Online ISSN : 2249 - 4626
Print ISSN : 0975 - 5896

Global Journal

of Science Frontier Research : F

Mathematics and Decision Sciences

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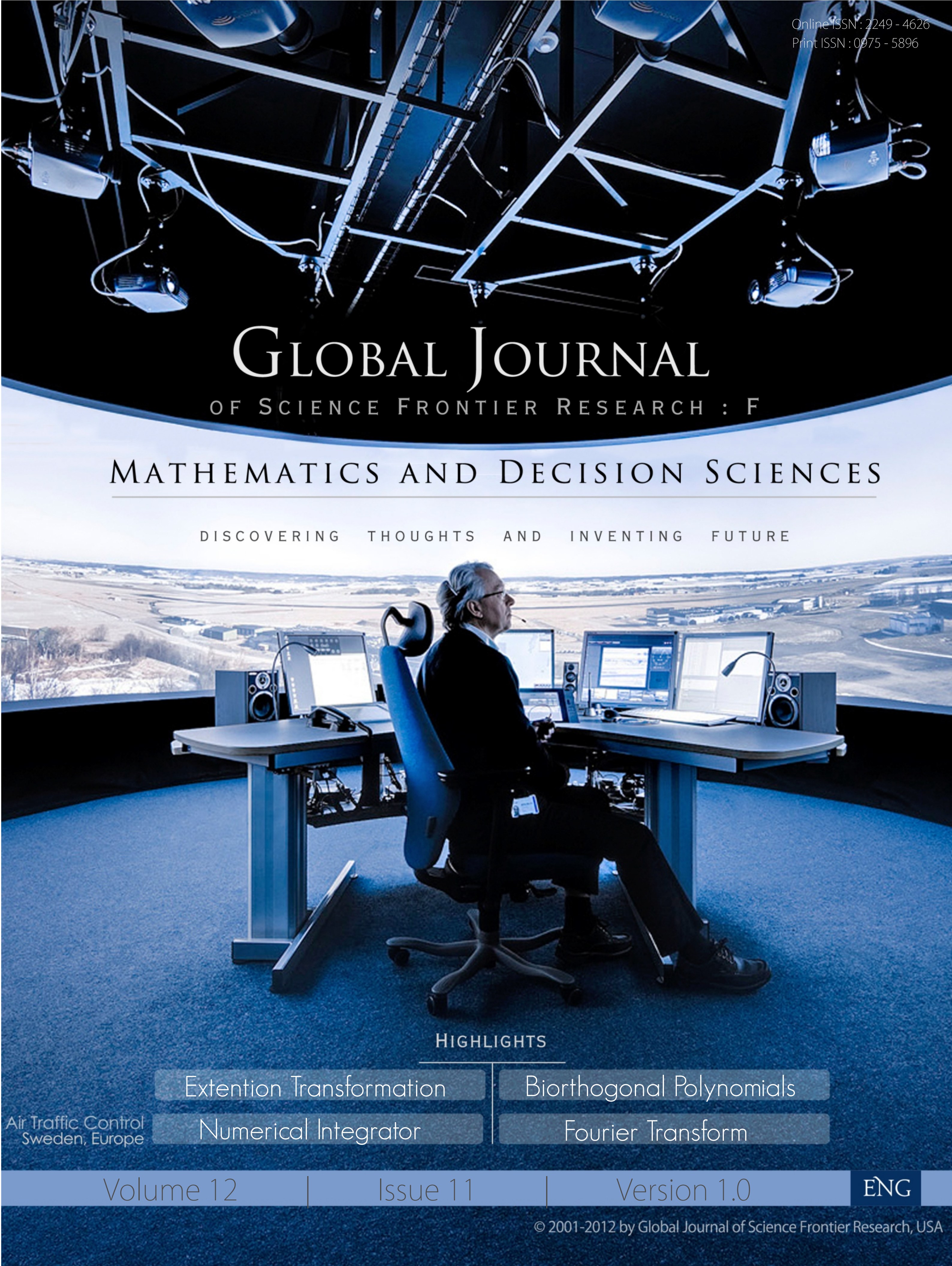

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Highlights

- Extention Transformation
- Biorthogonal Polynomials
- Numerical Integrator
- Fourier Transform

Volume 12 | Issue 11 | Version 1.0

ENG

Global Journal of Science Frontier Research: F
Mathematics & Decision Sciences

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Volume 12 Issue 11 (Ver. 1.0)

Open Association of Research Society

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USA Tax ID: 98-0673427

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Publisher's Headquarters office

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USA Toll Free: +001-888-839-7392
USA Toll Free Fax: +001-888-839-7392

Offset Typesetting

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Contents of the Volume

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 11 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Extention Transformation Used in I Ching

By Florentin Smarandache
University of New Mexico, USA

Abstract - In this paper we show how to using the extension transformation in I Ching in order to transforming a hexagram to another one. Each binary hexagram (and similarly the previous trigram) has a degree of Yang and a degree of Yin. As in neutrosophic logic and set, for each hexagram *<H>* there is corresponding an opposite hexagram *<antiH>*, while in between them all other hexagrams are neutralities denoted by *<neutH>*; a neutrality has a degree of *<H>* and a degree of *<antiH>*. A generalization of the trigram (which has three stacked horizontal lines) and hexagram (which has six stacked horizontal lines) to n-gram (which has n stacked horizontal lines) is provided. Instead of stacked horizontal lines one can consider stacked vertical lines - without changing the composition of the trigram/hexagram/n-gram. Afterwards, circular representations of the hexagrams and of the n-grams are given.

GJSFR-F Classification : MSC 2010: 97G50

EXTENTION TRANSFORMATION USED IN I CHING

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Notes

Extention Transformation Used in I Ching

Florentin Smarandache

Abstract - In this paper we show how to using the *extension transformation* in *I Ching* in order to transforming a hexagram to another one. Each binary hexagram (and similarly the previous trigram) has a degree of *Yang* and a degree of *Yin*. As in neutrosophic logic and set, for each hexagram <***H***> there is corresponding an opposite hexagram <***antiH***>, while in between them all other hexagrams are neutralities denoted by <***neutH***>*;* a neutrality has a degree of <***H***> and a degree of <***antiH***>*.* A generalization of the trigram (which has three stacked horizontal lines) and hexagram (which has six stacked horizontal lines) to *n*-gram (which has *n* stacked horizontal lines) is provided. Instead of stacked horizontal lines one can consider stacked vertical lines - without changing the composition of the trigram/hexagram/n-gram. Afterwards, circular representations of the hexagrams and of the *n*-grams are given.

I. Introduction

"I Ching", which means *The Book of Changes*, is one of the oldest classical Chinese texts. It is formed of *64* hexagrams.

I Ching is part of the Chinese culture, philosophy and divinization. According to *I Ching* everything is in a continuous change.

At the beginning, between *2800-2737 BC*, originating with the culture hero Fu Xi, there have been *8* trigrams, and within the time of the legendary Yu *(2194-2149 BC)* the trigrams were expanded into *64* hexagrams.

Each trigram was formed by three stacked horizontal lines. Then two trigrams formed a hexagram.

Therefore a hexagram is formed by six stacked horizontal lines; and each stacked horizontal line is either unbroken line (——), called **Yang**, or broken line (— —) , called **Yin**.

Yang is associated with MALE, positive, giving, creation, digit *1*, and Yin is associated with FEMALE, negative, receiving, reception, digit *0* in the Taoist philosophy. In Taoism, Yang and Yin complement each other, like in the *taijitu* symbol:

Figure 1

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The number of all possible trigrams formed with unbroken or broken lines is $2^3 = 8$.

And the number of all possible hexagrams also formed with unbroken or broken lines is

$2^6 = 64$.

A hexagram is formed by two trigrams: the first trigram (first three lines) is called *lower trigram* and represents the inner aspect of the change, while the second trigram (last three lines) is called *upper trigram* and represents the outer aspect of the change.

II. Analyzing the Hexagrams

As in neutrosophy (which is a philosophy that studies the nature of entities, their opposites, and the neutralities in between them), we have the following for the *I Ching* hexagrams:

- To each hexagram <*H*> an anti-hexagram <*antiH*> is corresponding, and *62* neutral hexagrams <*neutH*> are in between <*H*> and <*antiH*>.
- Each <*neutH*> has a degree of <*H*> and a degree of <*antiH*>. The degrees are among the numbers *1/6, 2/6, 3/6, 4/6, 5/6* and the sum of the degree of <*H*> and degree of <*antiH*> is *1*.
- Let's note the *62* neutral hexagrams by $\langle neutH_1\rangle$, $\langle neutH_2\rangle$, …, $\langle neutH_{62}\rangle$. For each neutral hexagram $\langle neutH_i\rangle$ there is a neutral hexagram $\langle neutH_j\rangle$, with $i \neq j$, which is the opposite of it.
- For each stacked horizontal line the **extension transformation** is the following:

$$T: \{Yang, Yin\} \rightarrow \{Yang, Yin\}$$

$$T(x) = \bar{x}, \text{ where } \bar{x} \text{ is the opposite of } x,$$

i.e.

T(Yang) = *Yin* or *T*(——) = — —

and

T(Yin) = *Yang* or *T*(— —) = ——

To transform a hexagram into another hexagram one uses this extension transformation once, twice, three times, four times, five, or six times. The maximum number of extension transformations used (six) occurs when we transform a hexagram into its opposite hexagram.

III. Hexagram Table

The below Hexagram Table is taken from Internet (*[1]* and *[2]*); instead of stacked horizontal lines one considers stacked vertical lines - without affecting the results of this article.

In this table one shows the modern interpretation of each hexagram, which is a retranslation of Richard Wilhelm's translation.

Hexagram Table

Hexagram	Modern Interpretation
01. ǀǀǀǀǀǀ Force (乾 qián)	Possessing Creative Power & Skill
02. ¦¦¦¦¦¦ Field (坤 kūn)	Needing Knowledge & Skill; Do not force matters and go with the flow

Ref.

1. Wilhelm (trans.), Richard; Cary Baynes (trans.), "The I Ching or Book of Changes", from Internet.

Notes

03. |¦¦¦|¦ Sprouting (屯 zhūn)	Sprouting
04. ¦|¦¦¦| Enveloping (蒙 méng)	Detained, Enveloped and Inexperienced
05. |||¦|¦ Attending (需 xū)	Uninvolvement (Wait for now), Nourishment
06. ¦|¦||| Arguing (訟 sòng)	Engagement in Conflict
07. ¦|¦¦¦¦ Leading (師 shī)	Bringing Together, Teamwork
08. ¦¦¦¦|¦ Grouping (比 bǐ)	Union
09. |||¦|| Small Accumulating (小畜 xiǎo chù)	Accumulating Resources
10. ||¦||| Treading (履 lǚ)	Continuing with Alertness
11. |||¦¦¦ Pervading (泰 tài)	Pervading
12. ¦¦¦||| Obstruction (否 pǐ)	Stagnation
13. |¦|||| Concording People (同人 tóng rén)	Fellowship, Partnership
14. ||||¦| Great Possessing (大有 dà yǒu)	Independence, Freedom
15. ¦¦|¦¦¦ Humbling (謙 qiān)	Being Reserved, Refraining
16. ¦¦¦|¦¦ Providing-For (豫 yù)	Inducement, New Stimulus
17. |¦¦||¦ Following (隨 suí)	Following
18. ¦||¦¦| Corrupting (蠱 gǔ)	Repairing
19. ||¦¦¦¦ Nearing (臨 lín)	Approaching Goal, Arriving
20. ¦¦¦¦|| Viewing (觀 guān)	The Withholding
21. |¦¦|¦| Gnawing Bite (噬嗑 shì kè)	Deciding
22. |¦|¦¦| Adorning (賁 bì)	Embellishing
23. ¦¦¦¦¦| Stripping (剝 bō)	Stripping, Flaying
24. |¦¦¦¦¦ Returning (復 fù)	Returning
25. |¦¦||| Without Embroiling (無妄 wú wàng)	Without Rashness
26. |||¦¦| Great Accumulating (大畜 dà chù)	Accumulating Wisdom
27. |¦¦¦¦| Swallowing (頤 yí)	Seeking Nourishment
28. ¦||||¦ Great Exceeding (大過 dà guò)	Great Surpassing
29. ¦|¦¦|¦ Gorge (坎 kǎn)	Darkness, Gorge
30. |¦||¦| Radiance (離 lí)	Clinging, Attachment
31. ¦¦|||¦ Conjoining (咸 xián)	Attraction
32. ¦|||¦¦ Persevering (恆 héng)	Perseverance

Hexagram	Modern Interpretation
33. ䷠ Retiring (遯 dùn)	Withdrawing
34. ䷡ Great Invigorating (大壯 dà zhuàng)	Great Boldness
35. ䷢ Prospering (晉 jìn)	Expansion, Promotion
36. ䷣ Brightness Hiding (明夷 míng yí)	Brilliance Injured
37. ䷤ Dwelling People (家人 jiā rén)	Family
38. ䷥ Polarising (睽 kuí)	Division, Divergence
39. ䷦ Limping (蹇 jiǎn)	Halting, Hardship
40. ䷧ Taking-Apart (解 xiè)	Liberation, Solution
41. ䷨ Diminishing (損 sǔn)	Decrease
42. ䷩ Augmenting (益 yì)	Increase
43. ䷪ Parting (夬 guài)	Separation
44. ䷫ Coupling (姤 gòu)	Encountering
45. ䷬ Clustering (萃 cuì)	Association, Companionship
46. ䷭ Ascending (升 shēng)	Growing Upward
47. ䷮ Confining (困 kùn)	Exhaustion
48. ䷯ Welling (井 jǐng)	Replenishing, Renewal
49. ䷰ Skinning (革 gé)	Abolishing the Old
50. ䷱ Holding (鼎 dǐng)	Establishing the New
51. ䷲ Shake (震 zhèn)	Mobilizing
52. ䷳ Bound (艮 gèn)	Immobility
53. ䷴ Infiltrating (漸 jiàn)	Auspicious Outlook, Infiltration
54. ䷵ Converting The Maiden (歸妹 guī mèi)	Marrying
55. ䷶ Abounding (豐 fēng)	Goal Reached, Ambition Achieved
56. ䷷ Sojourning (旅 lǚ)	Travel
57. ䷸ Ground (巽 xùn)	Subtle Influence
58. ䷹ Open (兌 duì)	Overt Influence
59. ䷺ Dispersing (渙 huàn)	Dispersal
60. ䷻ Articulating (節 jié)	Discipline
61. ䷼ Centre Confirming (中孚 zhōng fú)	Staying Focused, Avoid Misrepresentation
62. ䷽ Small Exceeding (小過 xiǎo guò)	Small Surpassing

Notes

63. ¦|¦|¦| Already Fording (既濟 jì jì) Completion
64. |¦|¦|¦ Not-Yet Fording (未濟 wèi jì) Incompletion

IV. Examples of Extension Transformations Used for Hexagrams

As an example of studying the above Hexagram Table, let's take the first hexagram and denote it by

$$<H> = ||||||$$

Then its opposite diagram happened to be its second hexagram:

$$<antiH> = ¦¦¦¦¦¦$$

Their modern interpretation is consistent with them, since <*H*> means "Possessing Creative Power & Skill", while <*antiH*> means the opposite, i.e. "Needing Knowledge & Skill" (because <*antiH*> doesn't have knowledge and skills).

Hexagram <*H*> is known as "Force", while <*antiH*> as "Field", or the Force works the Field.

As in Extenics founded and developed by Cai Wen [3, 4], to transform <*H*> into <*antiH*> one uses the extension transformation *T(Yang)=Yin* six times (for each stacked vertical line). The other *62* hexagrams have a percentage of <*H*> and a percentage of <*antiH*>.

There are:

$C_6^0 = 1$ hexagram that has $6/6 = 100\%$ percentage of <*H*> and $0/6 = 0\%$ percentage of <*antiH*>;

$C_6^1 = 6$ hexagrams that have *5/6* percentage of <*H*> and *1/6* percentage of <*antiH*>;

$C_6^2 = 15$ hexagrams that have *4/6* percentage of <*H*> and *2/6* percentage of <*antiH*>;

$C_6^3 = 20$ hexagrams that have *3/6* percentage of <*H*> and *3/6* percentage of <*antiH*>;

$C_6^4 = 15$ hexagrams that have *2/6* percentage of <*H*> and *4/6* percentage of <*antiH*>;

$C_6^5 = 6$ hexagrams that have *1/6* percentage of <*H*> and *5/6* percentage of <*antiH*>;

$C_6^6 = 1$ hexagram that has $0/6 = 0\%$ percentage of <*H*> and $6/6 = 100\%$ percentage of <*antiH*>.

The total number of hexagrams is:

$$\sum_{k=0}^{6} C_6^k = (1+1)^6 = 1+6+15+20+15+6+1 = 64 .$$

For the following neutral hexagram ("Gorge")

$$<neutH_{29}> = ¦|¦¦|¦$$

Ref.

3. Cai Wen. Extension Set and Non-Compatible Problems [J]. Journal of Scientific Exploration, 1983, (1): 83-97.

its opposite is another neutral hexagram ("Radiance")

$$<neutH_{30}> = ䷝.$$

$<neutH_{29}>$ can be obtained from the hexagram $<H>$ by using four times the extension transformation $T(Yang) = Yin$ for the first, third, fourth, and sixth stacked vertical lines.

Hexagram $<neutH_{29}>$ is $2/6 = 33\%$ $<H>$ and $4/6 = 67\%$ $<antiH>$.

$<neutH_{30}>$ can be obtained from the hexagram $<H>$ by using two times the extension transformation $T(Yang) = Yin$ for the second, and fifth stacked vertical lines.

Hexagram $<neutH_{30}>$ is $4/6 = 67\%$ $<H>$ and $2/6 = 33\%$ $<antiH>$.

V. Circular Representation of the Hexagrams

Shao Yung in the 11^{th} century has displayed the hexagrams in the formats of a circle and of a rectangle.

We represent the hexagrams in the format of a circle, but such that each hexagram $<H_i>$ is diametrically opposed to its opposite hexagram $<antiH_i>$. We may start with any hexagram $<H_0>$ as the main one:

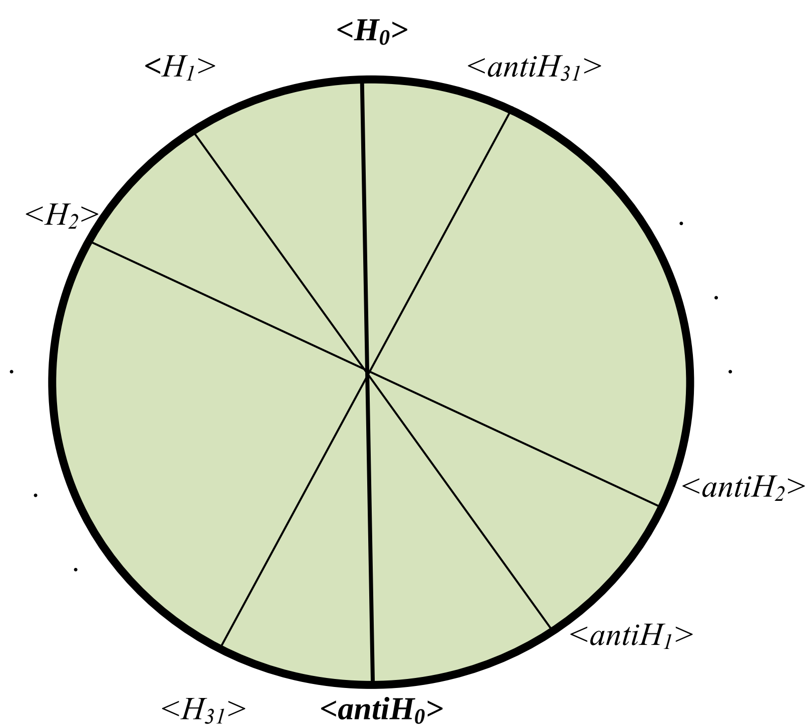

Figure 2

Notes

VI. Generalization of Hexa-Grams to N-Grams

Notes

The *3-gram* (or *trigram*) and the *6-gram* (or *hexagram*) can be generalized to an *n-gram*, where *n* is an integer greater than *1*.
We define the ***n*-gram** as formed by *n* stacked horizontal lines; and each stacked horizontal line is either unbroken line (——), called **Yang**, or broken line (— —), called **Yin**.
Therefore we talk about <u>binary *n*-grams</u>.

