} t_i f_n^i(z)=z-\sum_{k=2}^{\infty}(\sum_{i=1}^{\infty} t_i a_k^i)z^k+(-1)^{n-1}\sum_{k=1}^{\infty}(\sum_{i=1}^{\infty} t_i b_k^i)\overline{z}^k.$$

Then by (2.4)

$$\sum_{k=1}^{\infty}\frac{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+1)}}{1-\alpha}(\sum_{i=1}^{\infty} t_i a_k^i)+$$

$$+\sum_{k=1}^{\infty}\frac{k^{-n}(1+\beta)+(\beta+\alpha)k^{-(n+1)}}{1-\alpha}(\sum_{i=1}^{\infty} t_i b_k^i)=$$

$$=\sum_{i=1}^{\infty} t_i[\sum_{k=1}^{\infty}\frac{k^{-n}(1+\beta)-(\beta+\alpha)k^{-(n+1)}}{1-\alpha}a_k^i+$$

$$+\sum_{k=1}^{\infty}\frac{k^{-n}(1+\beta)+(\beta+\alpha)k^{-(n+1)}}{1-\alpha}b_k^i]\leq 2\sum_{i=1}^{\infty} t_i=2$$

and therefore $\sum_{i=1}^{\infty} t_i f_n^i(z)\in H^{-}(n,\alpha,\beta)$.

The beautiful results for harmonic functions, was obtained by P. T. Mocanu in [3].

Ref.

[3]P. T. Mocanu, *Three-cornered hat harmonic functions*, Complex Variables and Elliptic Equation, **12**(2009), 1079-1084.

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Notes

GLOBAL JOURNAL OF SCIENCE FRONTIER RESEARCH
MATHEMATICS AND DECISION SCIENCES
Volume 12 Issue 10 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Generalizations of Ranamujan's Results in Terms of q-product Identities

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Abstract - In this paper author has generalized Ramanujan's results and established four new relations on q-product identities with the help of Jacobi's triple product identity using elementary method.

Keywords : *Triple product identities, q-product identities.*

GJSFR-F Classification : *MSC 2010: Primary 05A17, 05A15; Secondary 11P83*

GENERALIZATIONS OF RANAMUJANS RESULTS IN TERMS OF Q-PRODUCT IDENTITIES

Strictly as per the compliance and regulations of :

Ref.

2. B.C. Berndt; *What is a q-series?*, preprint.

Generalizations of Ranamujan's Results in Terms of q-product Identities

M.P. Chaudhary [α] & Rahul Singh [σ]

Abstract - In this paper author has generalized Ramanujan's results and established four new relations on q-product identities with the help of Jacobi's triple product identity using elementary method.

Keywords : Triple product identities, q-product identities.

I. INTRODUCTION

For $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) \tag{1.1}$$

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{(n-1)}) \tag{1.2}$$

$$(a_1,a_2,a_3,...,a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty...(a_k;q)_\infty \tag{1.3}$$

Ramanujan [2, p.1(1.2)]has defined general theta function, as

$$f(a,b) = \sum_{-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \quad ; \quad |ab| < 1, \tag{1.4}$$

Jacobi's triple product identity [3, p.35] is given, as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{1.5}$$

Special cases of Jacobi's triple products identity are given, as

$$\phi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty \tag{1.6}$$

$$(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.7}$$

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.8}$$

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Equation (1.8) is known as Euler's pentagonal number theorem. Euler's another well known identity is as

$$(q;q^2)_\infty^{-1} = (-q;q)_\infty \tag{1.9}$$

Throughout this paper we use the following representations

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty = (q^a,q^b,q^c\cdots q^t;q^n)_\infty \tag{1.10}$$

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty = (q^a,q^b,q^c\cdots q^t;q^n)_\infty \tag{1.11}$$

$$(-q^a;q^n)_\infty(-q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty = (-q^a,-q^b,q^c\cdots q^t;q^n)_\infty \tag{1.12}$$

Computation of q-product identities:
Now we can have following q-products identities, as

$$\begin{aligned}(q^2;q^2)_\infty &= \prod_{n=0}^{\infty}(1-q^{2n+2})\\ &= \prod_{n=0}^{\infty}(1-q^{2(4n)+2})\times\prod_{n=0}^{\infty}(1-q^{2(4n+1)+2})\times\\ &\times\prod_{n=0}^{\infty}(1-q^{2(4n+2)+2})\times\prod_{n=0}^{\infty}(1-q^{2(4n+3)+2})\\ &= \prod_{n=0}^{\infty}(1-q^{8n+2})\times\prod_{n=0}^{\infty}(1-q^{8n+4})\times\prod_{n=0}^{\infty}(1-q^{8n+6})\times\prod_{n=0}^{\infty}(1-q^{8n+8})\end{aligned}$$

or

$$\begin{aligned}(q^2;q^2)_\infty &= (q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty\\ &= (q^2,q^4,q^6,q^8;q^8)_\infty\end{aligned} \tag{1.13}$$

also we can compute

$$\begin{aligned}(q^2;q^2)_\infty &= (q^2;q^4)_\infty(q^4;q^4)_\infty\\ &= (q^2,q^4;q^4)_\infty\end{aligned} \tag{1.14}$$

$$\begin{aligned}(q^4;q^4)_\infty &= \prod_{n=0}^{\infty}(1-q^{4n+4})\\ &= \prod_{n=0}^{\infty}(1-q^{4(3n)+4})\times\prod_{n=0}^{\infty}(1-q^{4(3n+1)+4})\times\prod_{n=0}^{\infty}(1-q^{4(3n+2)+4})\\ &= \prod_{n=0}^{\infty}(1-q^{12n+4})\times\prod_{n=0}^{\infty}(1-q^{12n+8})\times\prod_{n=0}^{\infty}(1-q^{12n+12})\end{aligned}$$

or

$$\begin{aligned}(q^4;q^4)_\infty &= (q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty\\ &= (q^4,q^8,q^{12};q^{12})_\infty\end{aligned} \tag{1.15}$$

Notes

$$
\begin{aligned}
(q^4;q^{12})_\infty &= \prod_{n=0}^{\infty}(1-q^{12n+4})\\
&= \prod_{n=0}^{\infty}(1-q^{12(5n)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+1)+4})\times\\
&\times \prod_{n=0}^{\infty}(1-q^{12(5n+2)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+3)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+4)+4})\\
&= \prod_{n=0}^{\infty}(1-q^{60n+4}) \times \prod_{n=0}^{\infty}(1-q^{60n+16}) \times \prod_{n=0}^{\infty}(1-q^{60n+28})\times\\
&\times \prod_{n=0}^{\infty}(1-q^{60n+40}) \times \prod_{n=0}^{\infty}(1-q^{60n+52})
\end{aligned}
$$

or

$$
\begin{aligned}
(q^4;q^{12})_\infty &= (q^4;q^{60})_\infty(q^{16};q^{60})_\infty(q^{28};q^{60})_\infty(q^{40};q^{60})_\infty(q^{52};q^{60})_\infty\\
&= (q^4,q^{16},q^{28},q^{40},q^{52};q^{60})_\infty \qquad (1.16)
\end{aligned}
$$

Similarly we can compute following q-product identities

$$
\begin{aligned}
(q^5;q^5)_\infty &= (q^5;q^{15})_\infty(q^{10};q^{15})_\infty(q^{15};q^{15})_\infty\\
&= (q^5,q^{10},q^{15};q^{15})_\infty \qquad (1.17)
\end{aligned}
$$

$$
\begin{aligned}
(q^6;q^6)_\infty &= (q^6;q^{24})_\infty(q^{12};q^{24})_\infty(q^{18};q^{24})_\infty(q^{24};q^{24})_\infty\\
&= (q^6,q^{12},q^{18},q^{24};q^{24})_\infty \qquad (1.18)
\end{aligned}
$$

$$
\begin{aligned}
(q^6;q^{12})_\infty &= (q^6;q^{60})_\infty(q^{18};q^{60})_\infty(q^{30};q^{60})_\infty(q^{42};q^{60})_\infty(q^{54};q^{60})_\infty\\
&= (q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty \qquad (1.19)
\end{aligned}
$$

The outline of this paper is as follows. In sections 2, some recent results obtained by the author [1], and also some well known results in [6;7] are recorded, those are useful to the rest of the paper. In section 3, we state and prove four q-product identities, which are new and not recorded in the literature of special functions.

II. Preliminaries

Recently author has established following identities [1],

$$(q^2,q^4,q^6;q^8)_\infty[(-q;q^2)^2_\infty + (q;q^2)^2_\infty] = 2(-q^4;q^8)^2_\infty \qquad (2.1)$$

$$(q^2,q^4,q^6,q^8;q^8)_\infty[(-q;q^2)^2_\infty - (q;q^2)^2_\infty] = 4q\frac{(q^{16},q^{32},q^{48};q^{48})_\infty}{(q^8,q^{24},q^{40};q^{48})_\infty} \qquad (2.2)$$

$$\frac{(-q;q^2)^2_\infty + (q;q^2)^2_\infty}{(-q;q^2)^2_\infty - (q;q^2)^2_\infty} = \frac{(-q^4;q^8)^2_\infty(q^8,q^8,q^{24},q^{24},q^{40},q^{40};q^{48})_\infty}{2q} \qquad (2.3)$$

Ref.