The number of all possible binary *n*-grams is equal to 2^n.

Similarly to hexagrams we have:

- To each *n-gram* <*G*> an anti-n-gram <*antiG*> is corresponding, and $2^n - 2$ neutral n-grams <*neutG*> are in between <*G*> and <*antiG*>.
- Each <*neutG*> has a degree of <*G*> and a degree of <*antiG*>. The degrees are among the numbers *1/n, 2/n, …, (n-1)/n* and the sum of the degree of <*G*> and degree of <*antiG*> is *1*.
- Let's note the $2^n - 2$ neutral *n*-grams by $<neutG_1>, <neutG_2>, \dots, <neutG_{2^n-1}>$. For each neutral *n*-gram $<neutG_i>$ there is a neutral *n*-gram $<neutG_j>$, with $i \neq j$, which is the opposite of it.
- For each stacked horizontal line the **extension transformation** is the same:

$$T: \{Yang, Yin\} \rightarrow \{Yang, Yin\}$$

$$T(x) = \bar{x}, \text{ where } \bar{x} \text{ is the opposite of } x,$$

i.e.

T(Yang) = Yin or *T*(——) = — —

and

T(Yin) = Yang or *T*(— —) = ——

To transform an *n*-gram into another n-gram one uses this extension transformation once, twice, three times, and so forth up to $2^n - 2$ times. The maximum number of extension transformations used ($2^n - 2$) occurs when we transform an *n*-gram into its opposite *n*-gram.

To transform an *n*-gram <*G*> into its opposite <*antiG*> one uses the extension transformation *T(Yang)=Yin* 2^n times (for each stacked vertical line). The other $2^n - 2$ *n*-grams have a percentage of <*G*> and a percentage of <*antiG*>.

There are:

$C_n^0 = 1$ *n*-gram that have *n/n = 100%* percentage of <*G*> and *0/n = 0%* percentage of <*antiG*>;

$C_n^1 = n$ *n*-grams that have *(n-1)/n* percentage of <*G*> and *1/n* percentage of <*antiG*>;

$C_n^2 = n(n-1)/2$ *n*-grams that have *(n-2)/n* percentage of <*G*> and *2/n* percentage of <*antiG*>;

.
.
.

$C_n^k = \frac{n!}{k!(n-k)!}$ *n*-grams that have *(n-k)/n* percentage of <*G*> and *k/n* percentage of <*antiG*>;

.

.

.

$C_n^n = 1$ *n*-gram that has *0/n = 0%* percentage of <*G*> and *n/n = 100%* percentage of <*antiG*>.

The total number of *n*-grams is:

$$\sum_{k=0}^{n} C_n^k = (1+1)^n = 1 + n + n(n-1)/2 + ... = 2^n.$$

VII. Circular Representation of the N-Grams

We represent the n-grams in the format of a circle, but such that each n-gram $<G_i>$ is diametrically opposed to its opposite n-gram $<antiG_i>$. We may start with any n-gram $<G_0>$ as the main one:

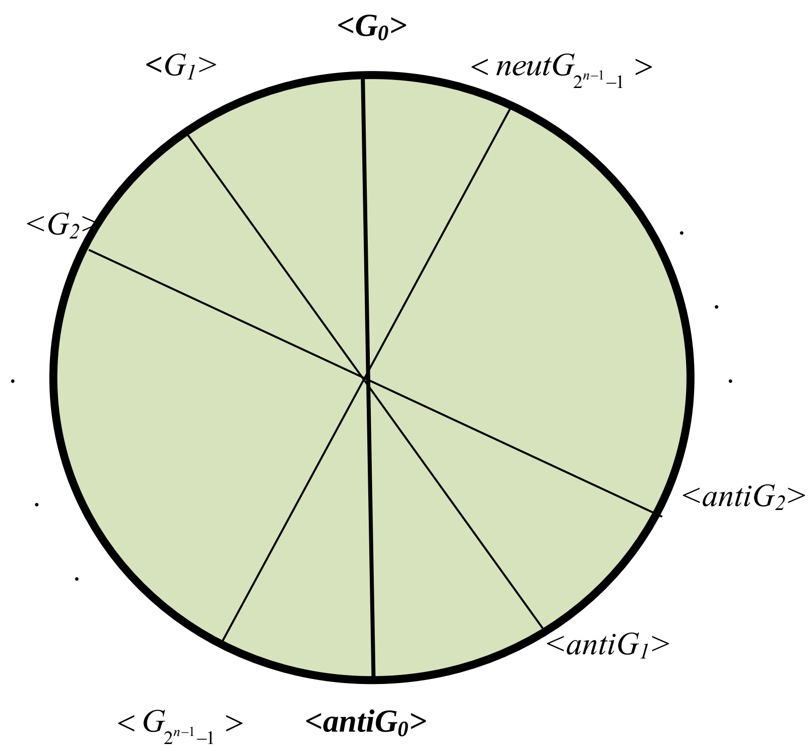

Figure 3

Notes

VIII. Conclusion

In this article the connection between *I Ching* (The Book of Change), Extenics, and neutrosophics has been made. Then a generalization from ancient trigrams and hexagrams to *n*-grams, $n \geq 1$, was presented at the end, together with the geometric interpretations of hexagrams and *n*-grams. An extension transformation is used to change from a hexagram to another one, and in general from an *n*-gram to another *n*-gram.

Notes

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 11 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Some Integrals Pertaining Biorthogonal Polynomials and Certain Product of Special Functions

By Poonia, M.S
NIMS University, India

Abstract - The main object of this paper is to obtain some integrals and series relation pertaining to biorthogonal polynomials, Fox's H-function, the general class of polynomials and the H-function of several complex variables.

Keywords : Fox's H-function, the general class of polynomials and the H-function of complex variables.

GJSFR-F Classification : MSC 2010: 14K20

SOME INTEGRALS PERTAINING BIORTHOGONAL POLYNOMIALS AND CERTAIN PRODUCT OF SPECIAL FUNCTIONS

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Ref.

7. **Prabhakar, T.R. and Tomar, R.C.**, Some integrals and series relations for biorthogonal polynomials suggested by the Legendre polynomials, Indian J. pure appl. Math., 11(7), (1980), 863-869.

Some Integrals Pertaining Biorthogonal Polynomials and Certain Product of Special Functions

Poonia, M.S

Abstract -The main object of this paper is to obtain some integrals and series relation pertaining to biorthogonal polynomials, Fox's H-function, the general class of polynomials and the H-function of several complex variables.

Keywords : Fox's H-function, the general class of polynomials and the H-function of complex variables.

I. Introduction

Integrals with Fox's H-function, the general class of polynomials and the H-function of complex variables were studied by many authors.
Prabhakar and Tomar [7] have given a biorthogonal pair of polynomial sets

$$U_n\,(x,k) \text{ and } V_n\,(x,k)$$

where

$$U_n(x,k)=\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{\left(\frac{j+1}{k}\right)_n}{(1/k)_n}\left(\frac{1-x)}{2}\right)^j, \tag{1}$$

and

$$V_n(x,k)=\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(1+n)_{kj}}{(1)_{kj}}\left(\frac{1-x}{2}\right)^{kj} \tag{2}$$

The general multivariable polynomials defined by Srivastava ([12], p.185, eq. (7)) is represented in the following manner:

$$S^{p_1,\ldots,p_s}_{q_1,\ldots,q_s}[x_1,\ldots,x_s]=\sum_{k_1=0}^{[q_1/p_1]}\cdots\sum_{k_s=0}^{[q_s/p_s]}\frac{(-q_1)_{p_1k_1}\ldots(-q_s)_{p_sk_s}}{k_1!\ldots k_s!}$$

Author : NIMS University, Jaipur, Rajsthan, India.

$$. \, L[q_1, k_1; ...; q_s k_s] x_1^{k_1} ... x_s^{k_s}$$

where $$q_m, p_m \, (m = 1, ..., s) \tag{3}$$

are non-zero arbitrary positive integers. The coefficients $L[q_1, k_1; ...; q_s, k_s]$ being arbitrary constants real or complex.

Taking s = 1, the equation (3) reduces to the well known general class of polynomials $S_q^p[x]$ due to Srivastava ([13], p.158, eq. (1.1)].

II. Main Integrals

The following integrals concerning the biorthogonal polynomials with certain products of special functions have been derived in the paper.

a) First Integral

$$\int_0^{\pi/2} \cos 2u\theta (\sin\theta)^v \, U_n(1 - 2x \sin^{2h}\theta; k)$$

$$. \, H_{P_1,Q_1}^{M_1,N_1}\left[z(\sin\theta)^{2\rho_1'} \middle| \begin{matrix} (a_{P_1}, e_{P_1}) \\ (b_{Q_1}, f_{Q_1}) \end{matrix} \right] {}_{P_2}\overset{\alpha'}{M}_{Q_2}[y(\sin\theta)^{2\rho_2'}]$$

$$H(z_1(\sin\theta)^{2\sigma_1}, ..., z_r(\sin\theta)^{2\sigma_r} \; . \; S_{q_1,...,q_s}^{p_1,...,p_s}[x_1(\sin\theta)^{2\rho_1} ... x_s(\sin\theta)^{2\rho_s}] \, d\theta$$

$$= \sum_{\tau_1=1}^{M_1} \sum_{\tau_2=0}^{n} \sum_{\substack{s'=0 \\ s''=0}}^{\infty} \sum_{k_1=0}^{[q_1/p_1]} ... \sum_{k_s=0}^{[q_s/p_s]} (-1)^{\tau_2} \binom{n}{\tau_2} \frac{\left(\frac{\tau_2+1}{k}\right) x^{\tau_2}}{(1/k)_n}$$

$$. \, \frac{(-1)^{s'} z^{\eta_{s'}} \phi(\eta_{s'}) \Gamma\left(\frac{1}{2} \pm u\right)}{\tau_1! f_{\tau_1} \, s'! \, 2^{v+2h\,\tau_2+2\rho_1'\eta_{s'}} + 2\rho_2' s'' + \sum_{i=1}^{s} 2\rho_i}$$

$$. \, \frac{(a_1)_{s''} ... (a\,P_2)_{s''} \, y^{s''}}{(b_1)_{s''} ... (b_{Q_2})_{s''} \, \Gamma(\alpha' s'' + 1)} \frac{(-q_1)_{p_1 k_1} ... (-q_s) p_s k_s}{k_1! \quad k_s!}$$

Notes

$$.\ L[q_1,k_1;...;q_s,k_s]x_1^{k_1}...x_s^{k_s}$$

$$.\ H^{0,\lambda+1\ \ :(u',v');...;(u^{(r)},v^{(r)})}_{A+1,C+2:(B',D');...;B^{(r)},D^{(r)}}\left[\begin{array}{l}[-V-2h\tau_2-2\rho_1'\eta_{s'}-2\rho_2's''-\sum_{\xi=1}^{s}2\rho_\xi k_\xi:2\sigma_1,...,2\sigma_r],\\ [(c):\psi',...,\psi^{(r)}],\end{array}\right.$$

$$\left.\begin{array}{ll}[(a):\theta',...,\theta^{(r)}], & [(b'):\phi'];...;[(b^{(r)}):\phi^{(r)}];\\ [-\frac{v}{2}\pm\frac{u}{2}-h\tau_2-\rho_1'\eta_{s'}-\rho_2's''-\sum_{\xi=1}^{s}2\rho_\xi k_\xi:\sigma_1,...,\sigma_s] & [(d'):\delta'];...;[(d^{(r)}):\delta^{(r)}];\end{array} z_1 2^{-\sigma_1},...,z_r 2^{-\sigma_r}\right], \quad (4)$$

where $u=0,1,2,...;\operatorname{Re}\left(V+2\rho_1'+2\sum_{i=1}^{r}\sigma_i\frac{d^{(i)}}{\delta_j^{(i)}}\right)>0, j'=1,...,M; j=1,...,u^{(i)}$

$$T_i>0,|\arg(z_i)|<\frac{1}{2}T_i\pi, T'>0,|\arg z|\frac{1}{2}T'\pi,$$

$\rho_1'>0,\rho_2'>0,\ \rho_1,...,\rho_s>0, p_2<Q_2, |y|<1, p_m\ (m=1\text{ to }s)$ are non-zero arbitrary positive integers and the coefficients L $[q_1,k_1;...;q_s,k_s]$ are arbitrary constants, real or complex.

b) Second Integral

$$\int_0^{\pi/2}\cos 2u\theta(\cos\theta)^v\, U_n(1-2x\cos^{2h}\theta;k)$$

$$.\,H^{M_1,N_1}_{P_1,Q_1}\left[z(\cos\theta)^{2\rho_1'}\middle|\begin{array}{l}(a_{P_1},e_{P_1})\\(b_{Q_1},f_{Q_1})\end{array}\right]$$

$$.\,H(z_1(\cos\theta)^{2\sigma_1},...,z_r(\cos\theta)^{2\sigma_r}\ {}_{P_2}M^{\alpha'}_{Q_2}[y(\cos\theta)^{2\rho_2'}]$$

$$.\ S^{p_1,...,p_s}_{q_1,...,q_s}[x_1(\cos\theta)^{2\rho_1}...x_s(\cos\theta)^{2\rho_s}]\,d\theta$$

Notes

$$= \sum_{\tau_2=0}^{n} \sum_{\tau=1}^{M_1} \sum_{s',s''=0}^{\infty} \prod_{m=1}^{s} \left[\sum_{m}^{[q_m/p_m]} \frac{(-q_m)_{p_m k_m} x_m^{k_m}}{k_m!} \right] (-1)^{\tau_2} \binom{n}{\tau_2} \frac{\left(\frac{\tau_2+1}{k}\right)_n x^{\tau_2}}{(1/k)_n}$$

$$. \frac{(-1)^{s'} z^{\eta_{s'}} \phi(\eta_{s'})}{f_{\tau_1} s'!} \frac{\pi \Gamma(u+1)}{2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2's''+\sum_{i=1}^{s} 2\rho_i k_i+1}}$$

$$. H^{0,\lambda \; :(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C+2:(B',D');\ldots;(B^{(r)},D^{(r)})} \left[\begin{matrix} [\text{----------}],[\text{----------------------------------}]: \\ [(c):\psi',\ldots,\psi^{(r)}],\left[-\frac{v}{2} \pm \frac{u}{2} - h\,\tau_2 - \rho_1'\eta_{s'} - \rho_2's'' - \sum_{i=1}^{s} \rho_i k_i : \sigma_1,\ldots,\sigma_r \right]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \; z_1 2^{-2\sigma_1},\ldots, z_r 2^{-2\sigma_s} \right], \qquad (5)$$

where $u = 0,1,2,\ldots,$ $\mathrm{Re}\left(V + 2\rho_1' \frac{b_{j'}}{f_j'} + 2\sum_{i=1}^{r} \sigma_i \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > 0, j' = 1,\ldots,Q_2; j = 1,\ldots,u^{(i)};$

$$T_i > 0, |\arg(z_i)| < \frac{1}{2} T_i \pi, T' > 0, |\arg(z)| < \frac{1}{2} T' \pi, \rho_1,\ldots,\rho_s > 0, P_2 < Q_2, |y| < 1, h > 0,$$

$p_m (m = 1,\ldots,s)$ are non-zero arbitrary positive integers and the coefficients $L[q_1,k_1,\ldots,q_s,k_s]$ are arbitrary constants, real or complex.

Proof of (4)

Expressing the polynomials *Un* as given (1), Fox's H-function in series , the generalized multivariable polynomials by (3), M-series and the H-function of several complex variables in Mellin - Barnes contour integral by, changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and then evaluating the resulting integral with the help of the following result,

$$\int_0^{\pi/2} \cos 2u\theta (\sin\theta)^v \, d\theta = \frac{\Gamma(v+1)\Gamma(\frac{1}{2} \pm u)}{2^{v+1} \Gamma\left(\frac{v}{2} \pm \frac{u}{2} + 1 \right)} \qquad (6)$$

Notes

Notes

where u = 0,1,2,..., and Re(v) > 0.

Finally interpreting the result thus obtained with the help of (1.2.1), we arrive at the required result (2.3.1).

The integrals from (2.3.2) to (2.3.4) can also be obtained in the similar manner with the help of the appropriate integral (2.3.5) and the following result

$$\int_0^{\pi/2} \cos u\theta (\cos\theta)^v \, d\theta = \frac{\pi \Gamma(u+1)}{2^{v+1}\Gamma\left(\frac{v}{2} \pm \frac{u}{2} + 1\right)} \tag{7}$$

where u = 0,1,2,..., and Re (v) > 0.

III. Special Cases

(i) Putting $\lambda = A, u^{(i)} = 1, v^{(i)} = B^{(i)}, D^{(i)} = D^{(i)} + 1, \forall i = 1,\ldots,r$ in (4), we find

$$\int_0^{\pi/2} \cos 2u\theta (\sin\theta)^v \, U_n(1 - 2x \sin^{2h}\theta; k)$$

$$. H^{M_1,N_1}_{P_1,Q_1}\left[z (\sin\theta)^{2\rho_1'} \middle| \begin{matrix} (a_{P_1}, e_{P_1}) \\ (b_{Q_1}, f_{Q_1}) \end{matrix} \right] S^{p_1,\ldots,p_s}_{q_1,\ldots,q_s}[x_1 (\sin\theta)^{2\rho_1} \ldots x_s (\sin\theta)^{2\rho_s}]$$

$$. {}_{P_2}M^{\alpha'}_{Q_2}[y (\sin\theta)^{2\rho_2'}]$$

$$. F^{A:B';\ldots;B^{(r)}}_{B:D';\ldots;D^{(r)}}\left(\begin{matrix} [1-(a):\theta',\ldots,\theta^{(r)}]:[1-(b'):\phi'];\ldots;[1-(b^{(r)}):\phi^{(r)}]; \\ [1-(c):\psi',\ldots,\psi^{(r)}]:[1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}]; \end{matrix} -z_1 (\sin\theta)^{2\sigma_1},\ldots,-z_r (\sin\theta)^{2\sigma_r} \right) d\theta$$

$$= \sum_{\tau_2=0}^{n} \sum_{\tau_1=1}^{M_1} \sum_{s',s''=0}^{\infty} \prod_{m=1}^{s} \left[\sum_{m}^{[q_m/p_m]} \frac{(-q_m)_{p_m k_m} x_m^{k_m}}{k_m!} \right] (-1)^{\tau_2} \binom{n}{\tau_2} \frac{\left(\frac{\tau_2+1}{k}\right)_n}{(1/k)_n} x^{\tau_2}.$$

$$\frac{(-1)^{s'} z^{\eta_{s'}} \phi(\eta_{s'})}{f_{\tau_1} s'!} \quad \frac{(a_1)_{s''}\ldots(a_{P_2})_{s''} y^{s''} \Gamma(\frac{1}{2} \pm u)}{(b_1)_{s''}\ldots(b_{Q_2})_{s''} \Gamma(\alpha' s''+1)\, 2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2' s''+\sum_{i=1}^{s} 2\rho_i k_i}}$$

$$. \, F^{A+1:B';\ldots;B^{(r)}}_{C+1:D';\ldots;D^{(r)}} \left[\begin{matrix} [-v-2h\tau_2-2\rho_1^{'}\eta_{s'}-2\rho_2^{'}s''-\sum_{\xi=1}^{s} :\rho_i k_i:2\sigma_1,\ldots,2\sigma_r], \\ [(c):\psi',\ldots,\psi^{(r)}], \end{matrix} \right.$$

$$\left. \begin{matrix} [1-(a):\theta',\ldots,\theta^{(r)}], & [1-(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [-\frac{v}{2}\pm\frac{u}{2}-hk\tau_2-\rho_1^{'}\eta_{s'}-\rho_2^{'}s''-\sum_{i=1}^{s} \rho_i k_i:\sigma_1,\ldots,\sigma_r] & [1-(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \; z_1 2^{-2\sigma_1},\ldots,z_r 2^{-2\sigma_r} \right],$$

(8)

provided that $u=0,1,2,\ldots, \mathrm{Re}\left(v+2\rho_1^{'}\frac{b_{j'}}{f_{j'}}\right)>0, j'=1,\ldots,Q_2, |\arg(z)|<\frac{1}{2}T^{'}\pi, T>0$ and the series on the right of (8) is absolutely convergent.