6. S. Ramanujan; *Notebooks (Volume I)*, Tata Institute of Fundamental Research, Bombay, 1957.

$$(-q;q^2)^2_\infty (q;q^2)^2_\infty (q^2;q^2)^2_\infty = (q^2,q^2,q^4;q^4)_\infty \tag{2.4}$$

$$\frac{(-q;q^2)_\infty(-q^3;q^6)_\infty - (q;q^2)_\infty(q^3;q^6)_\infty}{(-q;q^2)_\infty \times (-q^3;q^6)_\infty \times (q;q^2)_\infty \times (q^3;q^6)_\infty}$$
$$= \frac{2q(-q^2;q^4)^2_\infty(q^4,q^8,q^{16},q^{20},q^{24};q^{24})_\infty}{(q^2,q^4,q^6,q^8;q^8)_\infty(q^6,q^{12},q^{18};q^{24})_\infty} \tag{2.5}$$

$$\frac{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty - (q^3;q^6)_\infty(q^5;q^{10})_\infty}{(-q^3;q^6)_\infty \times (-q^5;q^{10})_\infty \times (q^3;q^6)_\infty \times (q^5;q^{10})_\infty}$$
$$= \frac{(q^4,q^8,q^{12};q^{12})_\infty}{(q^6,q^{12},q^{18},q^{24};q^{24})_\infty} \times$$
$$\times \frac{2q^3}{(q^2,q^6,q^{10};q^{12})_\infty(q^{10},q^{20},q^{30},q^{30},q^{40},q^{50};q^{60})_\infty} \tag{2.6}$$

$$\frac{[(q;q^2)_\infty(q^{15};q^{30})_\infty] + [(-q;q^2)_\infty(-q^{15};q^{30})_\infty]}{[(q;q^2)_\infty(q^{15};q^{30})_\infty][(-q;q^2)_\infty(-q^{15};q^{30})_\infty]}$$
$$= \frac{(q^{12},q^{20},q^{24},q^{36},q^{40},q^{48},q^{60},q^{60};q^{60})_\infty}{(q^{10},q^{30},q^{30},q^{50},q^{60};q^{60})_\infty} \times$$
$$\times \frac{2}{(q^2,q^4,q^6,q^8,q^8;q^8)_\infty(q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty} \tag{2.7}$$

In Ramanujan's notebook [6, p.240], the following entries are recorded as

$$f(-q,-q^5) = \psi(q^3)\left[\frac{\phi(-q)}{\psi(q)}\right]^{\frac{1}{3}} \tag{2.8}$$

$$\phi^4(q) - \phi^4(-q) = 16q\psi^4(q^2) \tag{2.9}$$

In Ramanujan's notebook [6, p.245], the following entry is recorded as

$$\psi(q)\psi(-q) = \psi(q^2)\phi(-q^2) \tag{2.10}$$

In Ramanujan's notebook [7, p.209], the following entry [8(xii)] is recorded as

$$\frac{f(-q)}{f(-q^4)} = \frac{\phi(-q^4)}{\psi(q)} \tag{2.11}$$

III. Main Results

In this section, we generalized four results recorded in Ramanujan's note books [6;7], and established following four new results with help of Jacobi's triple product identity or in more general language we can say that with the help of $\psi(.)$ and $\phi(.)$ functions as these are special cases of Jacobi;s triple identity, using elementary method. These results are not recorded in the literature of special functions

Ref.

7. S. Ramanujan; *Notebooks (Volume II)*, Tata Institute of Fundamental Research, Bombay, 1957.

$$(q;q^2)_\infty = (q,q^3,q^5;q^6)_\infty \tag{3.1}$$

$$\left[\frac{(-q;q^2)_\infty^8 - (q;q^2)_\infty^8}{q}\right]^{\frac{1}{4}} = \frac{2}{\left[(q^2;q^4)_\infty\right]^2} \tag{3.2}$$

$$\frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty} = (q,-q;q^2)_\infty \tag{3.3}$$

$$(q^2;q^2)_\infty = (q^2;q^4)_\infty(q^4;q^4)_\infty \tag{3.4}$$

For the first time, we are establishing (3.4) with the help of $\psi(.)$ and $\phi(.)$ functions. Earlier this result is derived by using q-product methods as given in (1.14).

Proof of (3.1): By substituting $a = -q$ and $b = -q^5$ in (1.5), we have

$$f(-q,-q^5) = (q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty$$

also, by substituting $q = q^3$ in (1.7), we get

$$\psi(q^3) = \frac{(q^6;q^6)_\infty}{(q^3;q^6)_\infty}$$

again, by substituting $q = -q$ in (1.6), we get

$$\phi(-q) = (q;q^2)_\infty^2(q^2;q^2)_\infty$$

now on substituting the values of $f(-q,-q^5)$, $\psi(q^3)$, $\phi(-q)$, and employing (1.7) in (2.8), after simplification, we get

$$(q;q^2)_\infty = (q;q^6)_\infty(q^5;q^6)_\infty(q^3;q^6)_\infty$$

or,

$$(q;q^2)_\infty = (q,q^3,q^5;q^6)_\infty$$

which established (3.1)

Proof of (3.2): By substituting $q = -q$ in (1.6), we get

$$\phi(-q) = (q;q^2)_\infty^2(q^2;q^2)_\infty$$

also, by substituting $q = q^2$ in (1.7), we get

$$\psi(q^2) = \frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty}$$

now on substituting the values of $\phi(-q)$, $\psi(q^2)$, and employing (1.6) in (2.9), we get

$$(q^2;q^2)_\infty^4[(-q;q^2)_\infty^8 - (q;q^2)_\infty^8] = 16q\frac{(q^4;q^4)_\infty^4}{(q^2;q^4)_\infty^4}$$

after little algebra, we get

$$\left[\frac{(-q;q^2)_\infty^8-(q;q^2)_\infty^8}{q}\right]^{\frac{1}{4}}=\frac{2}{\left[(q^2;q^4)_\infty\right]^2}$$

which established (3.2)

Proof of (3.3): By substituting $q=-q$ and $q=q^2$ respectively in (1.7), we get

$$\psi\,(-q)=\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}$$

and

$$\psi\,(q^2)=\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty}$$

again, by substituting $q=-q^2$ in (1.6), we get

$$\phi(-q^2)=(q^2;q^4)_\infty^2(q^4;q^4)_\infty$$

now on substituting the values of $\psi(-q)$, $\psi(q^2)$, $\phi(-q^2)$ and employing (1.7) in (2.10), then after simplification, we get

$$\frac{(q^2;q^2)_\infty^2}{(q^4;q^4)_\infty^2}=(q^2;q^4)_\infty(q;q^2)_\infty(-q;q^2)_\infty$$

now employing (1.14), and after simplification, we get

$$\frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty}=(q,-q;q^2)_\infty$$

which established (3.3)

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Notes

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 10 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

A Summation Formula Tangled with Hypergeometric Function and Recurrence Relation

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GJSFR-F Classification : MSC 2000: 33C05, 33C20, 33C45, 33C60, 33C70

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Abstract - The main object of the present paper is to establish a summation formula tangled with Hypergeometric function and recurrence relation.

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I. INTRODUCTION

a) *Generalized Hypergeometric Functions*

A generalized hypergeometric function ${}_pF_q(a_1, ...a_p; b_1, ...b_q; z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(K+b_2)...(k+b_q)(k+1)}\ z. \tag{1}$$

Where $k+1$ in the denominator is present for historical reasons of notation, and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{array}{ccc} a_1, a_2, \cdots, a_p & ; & \\ & & z \\ b_1, b_2, \cdots, b_q & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k z^k}{(b_1)_k (b_2)_k \cdots (b_q)_k k!} \tag{2}$$

or

$${}_pF_q\left[\begin{array}{ccc} (a_p) & ; & \\ & & z \\ (b_q) & ; & \end{array}\right] \equiv {}_pF_q\left[\begin{array}{ccc} (a_j)_{j=1}^{p} & ; & \\ & & z \\ (b_j)_{j=1}^{q} & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_p))_k z^k}{((b_q))_k k!} \tag{3}$$

where the parameters $b_1, b_2, \cdots, b_q$ are neither zero nor negative integers and p, q are non-negative integers.

The ${}_pF_q$ series converges for all finite z if $p \leq q$, converges for $\mid z \mid < 1$ if $p \neq q+1$, diverges for all z , $z \neq 0$ if $p > q+1$.

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The ${}_pF_q$ series absolutely converges for $\mid z \mid= 1$ if $R(\zeta) < 0$, conditionally converges for $\mid z \mid= 1, z \neq 0$ if $0 \leq R(\zeta) < 1$, diverges for $\mid z \mid= 1$, if $1 \leq R(\zeta)$, $\zeta = \sum\limits_{i=1}^{p} a_i - \sum\limits_{i=0}^{q} b_i$.

The function ${}_2F_1(a, b; c; z)$ corresponding to $p = 2, q = 1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function (Gauss 1812, Barnes 1908). To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

The hypergeometric functions are solutions of Gaussian hypergeometric linear differential equation of second order

$$z(1-z)y'' + [c-(a+b+1)z]y' - aby = 0 \tag{4}$$

The solution of this equation is

$$y = A_0\Big[1 + \frac{ab}{1!\ c}z + \frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2 + \cdots\cdots\Big] \tag{5}$$

This is the so-called regular solution,denoted

$${}_2F_1(a,b;c;z) = \Big[1 + \frac{ab}{1!\ c}z + \frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2 + \cdots\cdots\Big] = \sum_{k=0}^{\infty} \frac{(a)_k\ (b)_k z^k}{(c)_k k!} \tag{6}$$

which converges if c is not a negative integer for all of $\mid z \mid< 1$ and on the unit circle $\mid z \mid= 1$ if $R(c-a-b) > 0$.

It is known as Gauss hypergeometric function in terms of Pochhammer symbol $(a)_k$ or generalized factorial function.

Many of the common mathematical functions can be expressed in terms of the hypergeometric function, or as limiting cases of it. Some typical examples are

$$(1-z)^{-a} = z\ {}_2F_1(1,1;2;-z) \tag{7}$$

$$\sin^{-1} z = z\ {}_2F_1(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2) \tag{8}$$

The special case of (1.3.4) when $a = c$ and $b = 1$, or $a = 1$ and $b = c$, yields the elementary geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots \tag{9}$$

Hence the term "Hypergeometric" is given. The term hypergeometric was first used by Wallis in his work "Arithmetrica Infinitorum". Hypergeometric series or more precisely Gauss series is due to Carl Friedrich Gauss(1777-1855) who in year 1812 introduced and studied this series in his thesis presented at Gottingen and gave the F-notation for it.

Here z is a real or complex variable. If c is zero or negative integer, the series (6) does not exist and hence the function ${}_2F_1(a, b;\ c;\ z)$ is not defined unless one of the parameters

a or b is also a negative integer such that $-c < -a$. If either of the parameters a or b is a negative integer, say $-m$ then in this case (6) reduce to the hypergeometric polynomial defined as

$$ {}_2F_1(-m, b;\, c;\, z) = \sum_{n=0}^{m} \frac{(-m)_n (b)_n\, z^n}{(c)_n\, n!} \tag{10}$$

b) Generalized Ordinary Hypergeometric Function of One Variable

The generalized Gaussian hypergeometric function of one variable is defined as follows

$$ {}_AF_B\left[\begin{array}{lcc} a_1,\, a_2,\, a_3, \ldots, a_A & ; & \\ & & z \\ b_1,\, b_2,\, b_3, \ldots, b_B & ; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\,(a_2)_n\,(a_3)_n \cdots (a_A)_n\, z^n}{(b_1)_n\,(b_2)_n\,(b_3)_n \cdots (b_B)_n\, n!} \tag{11}$$

or,

$$ {}_AF_B\left[\begin{array}{lcc} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{[(a_A)]_n\, z^n}{[(b_B)]_n\, n!} \tag{12}$$

where for the sake of convenience (in the contracted notation), (a_A) denotes thearray of "A" number of parameters given by $a_1,\, a_2,\, a_3,\, \ldots,\, a_A$. The denominatorparameters are neither zero nor negative integers. The numerator parameters may be zero and negative integers. A and B are positive integers or zero. Empty sum is to be interpreted as zero and empty product as unity.

$$\sum_{n=a}^{b} \text{ and } \prod_{n=a}^{b} \text{ are empty if } b < a.$$

$$[(a_A)]_{-n} = \frac{(-1)^{nA}}{[1-(a_A)]_n} \tag{13}$$

$$[(a_A)]_n = (a_1)_n (a_2)_n (a_3)_n \cdots (a_A)_n = \prod_{m=1}^{A} (a_m)_n = \prod_{m=1}^{A} \frac{\Gamma(a_m+n)}{\Gamma(a_m)} \tag{14}$$

where $a_1,\, a_2,\, a_3,\, \ldots,\, a_A;\; b_1,\, b_2,\, b_3,\, \ldots,\, b_B$ and z may be real and complex numbers.

$$ {}_3F_2\left[\begin{array}{lcc} a,\, b,\, 1 & ; & \\ & & z \\ c,\, 2 & ; & \end{array}\right] = \frac{(c-1)}{(a-1)(b-1)\, z} \times $$

$$ \times \left\{ {}_2F_1\left[\begin{array}{lcc} a-1,\, b-1 & ; & \\ & & z \\ c-1 & ; & \end{array}\right] - 1 \right\} \tag{15}$$

The convergence conditions of ${}_AF_B$ are given below

Suppose that numerator parameters are neither zero nor negative integers (otherwise question of convergence will not arise).

(i) If $A \leq B$, then series ${}_AF_B$ is always convergent for all finite values of z(real or complex) i.e., $|z| < \infty$.

(ii) If $A = B + 1$ and $|z| < 1$, then series ${}_AF_B$ is convergent.

(iii) If $A = B + 1$ and $|z| > 1$, then series ${}_AF_B$ is divergent.

(iv) If $A = B + 1$ and $|z| = 1$, then series ${}_AF_B$ is absolutely convergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} > 0$$

(v) If $A = B + 1$ and $z = 1$, then series ${}_AF_B$ is convergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} > 0$$

(vi) If $A = B + 1$ and $z = 1$, then series ${}_AF_B$ is divergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} \leq 0$$

(vii) If $A = B + 1$ and $z = -1$, then series ${}_AF_B$ is convergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} > -1$$

(viii) If $A = B + 1$ and $|z| = 1$, but $z \neq 1$, then series ${}_AF_B$ is conditionally convergent, when

$$-1 < \mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} \leq 0$$

(ix) If $A > B + 1$, then series ${}_AF_B$ is convergent, when $z = 0$.

(x) If $A = B + 1$ and $|z| \geq 1$, then it is defined as an analytic continuation of this series.

(xi) If $A = B + 1$ and $|z| = 1$, then series ${}_AF_B$ is divergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} \leq -1$$

(xii) If $A > B+1$, then a meaningful independent attempts were made to define MacRobert's E-function, Meijer's G-function, Fox's H-function and its related functions.

(xiii) If one or more of the numerator parameters are zero or negative integers, then series ${}_AF_B$ terminates for all finite values of z i.e., ${}_AF_B$ will be a hypergeometric polynomial and the question of convergence does not enter the discussion.

Contiguous Relation is defined by
[E. D. p.51(10)]

$$(a-b)\ {}_2F_1\left[\begin{matrix} a,\ b\ ; & z \\ c\ \ ; & \end{matrix}\right] = a\ {}_2F_1\left[\begin{matrix} a+1,\ b\ ; & z \\ c\ \ ; & \end{matrix}\right] - b\ {}_2F_1\left[\begin{matrix} a,\ b+1\ ; & z \\ c\ ; & \end{matrix}\right] \tag{16}$$

Gauss second summation theorem is defined by [Prud., 491(7.3.7.8)]

$${}_2F_1\left[\begin{matrix} a,\ b\ ; & \frac{1}{2} \\ \frac{a+b+1}{2}\ ; & \end{matrix}\right] = \frac{\Gamma(\frac{a+b+1}{2})\ \Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\ \Gamma(\frac{b+1}{2})} \tag{17}$$

$$= \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{a+b+1}{2})}{\Gamma(b)\ \Gamma(\frac{a+1}{2})} \tag{18}$$

In a monograph of Prudnikov et al., a summation theorem is given in the form [Prud., p.491(7.3.7.8)]

$${}_2F_1\left[\begin{matrix} a,\ b\ ; & \frac{1}{2} \\ \frac{a+b-1}{2}\ ; & \end{matrix}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\ \Gamma(\frac{b+1}{2})} + \frac{2\ \Gamma(\frac{a+b-1}{2})}{\Gamma(a)\ \Gamma(b)}\right] \tag{19}$$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

$${}_2F_1\left[\begin{matrix} a,\ b\ ; & \frac{1}{2} \\ \frac{a+b-1}{2}\ ; & \end{matrix}\right] = \frac{2^{(b-1)}\ \Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right] \tag{20}$$

Recurrence relation is defined by

$$\Gamma(z+1) = z\ \Gamma(z) \tag{21}$$

II. MAIN SUMMATION FORMULA

$$
{}_2F_1\left[\begin{array}{ll} a, \quad b\ ; & \frac{1}{2} \\ \frac{a+b+41}{2}\ ; & 2 \end{array}\right] = \frac{2^b\ \Gamma(\frac{a+b+41}{2})}{(a-b)\ \Gamma(b)}\times
$$

$$
\times\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{524288a(-82007945326378915593 75+20125013723397976152375a)}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+\right.\right.
$$

$$
+\frac{524288a(-196889934876028678982 25a^2+107927000304718403007 45a^3)}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+
$$

$$
+\frac{524288a(-3824294822931302783964a^4+946995223404049011324a^5)}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+
$$

$$
+\frac{524288a(-171930790626988570804a^6+23615262213846406804a^7)}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+
$$

$$
+\frac{524288a(-2505874787291646498a^8+208251057899323218a^9-13663776163658478a^{10})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+
$$

$$
+\frac{524288a(710084079834558a^{11}-29186718196012a^{12}+942715036492a^{13})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+
$$

$$
+\frac{524288a(-23625216132a^{14}+449681892a^{15}-6278151a^{1}6+60591a^{17}-361a^{18}+a^{19})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+
$$

$$
+\frac{524288a(33222453094521656744625b-26784014367886904649150ab)}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+
$$

$$
+\frac{524288a(6656941611306022627516 5a^2b-1819726185841839737640 0a^3b)}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+
$$

$$
+\frac{524288a(1190764959319051136850 0a^4b-173272020448741914247 2a^5b)}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+
$$

$$
+\frac{524288a(50772486007480891246 8a^6b-44575549851700633584a^7b)}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+
$$

$$
+\frac{524288a(7096841648258109774a^8b-394251249479137908a^9b+37227237877945830a^{10}b)}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+
$$