(ii) Taking r = 2, the result in (8) reduces to the following integral

$$\int_0^{\pi/2} \cos 2u\theta(\sin\theta)^v \, U_n(1-2x\sin^{2h}\theta;k)$$

$$. \, H^{M_1,N_1}_{P_1,Q_1}\left[z(\sin\theta)^{2\rho_1^{'}} \middle| \begin{matrix} (a_{P_1},e_{P_1}) \\ (b_{Q_1},f_{Q_1}) \end{matrix} \right] S^{p_1,\ldots,p_s}_{q_1,\ldots,q_s}[x_1(\sin\theta)^{2\rho_1}\ldots x_s(\sin\theta)^{2\rho_s}]$$

$$S^{A:B';B''}_{B:D';D''}\left(\begin{matrix} [1-(a):\theta',\ldots,\theta'']:[1-(b'):\phi'];\ldots;[1-(b''):\phi'']; \\ [1-(c):\psi',\ldots,\psi'']:[1-(d'):\delta'];\ldots;[1-(d''):\delta'']; \end{matrix} -z_1(\sin\theta)^{2\sigma_1},-z_2(\sin\theta)^{2\sigma_2} \right) d\theta$$

$$. \, {}_{P_2}M^{\alpha'}_{Q_2}[y(\sin\theta)^{2\rho_1^{'}}] \, d\theta$$

$$= \sum_{\tau_2=0}^{n} \sum_{\tau_1=1}^{M_1} \sum_{s',s''=0}^{\infty} \prod_{m=1}^{s} \left[\sum_{m}^{[q_m/p_m]} \frac{(-q_m)_{p_m k_m} x_m^{k_m}}{k_m!} \right] (-1)^{\tau_2} \binom{n}{\tau_2} \frac{\left(\frac{\tau_2+1}{k}\right)_n}{(1/k)_n} x^{\tau_2}.$$

$$\frac{(-1)^{s'} z^{\eta_{s'}} \phi(\eta_{s'})}{f_{\tau_1} s'!} \; \frac{(a_1)_{s''}\ldots(a_{P_2})_{s''} y^{s''}\Gamma(\frac{1}{2}\pm u)}{2^{v+2h\tau_2+2\rho_1^{'}\eta_{s'}+2\rho_2^{'}s''+\sum_{i=1}^{s} 2\rho_i k_i+1}}$$

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$$S^{A+1:B';\dots;B''}_{C+1:D';\dots;D''}\left(\begin{matrix}[-v-2h\tau_2-2\rho_1'\eta_{s'}-2\rho_2's''-\sum_{i=1}^{s}\rho_i k_i:2\sigma_1,2\sigma_2], \\ [1-(c):\psi',\dots,\psi^{(r)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[1-(a):\theta'\theta''], & [1-(b'):\phi'];[(b''):\phi''];\\ [-\frac{v}{2}\pm\frac{u}{2}-hk\tau_2-\rho_1'\eta_{s'}-\rho_2's''-\sum_{i=1}^{s}\rho_i k_i:\sigma_1,\sigma_2],[1-(d'):\delta'];[(d''):\delta''];\end{matrix}\; -z_1 2^{-2\sigma_1}, -z_2 2^{-2\sigma_2}\right) \qquad (9)$$

where $u=0,1,2,\dots,\mathrm{Re}\left(v+2\rho_1'\frac{b_j}{f_j}\right)>0,\ j=1,\dots,M;|\arg(z)|<\frac{1}{2}T'\pi, T'>0$, and the series on the right of (9) converges absolutely.

(iii) Letting $\lambda = A = C = 0$ in (4), we get

$$\int_0^{\pi/2}\cos 2u\theta(\sin\theta)^v\, U_n(1-2x\sin^{2h}\theta;k)$$

$$.\ H^{M_1,N_1}_{P_1,Q_1}\left[z(\sin\theta)^{2\rho_1'}\left|\begin{matrix}(a_{P_1},e_{P_1})\\(b_{Q_1},f_{Q_1})\end{matrix}\right.\right]\ {}_{P_2}M^{\alpha'}_{Q_2}[y(\sin\theta)^{2\rho_2'}]$$

$$.\ S^{p_1,\dots,p_s}_{q_1,\dots,q_s}(x_1(\sin\theta)^{2\rho_1}\dots x_s(\sin\theta)^{2\rho_s})$$

$$.\ \prod_{i=1}^{r}\left\{H^{u^{(i)},v^{(i)}}_{B^{(i)},D^{(i)}}\left[z_1(\sin\theta)^{2\sigma_1}\left|\begin{matrix}[(b^{(i)}):(\phi^{(i)})]\\ [(d^{(i)}):(\delta^{i)})]\end{matrix}\right.\right]\right\}d\theta$$

$$=\sum_{\tau_2=0}^{n}\sum_{\tau_1=1}^{M_1}\sum_{s',s''=0}^{\infty}\prod_{m=1}^{s}\left[\sum_{m}^{[q_m/p_m]}\frac{(-q_m)_{p_m k_m}\, x_m^{k_m}}{k_m!}\right](-1)^{\tau_2}\binom{n}{\tau_2}\frac{\left(\frac{\tau_2+1}{k}\right)_n}{(1/k)_n}x^{\tau_2}.$$

$$\frac{(-1)^{s'}z^{\eta_{s'}}\phi(\eta_{s'})}{f_{\tau_1}s'!}\ \frac{\Gamma(\frac{1}{2}\pm u)}{2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2's''+\sum_{i=1}^{s}2\rho_i k_i+1}}.\frac{(a_1)_{s''}\dots(a_{P_2})_{s''}\,y^{s''}}{(b_1)_{s''}\dots(b_{Q_2})_{s''}\gamma(\alpha's''+1)}$$

$$. . H^{0,1:(u',v');...;(u^{(r)},v^{(r)})}_{1,2:(B',D');...;(B^{(r)},D^{(r)})}\left[\begin{array}{l}[-v-2h\tau_2-2\rho_1'\eta_{s'}-2\rho_2's''-\sum_{i=1}^{s}\rho_i k_i:2\sigma_1,...,2\sigma_r],\\ [(c):\psi',...,\psi^{(r)}],\end{array}\right.$$

$$\left.\begin{array}{ll}[1-(a):\theta',...,\theta^{(r)}], & [(b'):\phi'];...;[(b^{(r)}):\phi^{(r)}];\\ [-\frac{v}{2}\pm\frac{u}{2}-hk\tau_2-\rho_1'\eta_{s'}-\rho_2's''-\sum_{i=1}^{s}\rho_i k_i:\sigma_1,\sigma_2...\sigma_r],[1-(d'):\delta'];...;[(d^{(r)}):\delta^{(r)}]; & \end{array} z_1 2^{-2\sigma_1},...,z_2 2^{-2\sigma_2}\right), \tag{10}$$

valid under the same conditions as obtainable from result (4).

(iv) Setting r = 2 in equation (4), we find

$$\int_0^{\pi/2}\cos 2u\theta(\sin\theta)^v U_n(1-2x\sin^{2h}\theta;k)$$

$$. H^{M_1,N_1}_{P_1,Q_1}\left[z(\sin\theta)^{2\rho_1'}\left|\begin{array}{l}(a_{P_1},e_{P_1})\\(b_{Q_1},f_{Q_1})\end{array}\right.\right] {}_{P_2}M^{\alpha'}_{Q_2}[y(\sin\theta)^{2\rho_2'}]$$

$$. S^{p_1,...,p_s}_{q_1,...,q_s}(x_1(\sin\theta)^{2\rho_1}...x_s(\sin\theta)^{2\rho_s})$$

$$. H^{0,\lambda:(u',v');(u'',v'')}_{A,C:(B',D');(B'',D'')}\left[\begin{array}{l}[(a):\theta',\theta'']:[(b'):\phi'];[(b''):\phi''];\\ [(c):\psi',\psi'']:[(d'):\delta'];[(d''):\delta''];\end{array} z_1(\sin\theta)^{2\sigma_1},z_2(\sin\theta)^{2\sigma_2}\right]d\theta$$

$$=\sum_{\tau_2=0}^{n}\sum_{\tau_1=1}^{M_1}\sum_{s',s''=0}^{\infty}\prod_{m=1}^{s}\left[\sum_{m}^{[q_m/p_m]}\frac{(-q_m)_{p_m k_m}x_m^{k_m}}{k_m!}\right](-1)^{\tau_2}\binom{n}{\tau_2}\frac{\left(\frac{\tau_2+1}{k}\right)_n}{(1/k)_n}x^{\tau_2}.$$

$$\frac{(-1)^{s'}z^{\eta_s'}\phi(\eta_s')}{f_{\tau_1}s'!}\frac{\Gamma(\frac{1}{2}\pm u)}{2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2's''+\sum_{i=1}^{s}2\rho_i k_i+1}}.\frac{(a_1)_{s''}...(a_{P_2})_{s''}y^{s''}}{(b_1)_{s''}...(b_{Q_2})_{s''}\gamma(\alpha's''+1)}$$

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$$.. H^{0,\lambda+1:(u',v');u'',v''}_{A+1,C+2:(B',D');B'',D''}\left[\begin{matrix}[-v-2h\tau_2-2\rho_1'\eta_{s'}-2\rho_2's''-\sum_{i=1}^{s}:\rho_i k_i:2\sigma_1,2\sigma_2],\\ [(c):\psi',\psi''],\end{matrix}\right.$$

Notes

$$\left.\begin{matrix}[(a):\theta',...,\theta''], & [(b'):\phi'];[(b''):\phi''] \\ [-\frac{v}{2}\pm\frac{u}{2}-hk\tau_2-\rho_1'\eta_{s'}-\rho_2's''-\sum_{i=1}^{s}\rho_i k_i:2\rho_\xi k_\xi:\sigma_1,\sigma_2],[(d'):\delta'];[(d''):\delta''];\end{matrix} -z_1 2^{-2\sigma_1}, -z_2 2^{-2\sigma_2}\right], \tag{11}$$

where $u=0,1,2,...,\ \mathrm{Re}\left(v+2\rho_1'\frac{b_{j'}}{f_{j'}}+2\sigma_1\frac{d_j'}{\delta_j'}+2\sigma_2\frac{d_j''}{\delta_j''}\right)>0,\ j=1,...,M_1,\ j'=1,...,u',$

$j'=1,...,u'',T_1,T_2>0,|\arg(z_1)|<\frac{1}{2}T_1\pi,|\arg(z_2)|<\frac{1}{2}T_2\pi;|\arg(z)|<\frac{1}{2}\pi T',T>0,$

$\rho_m>0\,(m=1,...,s),\ P_2<Q_2,|y|<1,h>0,p_m\,(m=1,...,s)$ are positive coefficients and $L(q_1k_1,...,q_sk_s]$ are arbitrary constants, real or complex.

(v) Taking $\lambda=A,\ U^{(i)}=1, v^{(i)}=B^{(i)}, D^{(i)}=D^{(i)}+1\ \forall i=1,...,r$ in (5), we get

$$\int_0^{\pi/2}\cos 2u\theta(\cos\theta)^v\,U_n(1-2x\cos^{2h}\theta;k)$$

$$.\,H^{M_1,N_1}_{P_1,Q_1}\left[z(\cos\theta)^{2\rho_1'}\middle|\begin{matrix}(a_{P_1},e_{P_1})\\(b_{Q_1},f_{Q_1})\end{matrix}\right]\ {}_{P_2}\overset{\alpha'}{M}_{Q_2}[y(\cos\theta)^{2\rho_2'}]$$

$$.S^{p_1,...,p_s}_{q_1,...,q_s}(x_1(\cos\theta)^{2\rho_1}...x_s(\cos\theta)^{2\rho_s})$$

$$F^{A:B';...;B^{(r)}}_{B:D';...;D^{(r)}}\left(\begin{matrix}[1-(a):\theta',...,\theta^{(r)}]:[1-(b'):\phi'];...;[1-(b^{(r)}):\phi^{(r)}];\\ [1-(c):\psi',...,\psi^{(r)}]:[1-(d'):\delta'];...;[1-(d^{(r)}):\delta^{(r)}];\end{matrix} -z_1(\cos\theta)^{2\sigma_1},...,-z_r(\cos\theta)^{2\sigma_r}\right)d\theta$$

$$= \sum_{\tau_2=0}^{n} \sum_{\tau_1=1}^{M_1} \sum_{s',s''=0}^{\infty} \prod_{m=1}^{s} \left[\sum_{m}^{[q_m/p_m]} \frac{(-q_m)_{p_m k_m} \, x_m^{k_m}}{k_m!} \right] (-1)^{\tau_2} \binom{n}{\tau_2} \frac{\left(\frac{\tau_2+1}{k}\right)_n}{(1/k)_n} x^{\tau_2} \quad .$$

$$\frac{(-1)^{s'} z^{\eta_{s'}} \phi(\eta_{s'})}{f_{\tau_1} s'!} \frac{(a_1)_{s''} \ldots (a_{P_2})_{s''} \, y^{s''}}{(b_1)_{s''} \ldots (b_{Q_2})_{s''} \Gamma(\alpha' s''+1)} \frac{\pi \Gamma(u+1)}{2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2' s''+\sum_{i=1}^{s} 2\rho_i k_i+1}}$$

$$. F^{A\ :B';\ldots;B^{(r)}}_{C+2:D';\ldots;D^{(r)}} \left[\begin{matrix} [1-(a):\theta',\ldots,\theta^{(r)}]:[-----------------------], \\ [1-(c):\psi',\ldots,\psi^{(r)}]:\left[-\frac{v}{2}\pm\frac{u}{2}-hk\tau_2-\rho_1'\eta_{s'}-\rho_2' s''-\sum_{i=1}^{s}\rho_i k_i\right], \end{matrix} \right.$$

$$\left. \begin{matrix} [1-(b'):\phi'];\ldots;[1-(b^{(r)}):\phi^{(r)}]; \\ [1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}]; \end{matrix} -z_1 e^{-2\sigma_1},\ldots,-z_r e^{-2\sigma_r} \right], \qquad (12)$$

Provided $u = 0,1,2,\ldots, \mathrm{Re}\left(v + 2\rho_1' \frac{b_j}{f_j}\right) > 0, j = 1,\ldots,M_1, |\arg(z)| < \frac{1}{2} T'\pi, T' > 0,$

$|y| < 1, P_2 < Q_2$ and the series on the right of (12) converges absolutely.

(vi) Putting r = 2 in (12), we obtain

$$\int_0^{\pi/2} \cos 2u\theta (\cos\theta)^v \, U_n(1-2x\cos^{2h}\theta; k)$$

$$. \, H^{M_1,N_1}_{P_1,Q_1}\left[z(\cos\theta)^{2\rho_1'} \left| \begin{matrix} (a_{P_1}, e_{P_1}) \\ (b_{Q_1}, f_{Q_1}) \end{matrix} \right. \right] {}_{P_2}M^{\alpha'}_{Q_2}[y(\cos\theta)^{2\rho_2'}]$$

$$. S^{p_1,\ldots,p_s}_{q_1,\ldots,q_s}(x_1(\cos\theta)^{2\rho_1} \ldots x_s(\cos\theta)^{2\rho_s})$$

Notes

$$S^{A:B';B''}_{B:D';D''}\left(\begin{matrix}[1-(a):\theta',\theta'']:[1-(b'):\phi'];[1-(b''):\phi''];\\ [1-(c):\psi',\psi'']:[1-(d'):\delta'];[1-(d''):\delta''];\end{matrix} -z_1(\cos\theta)^{2\sigma_1},-z_2(\cos\theta)^{2\sigma_2}\right)d\theta$$

Notes

$$=\sum_{\tau_2=0}^{n}\sum_{\tau_1=1}^{M_1}\sum_{s',s''=0}^{\infty}(-1)^{\tau_2}\binom{n}{\tau_2}\frac{\left(\frac{\tau_2+1}{k}\right)_n}{(1/k)_n}x^{\tau_2}$$

$$.\frac{(-1)^{s'}z^{\eta_s'}\phi(\eta_s')}{f_{\tau_1}\,s'!}\;\frac{\pi\Gamma(u+1)}{2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2's''+\sum_{i=1}^{s}2\rho_i k_i+1}}\;\frac{(a_1)_{s''}\dots(a_{P_2})_{s''}\,y^{s''}}{(b_1)_{s''}\dots(b_{Q_2})_{s''}\,\Gamma(\alpha's''+1)}$$

$$.\prod_{m=1}^{s}\left[\sum_{m}^{[q_m/p_m]}\frac{(-q_m)_{p_m k_m}\,x_m^{k_m}}{k_m!}\right]$$

$$.S^{A\;:B';B''}_{C+2:D';D''}\left[\begin{matrix}[1-(a):\theta',\dots,\theta'']:[-------------------------],\\ [1-(c):\psi',\dots,\psi'']:\left[-\frac{v}{2}\pm\frac{u}{2}-hk\tau_2-\rho_1'\eta_{s'}-\rho_2's''-\sum_{i=1}^{s}\rho_i k_i:\sigma_1,\sigma_2\right],\end{matrix}\right.$$

$$\left.\begin{matrix}[1-(b'):\phi'];[1-(b''):\phi''];\\ [1-(d'):\delta'];[1-(d''):\delta''];\end{matrix}-z_1 2^{-2\sigma_1},-z_2 2^{-2\sigma_2}\right], \qquad (13)$$

provided that $u=0,1,2,\dots,\mathrm{Re}\left(v+2\rho_1'\frac{b_j}{f_j}\right)>0, j=1,\dots,M_1, |\arg(z)|<\frac{1}{2}T'\pi, T'>0,$

and the series on the right of (13) is absolutely convergent.