$$+\frac{524288a(-1316694562355952a^{11}b+76320137288772a^{12}b-1663027017288a^{13}b)}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(58788536196a^{14}b-716920464a^{15}b+14574729a^{16}b-75582a^{1}7b+741a^{18}b)}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(2464947339460964078175b^{2}+8970915492707914633855ab^{2})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(-16481181023683218686376a^{2}b^{2}+4035765117035231492296 8a^{3}b^{2})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(-4673222221480836168652a^{4}b^{2}+3176905101637503805348a^{5}b^{2})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(-2370220872204284306648a^{6}b^{2}+726162365123012302l6a^{7}b^{2})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(-35547137150588254 62a^{8}b^{2}+5888358008710706 10a^{9}b^{2})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(-18785390190548696a^{10}b^{2}+1828505864702504a^{11}b^{2}-36535526629420a^{12}b^{2})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(2155023393796a^{13}b^{2}-24404420040a^{14}b^{2}+861332472a^{15}b^{2}-4194801a^{16}b^{2})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(82251a^{17}b^{2}+2871822593783591482729 5b^{3}+33188945046817866714 72ab^{3})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(50291362269874511578728a^{2}b^{3}-34381521895875722337 12a^{3}b^{3})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(82085057685063976232 04a^{4}b^{3}-4794711985863170935 20a^{5}b^{3})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(3221777528433933421 68a^{6}b^{3}-1351791004842640190 4a^{7}b^{3})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(4118567530121081466a^{8}b^{3}-1177654981112095 20a^{9}b^{3})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(19370822163507672a^{10}b^{3}-356401367234640a^{11}b^{3}+34220151840420a^{12}b^{3})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(-363586707120a^{13}b^{3}+20899430760a^{14}b^{3}-96946512a^{15}b^{3}+3262623a^{16}b^{3})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(3743527666786355832228b^{4}+2373503933616846650583 6ab^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(10655980754578014827 40a^{2}b^{4}+9808909042980361520700a^{3}b^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(-3075403918796427345 40a^{4}b^{4}+711859291630188684892a^{5}b^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(-22863326090812082876a^{6}b^{4}+1483422896201781220 4a^{7}b^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(-362192635506367380a^{8}b^{4}+106961337355063620a^{9}b^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(-1770421086614180a^{10}b^{4}+281900758731956a^{11}b^{4}-2778732507460a^{12}b^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(256594423540a^{13}b^{4}-1127935380a^{14}b^{4}+61523748a^{15}b^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(3405266444028472415652b^{5}+1592826112836059973560ab^{5})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(5499330172367303710204a^{2}b^{5}+1277479220245873721 44a^{3}b^{5})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(828745355724256566596a^{4}b^{5}-132443262946906831 92a^{5}b^{5})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(29612710418417746620a^{6}b^{5}-5417114499085798 08a^{7}b^{5})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(3350369033944441 08a^{8}b^{5}-47017278502674 48a^{9}b^{5}+1325553122001108a^{10}b^{5})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(-11743720135056a^{11}b^{5}+1781300804556a^{12}b^{5}-7282174536a^{13}b^{5})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(635745396a^{14}b^{5}+30734042331963345 7676b^{6}+134038418895711247 1692ab^{6})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(211246684907825219016a^{2}b^{6}+51259332349154452068 0a^{3}b^{6})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(68552945474987453 48a^{4}b^{6}+33871331518519795300a^{5}b^{6})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(-2893822962180069 92a^{6}b^{6}+6232728532645128 80a^{7}b^{6})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(-6430101458336556a^{8}b^{6}+376072220682958 8a^{9}b^{6}-28033056115064a^{10}b^{6})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(7469623102760a^{11}b^{6}-27375582052a^{12}b^{6}+3910797436a^{13}b^{6})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(10826932741535343 5916b^{7}+70179445128686011664ab^{7})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(162371581902467831608a^{2}b^{7}+118731849484543952 80a^{3}b^{7})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(22318167470495565812a^{4}b^{7}+1785099899444347 20a^{5}b^{7})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{524288a(703969753138114320a^{6}b^{7}-3211397396599392a^{7}b^{7}+664743514741534 8a^{8}b^{7})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

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$$+\frac{524288a(-35833247976240a^{9}b^{7}+19709827528248a^{10}b^{7}-60338017584a^{11}b^{7})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(15084504396a^{1}2b^{7}+6586460453221363806b^{8}+23694863813913400290ab^{8})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(5175623316897888426a^{2}b^{8}+8294628633401516406a^{3}b^{8})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(3164114220615948860a^{4}b^{8}+4845864709414888916a^{5}b^{8}+2291343736653972a^{6}b^{8})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(743848555186649900a^{7}b^{8}-16894761676650a^{8}b^{8}+33539087889450a^{9}b^{8})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(-73204212510a^{10}b^{8}+37711260990a^{11}b^{8}+1190397299268527454b^{9})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(882649351319057036ab^{9}+1644205273478553214a^{2}b^{9}+162921387014111440a^{3}b^{9})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(200537843674548380a^{4}b^{9}+4108083073246152a^{5}b^{9}+5284711076616972a^{6}b^{9})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(13647662542800a^{7}b^{9}+37267793684550a^{8}b^{9}-32820602100a^{9}b^{9})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(62359143990a^{10}b^{9}+48653715410164722b^{10}+154157906385590250ab^{10})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(37967523155613480a^{2}b^{10}+48186270011142120a^{3}b^{10}+2366386284722460a^{4}b^{10})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(2360734480596492a^{5}b^{10}+24584628748680a^{6}b^{10}+27105250989960a^{7}b^{10})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(29538541890a^{8}b^{10}+68923264410a^{9}b^{10}+5046299073566322b^{11})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

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$$+\frac{524288a(3908213830318096ab^{11}+6287173301234072a^{2}b^{11}+671998070250416a^{3}b^{11})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(647217596189164a^{4}b^{11}+15334926887280a^{5}b^{11}+12807631555992a^{6}b^{11})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(53239427280a^{7}b^{11}+51021117810a^{8}b^{11}+136044645566804b^{12})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(390969013904092ab^{12}+96486711472788a^{2}b^{12}+104212054616124a^{3}b^{12})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(4990312383420a^{4}b^{12}+3859957069332a^{5}b^{12}+35197176924a^{6}b^{12})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(25140840660a^{7}b^{12}+8455024465236b^{13}+6364613182648ab^{13})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(9131066181020a^{2}b^{13}+886583500880a^{3}b^{13}+718310791660a^{4}b^{13})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(12634884024a^{5}b^{13}+8122425444a^{6}b^{13}+143249607228b^{14}+377940383964ab^{14})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(82385891640a^{2}b^{14}+78284308440a^{3}b^{14}+2600776620a^{4}b^{14}+1676056044a^{5}b^{14})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(5307418428b^{15}+3532333168ab^{15}+4577615432a^{2}b^{15}+300782768a^{3}b^{15})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(211915132a^{4}b^{15}+50652537b^{16}+122581407ab^{16}+18177471a^{2}b^{16})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(15380937a^{3}b^{16}+1047033b^{17}+493506ab^{17}+575757a^{2}b^{17})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288a(4199b^{18}+9139ab^{18}+39b^{19})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{524288b(-8200794532637891559375+33222453094521656744625a)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(24649473394609640781 75a^{2}+28718225937835914827295a^{3})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(3743527666786355832228a^{4}+3405266444028472415652a^{5})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(307340423319633457676a^{6}+108269327415353435916a^{7})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(6586460453221363806a^{8}+1190397299268527454a^{9}+48653715410164722a^{10})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(5046299073566322a^{11}+136044645566804a^{12}+8455024465236a^{13})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(143249607228a^{14}+5307418428a^{15}+50652537a^{16}+1047033a^{17}+4199a^{18})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(39a^{1}9+20125013723397976152375b-26784014367886904649150ab)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(8970915492707914633855 5a^{2}b+3318894504681786671472a^{3}b)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(2373503933616846650583 6a^{4}b+1592826112836059973560a^{5}b)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(1340384188957112471692a^{6}b+701794451286860116 64a^{7}b)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(23694863813913400290a^{8}b+882649351319057036a^{9}b+15415790638559025 0a^{10}b)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(3908213830318096a^{11}b+390969013904092a^{12}b+6364613182648a^{13}b)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

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$$+\frac{524288b(377940383964a^{14}b+3532333168a^{15}b+122581407a^{16}b+493506a^{17}b)}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(9139a^{1}8b-1968899348760286789822 5b^{2}+665694161130602262751 65ab^{2})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(-1648118102368321868637 6a^{2}b^{2}+5029136226987451157872 8a^{3}b^{2})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(106559807545780148274 0a^{4}b^{2}+549933017236730371020 4a^{5}b^{2})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(211246684907825219016a^{6}b^{2}+16237158190246783160 8a^{7}b^{2})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(5175623316897888426a^{8}b^{2}+1644205273478553214a^{9}b^{2})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(37967523155613480a^{10}b^{2}+6287173301234072a^{11}b^{2}+96486711472788a^{12}b^{2})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(9131066181020a^{13}b^{2}+82385891640a^{14}b^{2}+4577615432a^{15}b^{2}+18177471a^{16}b^{2})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(575757a^{17}b^{2}+1079270003047184030074 5b^{3}-1819726185841839737640 0ab^{3})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(4035765117035231492296 8a^{2}b^{3}-3438152189587572233712a^{3}b^{3})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(980890904298036152070 0a^{4}b^{3}+127747922024587372144a^{5}b^{3})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(51259332349154452068 0a^{6}b^{3}+118731849484543952 80a^{7}b^{3})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(8294628633401516406a^{8}b^{3}+1629213870141114 40a^{9}b^{3})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(4818627001114212 0a^{10}b^3+671998070250416a^{11}b^3+104212054616124a^{12}b^3)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(886583500880a^{13}b^3+78284308440a^{14}b^3+300782768a^{15}b^3+15380937a^{16}b^3)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(-3824294822931302783964b^4+1190764959319051136850 0ab^4)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(-4673222221480836168652a^2b^4+8208505768506397623204a^3b^4)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(-3075403918796427345 40a^4b^4+8287453557242565665 96a^5b^4)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(685529454749874534 8a^6b^4+223181674704955658 12a^7b^4)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(31641142206159486 0a^8b^4+20053784367454838 0a^9b^4+2366386284722460a^{10}b^4)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(647217596189164a^{11}b^4+4990312383420a^{12}b^4+718310791660a^{13}b^4)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(2600776620a^{14}b^4+211915132a^{15}b^4+946995223404049011324b^5)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(-1732720204487419142472ab^5+3176905101637503805348a^2b^5)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(-4794711985863170935 20a^3b^5+7118592916301886848 92a^4b^5)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(-1324432629469068319 2a^5b^5+3387133151851979530 0a^6b^5)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(17850998994443472 0a^7b^5+48458647094148891 6a^8b^5+4108083073246152a^9b^5)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