(vii) Letting $\lambda = A = C = 0$ in (5), we have

$$\int_0^{\pi/2}\cos 2u\theta(\cos\theta)^v\,U_n(1-2x\cos^{2h}\theta;k)$$

$$. H^{M_1,N_1}_{P_1,Q_1}\left[z(\cos\theta)^{2\rho_1'} \middle| \begin{matrix} (a_{P_1}, e_{P_1}) \\ (b_{Q_1}, f_{Q_1}) \end{matrix} \right] {}_{P_2}\overset{\alpha'}{M}_{Q_2}[y(\cos\theta)^{2\rho_2'}]$$

$$. S^{p_1,\dots,p_s}_{q_1,\dots,q_s}(x_1(\cos\theta)^{2\rho_1} \dots x_s(\cos\theta)^{2\rho_s})$$

$$\prod_{i=1}^{r} \left\{ H^{u^{(i)},v^{(i)}}_{B^{(i)},D^{(i)}}\left[z_1(\cos\theta)^{2\sigma_1} \middle| \begin{matrix} [(b^{(i)}):(\phi^{(i)})] \\ [(d^{(i)}):(\delta^{i)})] \end{matrix} \right] \right\} d\theta$$

$$= \sum_{\tau_2=0}^{n} \sum_{\tau_1=1}^{M_1} \sum_{s',s''=0}^{\infty} \prod_{m=1}^{s} \left[\sum_{m}^{[q_m/p_m]} \frac{(-q_m)_{p_m k_m}\, x_m^{k_m}}{k_m!} \right] (-1)^{\tau_2} \frac{\left(\frac{\tau_2+1}{k}\right)_n x^{\tau_2} \binom{n}{\tau_2}}{(1/k)_n}.$$

$$\frac{(-1)^{s'} z^{\eta_{s'}} \phi(\eta_{s'})}{f_{\tau_1}\, s'!} \frac{\Gamma(u+1)}{2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2' s''+\sum_{i=1}^{s} 2\rho_i k_i+1}} \frac{(a_1)_{s''} \dots (a_{P_2})_{s''}\, y^{s''}}{(b_1)_{s''} \dots (b_{Q_2})_{s''}\, \Gamma(\alpha' s''+1)}$$

$$H^{0,0:(u',v');\dots;(u^{(r)},v^{(r)})}_{0,2\,:(B',D');\dots;(B^{(r)},D^{(r)})} \left[\begin{matrix} [(a):\theta',\dots,\theta^{(r)}]:[\text{------------------------------}]: \\ [(c):\psi',\dots,\psi^{(r)}],\left[-\frac{v}{2}\pm\frac{u}{2}-h\,\tau_2-\rho_1'\eta_{s'}-\rho_2' s''-\sum_{i=1}^{s}\rho_i k_i:\sigma_1,\dots,\sigma_r\right], \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}]; \end{matrix}\ z_1 z^{-2\sigma_1},\dots,z_r z^{-2\sigma_r} \right], \qquad (14)$$

valid under the same conditions as stated for (5).

(viii) Putting r = 2 in (5), we get

$$\int_0^{\pi/2} \cos 2u\theta(\cos\theta)^v\, U_n(1-2x\cos^{2h}\theta;k)$$

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$$. H^{M_1,N_1}_{P_1,Q_1}\left[z(\cos\theta)^{2\rho_1'} \middle| \begin{matrix}(a_{P_1},e_{P_1})\\(b_{Q_1},f_{Q_1})\end{matrix}\right] {}_{P_2}M^{\alpha'}_{Q_2}[y(\cos\theta)^{2\rho_2'}]$$

$$.S^{p_1,\ldots,p_s}_{q_1,\ldots,q_s}(x_1(\cos\theta)^{2\rho_1}\ldots x_s(\cos\theta)^{2\rho_s})$$

$$. H^{0,\lambda:(u',v');(u'',v'')}_{A,C:(B',D');(B'',D'')}\left[\begin{matrix}[(a):\theta',\theta'']:[(b'):\phi'];[(b''):\phi''];\\ [(c):\psi',\psi'']:[(d'):\delta'];[(d''):\delta''];\end{matrix} z_1(\cos\theta)^{2\sigma_1}, z_2(\cos\theta)^{2\sigma_2})\right]d\theta$$

$$= \sum_{\tau_2=0}^{n}\sum_{\tau_1=1}^{M_1}\sum_{s',s''=0}^{\infty}(-1)^{\tau_2}\binom{n}{\tau_2}\frac{\left(\frac{\tau_2+1}{k}\right)_n x^{\tau_2}}{(1/k)_n}\frac{(-1)^{s'} z^{\eta_{s'}}\phi(\eta_{s'})}{f_{\tau_1} s'!}$$

$$\prod_{m=1}^{s}\left[\sum_{m}^{[q_m/p_m]}\frac{(-q_m)_{p_m k_m} x_m^{k_m}}{k_m!}\right]\frac{\pi\Gamma(u+1)}{2^{v+2h\tau_2+2\rho_1'\eta_{s'}+2\rho_2' s''+\sum_{i=1}^{s}2\rho_i k_i+1}}$$

$$.\frac{(a_1)_{s''}\ldots(a_{P_2})_{s''}\, y^{s''}}{(b_1)_{s''}\ldots(b_{Q_2})_{s''}\,\Gamma(\alpha' s''+1)}$$

$$H^{0,0:(u',v');\ldots;(u'',v'')}_{0,2\,:(B',D');\ldots;(B'',D'')}\left[\begin{matrix}[(a):\theta',\ldots,\theta'']:[\text{------------------------------}]:\\ [(c):\psi',\ldots,\psi''],\left[-\frac{v}{2}\pm\frac{u}{2}-h\,\tau_2-\rho_1'\eta_{s'}-\rho_2' s''-\sum_{i=1}^{s}\rho_i k_i:\sigma_1,\ldots,\sigma_2\right],\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b''):\phi''];\\ [(d'):\delta'];\ldots;[(d''):\delta''];\end{matrix} z_1 2^{-2\sigma_1}, z_2 2^{-2\sigma_2}\right], \qquad (15)$$

where $u = 0,1,2,\ldots,\ \mathrm{Re}\left(v + 2\rho_1'\frac{b_{j'}}{f_{j'}} + 2\sigma_1\frac{d_j'}{\delta_j'} + 2\sigma_2\frac{d_{j''}''}{\delta_{j''}''}\right) > 0,\ j = 1,\ldots,M_1;\ j' = 1,\ldots,u';$

$j''=1,\ldots,u'';|\arg(z)|<\frac{1}{2}T'\pi, T', T_1, T_2>0, |y|<1, P_2<Q_2, h>0, \rho_1', \rho_2', \rho_m\,(m=1\text{ to }s),$

$(i=1,\ldots,r)>0.$

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 11 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

A New Self-Adjusting Numerical Integrator for the Numerical Solutions of Ordinary Differential Equations

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Abstract - In this work, we consider a class of formulae for the numerical solution of IVP, in ordinary differential equations with point of singularity, in which the underlying interpolant is a rational function. This is in contrast with the classical formulae which are in general based on polynomial approximation. The proof of convergence and consistency for the scheme are also given. There are two parameters that control the position and the nature of singularity. The values of these parameters are automatically chosen and revised, during the computation.

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GJSFR-F Classification : *MSC 2010: 49K15*

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Strictly as per the compliance and regulations of :

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O. O.A. Enoch [α] & A. A. Olatunji [σ]

Abstract - In this work, we consider a class of formulae for the numerical solution of IVP, in ordinary differential equations with point of singularity, in which the underlying interpolant is a rational function. This is in contrast with the classical formulae which are in general based on polynomial approximation. The proof of convergence and consistency for the scheme are also given. There are two parameters that control the position and the nature of singularity. The values of these parameters are automatically chosen and revised, during the computation.

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I. Introduction

Authors like Lambert and Shaw (1965) [1, 15] considered a class of formulae for the numerical solution of

$$y' = f(x,y); y(x) = y \tag{1}$$

in which the underlying interpolant was a rational function, which was in contrast with the classical formulae. The numerical methods that resulted from the works of the above mentioned authors afforded an improved numerical solution which was closed to a singularity of the theoretical solution of (1), since they locally represented the numerical solution of (1) by an interpolant which can possess a simple pole.

II. Determination of the Undetermined Coefficients

The Interpolant considered in this work is presented as:

$$F(x_n) = \sum_{J=0}^{L} a_j x_n^j + b\,|A + x_n|^N, N \notin \{0,1,2,\ldots,L\} \tag{2}$$

where a_n, b, A and N are real, L is a positive integers.

Assuming that

$$F(x_n) = y_n \text{ and } F(x_{n+1}) = y_{n+1}; x_{n+1} = x_n + h \text{ for which } x_n = a + nh$$

Ref.

1. C. W. Gear.(1971) Numerical Initial Value Problems in Ordinary Differential Equations. Prentice Hall, Englewood Cliffs, NJ.

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$$F(x_{n+1}) - F(x_n) = y_{n+1} - y_n \tag{3}$$

Let $f^{(i)}$ denotes the i^{th} total derivative of f(x,y) with respect to x such that

$$F^{(1)}(x_n) = f(x_n, y_n) = f_n \text{ and} \tag{4}$$

$$F^{(2)}(x_n) = f^{(1)}(x_n, y_n) = f_n^{(1)} \tag{5}$$

$$F^{(m)}(x_n) = f^{(m-1)}(x_n, y_n) = f_n^{(m-1)} \tag{6}$$

It follows thus;

$$y_{n+1} - y_n = \sum_{j=0}^{L} a_j \left[x_{n+1}^j - x_n^j\right] + b\left[(A + x_{n+1})^N - (A + x_n)^N\right] \tag{7}$$

The above expressions hold provided all the derivatives concerned exist.
Elimination of the undetermined coefficients from (7) then gives the required algorithm:

When L = 1 (i.e. the polynomial $P_j(x)$ is linear)

$$P_j(x) = \sum_{j=0}^{1} a_j x^j = a_0 x_0 + a_1 x_1 = a_0 + a_1 x \tag{8}$$

$$\mathrm{F}(x_n) = a_0 + a_1 x_n + b(A + x_n)^N \tag{9}$$

$$\mathrm{F}(x_{n+1}) = a_0 + a_1 x_{n+1} + b(A + x_{n+1})^N \tag{10}$$

$$\text{Let } y_n = F(x_n) \text{ and } y_{n+1} = F(x_{n+1}) \tag{11}$$

$$\Rightarrow \mathrm{F}(x_{n+1}) - F(x_n) = y_{n+1} - y_n \; (12)$$

$$y_{n+1} - y_n = a_1(x_{n+1} - x_n) + b\left[(A + x_{n+1})^N - (A + x_n)^N\right] \tag{13}$$

$$y_{n+1} - y_n = a_1 h + b\left[(A + x_n + h)^N - (A + x_n)^N\right] \tag{14}$$

Differentiate $\mathrm{F}(x_n) = a_0 + a_1 x_n + b(A + x_n)^N$ to eliminate the undetermined coefficients

$$a_1 = f_n - \left[Nb(A + x_n)^{N-1}\right] \tag{15}$$

$$b = \frac{f_n^{(1)}}{N(N-1)(A + x_n)^{N-2}} \tag{16}$$

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Therefore

$$y_{n+1}-y_n=hf_n+\frac{(A+x_n)^2}{N(N-1)}\left[\left(1+\frac{h}{A+x_n}\right)^N-1-\frac{Nh}{A+x_n}\right]f_n^{(1)}$$

Let us introduce $\frac{N(A+x_n)}{N(A+x_n)}$ to the third term in the bracket to have;

$$\Rightarrow \quad hf_n+\left[\frac{(A+x_n)^2}{N(N-1)}\left(1+\frac{h}{A+x_n}\right)^N-\frac{(A+x_n)^2}{N(N-1)}-\frac{N(A+x_n)(A+x_n)h}{N(N-1)(A+x_n)}\right]f_n^{(1)} \tag{17}$$

$$\Rightarrow \quad y_{n+1}=y_n+hf_n+\frac{(A+x_n)^2f_n^{(1)}}{N(N-1)}\left[\left(1+\frac{h}{A+x_n}\right)^N-1-\frac{Nh}{A+x_n}\right] \tag{18}$$

When L=2 (i.e. the polynomial $P_j(x)$ is a quadratic):

$$P_j(x)=\sum_{j=0}^{2}a_jx^j=a_0x^0+a_1x^1+a_2x^2=a_0+a_1x+a_2x^2 \tag{19}$$

$$\mathrm{F}(x_n)=a_0+a_1x_n+a_2x_n^2+b(A+x_n)^N \tag{20}$$

By applying the above assumptions, one obtains the undetermined coefficients as;

$$b=\frac{(A+x_n)^3f_n^{(2)}}{N(N-1)(N-2)(A+x_n)^N} \qquad a_2=\frac{1}{2}\left[f_n^{(1)}-\frac{(A+x_n)}{(N-2)}f_n^{(2)}\right] \tag{22}$$

$$a_1=f_n-\left\{x_nf_n^{(1)}-x_n\frac{(A+x_n)f_n^{(2)}}{(N-2)}+\frac{(A+x_n)^3f_n^{(2)}}{(N-1)(N-1)}\right\} \tag{23}$$

Thus

$$y_{n+1}-y_n=hf_n+\frac{h^2}{2}f_n^{(1)}+\frac{(A+x_n)^3f_n^{(2)}}{N(N-1)(N-2)}\left[\left(1+\frac{h}{A+x_n}\right)^N-1-\left(Nh+\frac{N(N-1)}{2}\right)\left(\frac{h}{A+x_n}\right)^2\right]$$

Let us introduce $\frac{N(A+x_n)}{N(A+x_n)}$ to the third term in the bracket to have;

$$hf_n+\frac{h^2}{2}f_n^{(1)}+\frac{(A+x_n)^3f_n^{(2)}}{N(N-1)(N-2)}\left[\left(1+\frac{h}{A+x_n}\right)^N-1-\left(Nh+\frac{N(N-1)}{2}\left(\frac{h}{A+x_n}\right)\right)\right] \tag{24}$$

Notes

To generalize this integrator, we let

$$\mathrm{F(x)} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + b(A + x)^N \quad (25)$$

$$F(x_n) = a_0 + a_1 x_n^1 + a_2 x_n^2 + a_3 x_n^3 + \cdots + a_n x_n^n + b(A + x_n)^N \quad (26)$$

Notes

Let $(\mathrm{A}+x_n) = \emptyset_n and (\mathrm{A} + x_{n+1}) = \phi_{n+1}$ (27)

$$F(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + \cdots + a_n x_n^N + b\emptyset_n^N \quad (28)$$

And $F(x_{n+1}) = a_0 + a_1 x_n^1 + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + \cdots + a_n x_{n+1}^n + b\emptyset_{n+1}^N$ (27)

It follows (3) that

$$y_n = a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + \cdots + a_n x_n^n + b[\phi(n)]^N \quad (29)$$

And so

$$y_{n+1} = a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + \cdots + a_n x_{n+1}^n + b[\phi(x_{n+1})]^N \quad (30)$$

Subtraction equation (29) from (30) we have

$$y_{n+1} = y_n + a_1(x_{n+1} - x_n) + a_2(x_{n+1}^2 - x_n{}^2) + \ldots + a_n(x_{n+1}^i - x_n^i) + b[\phi(x_{n+1})]^N - [\phi(x_n)]^N \quad (31)$$

Since the mesh size is defined as $x_t = a + th$ and Continuing untox_t^n;

$x_t^n = (a + th)^n usingbinomialexpansion$

We obtain

$$x_{t+1}^n - x_t{}^n = na^{n-1}h + n(n-1)a^{n-2}th^2 + \frac{n(n-1)a^{n-2}h^2}{2!} + \frac{3n(n-1)(n-2)a^{n-3}t^3h^2}{3!} + \frac{3n(n-1)(n-2)a^{n-3}th^3}{3!} + \frac{(n-1)(n-2)a^{n-3}h^3}{3!} \quad (36)$$

Thus, one obtains:

$$y_{t+1} - y_t = a_0 + a_1 h + a_2 \left(2ah + h^2(1 + 2t)\right) + a_3 (3a^2h + 3a^2h(1 + 2t) + h^3 (3t^3 + 3t + 1)) + \cdots + a_n (x_{t+1}^n - x_t{}^n) \quad (37)$$

Also with the generalized interpolant;

$$F(x_t) = a_0 + a_1 x_t + a_2 x_t^2 + a_3 x_t^3 + \cdots + a_n x_t^n + b[\phi(x_t)]^N \quad (38)$$

This can be written as;

$$F(x) = \sum_{i=0}^{n} a_i\, x_t^i + b[\phi(x_t)]^N \quad (39)$$

By differentiating 6.1.29 nth times, one obtains;

Notes

$$F^{\,1}(x_t) = a_1 + 2a_2 x_{2t}^2 + \; 3a_3 x_t^2 + \cdots + n\, a_n x_t^{n-1} + bN[\phi(x_t)]^{N-1} = \; f_t \quad (40)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$F^{(n-1)} \; = (n-1)!\, a_{n-1} + n!\; a_n\, x_t + \cdots + n(n-1)(n-2) \ldots\, (n - [(n-1)-1]) a_n x_t^{n-(n-1)}$$
$$+ bN\,(N-1)(N-2)\; \ldots\; (N - [(n-1)-1])\; \phi\,(x_t)^{N-(n-1)} = \; f_t^{(n-1)-1} \quad (41)$$

$$F^{\,n} \; = n!\, a_n + bN\,(N-1)(N-2)\; \ldots\; (N - [(n-1)-1])\; \phi\,(x_t)^{N-n} = \; f_t^{(n-1)-1} \quad (42)$$

$$F^{\,n} = n(n+1)(n-2) \ldots\, (n - [(n-1)])\;\; a_n \;\; + \cdots$$
$$+ bN\,(N-1)(N-2)\; \ldots\; (N-n)\; \phi\,(x_t)^{N-(n+1)} = \; f_t^{(n-1)-1} \quad (43)$$

$$f_t^{\,(n)} = bN\,(N-1)(N-2)(N-3)\; \ldots\; (N-n)[A + x_t]^{N-(n+1)} \quad (44)$$

Thus, the undetermined coefficients are obtained asfollows:

$$b \; = \frac{[A + x_t]^{n+1} f_t^{\,(n)}}{N\,(N-1)(N-2)(N-3)\; \ldots\; (N-n)[A + x_t]^N} \quad (45)$$

$$a_n = \frac{1}{n!}\left[f_t^{(n-1)} \; - \frac{[A + x_t]}{(N-n)} f_t^{\,(n)} \right] \quad (46)$$

$$a_{n-1} \; = \frac{1}{(n-1)!}\left(f_t^{\,(n-2)} \; - x_t f_t^{(n-1)} - \; \left[\frac{(N-n+2)^{(A+x_t)^2}}{(N-n)(N-(n-1)} - \frac{x_t(A + \; x_t)}{(N-n)} \right] f_t^{\,(n)} \right) \quad (47)$$

$$a_{n-2} = \frac{1}{(n-2)!}\left[\begin{array}{c} f_t^{(n-3)} - x_t\, f_t^{(n-2)} + x_t^2 f_t^{(n-1)} \\ + f_t^{(n)} \left\{ \dfrac{x_t\,(A + \; x_t)^2}{(N-(n-1))(N-n)} - \dfrac{x_t^{\,2}(A + \; x_t)}{(N-n)} \dfrac{(A + \; x_t)^3}{(N-(n-2))\,(N-(n-1))(N-n)} \right\} \end{array} \right] \quad (48)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$a_5 = \frac{1}{5!}\left[\begin{array}{c} f_t^{(4)} - 720\; a_6 x_t - \cdots - n(n-1) \ldots (n-4) a_n\, x^{n-5} \\ - bN(N-1) \ldots (n-4)[A + x_t]^{N-5} \end{array} \right] \quad (49)$$

$$a_4 = \frac{1}{4!}\begin{bmatrix} f_t^{(3)} - 120\, a_5 x_t - \cdots - n(n-1)\ldots(n-4)a_n\, x^{n-4} \\ -bN(N-1)\ldots(N-3)[A+x_t]^{N-4} \end{bmatrix} \tag{50}$$

$$a_3 = \frac{1}{3!}\begin{bmatrix} f_t^{(2)} - 24\, a_4 x_t - \cdots - \; n\,(n-1)(n-2)a_n\, x^{n-3} \\ -bN\,(N-1)(N-2)[A+x_t]^{N-3} \end{bmatrix} \tag{51}$$

$$a_2 = \frac{1}{2!}\left[f_t^{(1)} - \; 6\, a_3 x_t - \cdots - \; n\,(n-1)a_n\, {x_t}^{n-2} - \; bN\,(N-1)[A+x_t]^{N-2}\right] \tag{52}$$

$$a_1 = \; [f_t - \; 2a_2 x_t - 3a_3\, {x_t}^2 - \cdots - \; n\, a_n\, {x_t}^{n-1} - \; bN\, [A+x_t]^{N-1}] \tag{53}$$

In all, by substituting the undetermined coefficients appropriately, one obtains;

$$y_{n+1} - y_n = \sum_{K=1}^{L} \frac{h^k}{k!} f_n^{(k-1)} + \frac{(A+x_n)^{L-1}}{\alpha_L^N} f_n^L \left[\left(1+\frac{h}{A+x_n}\right)^N - 1 - \sum_{K=1}^{L=N} \frac{K-1}{K!}\left(\frac{h}{A+X_n}\right)\right]$$

Prove of Convergence for the Scheme

According to Henrici (1962): we define any algorithm for solving a differentialequation in which the approximation y_{t+1} to the solution at the x_{t+1} can be calculated if only x_t, y_t *and* h are known as a ONE-STEP METHOD. We proceed to establish that our numerical algorithm is one step methods. From (2), the numerical integrator generated is given by (). If we expand $\left(\left[1+\frac{h}{A+x_n}\right]\right)^N$ by binomial expansion and taking N as a real, we shall have

$$= h\left\{\frac{1}{h} + \frac{N}{A+x_n} + \sum_{i=1}^{\infty} \frac{N!}{(N-(i+1))!}\left(\frac{h^i}{(i+1)!\,(A+x_n)^{(i+1)}}\right)\right\}$$

This implies

$$y_{n+1} = y_n + \mathrm{h}\left(\left(\sum_{K=1}^{L} \frac{h^{K-1}}{K!} f_n^{(k-1)}\right) + \frac{(A+x_n)}{\alpha_L^N} f_n^{(L)} \left\{\frac{N}{A+x_n} + \beta - \sum_{K=1}^{L} \Psi\left(\frac{h^{K-1}}{(A+x)^K}\right)\right\}\right) \tag{56}$$

Thus

$$y_{n+1} = y_n + h\left\{\sum_{k=1}^{L} \left(G f_n^{(k-1)} + \gamma f_n^{(L)}\right)\right\} \tag{57}$$

$$y_{n+1} = y_n + h\theta(x_n, y_n; h) \tag{58}$$

$$\phi(x_n, y_n; h) = \sum_{k=1}^{L} \left(G f_{(x_n, y_n)}^{(k-1)} + \gamma f_{(x_n, y_n)}^{(L)}\right) \tag{59}$$

where

$$G = \frac{h^{K-1}}{K!}\;;\quad \gamma = \frac{(A+x_n)^1}{\alpha_L^N}\left\{\left(\frac{N!}{(A+x_n)^1}\right) + \beta - \sum_{K=1}^{L} \Psi\left(\frac{h^{K-1}}{(A+x)^K}\right)\right\}\;;\Psi = \frac{\alpha_{K-1}^N}{K!}\;;\beta = \sum_{i=1}^{\infty} \frac{N!}{(N-(i+1))!}\left(\frac{h^i}{(i+1)!(A+x_n)^{(i+1)}}\right)$$

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where $\vartheta(x_t, y_t; h)$ is called the increment function.

Derivationofthelocation andnatureofthepointofsingularity

To derive A(n) and N(n), we make use of the Taylor series expansion of (55). This gives the following expression for the truncation error:

$$T.E = y_{n+1} - y(x_{n+1}) \tag{63}$$

$$T.E = \sum_{q=1}^{\infty}[-f_n^{(L+q)} + \frac{\alpha_{q-1}^{N-L-1}}{(A+x_n)^q} f_n^{(L)}]\frac{h^{L+q+1}}{(L+q+1)!} \tag{64}$$

$$T_q = -f_n^{(L+q)} + \frac{\alpha_{q-1}^{N-L-1}}{(A+x_n)^q} f_n^{(L)}$$

The values of the parameters A(n) and N(n) are now chosen to satisfy

$$T_1 = T_2 = 0$$

So that :

$$T.E_1 = -f_n^{(L+1)} + \frac{\alpha_0^{N-L-1}}{(A+x_n)^0} f_n^{(L)} = 0 \tag{65}$$

$$T.E_2 = -f_n^{(L+2)} + \frac{\alpha_1^{N-L-1}}{(A+x_n)^2} f_n^{(L)} = 0 \tag{66}$$

$$\frac{-(A+x_n)^1 f_n^{(L+1)} + \alpha_0^{N-L-1} f_n^{(l)}}{(A+x_n)^1} = 0 \tag{67}$$

It can be shown that;

$$-Af_n^{(L+1)} = x_n f_n^{(L+1)} - \alpha_0^{N-L-1} f_n^{(L)} \tag{68}$$

$$-A(n) = x_n - \frac{\alpha_0^{N-L-1} f_n^{(L)}}{f_n^{(L+1)}} \tag{69}$$

$$x_n^2 f_n^{(L+2)} - \frac{4x_n\alpha_0^{N-L-1} f_n^{(L)} f_n^{(L+2)}}{f_n^{(L+1)}} + \left(\frac{\alpha_0^{N-L-1} f_n^{(L)}}{f_n^{(L+1)}}\right)^2 f_n^{(L+2)} = -\alpha_1^{N-L-1} f_n^{(L)} \tag{70}$$

From the above, one obtains;

$$\left[x_n^2 - 2x_n\left(\frac{\alpha_0^{N-L-1} f_n^{(L)}}{f_n^{(L+1)}}\right) + \left(\frac{\alpha_0^{N-L-1} f_n^{(L)}}{f_n^{L+1}}\right)^2 + 2x_n^2 - \frac{2x_n\alpha_0^{N-L-1} f_n^{(L)}}{f_n^{L+1}}\right] f_n^{L+1} - x_n^2 f_n^{(L+2)} = -\alpha_0^{N-L-1} f_n^{(L)} \tag{71}$$

$$x_n^2\left(f_n^{(L+1)}\right)^2 f_n^{(L+2)} - \left(4x_n\alpha_0^{N-L-1} f_n^{(L+1)} - \left(\alpha_0^{N-L-1}\right)^2 f_n^{(L)}\right) f_n^{(L)} f_n^{(L+2)} = -\alpha_1^{N-L-1} f_n^{(L)}\left(f_n^{(L+2)}\right)^2 \tag{72}$$

$$\frac{(N-L-2)}{(N-L-1)^1} = \left(\frac{f_n^{(L)}}{(f_n^{(L+1)})^2}\right)^1 f_n^{(L+2)} \tag{73}$$

$$N(f_n^{(L+1)})^2 - (L+2)(f_n^{(L+1)})^2 = Nf_n^{(L)} f_n^{(L+2)} - (L+1) f_n^{(L)} f_n^{(L+2)} \tag{74}$$

This result to;

$$N(n)=\frac{L\left[(f_n^{(L+1)})^2-f_n^{(L)}f_n^{(L+2)}\right]+(f_n^{(L+1)})^2-f_n^{(L)}f_n^{(L+2)}}{\left[(f_n^{(L+1)})^2\ f_n^{(L)}\ f_n^{(L+2)}\right]} \quad (75)$$

$$N(n)=(L+1)\frac{\left[(f_n^{(L+1)})^2\right]}{\left[(f_n^{(L+1)})^2\ f_n^{(L)}\ f_n^{(L+2)}\right]} \quad (76)$$

Substitude (76) into (69) to obtain the value of A(n) as follow:

$$-A(n)=x_n-\left[[L+1]+\frac{\left[(f_n^{(L+1)})^2\right]}{\left[(f_n^{(L+1)})^2\ f_n^{(L)}\ f_n^{(L+2)}\right]}-L-1\right]\frac{f_n^{(L)}}{f_n^{(L+1)}} \quad (77)$$

This gives;

$$-A(n)=x_n-\frac{\left[f_n^{(L+1)}\right]\ f_n^{(L)}}{\left[(f_n^{(L+1)})^2-f_n^{(L)}\ f_n^{(L+2)}\right]} \quad (78)$$

In the above derivation,N(n) is the nature of singularity and A(n) is the location of singularity.

III. Convergence Theorem

Let the function Φ(x,y;h) be continuous (jointly as a function of its three arguments) in the region defined by x $x\epsilon[a,b]$, y ϵ (a,x) $0\le h\le h_0$, where $h_0>0$, and let there exist a constant L such that

$$\left|\Phi(x,y^*;h)-\Phi(x,y;h)\right|\le L\left|y^*-y\right|, \quad (79)$$

for all (x,y;h) and (x,y*;h) in the region just defined. Then the relation $\Phi(x,y;0)=f(x,y)$ is a necessary and sufficient condition for theconvergence of the method defined by the incrementfunction, Φ. With the increment function deducted from the formula or scheme.

$$\phi\left(x_n,y_n^*;h\right)=\sum_{k=1}^{L}\left[Af^{(k-1)}_{(x_n,y_n^*)}\right]+Bf^{(L)}_{(x_n,y_n^*)}+Cf^{(L)}_{(x_n,y_n^*)}+\sum_{k=1}^{L}\left[Df^{(L)}_{(x_n,y_n^*)}\right] \quad (81)$$

Hence

$$\phi\left(x_n,y_n^*;h\right)-\phi\left(x_n y_n;h\right)=\sum_{k=1}^{L}\left[Af^{(k-1)}_{(x_n,y_n^*)}\right]+\sum_{k=1}^{L}\left[Af^{(k-1)}_{(x_n,y_n)}\right]+Bf^{(L)}_{(x_n,y_n^*)}-Bf^{(L)}_{(x_n,y_n)}+Cf^{(L)}_{(x_n,y_n^*)}-Cf^{(L)}_{(x_n,y_n)}+\sum_{k=1}^{L}Df^{(L)}_{(x_n,y_n^*)}-\sum_{k=1}^{L}Df^{(L)}_{(x_n,y_n)} \quad (82)$$

$$=\sum_{K=1}^{L}\left[A\left(f^{(k-1)}_{(x_n,y_n^*)}\right)-f^{(k-1)}_{(x_n,y_n)}\right]+B\left(f^{(L)}_{(x_n,y_n^*)}-f^{(L)}_{(x_n,y_n)}\right)+C\left(f^{(L)}_{(x_n,y_n^*)}-f^{(L)}_{(x_n,y_n)}\right)+\sum_{K=1}^{L}\left[D\left(f^{(L)}_{(x_n,y_n^*)}\right)-f^{(L)}_{(x_n,y_n)}\right] \quad (83)$$

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Let y_t be defined as a point in the interior of the interval whose endpoints are y and y*, if we apply the mean value, we have

$$f(x_n, y_n^*) - f(x_n, y_n) = \frac{\partial f(x_n, y)}{\partial y_n}(y_n^* - y_n),\ f^{(1)}(x_n, y_n^*) - f^{(1)}(x_n, y_n) = \frac{\partial f^{(1)}(x_n, y)}{\partial y_n}(y_n^* - y_n),\ldots,$$

$$f^{(L)}_{(x_n, y_n^*)} - f^{(L)}_{(x_n, y_n)} = \frac{\partial f^{(L)}_{(x_n, y)}}{\partial y_n}(y_n^* - y_n) \text{ And } f^{(k-1)}_{(x_n, y_n^*)} - f^{(k-1)}_{(x_n, y_n)} = \frac{\partial f^{(k-1)}_{(x_n, y)}}{\partial y_n}(y_n^* - y_n) \tag{84}$$

If we defined

$$L_1 = \sup_{1(x_n, \bar{l}_n)\in Dom} \frac{\partial f(x_n, \bar{y}_n)}{\partial y_n} \ldots L_K = \sup_{(x_n, \bar{l}_n)\in Dom} \frac{\partial f^{(L)}_{(x_n, \bar{y}_n)}}{\partial y_n} \text{ and } L_L = \sup_{(x_n, \bar{l}_n)\in Dom} \frac{\partial f^{(L-1)}_{(x_n, \bar{y}_n)}}{\partial y_n}$$

Put equations

$$\phi(x_n, y_n^*; h) - \phi(x_n y_n; h) = \sum_{k=1}^{L} A\left(\frac{\partial f^{(k-1)}_{(x_n, y)}}{\partial y_n}(y_n^* - y_n)\right) + B\left(\frac{\partial f^{(L)}_{(x_n, y)}}{\partial y_n}(y_n^* - y_n)\right) + C\left(\frac{\partial f^{(L)}_{(x_n, y)}}{\partial y_n}(y_n^* - y_n)\right) + \sum_{k=1}^{L} D\left(\frac{\partial f^{(L)}_{(x_n, y)}}{\partial y_n}(y_n^* - y_n)\right)$$

$$= \phi(x_n, y_n^*; h) - \phi(x_n y_n; h)\left[L_L\left(B + C + \sum_{k=1}^{L} D\right) + L_k \sum_{k=1}^{L} A\right](y_n^* - y_n) \tag{85}$$

Taking the absolute value of both sides, we have

$$\left|\phi(x_n, y_n^*; h) - \phi(x_n y_n; h)\right| \le \left|L_L\left(B + C + \sum_{K=1}^{L} D\right) + L_K \sum_{K=1}^{L} A\right|\left|(y_n^* - y_n)\right| \tag{86}$$

$$\text{Let } K = \left|L_L\left(B + C + \sum_{K=1}^{L} D\right) + L_K \sum_{K=1}^{L} A\right| \tag{87}$$

$$\text{Thus } \left|\phi(x_n, y_n^*; h) - \phi(x_n y_n; h)\right| \le K\left|(y_n^* - y_n)\right| \tag{88}$$

which is the condition for convergence.

IV. Consistency

$$\phi(x_n, y_n; 0) = f(x_n, y_n) \tag{89}$$

If put $h = 0$

$$y_{n+1} = y_n + \sum_{K=1}^{L} \frac{0^k}{K!} f_n^{(k-1)} + \frac{(A + x_n)^{L+1}}{\alpha_L^N} f_n^{(L)}\{1^N + 0 - 1 - 0\} \tag{90}$$

$$y_{n+1} = y_n \Rightarrow f(x_n, y_n) \tag{91}$$

$$y_{n+1} = y_n + h\left\{\left(B + C + \sum_{k=1}^{L} D\right) f^{(L)}_{(x_n, y_n)} + \left(\sum_{K=1}^{L} A\right) f^{(K-1)}_{(x_n, y_n)}\right\} \tag{93}$$

$$l_{n+1} = l_n + h\left\{\left(B + C + \sum_{k=1}^{L} D\right) f^{(L)}_{(x_n, l_n)} + \left(\sum_{K=1}^{L} A\right) f^{(K-1)}_{(x_n, l_n)}\right\} \tag{97}$$

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The application of mean value theorem and the subtraction of 4.6 and 4.6, one obtains;

$$y_{n+1}-l_{n+1}=y_n-l_n+h\left[\sup_{(x_n,\bar{l}_n)\in Dom}\frac{\partial f^{(L)}}{\partial l_n}(x_n,\bar{l}_n)\left(B+C+\sum_{K=1}^{L}D\right)+\sup_{(x_n,\bar{l}_n)\in Dom}\frac{\partial f^{(K-1)}}{\partial l_n}(x_n,\bar{l}_n)\left(\sum_{K=1}^{L}A\right)\right]$$

$$=y_n-l_n+h\left[\left(B+C+\sum_{K=1}^{L}D\right)L_L+\left(\sum_{K=1}^{L}A\right)L_{K-1}\right](y_n-l_n) \quad (98)$$

$$|y_{n+1}-l_{n+1}|\le|y_n-l_n|+|h|\|PL_L+ML_{K-1}\||y_n-l_n| \quad (99)$$

If $\quad 1+hS=R$, $S=|PL_L+ML_{K-1}|$, $y_n=\lambda^*$ and $l_n=\lambda$

then, $\quad |y_{n+1}-l_{n+1}|\le R|\lambda^*-\lambda|\Rightarrow|y_{n+1}-l_{n+1}|\le[1+hS]|y_n-l_n|\Rightarrow|y_{n+1}-l_{n+1}|\le R|y_n-l_n| \quad (100)$

V. Conclusion

If in (2), the parameter A is regarded as undetermined coefficients and eliminated in the same way as b *and* $a_p(p=0,1,\ldots L)$, another class of formulae would emerge, which is given as:

$$y_{n+1}-y_n=\frac{hf_{n+1}^{(N/(N-1))}-f_n^{(N/(N-1))}}{Nf_{n+1}^{(1/(N-1))}-f_n^{(1/(N-1))}}\ ,N\neq 0 \quad (60)$$

This shall be used to construct a subroutine called **GENFOR**, which shall be able to jump the point of singularity.

Ibijola, et (2004) constructed a one-step method, which was based on the non-linear interpolant:

$$F(x)=\frac{C}{1+ae^{\lambda x}}\ , \quad (61)$$

where C and a are real constants.

The resulting integrator is:

$$y_{n+1}=\frac{\lambda y_n^2}{\lambda y_n+(e^{\lambda x}-1)hy_n'}. \quad (62)$$

This is capable of skipping the point of singularity if the mesh size is carefully selected. This scheme can't give any information concerning the location and nature of singularity. However, it will be used for the construction of another subroutine called GENDOR, which could be preferred where GENFOR might not be strong enough to give a better approximation, hereafter, the programme retunes to (55) for a continuation after the point of singularity.

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Global Journal of Science Frontier Research (F) Volume XII Issue XI Version I Year 2012

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 11 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Certain Indefinite Integrals Involving Lucas Polynomials and Harmonic Number

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Abstract - In this paper we have established certain indefinite integrals involving Harmonic number and Lucas Polynomials. The results represent here are assume to be new.

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I. Introduction and Preliminaries

a) Harmonic Number

The n^{th} harmonic number is the sum of the reciprocals of the first n natural numbers:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \tag{1.1}$$

Harmonic numbers were studied in antiquity and are important in various branches of number theory. They are sometimes loosely termed harmonic series, are closely related to the Riemann zeta function, and appear in various expressions for various special functions.
An integral representation is given by Euler

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx \tag{1.2}$$

The equality above is obvious by the simple algebraic identity below

$$\frac{1-x^n}{1-x} = 1 + x + ... + x^n \tag{1.3}$$

An elegant combinatorial expression can be obtained for H_n using the simple integral transform $x = 1 - u$:

$$H_n = \int_0^1 \frac{1-x^n}{1-x} = -\int_1^0 \frac{1-(1-u)^n}{u} du = \int_0^1 \frac{1-(1-u)^n}{u} du$$

$$= \int_0^1 \left[\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} u^{k-1} \right] du$$

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$$= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \int_0^1 u^{k-1} du$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} \binom{n}{k} \tag{1.4}$$

b) *Lucas polynomials*

The sequence of Lucas polynomials is a sequence of polynomials defined by the recurrence relation

$$L_n(x) = \begin{cases} 2x^0 = 2 & , \text{ if } n = 0 \\ 1x^1 = x & , \text{ if } n = 1 \\ x^1 L_{n-1}(x) + x^0 L_{n-2}(x) & , \text{ if } n \geq 2 \end{cases} \tag{1.5}$$

The first few Lucas polynomials are:

$$L_0(x) = 2$$

$$L_1(x) = x$$

$$L_2(x) = x^2 + 2$$

$$L_3(x) = x^3 + 3x$$

$$L_4(x) = x^4 + 4x^2 + 2$$

The ordinary generating function of the Lucas polynomials is

$$G_{\{L_n(x)\}}(t) = \sum_{n=0}^{\infty} L_n(x) t^n = \frac{2 - xt}{1 - t(x + t)}. \tag{1.6}$$

c) *Polylogarithm*

The polylogarithm (also known as Jonquire's function) is a special function $Li_s(z)$ that is defined by the infinite sum, or power series:

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \tag{1.7}$$

It is in general not an elementary function, unlike the related logarithm function. The above definition is valid for all complex values of the order s and the argument z where$| z |< 1$.The polylogarithm is defined over a larger range of z than the above definition allows by the process of analytic continuation.