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$$+\frac{524288b(2360734480596492a^{10}b^{5}+15334926887280a^{11}b^{5}+3859957069332a^{12}b^{5})}{\Big[\prod\limits_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod\limits_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{524288b(12634884024a^{13}b^{5}+1676056044a^{14}b^{5}-1719307906269885708804b^{6})}{\Big[\prod\limits_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod\limits_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{524288b(5077248600748089124 68ab^{6}-2370220872204284306 48a^{2}b^{6})}{\Big[\prod\limits_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod\limits_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{524288b(3221777528433933421 68a^{3}b^{6}-2286332609081208287 6a^{4}b^{6})}{\Big[\prod\limits_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod\limits_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{524288b(296127104184177466 20a^{5}b^{6}-289382296218006992a^{6}b^{6})}{\Big[\prod\limits_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod\limits_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{524288b(70396975313811432 0a^{7}b^{6}+2291343736653972a^{8}b^{6}+528471107661697 2a^{9}b^{6})}{\Big[\prod\limits_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod\limits_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{524288b(24584628748680a^{10}b^{6}+12807631555992a^{11}b^{6}+35197176924a^{12}b^{6})}{\Big[\prod\limits_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod\limits_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{524288b(8122425444a^{13}b^{6}+236152622138464068 04b^{7}-4457554985170063358 4ab^{7})}{\Big[\prod\limits_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod\limits_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{524288b(726162365123012302 16a^{2}b^{7}-1351791004842640190 4a^{3}b^{7})}{\Big[\prod\limits_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod\limits_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{524288b(148342289620178122 04a^{4}b^{7}-541711449908579808a^{5}b^{7})}{\Big[\prod\limits_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod\limits_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{524288b(62327285326451288 0a^{6}b^{7}-3211397396599392a^{7}b^{7}+7438485518649900a^{8}b^{7})}{\Big[\prod\limits_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod\limits_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{524288b(13647662542800a^{9}b^{7}+27105250989960a^{10}b^{7}+53239427280a^{11}b^{7})}{\Big[\prod\limits_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod\limits_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{524288b(25140840660a^{12}b^{7}-2505874787291646498b^{8}+7096841648258109774ab^{8})}{\Big[\prod\limits_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod\limits_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{524288b(-3554713715058825462a^{2}b^{8}+4118567530121081466a^{3}b^{8})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(-3621926355063673800a^{4}b^{8}+335036903394444108a^{5}b^{8}-6430101458336556a^{6}b^{8})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(6647435147415348a^{7}b^{8}-168947616766650a^{8}b^{8}+37267793684550a^{9}b^{8})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(29538541890a^{10}b^{8}+51021117810a^{11}b^{8}+208251057899323218b^{9})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(-394251249479137908ab^{9}+588835800871070610a^{2}b^{9})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(-117765498111209520a^{3}b^{9}+106961337355063620a^{4}b^{9}-4701727850267448a^{5}b^{9})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(3760722206829588a^{6}b^{9}-35833247976240a^{7}b^{9}+33539087889450a^{8}b^{9})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(-32820602100a^{9}b^{9}+68923264410a^{10}b^{9}-13663776163658478b^{10})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(37227237877945830ab^{10}-18785390190548696a^{2}b^{10}+19370822163507672a^{3}b^{10})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(-1770421086614180a^{4}b^{10}+1325553122001108a^{5}b^{10}-28033056115064a^{6}b^{10})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(19709827528248a^{7}b^{10}-73204212510a^{8}b^{10}+62359143990a^{9}b^{10})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(7100840798834558b^{11}-1316694562355952ab^{11}+1828505864702504a^{2}b^{11})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{524288b(-356401367234640a^{3}b^{11}+281900758731956a^{4}b^{11}-11743720135056a^{5}b^{11})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

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$$+\frac{524288b(7469623102760a^{6}b^{11}-60338017584a^{7}b^{11}+37711260990a^{8}b^{11})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(-29186718196012b^{12}+76320137288772ab^{12}-36535526629420a^{2}b^{12})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(34220151840420a^{3}b^{12}-2778732507460a^{4}b^{12}+1781300804556a^{5}b^{12})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(-27375582052a^{6}b^{12}+15084504396a^{7}b^{12}+942715036492b^{13})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(-1663027017288ab^{13}+2155023393796a^{2}b^{13}-363586707120a^{3}b^{13})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(256594423540a^{4}b^{13}-7282174536a^{5}b^{13}+3910797436a^{6}b^{13}-23625216132b^{14})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(58788536196ab^{14}-24404420040a^{2}b^{14}+20899430760a^{3}b^{14}-1127935380a^{4}b^{14})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(635745396a^{5}b^{14}+449681892b^{15}-716920464ab^{15}+861332472a^{2}b^{15})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(-96946512a^{3}b^{15}+61523748a^{4}b^{15}-6278151b^{16}+14574729ab^{16})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(-4194801a^{2}b^{16}+3262623a^{3}b^{16}+60591b^{17}-75582ab^{17}+82251a^{2}b^{17})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{524288b(-361b^{18}+741ab^{18}+b^{19})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}\Bigg\}-$$

$$-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})}\Bigg\{\frac{1048576(8200794532637891559375+33222453094521656744625a)}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{1048576(-2464947339460964078175a^{2}+28718225937835914827295a^{3})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{1048576(-37435276667863558322228a^{4}+34052664440284724156520a^{5})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-3073404233196334576760a^{6}+108269327415353435916a^{7})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-6586460453221363806a^{8}+1190397299268527454a^{9})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-48653715410164722a^{10}+5046299073566322a^{11}-136044645566804a^{12})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(8455024465236a^{13}-143249607228a^{14}+5307418428a^{15}-50652537a^{16})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(1047033a^{17}-4199a^{18}+39a^{19}+20125013723397976152375b)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(26784014367886904649150ab+89709154927079146338555a^{2}b)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-33188945046817866714720a^{3}b+23735039336168466505836a^{4}b)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-1592826112836059973560a^{5}b+1340384188957112471692a^{6}b)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-70179445128686011664a^{7}b+23694863813913400290a^{8}b)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-882649351319057036a^{9}b+154157906385590250a^{10}b-3908213830318096a^{11}b)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(390969013904092a^{12}b-6364613182648a^{13}b+377940383964a^{14}b)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-3532333168a^{15}b+122581407a^{16}b-493506a^{17}b+9139a^{18}b)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(1968899348760286789822 5b^2+665694161130602262751 65ab^2)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(1648118102368321868637 6a^2b^2+502913622698745115787 28a^3b^2)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-1065598075457801482740a^4b^2+5499330172367303710204a^5b^2)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-211246684907825219016a^6b^2+162371581902467831608a^7b^2)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-5175623316897888426a^8b^2+1644205273478553214a^9b^2)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-37967523155613480a^{10}b^2+6287173301234072a^{11}b^2-96486711472788a^{12}b^2)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(9131066181020a^{13}b^2-82385891640a^{14}b^2+4577615432a^{15}b^2-18177471a^{16}b^2)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(575757a^{17}b^2+1079270003047184030074 5b^3+1819726185841839737640 0ab^3)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(4035765117035231492296 8a^2b^3+343815218958757223371 2a^3b^3)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(9808909042980361520700a^4b^3-1277479220245873721 44a^5b^3)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(512593323491544520680a^6b^3-118731849484543952 80a^7b^3)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(8294628633401516406a^8b^3-1629213870141114 40a^9b^3)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(48186270011142120a^{10}b^3-671998070250416a^{11}b^3+104212054616124a^{12}b^3)}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-886583500880a^{13}b^{3}+78284308440a^{14}b^{3}-300782768a^{15}b^{3}+15380937a^{16}b^{3})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(3824294822931302783964b^{4}+11907649593190511368500ab^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(4673222221480836168652a^{2}b^{4}+8208505768506397623204a^{3}b^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(3075403918796427345540a^{4}b^{4}+828745355724256566596a^{5}b^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-68552945474987453348a^{6}b^{4}+22318167470495565812a^{7}b^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-3164114220615948 60a^{8}b^{4}+200537843674548380a^{9}b^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-2366386284722460a^{10}b^{4}+647217596189164a^{11}b^{4}-4990312383420a^{12}b^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(718310791660a^{13}b^{4}-2600776620a^{14}b^{4}+211915132a^{15}b^{4})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(946995223404049011324b^{5}+1732720204487419142472ab^{5})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(3176905101637503805348a^{2}b^{5}+479471198586317093520a^{3}b^{5})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(711859291630188684892a^{4}b^{5}+132443262946906831 92a^{5}b^{5})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(33871331518519795300a^{6}b^{5}-17850998994434720a^{7}b^{5})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(484586470941488916a^{8}b^{5}-41080830732461 52a^{9}b^{5}+2360734480596492a^{10}b^{5})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