The special case s = 1 involves the ordinary natural logarithm ($Li_1(z) = -ln(1 - z)$) while the special cases s = 2 and s = 3 are called the dilogarithm (also referred to as Spence's function) and trilogarithm respectively. The name of the function comes from the fact that it may alternatively be defined as the repeated integral of itself, namely that

$$Li_{s+1}(z) = \int_0^z \frac{Li_s(t)}{t} dt \tag{1.8}$$

Notes

Thus the dilogarithm is an integral of the logarithm, and so on. For nonpositive integer orders s, the polylogarithm is a rational function.

The polylogarithm also arises in the closed form of the integral of the FermiDirac distribution and the Bose-Einstein distribution and is sometimes known as the Fermi- Dirac integral or the Bose-Einstein integral. Polylogarithms should not be confused with polylogarithmic functions nor with the offset logarithmic integral which has a similar notation.

Notes

d) *Generalized Gaussian Hypergeometric Function*

Generalized ordinary hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{lcc} a_1, a_2, \cdots, a_A & ; & \\ & & z \\ b_1, b_2, \cdots, b_B & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} $$

or

$$ {}_AF_B\left[\begin{array}{lcc} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{lcc} (a_j)_{j=1}^A & ; & \\ & & z \\ (b_j)_{j=1}^B & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1.9} $$

where denominator parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers.

II. Main Indefinite Integrals

$$ \int \frac{\sinh x \; H_1^{(x)} L_1(x)}{\sqrt{1-\cos x}} \, \mathrm{dx} = -\frac{1}{\sqrt{1-\cos x}} \left(\frac{8}{25} - \frac{6\iota}{25}\right) e^{(-1-\frac{\iota}{2})x} \sin\frac{x}{2} \times $$
$$ \times \Big[2e^{2x} {}_3F_2\Big(-\frac{1}{2}-\iota, -\frac{1}{2}-\iota, 1; \frac{1}{2}-\iota, \frac{1}{2}-\iota; e^{\iota x}\Big) - 2e^{\iota x} {}_3F_2\Big(\frac{1}{2}+\iota, \frac{1}{2}+\iota, 1; \frac{3}{2}+\iota, \frac{3}{2}+\iota; e^{\iota x}\Big) - $$
$$ -(2-\iota) x e^{2x} {}_2F_1\Big(-\frac{1}{2}-\iota, 1; \frac{1}{2}-\iota; e^{\iota x}\Big) - (2-\iota) x e^{\iota x} {}_2F_1\Big(\frac{1}{2}+\iota, 1; \frac{3}{2}+\iota; e^{\iota x}\Big) + $$
$$ +(2-\iota) x e^{2x} - 2e^{2x} \Big] + Constant \tag{2.1} $$

$$ \int \frac{\sin x \; H_1^{(x)} L_1(x)}{\sqrt{1-\cosh x}} \, \mathrm{dx} = \frac{1}{25\sqrt{1-\cosh x}} e^{-\iota x}(e^x - 1) \Big[-(8+6\iota) {}_3F_2\Big(\frac{1}{2}-\iota, \frac{1}{2}-\iota, 1; \frac{3}{2}-\iota, \frac{3}{2}-\iota; e^x\Big) - $$
$$ -(8-6\iota) e^{2\iota x} {}_3F_2\Big(\frac{1}{2}+\iota, \frac{1}{2}+\iota, 1; \frac{3}{2}+\iota, \frac{3}{2}+\iota; \cosh x + \sinh x\Big) + 5x \Big\{ (2-\iota) {}_2F_1\Big(\frac{1}{2}-\iota, 1; \frac{3}{2}-\iota; e^x\Big) + $$
$$ +(2+\iota) e^{2\iota x} {}_2F_1\Big(\frac{1}{2}+\iota, 1; \frac{3}{2}+\iota; \cosh x + \sinh x\Big) \Big\} \Big] + Constant \tag{2.2} $$

$$ \int \frac{\cos x \; H_1^{(x)} L_1(x)}{\sqrt{1-\cosh x}} \, \mathrm{dx} = -\frac{1}{25\sqrt{1-\cosh x}} e^{-\iota x}(e^x - 1) \Big[(6-8\iota) {}_3F_2\Big(\frac{1}{2}-\iota, \frac{1}{2}-\iota, 1; \frac{3}{2}-\iota, \frac{3}{2}-\iota; e^x\Big) + $$
$$ +(6+8\iota) e^{2\iota x} {}_3F_2\Big(\frac{1}{2}+\iota, \frac{1}{2}+\iota, 1; \frac{3}{2}+\iota, \frac{3}{2}+\iota; \cosh x + \sinh x\Big) + 5x \Big\{ (1+2\iota) {}_2F_1\Big(\frac{1}{2}-\iota, 1; \frac{3}{2}-\iota; e^x\Big) + $$
$$ +(1-2\iota) e^{2\iota x} {}_2F_1\Big(\frac{1}{2}+\iota, 1; \frac{3}{2}+\iota; \cosh x + \sinh x\Big) \Big\} \Big] + Constant \tag{2.3} $$

$$\int \frac{\sin x\ H_1^{(x)} L_1(x)}{\sqrt{1-\sin x}}\,\mathrm{dx} = \frac{2}{\sqrt{1-\sin x}}\Big(\cos\frac{x}{2}-\sin\frac{x}{2}\Big)\Bigg[\frac{1}{\sqrt{2}}\Bigg\{\pi\tanh^{-1}\Bigg(\frac{\tan\frac{x}{4}+1}{\sqrt{2}}\Bigg)+$$
$$+\frac{1}{2}\Big(4\iota Li_2\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-4\iota Li_2\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-(\pi-2x)\Big(\log\Big(1-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-$$
$$-\log\Big(1+(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)\Big)\Big)\Bigg\}-(x-2)\sin\frac{x}{2}-(x+2)\cos\frac{x}{2}\Bigg]+Constant \qquad (2.4)$$

$$\int \frac{\cot x\ H_1^{(x)} L_1(x)}{\sqrt{1-\sin x}}\,\mathrm{dx} = \frac{1}{2\sqrt{1-\sin x}}\Big(\cos\frac{x}{2}-\sin\frac{x}{2}\Big)\Big[4\iota Li_2(-e^{\frac{\iota x}{2}})+4\iota Li_2(-\iota e^{\frac{\iota x}{2}})-$$
$$-4\iota Li_2(\iota e^{\frac{\iota x}{2}})-4\iota Li_2(e^{\frac{\iota x}{2}})-\iota\pi x+2x\log(1-e^{\frac{\iota x}{2}})+2x\log(1-\iota e^{\frac{\iota x}{2}})-2x\log(1+e^{\frac{\iota x}{2}})-$$
$$-2x\log(1+\iota e^{\frac{\iota x}{2}})+2\pi\log(1-\iota e^{\frac{\iota x}{2}})+2\pi\log(1+\iota e^{\frac{\iota x}{2}})-2\pi\log\Big(\sin\frac{x+\pi}{4}\Big)-$$
$$-2\pi\log\Big(-\cos\frac{x+\pi}{4}\Big)\Big]+Constant \qquad (2.5)$$

$$\int \frac{\tan x\ H_1^{(x)} L_1(x)}{\sqrt{1-\sec x}}\,\mathrm{dx} = \frac{1}{8\sqrt{\frac{(-1+e^{\iota x})^2}{1+e^{2\iota x}}}\sqrt{1+e^{2\iota x}}}\Bigg[-\Big(\iota(-1+e^{\iota x})\Big(4Li_2\Big(\frac{1}{2}-\frac{1}{2}\sqrt{1+e^{2\iota x}}\Big)-$$
$$-4Li_2\Big(e^{-2\sinh^{-1}(e^{\iota x})}\Big)-\log^2(-e^{2\iota x})-2\log^2\Big(\frac{1}{2}\Big(1+\sqrt{1+e^{2\iota x}}\Big)\Big)+$$
$$+4\log(-e^{2\iota x})\log\Big(\frac{1}{2}\Big(1+\sqrt{1+e^{2\iota x}}\Big)\Big)-8\iota x\log\Big(\sqrt{1+e^{2\iota x}}+e^{\iota x}\Big)+4\sinh^{-1}(e^{\iota x})^2+$$
$$+8\iota x\tanh^{-1}\Big(\sqrt{1+e^{2\iota x}}\Big)+8\sinh^{-1}(e^{\iota x})\log\Big(1-e^{-2\sinh^{-1}(e^{\iota x})}\Big)-$$
$$-4\log(-e^{2\iota x})\tanh^{-1}\Big(\sqrt{1+e^{2\iota x}}\Big)\Big)\Big)\Bigg]+Constant \qquad (2.6)$$

$$\int \frac{\cot x\ H_1^{(x)} L_1(x)}{\sqrt{1-\operatorname{cosec} x}}\,\mathrm{dx} = \frac{1}{8(e^{\iota x}-\iota)}\sqrt{\frac{(e^{\iota x}-\iota)^2}{-1+e^{2\iota x}}}\Bigg[\sqrt{-1+e^{2\iota x}}\Bigg\{\frac{1}{\sqrt{-1+e^{2\iota x}}}\sqrt{1-e^{2\iota x}}\times$$
$$\times\Big(-4Li_2\Big(\frac{1}{2}-\frac{1}{2}\sqrt{1-e^{2\iota x}}\Big)+\log^2(e^{2\iota x})+2\log^2\Big(\frac{1}{2}\Big(1+\sqrt{1-e^{2\iota x}}\Big)\Big)-$$
$$-4\log\Big(\frac{1}{2}\Big(1+\sqrt{1-e^{2\iota x}}\Big)\Big)\log(e^{2\iota x})\Big)+4(2\iota x-\log(e^{2\iota x}))\tan^{-1}\sqrt{-1+e^{2\iota x}}\Bigg\}-$$
$$-4\sqrt{1-e^{2\iota x}}\Big(Li_2\Big(e^{-2\iota\sin^{-1}(e^{\iota x})}\Big)+2\iota x\log(\sqrt{1-e^{2\iota x}}+\iota e^{\iota x})+\sin^{-1}(e^{\iota x})^2-$$
$$-2\iota\sin^{-1}(e^{\iota x})\log\Big(1-e^{-2\iota\sin^{-1}(e^{\iota x})}\Big)\Big)\Bigg]+Constant \qquad (2.7)$$

$$\int \frac{\tan x\ H_1^{(x)} L_1(x)}{\sqrt{1-\cos x}}\,\mathrm{dx} = \frac{1}{2\sqrt{2-2\cos x}}\sin\frac{x}{2}\Bigg[8\iota Li_2\Big(-\frac{(1+\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)+8\iota Li_2\Big(\frac{(1-\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)+$$
$$+8\iota Li_2\Big(-\frac{(1+\iota)(\cos\frac{x}{2}+\iota\sin\frac{x}{2})}{\sqrt{2}}\Big)+8\iota Li_2\Big(\frac{(1+\iota)(\sin\frac{x}{2}-\iota\cos\frac{x}{2})}{\sqrt{2}}\Big)+2\iota x^2-2\iota\pi x+$$
$$+4x\log\Big(1-\frac{(1-\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)+4x\log\Big(1+\frac{(1+\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)-4\pi\log\Big(1-\frac{(1-\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)-$$

Notes

$$-4\pi\log\Big(1+\frac{(1+\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)+16\sin^{-1}\Big(\frac{\sqrt{2+\sqrt{2}}}{2}\Big)\log\Big(1-\frac{(1-\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)-$$

$$-16\sin^{-1}\Big(\frac{\sqrt{2+\sqrt{2}}}{2}\Big)\log\Big(1+\frac{(1+\iota)e^{-\frac{\iota x}{2}}}{\sqrt{2}}\Big)+4\pi\log\Big(2\sin\frac{x}{2}+\sqrt{2}\Big)-$$

$$-4x\log\Big(-\frac{(1+\iota)\sin\frac{x}{2}}{\sqrt{2}}-\frac{(1-\iota)\cos\frac{x}{2}}{\sqrt{2}}+1\Big)-4x\log\Big(-\frac{(1-\iota)\sin\frac{x}{2}}{\sqrt{2}}+\frac{(1+\iota)\cos\frac{x}{2}}{\sqrt{2}}+1\Big)-$$

$$-32\sin^{-1}\Big(\frac{\sqrt{2+\sqrt{2}}}{2}\Big)\tanh^{-1}\Big(\frac{(\sqrt{2}-2)\cot\frac{x+\pi}{4}}{\sqrt{2}}\Big)+\iota\pi^2\Big]+Constant \qquad (2.8)$$

Notes

III. Derivation of the Integrals

Involving the same parallel method of ref[8] , one can derive the integrals.

IV. Conclusion

In our work we have established certain indefinite integrals involving Lucas Polynomials , Harmonic Number , and Hypergeometric function . However, one can establish such type of integrals which are very useful for different field of engineering and sciences by involving these integrals.Thus we can only hope that the development presented in this work will stimulate further interest and research in this important area of classical special functions.

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Notes

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 11 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Some Remarks on Product Summability of Sequences

By Suyash Narayan Mishra
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Abstract - In [4], the definition of product summability method $(\boldsymbol{DD},\boldsymbol{kk})$(C,l) for functions was given and some of its properties were investigated. In [2], $(\boldsymbol{D},\boldsymbol{k})\ (\boldsymbol{C},\boldsymbol{\alpha},\boldsymbol{\beta})\ (k>0, \alpha>0, \beta>-1)$ summability for functions are defined and some of its properties are investigated. In this paper $(\boldsymbol{D},\boldsymbol{k})(\boldsymbol{C},\boldsymbol{\alpha},\boldsymbol{\beta})\ (k>0,\alpha>0,\beta>-1)$ summability for sequences are defined and some of its properties investigated.

GJSFR-F Classification : MSC 2010: 28A20

SOME REMARKS ON PRODUCT SUMMABILITY OF SEQUENCES

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Some Remarks on Product Summability of Sequences

Suyash Narayan Mishra

Abstract - In [4], the definition of product summability method $(\boldsymbol{D},\boldsymbol{k})$(C,l) for functions was given and some of its properties were investigated . In [2], $(\boldsymbol{D},\boldsymbol{k})(\boldsymbol{C},\boldsymbol{\alpha},\boldsymbol{\beta})$ $(k>0,\alpha>0,\beta>-1)$ summability for functions are defined and some of its properties are investigated. In this paper $(\boldsymbol{D},\boldsymbol{k})(\boldsymbol{C},\boldsymbol{\alpha},\boldsymbol{\beta})$ $(k>0,\alpha>0,\beta>-1)$ summability for sequences are defined and some of its properties investigated.

I. Introduction

Kuttner [1], introduced the summability method for functions and investigated some of its properties. Pathak [4], defined the summability method for functions and investigated some of its properties. Mishra and Srivastava [3], introduced the summability method for functions by generalizing summability method. Mishra and Mishra [2], introduced the summability method for functions and investigated some of its properties. In this paper we define summability method for sequences and investigate some of its properties.

II. Some Relations and Definitions

Let $f(x)$ be any function which is Lebesgue-measurable, and that $f:[0,+\infty)\rightarrow R$, and integrable in $(0,\ x)$, for any finite x and which is bounded in some right hand neighbourhood of origin. Integrals of the form $\int_0^\infty$ are throughout to be taken as $\lim_{x\to\infty}\int_0^x$, $\int_0^x$ being a Lebesgue integral. For any n¿o, we write $a_n(x)$ for the n^{th} integral,

$$a_n(x) = \frac{1}{\Gamma(n)}\int_0^x (x-y)^{n-1}\, a(y)dy\,,$$

$$\boldsymbol{a\_(0)\,(x) = a(x)}$$

The (C,α,β) transform of $a\,(t)$, which we denote by $\partial_{\alpha,\beta}(t)$ is given by

$$a(t) \qquad (\alpha=0)$$

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Ref.

4. **S.N.Pathak,** Some investigations of summability of functions, Ph.D. Thesis,Gorakhpur University,Gorakhpur (1986).

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\frac{1}{t^{\alpha+\beta}}\int_0^x (t-u)^{\alpha-1}u^{\beta}a(y)dy\,,\ (\alpha>0,\beta>-1)\,, \tag{2.1}$$

If, for t > 0, the integral defining $\partial_{\alpha,\beta}(t)$ exists and if $\partial_{\alpha,\beta}(t)\to s$ as $t\to\infty$, we say that a (x) is summable (C,α,β) to s, and we write $\boldsymbol{a(x)}\to$ s (C, α, β) . We write

$$g(t)=g^{(k)}(t)=kt\int_0^{\infty}\frac{x^{k-1}}{(x+t)^{k+1}}a(x)dx\ ,(\ k>0\ ) \qquad (2.2)$$ if this exists ,We also write

$$U_{k,\alpha,\beta}(t)=kt\int_0^{\infty}\frac{x^{k-1}}{(x+t)^{k+1}}\partial_{\alpha,\beta}(x)dx\,, \qquad (2.3)$$ if this exists.

With the usual terminology, we say that the sequence a_n is summable,

(I) (D, k) to the sum s, if g (t) tends to a limit s as t→ ∞,

(II) (D, k) (C,α,β) to s, if $U_{k,\alpha,\beta}(t)$ tends to s as t→ ∞. When $\beta=0$,$(D,k)(C,\alpha,\beta)$ and $(D,k)(C,\alpha)$ denote the same method . The case $\beta=0$ is due to Pathak [5]. We know that for any fixed t > 0, k > 0, it is necessary and sufficient for the convergence of (2.3) that

$$\int_1^{\infty}\frac{\partial_{\alpha,\beta}(x)}{x^2}\,dx \text{ should converge.} \tag{2.4}$$

If (2.4) converges, write for x > 0, $F_{\alpha,\beta}(x)=\int_x^{\infty}\frac{\partial_{\alpha,\beta}(t)}{t^2}\,dt$.

We note that $F_{\alpha,\beta}(x)=o(1)$ as x→ ∞ . Further, (since f(x) is bounded in some right hand neighbourhood of the origin) we have,

$$F_{\alpha,\beta}(x)=o\left(\frac{1}{x}\right) \text{ as } x\to 0+.$$

III. Main Results

In this section, we have following theorems for sequences analogous to [2].

Theorem 3.1: If $\alpha>\gamma\geq 1,\ k>0$ then $a\,(x)\to s\,(D,k)(C,\alpha-1,\beta)$, whenever $a(x)\to s(D,k)(C,\gamma-1,\beta)$.

Theorem 3.2: Let $\alpha>\gamma\geq 0,\ \beta>-1$, and suppose that a(x) is summable (C,γ,β) to s and that $\int_1^{\infty}\frac{\partial_{\gamma,\beta}(x)}{x^2}\,dx$ converges. Then a(x) is summable $(D,k)(C,\alpha,\beta)$ to s.

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 11 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Existence of Classical Solutions for a Class Nonlinear Wave Equations

By Svetlin Georgiev Georgiev
University of Sofia, Bulgaria

Abstract - In this article we investigate the Cauchy problem for the equation $u_{tt} - u_{xx} = |u|^l$, $t \in [0, \infty)$, $x \in \mathbb{R}$, $l \in [0,1)$. At this moment, the cases $l \geq 1, l = 0$ are well studied. Here we answer of the open problem $l \in (0,1)$ using approach.

Keywords and phrases : *wave equation, existence.*

GJSFR-F Classification : *MSC 2010: 34G20*

EXISTENCE OF CLASSICAL SOLUTIONS FOR A CLASS NONLINEAR WAVE EQUATIONS

Strictly as per the compliance and regulations of :

Ref.

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Existence of Classical Solutions for a Class Nonlinear Wave Equations

Svetlin Georgiev Georgiev

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Keywords and phrases : *wave equation, existence.*

I. Introduction

In this article we investigate the Cauchy problem

$$u_{tt} - u_{xx} = |u|^l, \quad t > 0, x \in \mathbb{R}, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $u_0 \in \mathcal{C}^2(\mathbb{R})$, $u_1 \in \mathcal{C}^1(\mathbb{R})$ are given functions for which $|u_0(x)| \leq D$, $|u_1(x)| \leq D$ for every $x \in \mathbb{R}$, D is given positive constant, $l \in [0,1)$ is fixed, u is unknown function.

The problem (1.1), (1.2) was considered in the cases when $l \geq 1$, $l = 0$, for local existence, global existence, blow up and etc, see for instance [2] and references therein. The case $l \in (0,1)$ was opened. Our aim in this article is to give an answer in this case. We give an answer for local existence of classical solutions. The problem for uniqueness of the classical solutions(twice continuous - differentiable in x and in t) is opened yet.

For $M, N \subseteq \mathbb{R}$ with $\mathcal{C}^2(M, \mathcal{C}^2(N))$ we will denote the space of the functions u which are twice continuous - differentiable in $t \in M$ and twice continuous - differentiable in $x \in N$.

Our main results are as follows.

Theorem 1.1. *Let D be fixed positive constant and $u_0 \in \mathcal{C}^2(\mathbb{R})$, $u_1 \in \mathcal{C}^1(\mathbb{R})$ be fixed so that $|u_0(x)| \leq D$, $|u_1(x)| \leq D$ for every $x \in \mathbb{R}$, let also $l \in [0,1)$ be fixed. Then there exist positive constants A and B so that the Cauchy problem (1.1), (1.2) has a solution $u \in \mathcal{C}^2([0,A], \mathcal{C}^2([0,B]))$.*

Theorem 1.2. *Let D be fixed positive constant and $u_0 \in \mathcal{C}^2(\mathbb{R})$, $u_1 \in \mathcal{C}^1(\mathbb{R})$ be fixed so that $|u_0(x)| \leq D$, $|u_1(x)| \leq D$ for every $x \in \mathbb{R}$, let also $l \in [0,1)$ be fixed. Then there exists positive constant A so that the Cauchy problem (1.1), (1.2) has a solution $u \in \mathcal{C}^2([0,A], \mathcal{C}^2([0,\infty)))$.*

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Theorem 1.3. *Let D be fixed positive constant and $u_0 \in \mathcal{C}^2(\mathbb{R})$, $u_1 \in \mathcal{C}^1(\mathbb{R})$ be fixed so that $|u_0(x)| \leq D$, $|u_1(x)| \leq D$ for every $x \in \mathbb{R}$, let also $l \in [0,1)$ be fixed. Then there exists positive constant A so that the Cauchy problem* (1.1), (1.2) *has a solution $u \in \mathcal{C}^2([0,A], \mathcal{C}^2(\mathbb{R}))$.*

We note that when u_0 or u_1 is not identically equal to zero the Cauchy problem (1.1), (1.2) has a nontrivial solution.

To prove our main results we will use a new approach which is used in the author's article [1] for another class of nonlinear wave equations.

The article is organized as follows: in the next section we will prove Theorem 1.1, in the section 3 we will prove Theorem 1.2, in the section 4 we will prove Theorem 1.3. In the appendix we will prove some facts which are used in the proof of basic results.

II. Proof of Theorem 1.1

Let $\epsilon \in (0,1)$ be fixed. We choose enough small positive constants A and B so that

$$\epsilon D + B^2 D(2+A) + (2+B)A^2 D + A^2 B^2 D^l \leq D,$$

$$\epsilon D + BD(2+A) + (2+B)A^2 D + A^2 B D^l \leq D, \tag{2.1}$$

$$\epsilon D + 2B^2 D + (2+B)AD + AB^2 D^l \leq D.$$

For example $\epsilon = \frac{1}{2}$, $D = 100$, $A = B = \frac{1}{1000000}$.

In this section we will prove that the Cauchy problem

$$u_{tt} - u_{xx} = |u|^l, \quad t \in [0,A], x \in [0,B], \tag{2.2}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in [0,B], \tag{2.3}$$

has a solution $u^{+1} \in \mathcal{C}^2([0,A], \mathcal{C}^2([0,B]))$.

We define the sets

$$N_{+1} = \Big\{u \in \mathcal{C}^2([0,A], \mathcal{C}^2([0,B])) : |u(t,x)| \leq D, |u_t(t,x)| \leq D, |u_x(t,x)| \leq D$$

$$\forall t \in [0,A], \quad \forall x \in [0,B]\Big\},$$

$$N^*_{+1} = \Big\{u \in \mathcal{C}^2([0,A], \mathcal{C}^2([0,B])) : |u(t,x)| \leq (1+\epsilon)D, |u_t(t,x)| \leq (1+\epsilon)D,$$

$$|u_x(t,x)| \leq (1+\epsilon)D \quad \forall t \in [0,A], \quad \forall x \in [0,B]\Big\},$$

in these sets we define a norm as follows

$$||u||_1 = \max_{t \in [0,A], x \in [0,B]} |u(t,x)|.$$

Lemma 2.1. *The sets N_{+1} and N^*_{+1} are closed, compact and convex spaces in $\mathcal{C}([0,A] \times [0,B])$ in the sense of norm $||\cdot||_1$.*

Proof. We will prove our assertion for N_{+1}.

Ref.

[1] Georgiev, S. Global existence for nonlinear wave equation. Dyn. PDE, Vol. 7, No. 3, pp. 207-216.

Firstly we will prove that N_{+1} is a closed space with respect $||\cdot||_1$. For this we will propose two ways, the first one is to be proved that N_{+1} is a completely normed space with respect the norm $||\cdot||_1$, using Weierstrass - Stone theorem, the second one is based on the definition - in other words we will prove that it contains its limit points.

First proof. Let $\{u_n\}$ is a sequence of elements of the space N_{+1} which is a fundamental sequence and it is well known that there exists $U \in \mathcal{C}([0,A]\times[0,B])$ so that $\lim_{n\longrightarrow\infty} u_n = U$ with respect the norm $||\cdot||_1$. Then for every $\epsilon > 0$ there exists $N_1 = N_1(\epsilon) > 0$ so that for every $n > N_1$ we have

$$||u_n - U||_1 < \epsilon.$$

From Weierstrass - Stone theorem there exists a sequence $\{p_l\}$ of trigonometric polynomials so that $||p_l - U||_1 \longrightarrow 0$ when $l \longrightarrow \infty$. We have $p_l \in \mathcal{C}^2([0,A]\times[0,B])$ and there exists $L_1 = L_1(\epsilon) > 0$ so that for every $L > L_1$ we have

$$||p_L - U||_1 < \epsilon.$$

We fix $L > L_1$ and put $u = p_L$. From here, for every $n > N_1$,

$$||u_n - u||_1 \leq ||u_n - U||_1 + ||U - u||_1 < 2\epsilon.$$

Consequently the fundamental sequence u_n of elements of N_{+1} is convergent to the element $u \in \mathcal{C}^2([0,A]\times[0,B])$ with respect the norm $||\cdot||_1$. Now we will prove that $u \in N_{+1}$. Evidently $|u(t,x)| \leq D$ for every $(t,x) \in [0,A]\times[0,B]$. Now we suppose that there exists $(\tilde{t},\tilde{x}) \in [0,A]\times[0,B]$ so that

$$|u_t(\tilde{t},\tilde{x})| > D.$$

Then there exists $\epsilon_2 > 0$ so that

$$|u_t(\tilde{t},\tilde{x})| \geq D + \epsilon_2.$$

From here there exists $\delta_5 = \delta_5(\epsilon_2) > 0$ such that from $|h| < \delta_5$, $h \neq 0$, $(\tilde{t},\tilde{x}) \in [0,A]\times[0,B]$ we have

$$\left|\frac{u(\tilde{t},\tilde{x}+h) - u(\tilde{t},\tilde{x})}{h}\right| \geq D + \epsilon_2.$$

On the other hand, since $u_n(\tilde{t},\tilde{x}) \longrightarrow u(\tilde{t},\tilde{x})$ in sense of $||\cdot||_1$, as $n \longrightarrow \infty$, follows that there exists $\delta_6 = \delta_6(\epsilon_2) > 0$ so that we have from $|h| < \delta_6$, $h \neq 0$, $(\tilde{t}+h,\tilde{x}) \in [0,A]\times[0,B]$

$$\left|\tfrac{u_n(\tilde{t}+h,\tilde{x}) - u_n(\tilde{t},\tilde{x})}{h} - \tfrac{u(\tilde{t}+h,\tilde{x}) - u(\tilde{t},\tilde{x})}{h}\right| < \epsilon_2$$

and since $|(u_n)_t| \leq D$ in $[0,A]\times[0,B]$

$$\left|\frac{u_n(\tilde{t}+h,\tilde{x}) - u_n(\tilde{t},\tilde{x})}{h}\right| \leq D$$

for enough large n. From here, for enough large n and for $|h| < \min\{\delta_5,\delta_6\}$, $h \neq 0$, $(\tilde{t}+h,\tilde{x}) \in [0,A]\times[0,B]$ we have

$$\epsilon_2 = D + \epsilon_2 - D$$

Notes

$$\leq \left|\frac{u(\tilde{t}+h,\tilde{x})-u(\tilde{t},\tilde{x})}{h}\right| - \left|\frac{u_n(\tilde{t}+h,\tilde{x})-u_n(\tilde{t},\tilde{x})}{h}\right|$$

$$\leq \left|\frac{u(\tilde{t}+h,\tilde{x})-u(\tilde{t},\tilde{x})}{h} - \frac{u_n(\tilde{t}+h,\tilde{x})-u_n(\tilde{t},\tilde{x})}{h}\right| < \epsilon_2,$$

which is a contradiction. Therefore $|u_t| \leq D$ in $[0, A] \times [0, B]$. As in above we can prove that $|u_x(t, x)| \leq D$ in $[0, A] \times [0, B]$. Consequently $u \in N_{+1}$ and N_{+1} is closed in $\mathcal{C}([0, A] \times [0, B])$ in sense of $|| \cdot ||_1$.

Second proof. Let u is a limit point of N_{+1}.

Then there exists a sequence $\{u_n\}$ of elements of N_{+1} and $u_n \longrightarrow_{n\longrightarrow\infty} u$ in the sense of the norm $|| \cdot ||_1$.

Evidently $|u(t, x)| \leq D$ for every $(t, x) \in [0, A] \times [0, B]$.

We suppose that $u \notin \mathcal{C}^1([0, A] \times [0, B])$. Without loss of generality we suppose that u_t does not exist in a point $(t, x) \in [0, A] \times [0, B]$. Then there exists $\epsilon > 0$ so that for every $\delta_1 = \delta_1(\epsilon) > 0$ and $|h| < \delta_1$, $h \neq 0$, $(t + h, x) \in [0, A] \times [0, B]$, we have

$$\left|\frac{u(t + h, x) - u(t, x)}{h}\right| > \epsilon. \tag{2.4}$$

On the other hand since $u_n \in \mathcal{C}^2([0, A], \mathcal{C}^2([0, B]))$ we have that there exists $\delta_2 = \delta_2(\epsilon) > 0$ so that from $|h| < \delta_2$, $h \neq 0$, $(t + h, x) \in [0, A] \times [0, B]$, we have

$$\left|\frac{u_n(t + h, x) - u_n(t, x)}{h}\right| < \frac{\epsilon}{3}. \tag{2.5}$$

Also, from $u_n \longrightarrow_{n\longrightarrow\infty} u$ in the sense of the norm $|| \cdot ||_1$ we have for enough large n and $|h| < \min\{\delta_1, \delta_2\}$, $h \neq 0$, $(t + h, x) \in [0, A] \times [0, B]$ that

$$\left|\frac{u_n(t+h,x)-u_n(t,x)}{h} - \frac{u(t+h,x)-u(t,x)}{h}\right| < \frac{\epsilon}{3}. \tag{2.6}$$

Then from (2.6), (2.5), (2.4) we obtain for $|h| < \min\{\delta_1, \delta_2\}$, $h \neq 0$, $(t+h, x) \in [0, A]\times[0, B]$, for enough large n,

$$\epsilon < \left|\frac{u(t+h,x)-u(t,x)}{h}\right| \leq \left|\frac{u_n(t+h,x)-u_n(t,x)}{h}\right| + \left|\frac{u_n(t+h,x)-u_n(t,x)}{h} - \frac{u(t+h,x)-u(t,x)}{h}\right| < 2\frac{\epsilon}{3},$$

which is a contradiction with our assumption that $u_t(t, x)$ does not exist. If we suppose that $u_x(t, x)$ does not exist in a point $(t, x) \in [0, A] \times [0, B]$, as in above we will go to a contradiction. Therefore $u \in \mathcal{C}^1([0, A], \mathcal{C}^1([0, B]))$.

We note that from $u_n \longrightarrow u$ as $n \longrightarrow \infty$ and $u_n, u \in \mathcal{C}^1([0, A], \mathcal{C}^1([0, B]))$ follows that for every $\tilde{\epsilon}$ there exists $\tilde{\delta}(\tilde{\epsilon}) > 0$ so that from $|h| < \tilde{\delta}$, $h \neq 0$, $(t + h, x) \in [0, A] \times [0, B]$, we have

$$\left|\frac{u_n(t + h, x) - u_n(t, x)}{h} - \frac{u(t + h, x) - u(t, x)}{h}\right| < \tilde{\epsilon},$$

from where we conclude that $u_{nt} \longrightarrow u_t$ in sense of $|| \cdot ||_1$ as $n \longrightarrow \infty$. In the same way we have $u_{nx} \longrightarrow u_x$ when $n \longrightarrow \infty$ in sense of $|| \cdot ||_1$.

We suppose that $u \notin \mathcal{C}^2([0, A], \mathcal{C}^2([0, B])$. Without loss of generality we suppose that u_{tt} does not exist in a point $(t, x) \in [0, A] \times [0, B]$. Then there exists $\epsilon_1 > 0$ so that for every $\delta_3 = \delta_3(\epsilon_1) > 0$ and $|h| < \delta_3$, $h \neq 0$, $(t + h, x) \in [0, A] \times [0, B]$ we have

Notes

$$\Big|\frac{u_t(t+h,x)-u_t(t,x)}{h}\Big| > \epsilon_1. \tag{2.7}$$

On the other hand since $u_n \in \mathcal{C}^2([0,1],\mathcal{C}^2(B_1))$ we have that there exists $\delta_4=\delta_4(\epsilon_1)>0$ so that from $|h|<\delta_4$, $h\neq 0$, $(t+h,x)\in[0,A]\times[0,B]$ we have

$$\Big|\frac{(u_n)_t(t+h,x)-(u_n)_t(t,x)}{h}\Big| < \frac{\epsilon_1}{3}. \tag{2.8}$$

Notes

Also, from $u_{nt} \longrightarrow_{n\longrightarrow\infty} u_t$ in the sense of the norm $||\cdot||_1$ we have for enough large n and $|h|<\min\{\delta_3,\delta_4\}$, $h\neq 0$, $(t+h,x)\in[0,A]\times[0,B]$ that

$$\Big|\tfrac{(u_n)_t(t+h,x)-(u_n)_t(t,x)}{h}-\tfrac{u_t(t+h,x)-u_t(t,x)}{h}\Big| < \tfrac{\epsilon_1}{3}. \tag{2.9}$$

Then from (2.9), (2.8), (2.7) we obtain for $|h|<\min\{\delta_3,\delta_4\}$, $h\neq 0$, $(t+h,x)\in[0,A]\times[0,B]$, for enough large n,

$$\begin{aligned}\epsilon_1 &< \Big|\tfrac{u_t(t+h,x)-u_t(t,x)}{h}\Big|\\ &\le \Big|\tfrac{(u_n)_t(t+h,x)-(u_n)_t(t,x)}{h}\Big| + \Big|\tfrac{(u_n)_t(t+h,x)-(u_n)_t(t,x)}{h}-\tfrac{u_t(t,x)-u_t(t,x)}{h}\Big| < 2\tfrac{\epsilon_1}{3},\end{aligned}$$

which is a contradiction with our assumption that u_{tt} does not exists in a point $(t,x)\in[0,A]\times[0,B]$. If we suppose that u_{xx} does not exist in a point $(t,x)\in[0,A]\times[0,B]$ as in above we can go to a contradiction. Therefore $u\in\mathcal{C}^2([0,A],\mathcal{C}^2([0,B]))$. As in the first proof(above) we have that $|u(t,x)|\le D$, $|u_t(t,x)|\le D$, $|u_x(t,x)|\le D$ for every $(t,x)\in[0,A]\times[0,B]$, i.e. $u\in N_{+1}$. Consequently N_{+1} contains its limit points.

Using Arzela - Ascoli Theorem the set N_{+1} is a compact set in $\mathcal{C}([0,A]\times[0,B])$ in sense of $||\cdot||_1$.

Let now $\lambda\in[0,1]$ is arbitrary chosen and fixed and $u_1,u_2\in N_{+1}$. Then for $(t,x)\in[0,A]\times[0,B]$ we have $\lambda u_1(t,x)+(1-\lambda)u_2(t,x)\in\mathcal{C}^2([0,A],\mathcal{C}^2([0,B]))$ and

$$|u_i(t,x)|\le D, |u_{it}(t,x)|\le D \quad |u_{ix}(t,x)|\le D \quad \text{for} \quad i=1,2,$$

$$|\lambda u_1(t,x)+(1-\lambda)u_2(t,x)|\le\lambda|u_1(t,x)|+(1-\lambda)|u_2(t,x)|\le\lambda D+(1-\lambda)D=D,$$

$$|\lambda u_{1t}(t,x)+(1-\lambda)u_{2t}(t,x)|\le\lambda|u_{1t}(t,x)|+(1-\lambda)|u_{2t}(t,x)|\le\lambda D+(1-\lambda)D=D,$$

$$|\lambda u_{1x}(t,x)+(1-\lambda)u_{2x}(t,x)|\le\lambda|u_{1x}(t,x)|+(1-\lambda)|u_{2x}(t,x)|\le\lambda D+(1-\lambda)D=D.$$

Therefore N_{+1} is convex.

As in above we can prove that N^*_{+1} is closed, compact and convex in $\mathcal{C}([0,A]\times[0,B])$ in sense of $||\cdot||_1$.

For $u\in N^*_{+1}$ we define the operators

$$K_{+1}(u)(t,x)=(1+\epsilon)u(t,x),$$

$$L_{+1}(u)(t,x)=-\epsilon u(t,x)+\int_0^x\int_0^\sigma u(t,y)dyd\sigma-\int_0^x\int_0^\sigma\Big(u_0(y)+tu_1(y)\Big)dyd\sigma$$

$$-\int_0^t\int_0^\tau u(s,x)dsd\tau-\int_0^t\int_0^\tau\int_0^x\int_0^\sigma |u|^l(s,y)dyd\sigma dsd\tau,$$

$$P_{+1}(u)(t,x)=K_{+1}(u)(t,x)+L_{+1}(u)(t,x).$$

Our first observation is as follows.