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$$+\frac{1048576(-15334926887280a^{11}b^{5}+3859957069332a^{12}b^{5}-12634884024a^{13}b^{5})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(1676056044a^{14}b^{5}+1719307906269885708O4b^{6}+5077248600748O8912468ab^{6})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(23702208722042843O648a^{2}b^{6}+322177752843393342168a^{3}b^{6})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(228633260908120828 76a^{4}b^{6}+296127104184177466 20a^{5}b^{6})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(2893822962180O6992a^{6}b^{6}+703969753138114320a^{7}b^{6}-2291343736653972a^{8}b^{6})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(5284711076616972a^{9}b^{6}-24584628748680a^{10}b^{6}+12807631555992a^{11}b^{6})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-35197176924a^{12}b^{6}+8122425444a^{13}b^{6}+236152622138464068 04b^{7})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(445755498517006335 84ab^{7}+726162365123012302 16a^{2}b^{7})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(135179100484264019 04a^{3}b^{7}+148342289620178122 04a^{4}b^{7})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(54171144990857980 8a^{5}b^{7}+623272853264512880a^{6}b^{7}+3211397396599392a^{7}b^{7})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(7438485518649900a^{8}b^{7}-13647662542800a^{9}b^{7}+27105250989960a^{10}b^{7})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(-53239427280a^{11}b^{7}+25140840660a^{12}b^{7}+2505874787291646498b^{8})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(7096841648258109774ab^{8}+3554713715058825462a^{2}b^{8})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(335036903394444108a^{5}b^{8}+643010145833656a^{6}b^{8}+664743514741534 8a^{7}b^{8})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(16894761676650a^{8}b^{8}+37267793684550a^{9}b^{8}-29538541890a^{10}b^{8})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(51021117810a^{11}b^{8}+208251057899323218b^{9}+394251249479137908ab^{9})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(588835800871070610a^{2}b^{9}+117765498111209520a^{3}b^{9}+106961337355063620a^{4}b^{9})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(4701727850267448a^{5}b^{9}+3760722206829588a^{6}b^{9}+35833247976240a^{7}b^{9})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(33539087889450a^{8}b^{9}+32820602100a^{9}b^{9}+68923264410a^{10}b^{9})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(13663776163658478b^{10}+37227237877945830ab^{10}+18785390190548696a^{2}b^{10})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(19370822163507672a^{3}b^{10}+1770421086614180a^{4}b^{10}+1325553122001108a^{5}b^{10})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(28033056115064a^{6}b^{10}+19709827528248a^{7}b^{10}+73204212510a^{8}b^{10})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(62359143990a^{9}b^{10}+7100840798345558b^{11}+1316694562355952ab^{11})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(1828505864702504a^{2}b^{11}+356401367234640a^{3}b^{11}+281900758731956a^{4}b^{11})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(11743720135056a^{5}b^{11}+7469623102760a^{6}b^{11}+60338017584a^{7}b^{11})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

$$+\frac{1048576(37711260990a^{8}b^{11}+29186718196012b^{12}+76320137288772ab^{12})}{\left[\prod_{\Theta=1}^{19}\left\{a-b-(2\Theta-1)\right\}\right]\left[\prod_{\Upsilon=1}^{20}\left\{a-b+(2\Upsilon-1)\right\}\right]}+$$

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$$+\frac{1048576(36535526629420a^2b^{12}+34220151840420a^3b^{12}+2778732507460a^4b^{12})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{1048576(1781300804556a^5b^{12}+27375582052a^6b^{12}+15084504396a^7b^{12})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{1048576(942715036492b^{13}+1663027017288ab^{13}+2155023393796a^2b^{13})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{1048576(363586707120a^3b^{13}+256594423540a^4b^{13}+7282174536a^5b^{13}+3910797436a^6b^{13})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{1048576(23625216132b^{14}+58788536196ab^{14}+24404420040a^2b^{14}+20899430760a^3b^{14})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{1048576(1127935380a^4b^{14}+635745396a^5b^{14}+449681892b^{15}+716920464ab^{15})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{1048576(861332472a^2b^{15}+96946512a^3b^{15}+61523748a^4b^{15}+6278151b^{16})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{1048576(14574729ab^{16}+4194801a^2b^{16}+3262623a^3b^{16}+60591b^{17}+75582ab^{17})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{1048576(82251a^2b^{17}+361b^{18}+741ab^{18}+b^{19})}{\left[\prod_{\Theta=1}^{19}\{a-b-(2\Theta-1)\}\right]\left[\prod_{\Upsilon=1}^{20}\{a-b+(2\Upsilon-1)\}\right]}+$$

$$+\frac{1048576(820079453263789155937\,5+20125013723397976152375a)}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(19688993487602867898225a^2+10792700030471840300745a^3)}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(3824294822931302783964a^4+946995223404049011324a^5)}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(171930790626988570804a^6+23615262213846406804a^7)}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(2505874787291646498a^{8}+208251057899323218a^{9}+13663776163658478a^{10})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(710084079834558a^{11}+29186718196012a^{12}+942715036492a^{13})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(23625216132a^{14}+449681892a^{15}+6278151a^{1}6+60591a^{17}+361a^{18}+a^{19})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(33222453094521656744625b+26784014367886904649150ab)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(66569416113060226275165a^{2}b+18197261858418397376400a^{3}b)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(11907649593190511368500a^{4}b+1732720204487419142472a^{5}b)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(507724860074808912468a^{6}b+44575549851700633584a^{7}b)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(7096841648258109774a^{8}b+394251249479137908a^{9}b+37227237877945830a^{10}b)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(1316694562355952a^{11}b+76320137288772a^{12}b+1663027017288a^{13}b)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(58788536196a^{14}b+716920464a^{15}b+14574729a^{16}b+75582a^{17}b+741a^{18}b)}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(-2464947339460964078175b^{2}+89709154927079146338555ab^{2})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(16481181023683218686376a^{2}b^{2}+40357651170352314922968a^{3}b^{2})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(4673222221480836168652a^{4}b^{2}+3176905101637503805348a^{5}b^{2})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(23702208722042843064 8a^{6}b^{2}+72616236512301230216a^{7}b^{2})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(3554713715058825462a^{8}b^{2}+588835800871070610a^{9}b^{2}+18785390190548696a^{10}b^{2})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(1828505864702504a^{11}b^{2}+36535526629420a^{12}b^{2}+2155023393796a^{13}b^{2})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(24404420040a^{14}b^{2}+861332472a^{15}b^{2}+4194801a^{16}b^{2}+82251a^{17}b^{2})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(287182259378359148272295b^{3}-331889450468178667147 2ab^{3})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(502913622698745115787 28a^{2}b^{3}+343815218958757223371 2a^{3}b^{3})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(820850576850639762320 4a^{4}b^{3}+479471198586317093520a^{5}b^{3})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(322177752843393342168a^{6}b^{3}+13517910048426401904a^{7}b^{3})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(411856753012108146 6a^{8}b^{3}+117765498111209520a^{9}b^{3})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(19370822163507672a^{10}b^{3}+356401367234640a^{11}b^{3}+34220151840420a^{12}b^{3})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(363586707120a^{13}b^{3}+20899430760a^{14}b^{3}+96946512a^{15}b^{3}+3262623a^{16}b^{3})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(-374352766678635583222 8b^{4}+2373503933616846650583 6ab^{4})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(-106559807545780148274 0a^{2}b^{4}+980890904298036152070 0a^{3}b^{4})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

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$$+\frac{1048576(30754039187964273454 0a^{4}b^{4}+711859291630188684892a^{5}b^{4})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(228633260908120828 76a^{6}b^{4}+148342289620178122 04a^{7}b^{4})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(36219263550636738 0a^{8}b^{4}+10696133735506362 0a^{9}b^{4}+1770421086614180a^{10}b^{4})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(281900758731956a^{11}b^{4}+2778732507460a^{12}b^{4}+256594423540a^{13}b^{4})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(1127935380a^{14}b^{4}+61523748a^{15}b^{4}+340526644402847241565 2b^{5})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(-159282611283605997356 0ab^{5}+549933017236730371020 4a^{2}b^{5})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(-127747922024587372144a^{3}b^{5}+828745355724256566596a^{4}b^{5})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(132443262946906831 92a^{5}b^{5}+296127104184177466 20a^{6}b^{5})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(54171144990857980 8a^{7}b^{5}+33503690339444410 8a^{8}b^{5}+4701727850267448a^{9}b^{5})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(1325553122001108a^{10}b^{5}+11743720135056a^{11}b^{5}+1781300804556a^{12}b^{5})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(7282174536a^{13}b^{5}+635745396a^{14}b^{5}-307340423319633457676b^{6})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(134038418895711247169 2ab^{6}-211246684907825219016a^{2}b^{6})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

$$+\frac{1048576(512593323491544520680a^{3}b^{6}-68552945474987453 48a^{4}b^{6})}{\left[\prod_{\Xi=1}^{20}\left\{a-b-2\Xi)\right\}\right]\left[\prod_{\Psi=1}^{19}\left\{a-b+2\Psi)\right\}\right]}+$$

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$$+\frac{1048576(338713315185197953 00a^{5}b^{6}+289382296218006992a^{6}b^{6})}{\Big[\prod_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{1048576(62327285326451288 0a^{7}b^{6}+6430101458336556a^{8}b^{6}+376072220682958 8a^{9}b^{6})}{\Big[\prod_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{1048576(28033056115064a^{10}b^{6}+7469623102760a^{11}b^{6}+27375582052a^{12}b^{6})}{\Big[\prod_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{1048576(3910797436a^{13}b^{6}+10826932741535343591 6b^{7}-70179445128686011664ab^{7})}{\Big[\prod_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{1048576(16237158190246783160 8a^{2}b^{7}-1187318494845439528 0a^{3}b^{7})}{\Big[\prod_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{1048576(223181674704955658 12a^{4}b^{7}-178509989944434720a^{5}b^{7})}{\Big[\prod_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{1048576(70396975313811432 0a^{6}b^{7}+3211397396599392a^{7}b^{7}+6647435147415348a^{8}b^{7})}{\Big[\prod_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{1048576(35833247976240a^{9}b^{7}+19709827528248a^{10}b^{7}+60338017584a^{11}b^{7})}{\Big[\prod_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{1048576(15084504396a^{1}2b^{7}-6586460453221363806b^{8}+23694863813913400290ab^{8})}{\Big[\prod_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{1048576(-5175623316897888426a^{2}b^{8}+8294628633401516406a^{3}b^{8})}{\Big[\prod_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{1048576(-316411422061594860a^{4}b^{8}+484586470941488916a^{5}b^{8}-22913437366539 72a^{6}b^{8})}{\Big[\prod_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{1048576(743848551864990 0a^{7}b^{8}+16894761676650a^{8}b^{8}+33539087889450a^{9}b^{8})}{\Big[\prod_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{1048576(73204212510a^{10}b^{8}+37711260990a^{11}b^{8}+11903972992685274 54b^{9})}{\Big[\prod_{\Xi=1}^{20}\big\{a-b-2\Xi)\big\}\Big]\Big[\prod_{\Psi=1}^{19}\big\{a-b+2\Psi)\big\}\Big]}+$$

$$+\frac{1048576(-882649351319057036ab^{9}+1644205273478553214a^{2}b^{9})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(-1629213870141111440a^{3}b^{9}+2005378436745483 80a^{4}b^{9}-41080830732461 52a^{5}b^{9})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(5284711076616972a^{6}b^{9}-13647662542800a^{7}b^{9}+37267793684550a^{8}b^{9})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(32820602100a^{9}b^{9}+62359143990a^{10}b^{9}-486537154101647 22b^{10})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(1541579063855902 50ab^{10}-379675231556134 80a^{2}b^{10}+481862700111421 20a^{3}b^{10})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(-236638628472246 0a^{4}b^{10}+236073448059649 2a^{5}b^{10}-24584628748680a^{6}b^{10})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(27105250989960a^{7}b^{10}-29538541890a^{8}b^{10}+68923264410a^{9}b^{10})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(5046299073566322b^{11}-3908213830318096ab^{11}+6287173301234072a^{2}b^{11})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(-671998070250416a^{3}b^{11}+647217596189164a^{4}b^{11}-15334926887280a^{5}b^{11})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(12807631555992a^{6}b^{11}-53239427280a^{7}b^{11}+51021117810a^{8}b^{11})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(-136044645566804b^{12}+390969013904092ab^{12}-96486711472788a^{2}b^{12})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(104212054616124a^{3}b^{12}-4990312383420a^{4}b^{12}+3859957069332a^{5}b^{12})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(-35197176924a^{6}b^{12}+25140840660a^{7}b^{12}+8455024465236b^{13})}{\left[\prod_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

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$$+\frac{1048576(-6364613182648ab^{13}+9131066181020a^{2}b^{13}-886583500880a^{3}b^{13})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(718310791660a^{4}b^{13}-12634884024a^{5}b^{13}+8122425444a^{6}b^{13}-143249607228b^{14})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(377940383964ab^{14}-82385891640a^{2}b^{14}+78284308440a^{3}b^{14}-2600776620a^{4}b^{14})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(1676056044a^{5}b^{14}+5307418428b^{15}-3532333168ab^{15}+4577615432a^{2}b^{15})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(-300782768a^{3}b^{15}+211915132a^{4}b^{15}-50652537b^{16}+122581407ab^{16})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$+\frac{1048576(-18177471a^{2}b^{16}+15380937a^{3}b^{16}+1047033b^{17}-493506ab^{17}+575757a^{2}b^{17})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}+$$

$$\left.\left.+\frac{1048576(-4199b^{18}+9139ab^{18}+39b^{19})}{\left[\prod\limits_{\Xi=1}^{20}\{a-b-2\Xi)\}\right]\left[\prod\limits_{\Psi=1}^{19}\{a-b+2\Psi)\}\right]}\right\}\right] \tag{22}$$

III. DERIVATION OF THE MAIN SUMMATION FORMULA

Proceeding on the same way of Ref[8],we get the main result.

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Page Size: 8.27" X 11'"

- Left Margin: 0.65
- Right Margin: 0.65
- Top Margin: 0.75
- Bottom Margin: 0.75
- Font type of all text should be Swis 721 Lt BT.
- Paper Title should be of Font Size 24 with one Column section.
- Author Name in Font Size of 11 with one column as of Title.
- Abstract Font size of 9 Bold, “Abstract” word in Italic Bold.
- Main Text: Font size 10 with justified two columns section
- Two Column with Equal Column with of 3.38 and Gaping of .2
- First Character must be three lines Drop capped.
- Paragraph before Spacing of 1 pt and After of 0 pt.
- Line Spacing of 1 pt
- Large Images must be in One Column
- Numbering of First Main Headings (Heading 1) must be in Roman Letters, Capital Letter, and Font Size of 10.
- Numbering of Second Main Headings (Heading 2) must be in Alphabets, Italic, and Font Size of 10.

You can use your own standard format also.
Author Guidelines:

1. General,

2. Ethical Guidelines,

3. Submission of Manuscripts,

4. Manuscript’s Category,

5. Structure and Format of Manuscript,

6. After Acceptance.

1. GENERAL

Before submitting your research paper, one is advised to go through the details as mentioned in following heads. It will be beneficial, while peer reviewer justify your paper for publication.

Scope

The Global Journals Inc. (US) welcome the submission of original paper, review paper, survey article relevant to the all the streams of Philosophy and knowledge. The Global Journals Inc. (US) is parental platform for Global Journal of Computer Science and Technology, Researches in Engineering, Medical Research, Science Frontier Research, Human Social Science, Management, and Business organization. The choice of specific field can be done otherwise as following in Abstracting and Indexing Page on this Website. As the all Global

Journals Inc. (US) are being abstracted and indexed (in process) by most of the reputed organizations. Topics of only narrow interest will not be accepted unless they have wider potential or consequences.

2. ETHICAL GUIDELINES

Authors should follow the ethical guidelines as mentioned below for publication of research paper and research activities.

Papers are accepted on strict understanding that the material in whole or in part has not been, nor is being, considered for publication elsewhere. If the paper once accepted by Global Journals Inc. (US) and Editorial Board, will become the copyright of the Global Journals Inc. (US).

Authorship: The authors and coauthors should have active contribution to conception design, analysis and interpretation of findings. They should critically review the contents and drafting of the paper. All should approve the final version of the paper before submission

The Global Journals Inc. (US) follows the definition of authorship set up by the Global Academy of Research and Development. According to the Global Academy of R&D authorship, criteria must be based on:

1) Substantial contributions to conception and acquisition of data, analysis and interpretation of the findings.

2) Drafting the paper and revising it critically regarding important academic content.

3) Final approval of the version of the paper to be published.

All authors should have been credited according to their appropriate contribution in research activity and preparing paper. Contributors who do not match the criteria as authors may be mentioned under Acknowledgement.

Acknowledgements: Contributors to the research other than authors credited should be mentioned under acknowledgement. The specifications of the source of funding for the research if appropriate can be included. Suppliers of resources may be mentioned along with address.

Appeal of Decision: The Editorial Board's decision on publication of the paper is final and cannot be appealed elsewhere.

Permissions: It is the author's responsibility to have prior permission if all or parts of earlier published illustrations are used in this paper.

Please mention proper reference and appropriate acknowledgements wherever expected.

If all or parts of previously published illustrations are used, permission must be taken from the copyright holder concerned. It is the author's responsibility to take these in writing.

Approval for reproduction/modification of any information (including figures and tables) published elsewhere must be obtained by the authors/copyright holders before submission of the manuscript. Contributors (Authors) are responsible for any copyright fee involved.

3. SUBMISSION OF MANUSCRIPTS

Manuscripts should be uploaded via this online submission page. The online submission is most efficient method for submission of papers, as it enables rapid distribution of manuscripts and consequently speeds up the review procedure. It also enables authors to know the status of their own manuscripts by emailing us. Complete instructions for submitting a paper is available below.

Manuscript submission is a systematic procedure and little preparation is required beyond having all parts of your manuscript in a given format and a computer with an Internet connection and a Web browser. Full help and instructions are provided on-screen. As an author, you will be prompted for login and manuscript details as Field of Paper and then to upload your manuscript file(s) according to the instructions.

To avoid postal delays, all transaction is preferred by e-mail. A finished manuscript submission is confirmed by e-mail immediately and your paper enters the editorial process with no postal delays. When a conclusion is made about the publication of your paper by our Editorial Board, revisions can be submitted online with the same procedure, with an occasion to view and respond to all comments.

Complete support for both authors and co-author is provided.

4. MANUSCRIPT'S CATEGORY

Based on potential and nature, the manuscript can be categorized under the following heads:

Original research paper: Such papers are reports of high-level significant original research work.

Review papers: These are concise, significant but helpful and decisive topics for young researchers.

Research articles: These are handled with small investigation and applications

Research letters: The letters are small and concise comments on previously published matters.

5.STRUCTURE AND FORMAT OF MANUSCRIPT

The recommended size of original research paper is less than seven thousand words, review papers fewer than seven thousands words also.Preparation of research paper or how to write research paper, are major hurdle, while writing manuscript. The research articles and research letters should be fewer than three thousand words, the structure original research paper; sometime review paper should be as follows:

Papers: These are reports of significant research (typically less than 7000 words equivalent, including tables, figures, references), and comprise:

(a)Title should be relevant and commensurate with the theme of the paper.

(b) A brief Summary, "Abstract" (less than 150 words) containing the major results and conclusions.

(c) Up to ten keywords, that precisely identifies the paper's subject, purpose, and focus.

(d) An Introduction, giving necessary background excluding subheadings; objectives must be clearly declared.

(e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition; sources of information must be given and numerical methods must be specified by reference, unless non-standard.

(f) Results should be presented concisely, by well-designed tables and/or figures; the same data may not be used in both; suitable statistical data should be given. All data must be obtained with attention to numerical detail in the planning stage. As reproduced design has been recognized to be important to experiments for a considerable time, the Editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned un-refereed;

(g) Discussion should cover the implications and consequences, not just recapitulating the results; conclusions should be summarizing.

(h) Brief Acknowledgements.

(i) References in the proper form.

Authors should very cautiously consider the preparation of papers to ensure that they communicate efficiently. Papers are much more likely to be accepted, if they are cautiously designed and laid out, contain few or no errors, are summarizing, and be conventional to the approach and instructions. They will in addition, be published with much less delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and to make suggestions to improve briefness.

It is vital, that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

Format

Language: The language of publication is UK English. Authors, for whom English is a second language, must have their manuscript efficiently edited by an English-speaking person before submission to make sure that, the English is of high excellence. It is preferable, that manuscripts should be professionally edited.

Standard Usage, Abbreviations, and Units: Spelling and hyphenation should be conventional to The Concise Oxford English Dictionary. Statistics and measurements should at all times be given in figures, e.g. 16 min, except for when the number begins a sentence. When the number does not refer to a unit of measurement it should be spelt in full unless, it is 160 or greater.

Abbreviations supposed to be used carefully. The abbreviated name or expression is supposed to be cited in full at first usage, followed by the conventional abbreviation in parentheses.

Metric SI units are supposed to generally be used excluding where they conflict with current practice or are confusing. For illustration, 1.4 l rather than 1.4 × 10-3 m3, or 4 mm somewhat than 4 × 10-3 m. Chemical formula and solutions must identify the form used, e.g. anhydrous or hydrated, and the concentration must be in clearly defined units. Common species names should be followed by underlines at the first mention. For following use the generic name should be constricted to a single letter, if it is clear.

Structure

All manuscripts submitted to Global Journals Inc. (US), ought to include:

Title: The title page must carry an instructive title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) wherever the work was carried out. The full postal address in addition with the e-mail address of related author must be given. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining and indexing.

Abstract, used in Original Papers and Reviews:

Optimizing Abstract for Search Engines

Many researchers searching for information online will use search engines such as Google, Yahoo or similar. By optimizing your paper for search engines, you will amplify the chance of someone finding it. This in turn will make it more likely to be viewed and/or cited in a further work. Global Journals Inc. (US) have compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Key Words

A major linchpin in research work for the writing research paper is the keyword search, which one will employ to find both library and Internet resources.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy and planning a list of possible keywords and phrases to try.

Search engines for most searches, use Boolean searching, which is somewhat different from Internet searches. The Boolean search uses "operators," words (and, or, not, and near) that enable you to expand or narrow your affords. Tips for research paper while preparing research paper are very helpful guideline of research paper.

Choice of key words is first tool of tips to write research paper. Research paper writing is an art.A few tips for deciding as strategically as possible about keyword search:

- One should start brainstorming lists of possible keywords before even begin searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in research paper?" Then consider synonyms for the important words.
- It may take the discovery of only one relevant paper to let steer in the right keyword direction because in most databases, the keywords under which a research paper is abstracted are listed with the paper.
- One should avoid outdated words.

Keywords are the key that opens a door to research work sources. Keyword searching is an art in which researcher's skills are bound to improve with experience and time.

Numerical Methods: Numerical methods used should be clear and, where appropriate, supported by references.

Acknowledgements: Please make these as concise as possible.

References

References follow the Harvard scheme of referencing. References in the text should cite the authors' names followed by the time of their publication, unless there are three or more authors when simply the first author's name is quoted followed by et al. unpublished work has to only be cited where necessary, and only in the text. Copies of references in press in other journals have to be supplied with submitted typescripts. It is necessary that all citations and references be carefully checked before submission, as mistakes or omissions will cause delays.

References to information on the World Wide Web can be given, but only if the information is available without charge to readers on an official site. Wikipedia and Similar websites are not allowed where anyone can change the information. Authors will be asked to make available electronic copies of the cited information for inclusion on the Global Journals Inc. (US) homepage at the judgment of the Editorial Board.

The Editorial Board and Global Journals Inc. (US) recommend that, citation of online-published papers and other material should be done via a DOI (digital object identifier). If an author cites anything, which does not have a DOI, they run the risk of the cited material not being noticeable.

The Editorial Board and Global Journals Inc. (US) recommend the use of a tool such as Reference Manager for reference management and formatting.

Tables, Figures and Figure Legends

Tables: Tables should be few in number, cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g. Table 4, a self-explanatory caption and be on a separate sheet. Vertical lines should not be used.

Figures: Figures are supposed to be submitted as separate files. Always take in a citation in the text for each figure using Arabic numbers, e.g. Fig. 4. Artwork must be submitted online in electronic form by e-mailing them.

Preparation of Electronic Figures for Publication

Even though low quality images are sufficient for review purposes, print publication requires high quality images to prevent the final product being blurred or fuzzy. Submit (or e-mail) EPS (line art) or TIFF (halftone/photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Do not use pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings) in relation to the imitation size. Please give the data for figures in black and white or submit a Color Work Agreement Form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution (at final image size) ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs) : >350 dpi; figures containing both halftone and line images: >650 dpi.

Color Charges: It is the rule of the Global Journals Inc. (US) for authors to pay the full cost for the reproduction of their color artwork. Hence, please note that, if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a color work agreement form before your paper can be published.

Figure Legends: Self-explanatory legends of all figures should be incorporated separately under the heading 'Legends to Figures'. In the full-text online edition of the journal, figure legends may possibly be truncated in abbreviated links to the full screen version. Therefore, the first 100 characters of any legend should notify the reader, about the key aspects of the figure.

6. AFTER ACCEPTANCE

Upon approval of a paper for publication, the manuscript will be forwarded to the dean, who is responsible for the publication of the Global Journals Inc. (US).

6.1 Proof Corrections

The corresponding author will receive an e-mail alert containing a link to a website or will be attached. A working e-mail address must therefore be provided for the related author.

Acrobat Reader will be required in order to read this file. This software can be downloaded

(Free of charge) from the following website:

www.adobe.com/products/acrobat/readstep2.html. This will facilitate the file to be opened, read on screen, and printed out in order for any corrections to be added. Further instructions will be sent with the proof.

Proofs must be returned to the dean at dean@globaljournals.org within three days of receipt.

As changes to proofs are costly, we inquire that you only correct typesetting errors. All illustrations are retained by the publisher. Please note that the authors are responsible for all statements made in their work, including changes made by the copy editor.

6.2 Early View of Global Journals Inc. (US) (Publication Prior to Print)

The Global Journals Inc. (US) are enclosed by our publishing's Early View service. Early View articles are complete full-text articles sent in advance of their publication. Early View articles are absolute and final. They have been completely reviewed, revised and edited for publication, and the authors' final corrections have been incorporated. Because they are in final form, no changes can be made after sending them. The nature of Early View articles means that they do not yet have volume, issue or page numbers, so Early View articles cannot be cited in the conventional way.

6.3 Author Services

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6.4 Author Material Archive Policy

Please note that if not specifically requested, publisher will dispose off hardcopy & electronic information submitted, after the two months of publication. If you require the return of any information submitted, please inform the Editorial Board or dean as soon as possible.

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the search? Will I be able to find all information in this field area? If the answer of these types of questions will be "Yes" then you can choose that topic. In most of the cases, you may have to conduct the surveys and have to visit several places because this field is related to Computer Science and Information Technology. Also, you may have to do a lot of work to find all rise and falls regarding the various data of that subject. Sometimes, detailed information plays a vital role, instead of short information.

2. Evaluators are human: First thing to remember that evaluators are also human being. They are not only meant for rejecting a paper. They are here to evaluate your paper. So, present your Best.

3. Think Like Evaluators: If you are in a confusion or getting demotivated that your paper will be accepted by evaluators or not, then think and try to evaluate your paper like an Evaluator. Try to understand that what an evaluator wants in your research paper and automatically you will have your answer.

4. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

5. Ask your Guides: If you are having any difficulty in your research, then do not hesitate to share your difficulty to your guide (if you have any). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work then ask the supervisor to help you with the alternative. He might also provide you the list of essential readings.

6. Use of computer is recommended: As you are doing research in the field of Computer Science, then this point is quite obvious.

7. Use right software: Always use good quality software packages. If you are not capable to judge good software then you can lose quality of your paper unknowingly. There are various software programs available to help you, which you can get through Internet.

8. Use the Internet for help: An excellent start for your paper can be by using the Google. It is an excellent search engine, where you can have your doubts resolved. You may also read some answers for the frequent question how to write my research paper or find model research paper. From the internet library you can download books. If you have all required books make important reading selecting and analyzing the specified information. Then put together research paper sketch out.

9. Use and get big pictures: Always use encyclopedias, Wikipedia to get pictures so that you can go into the depth.

10. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right! It is a good habit, which helps to not to lose your continuity. You should always use bookmarks while searching on Internet also, which will make your search easier.

11. Revise what you wrote: When you write anything, always read it, summarize it and then finalize it.

12. Make all efforts: Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

13. Have backups: When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

14. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several and unnecessary diagrams will degrade the quality of your paper by creating "hotchpotch." So always, try to make and include those diagrams, which are made by your own to improve readability and understandability of your paper.

15. Use of direct quotes: When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

16. Use proper verb tense: Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

17. Never use online paper: If you are getting any paper on Internet, then never use it as your research paper because it might be possible that evaluator has already seen it or maybe it is outdated version.

18. **Pick a good study spot:** To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

19. Know what you know: Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

20. Use good quality grammar: Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

21. Arrangement of information: Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

23. Multitasking in research is not good: Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

24. Never copy others' work: Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

25. Take proper rest and food: No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

26. Go for seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

27. Refresh your mind after intervals: Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

29. **Think technically:** Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

30. Think and then print: When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

31. Adding unnecessary information: Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

32. Never oversimplify everything: To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

33. Report concluded results: Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

34. After conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

INFORMAL GUIDELINES OF RESEARCH PAPER WRITING

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page

- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.

Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.
- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.

- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

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Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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ISSN 9755896