Lemma 2.2. *Let $u\in N^*_{+1}$ be a fixed point of the operator P_{+1}. Then u is a solution to the Cauchy problem (2.2), (2.3).*

Proof. Since $u\in N^*_{+1}$ is a fixed point of the operator P_{+1} we have for every $t\in[0,A]$ and $x\in[0,B]$

$$u(t,x)=P_{+1}(u)(t,x)=K_{+1}(u)(t,x)+L_{+1}(u)(t,x)$$

$$=(1+\epsilon)u(t,x)-\epsilon u(t,x)+\int_0^x\int_0^\sigma u(t,y)dyd\sigma-\int_0^x\int_0^\sigma\Big(u_0(y)+tu_1(y)\Big)dyd\sigma$$

$$-\int_0^t\int_0^\tau u(s,x)dsd\tau-\int_0^t\int_0^\tau\int_0^x\int_0^\sigma |u|^l(s,y)dyd\sigma dsd\tau$$

$$=u(t,x)+\int_0^x\int_0^\sigma u(t,y)dyd\sigma-\int_0^x\int_0^\sigma\Big(u_0(y)+tu_1(y)\Big)dyd\sigma$$

$$-\int_0^t\int_0^\tau u(s,x)dsd\tau-\int_0^t\int_0^\tau\int_0^x\int_0^\sigma |u|^l(s,y)dyd\sigma dsd\tau,$$

whereupon for every $t\in[0,A]$ and every $x\in[0,B]$ we have

$$\begin{aligned}0&=\int_0^x\int_0^\sigma u(t,y)dyd\sigma-\int_0^x\int_0^\sigma\Big(u_0(y)+tu_1(y)\Big)dyd\sigma\\&-\int_0^t\int_0^\tau u(s,x)dsd\tau-\int_0^t\int_0^\tau\int_0^x\int_0^\sigma |u|^l(s,y)dyd\sigma dsd\tau.\end{aligned}\tag{2.10}$$

Now we differentiate the last equality with respect t and we get, for $t\in[0,A]$, $x\in[0,B]$,

$$0=\int_0^x\int_0^\sigma\Big(u_t(t,y)-u_1(y)\Big)dyd\sigma-\int_0^t u(s,x)ds-\int_0^t\int_0^x\int_0^\sigma |u|^l(s,y)dyd\sigma ds.\tag{2.11}$$

We differentiate the last equality with respect the time variable t and we obtain

$$0=\int_0^x\int_0^\sigma u_{tt}(t,y)dyd\sigma-u(t,x)-\int_0^x\int_0^\sigma |u|^l(t,y)dyd\sigma,\quad t\in[0,A],x\in[0,B].$$

Now we differentiate the last equality with respect x we find

$$0=\int_0^x u_{tt}(t,y)dy-u_x(t,x)-\int_0^x |u|^l(t,y)dy,\quad t\in[0,A],x\in[0,B].$$

After we differentiate the last equality in x we obtain

$$0=u_{tt}-u_{xx}-|u|^l,\quad t\in[0,A],x\in[0,B],$$

in other words u satisfies the equation (2.2).

Now we put $t=0$ in (2.10) and we find

$$0=\int_0^x\int_0^\sigma\Big(u(0,y)-u_0(y)\Big)dyd\sigma,\quad x\in[0,B],$$

Notes

after which we differentiate it twice in x and we get

$$u(0,x) = u_0(x), \quad x \in [0,B].$$

We put $t = 0$ in (2.11) and we have

$$0 = \int_0^x \int_0^\sigma \Big(u_t(0,y) - u_1(y) \Big) dy d\sigma, \quad x \in [0,B],$$

Notes

we differentiate twice the last equality with respect x and we find

$$u_t(0,x) = u_1(x), \quad x \in [0,B].$$

Consequently u satisfies the initial conditions (2.3).

The above lemma motivate us to search fixed points of the operator P_{+1}. For this purpose we will use the following fixed point theorem.

Theorem 2.3. *(see [3], Corollary 2.4, pp. 3231) Let X be a nonempty closed convex subset of a Banach space Y. Suppose that T and S map X into Y such that*
(i) *S is continuous, $S(X)$ resides in a compact subset of Y;*
(ii) *$T : X \longrightarrow Y$ is expansive and onto.*
Then there exists a point $x^\star \in X$ with $Sx^\star + Tx^\star = x^\star$.

Here we will use the following definition for expansive operator.

Definition. (see [3], pp. 3230) Let (X, d) be a metric space and M be a subset of X. The mapping $T : M \longrightarrow X$ is said to be expansive, if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq hd(x,y) \quad \forall x, y \in M.$$

Lemma 2.4. *The operator $K_{+1} : N_{+1} \longrightarrow N^*_{+1}$ is an expansive operator and onto.*

Proof. Firstly we will see that $K_{+1} : N_{+1} \longrightarrow N^*_{+1}$. Let $u \in N_{+1}$. Then $u \in \mathcal{C}^2([0,A], \mathcal{C}^2([0,B]))$ and $|u(t,x)| \leq D$, $|u_t(t,x)| \leq D$, $|u_x(t,x)| \leq D$ for every $t \in [0,A]$ and $x \in [0,B]$. From here $K_{+1}(u) = (1+\epsilon)u \in \mathcal{C}^2([0,A], \mathcal{C}^2([0,B]))$ and $|K_{+1}(u)(t,x)| = (1+\epsilon)|u(t,x)| \leq (1+\epsilon)D$, $\left|\frac{\partial}{\partial t} K_{+1}(u)(t,x)\right| = (1+\epsilon)|u_t(t,x)| \leq (1+\epsilon)D$, $\left|\frac{\partial}{\partial x} K_{+1}(u)(t,x)\right| = (1+\epsilon)|u_x(t,x)| \leq (1+\epsilon)D$ for every $t \in [0,A]$ and $x \in [0,B]$. Consequently $K_{+1} : N_{+1} \longrightarrow N^*_{+1}$.

Let now $u, v \in N_{+1}$. Then

$$||K_{+1}(u) - K_{+1}(v)|| = (1+\epsilon)||u - v||,$$

i.e. the operator $K_{+1} : N_{+1} \longrightarrow N^*_{+1}$ is an expansive operator with a constant $h = 1 + \epsilon$.

Now we will see that the operator $K_{+1} : N_{+1} \longrightarrow N^*_{+1}$ is onto. Indeed, let $v \in N^*_{+1}$. Then $u = \frac{v}{1+\epsilon} \in N_{+1}$ and $K_{+1}(u)(t,x) = v(t,x)$ for every $t \in [0,A]$ and $x \in [0,B]$. Therefore $K_{+1} : N_{+1} \longrightarrow N^*_{+1}$ is onto.

Lemma 2.5. *The operator $L_{+1} : N_{+1} \longrightarrow N_{+1}$ is a continuous operator.*

Proof. Let $u \in N_{+1}$, from where $|u(t,x)| \leq D$, $|u_t(t,x)| \leq D$, $|u_x(t,x)| \leq D$ for every $t \in [0,A]$ and $x \in [0,B]$, also $|u_0(x)| \leq D$, $|u_1(x)| \leq D$ for every $x \in [0,B]$. From the definition of the operator L_{+1}, for $t \in [0,A]$, $x \in [0,B]$, we have

$$|L_{+1}(u)(t,x)| \leq \epsilon |u(t,x)| + \int_0^x \int_0^\sigma |u(t,y)| dy d\sigma + \int_0^x \int_0^\sigma \Big(|u_0(y)| + t|u_1(y)| \Big) dy d\sigma$$

$$+\int_0^t\int_0^\tau|u(s,x)|dsd\tau+\int_0^t\int_0^\tau\int_0^x\int_0^\sigma|u|^l(s,y)dyd\sigma dsd\tau$$

$$\leq \epsilon D+B^2D(2+A)+A^2D+A^2B^2D^l\leq D,$$

in the last inequality we use the first inequality of (2.1).

For $t\in[0,A]$, $x\in[0,B]$, we have

$$\frac{\partial}{\partial x}L_{+1}(u)(t,x)=-\epsilon u_x(t,x)+\int_0^x u(t,y)dy-\int_0^x\Big(u_0(y)+tu_1(y)\Big)dy$$

$$-\int_0^t\int_0^\tau u_x(s,x)dsd\tau-\int_0^t\int_0^\tau\int_0^x|u|^l(s,y)dydsd\tau$$

and from here, for $t\in[0,A]$ and $x\in[0,B]$, we get

$$\left|\frac{\partial}{\partial x}L_{+1}(u)(t,x)\right|\leq\epsilon|u_x(t,x)|+\int_0^x|u(t,y)|dy+\int_0^x\Big(|u_0(y)|+t|u_1(y)|\Big)dy$$

$$+\int_0^t\int_0^\tau|u_x(s,x)|dsd\tau+\int_0^t\int_0^\tau\int_0^x|u|^l(s,y)dydsd\tau$$

$$\leq\epsilon D+BD(2+A)+A^2D+A^2BD^l\leq D,$$

in the last inequality we use the second inequality of (2.1).

Also, for $t\in[0,A]$, $x\in[0,B]$, we have

$$\frac{\partial}{\partial t}L_{+1}(u)(t,x)=-\epsilon u_t(t,x)+\int_0^x\int_0^\sigma u_t(t,y)dyd\sigma-\int_0^x\int_0^\sigma u_1(y)dyd\sigma$$

$$-\int_0^t u(s,x)ds-\int_0^t\int_0^x\int_0^\sigma|u|^l(s,y)dyd\sigma ds$$

and

$$\left|\frac{\partial}{\partial t}L_{+1}(u)(t,x)\right|\leq\epsilon|u_t(t,x)|+\int_0^x\int_0^\sigma|u_t(t,y)|dyd\sigma+\int_0^x\int_0^\sigma|u_1(y)|dyd\sigma$$

$$+\int_0^t|u(s,x)|ds+\int_0^t\int_0^x\int_0^\sigma|u|^l(s,y)dyd\sigma ds$$

$$\leq\epsilon D+2B^2D+AD+AB^2D^l\leq D,$$

in the last inequality we use the third inequality of (2.1).

From the above estimates follows that $L_{+1}:N_{+1}\longrightarrow N_{+1}$.

Let now $\{u_n\}$ is a sequence of elements of N_{+1} and $u\in N_{+1}$ and $u_n\longrightarrow u$ when $n\longrightarrow\infty$ in the sense of the topology of the set N_{+1}, i.e. for every $\epsilon_1>0$ there exists $N_1=N_1(\epsilon_1)>0$ so that for every $n>N_1$ and $t\in[0,A]$, $x\in[0,B]$, we have

$$|u_n(t,x)-u(t,x)|<\epsilon_1,|(u_n)_x(t,x)-u_x(t,x)|<\epsilon_1,|(u_n)_t(t,x)-u_t(t,x)|<\epsilon_1.$$

From here, for every $\epsilon_2>0$ there exists $N_2=N_2(\epsilon_2)>0$ so that for every $n>N_2$ and for every $t\in[0,A]$, $x\in[0,B]$, we have $\Big||u_n|^l(t,x)-|u|^l(t,x)\Big|<\epsilon_2$ and

$$|u_n(t,x)-u(t,x)|<\epsilon_2,|(u_n)_x(t,x)-u_x(t,x)|<\epsilon_2,|(u_n)_t(t,x)-u_t(t,x)|<\epsilon_2,$$

$$|L_{+1}(u_n)(t,x)-L_{+1}(u)(t,x)|\leq\epsilon|u_n(t,x)-u(t,x)|+\int_0^x\int_0^\sigma|u_n(t,y)-u(t,y)|dyd\sigma$$

Notes

$$+\int_0^t\int_0^\tau |u_n(s,x)-u(s,x)|dsd\tau+\int_0^t\int_0^\tau\int_0^x\int_0^\sigma\Big||u_n|^l(s,y)-|u|^l(s,y)\Big|dyd\sigma dsd\tau$$

$$<\epsilon_2\Big(\epsilon+B^2+A^2+A^2B^2\Big),$$

$$\Big|\frac{\partial}{\partial x}L_{+1}(u_n)(t,x)-\frac{\partial}{\partial x}L_{+1}(u)(t,x)\Big|\le\epsilon|(u_n)_x(t,x)-u_x(t,x)|+\int_0^x|u_n(t,y)-u(t,y)|dy$$

$$+\int_0^t\int_0^\tau|(u_n)_x(s,x)-u_x(s,x)|dsd\tau+\int_0^t\int_0^\tau\int_0^x\Big||u_n|^l(s,y)-|u|^l(s,y)\Big|dydsd\tau$$

$$<\epsilon_2\Big(\epsilon+B+A^2+A^2B\Big),$$

$$\Big|\frac{\partial}{\partial t}L_{+1}(u_n)(t,x)-\frac{\partial}{\partial t}L_{+1}(u)(t,x)\Big|\le\epsilon|(u_n)_t(t,x)-u_t(t,x)|+\int_0^x\int_0^\sigma|(u_n)_t(t,y)-u_t(t,y)|dyd\sigma$$

$$+\int_0^t|u_n(s,x)-u(s,x)|ds+\int_0^t\int_0^x\int_0^\sigma\Big||u_n|^l(s,y)-|u|^l(s,y)\Big|dyd\sigma ds$$

$$<\epsilon_2\Big(\epsilon+B^2+A+AB^2\Big),$$

Therefore $L_{+1}(u_n)\longrightarrow L_{+1}(u)$ when $n\longrightarrow\infty$ in the sense of the topology of the space N_{+1}, i.e. the operator $L_{+1}:N_{+1}\longrightarrow N_{+1}$ is a continuous operator.

Using Lemma 2.1, Lemma 2.4, Lemma 2.5 we apply Theorem 2.3 as the operator T in Theorem 2.3 corresponds of the operator K_{+1}, the operator S in Theorem 2.3 corresponds of L_{+1}, the set X in Theorem 2.3 corresponds of N_{+1}, Y in Theorem 2.3 corresponds of N^*_{+1} and follows that the operator P_{+1} has a fixed point $u^{+1}\in N_{+1}$. From here and from Lemma 2.2 follows that u^{+1} is a solution to the Cauchy problem (2.2), (2.3).

III. Proof of Theorem 1.2

In the previous section we prove that if the positive constants A and B satisfy the conditions (2.1) then the Cauchy problem

$$u_{tt}-u_{xx}=|u|^l,\quad t\in[0,A],x\in[0,B],$$

$$u(0,x)=u_0(x),u_t(0,x)=u_1(x),\quad x\in[0,B],$$

has a solution $u^{+1}\in\mathcal{C}^2([0,A],\mathcal{C}^2([0,B]))$.

Let A and B be the same constants as in the Section 2. We consider the Cauchy problem

$$\begin{aligned}&u_{tt}-u_{xx}=|u|^l,\quad t\in[0,A],\quad x\in[B,2B],\\&u(0,x)=u_0(x),u_t(0,x)=u_1(x),\quad x\in[B,2B].\end{aligned}\tag{3.1}$$

We define the sets

$$N_{+2}=\Big\{u\in\mathcal{C}^2([0,A],\mathcal{C}^2([B,2B])):|u(t,x)|\le D,|u_t(t,x)|\le D,|u_x(t,x)|\le D$$

$$\forall t\in[0,A],\quad\forall x\in[B,2B]\Big\},$$

Notes

$$N^*_{+2} = \Big\{u \in \mathcal{C}^2([0,A], \mathcal{C}^2([B,2B])) : |u(t,x)| \le (1+\epsilon)D, |u_t(t,x)| \le (1+\epsilon)D,$$

$$|u_x(t,x)| \le (1+\epsilon)D \quad \forall t \in [0,A], \quad \forall x \in [B,2B]\Big\},$$

in these sets we define a norm as follows

$$||u||_1 = \max_{t\in[0,A], x\in[B,2B]} |u(t,x)|,$$

in this way the sets N_{+2} and N^*_{+2} are closed, convex and compact sets in $\mathcal{C}([0,A]\times[B,2B])$. For $u \in N^*_{+2}$ we define the operators

$$K_{+2}(u)(t,x) = (1+\epsilon)u(t,x),$$

$$L_{+2}(u)(t,x) = -\epsilon u(t,x) + \int_B^x \int_B^\sigma u(t,y)dyd\sigma - \int_B^x \int_B^\sigma \Big(u_0(y) + tu_1(y)\Big)dyd\sigma$$

$$-\int_0^t \int_0^\tau \Big(u(s,x) - u^{+1}(s,B) - (x-B)u_x^{+1}(s,B)\Big)dsd\tau - \int_0^t \int_0^\tau \int_B^x \int_B^\sigma |u|^l(s,y)dyd\sigma dsd\tau,$$

$$P_{+2}(u)(t,x) = K_{+2}(u)(t,x) + L_{+2}(u)(t,x).$$

As in the Section 2 we prove that the Cauchy problem (3.1) has a solution $u^{+2} \in \mathcal{C}^2([0,A], \mathcal{C}^2([B,2B]))$ for which we have, for $t \in [0,A]$, $x \in [B,2B]$,

$$0 = \int_B^x \int_B^\sigma u^{+2}(t,y)dyd\sigma - \int_B^x \int_B^\sigma \Big(u_0(y) + tu_1(y)\Big)dyd\sigma$$

$$-\int_0^t \int_0^\tau \Big(u^{+2}(s,x) - u^{+1}(s,B) - (x-B)u_x^{+1}(s,B)\Big)dsd\tau - \int_0^t \int_0^\tau \int_B^x \int_B^\sigma \Big|u^{+2}\Big|^l(s,y)dyd\sigma dsd\tau \tag{3.2}$$

Now we put $x = B$ in (3.2) and we obtain

$$0 = \int_0^t \int_0^\tau \Big(u^{+2}(s,B) - u^{+1}(s,B)\Big)dsd\tau, \quad t \in [0,A],$$

after we differentiate twice in t the last equality we get

$$u^{+2}(t,B) = u^{+1}(t,B), \quad t \in [0,A]. \tag{3.3}$$

Now we differentiate in x the equality (3.2), after which we put $x = B$ and we find

$$0 = \int_0^t \int_0^\tau \Big(u_x^{+2}(s,B) - u_x^{+1}(s,B)\Big)dsd\tau, \quad t \in [0,A],$$

after we differentiate the last equality twice in t we obtain

$$u_x^{+2}(t,B) = u_x^{+1}(t,B), \quad t \in [0,A].$$

From (3.3) we have

$$u_t^{+1}(t,B) = u_t^{+2}(t,B), u_{tt}^{+1}(t,B) = u_{tt}^{+2}(t,B), \quad t \in [0,A].$$

From here, from (3.3) and from

$$u_{tt}^{+2}(t,B) - u_{xx}^{+2}(t,B) = \Big|u^{+2}\Big|^l(t,B), \quad t \in [0,A],$$

Notes

$$u_{tt}^{+1}(t,B) - u_{xx}^{+1}(t,B) = \Big|u^{+1}\Big|^{l}(t,B), \quad t \in [0,A],$$

we conclude that

$$u_{xx}^{+2}(t,B) = u_{xx}^{+1}(t,B), \quad t \in [0,A].$$

Consequently the function

Notes

$$\tilde{u} = \begin{cases} u^{+1} & t \in [0,A], x \in [0,B], \\ u^{+2} & t \in [0,A], x \in [B,2B], \end{cases}$$

is a solution to the Cauchy problem

$$u_{tt} - u_{xx} = |u|^{l}, \quad t \in [0,A], x \in [0,2B],$$

$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), \quad x \in [0,2B],$$

which belongs in the space $\mathcal{C}^2([0,A],\mathcal{C}^2([0,2B]))$.
Now consider the Cauchy problem

$$u_{tt} - u_{xx} = |u|^{l}, \quad t \in [0,A], \quad x \in [2B,3B],$$

$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), \quad x \in [2B,3B].$$

We define the sets

$$N_{+3} = \Big\{u \in \mathcal{C}^2([0,A],\mathcal{C}^2([2B,3B])) : |u(t,x)| \le D, |u_t(t,x)| \le D, |u_x(t,x)| \le D$$

$$\forall t \in [0,A], \quad \forall x \in [2B,3B]\Big\},$$

$$N_{+3}^{*} = \Big\{u \in \mathcal{C}^2([0,A],\mathcal{C}^2([2B,3B])) : |u(t,x)| \le (1+\epsilon)D, |u_t(t,x)| \le (1+\epsilon)D,$$

$$|u_x(t,x)| \le (1+\epsilon)D \quad \forall t \in [0,A], \quad \forall x \in [2B,3B]\Big\},$$

in these sets we define a norm as follows

$$||u||_1 = \max_{t\in[0,A],x\in[2B,3B]} |u(t,x)|,$$

in this way the sets N_{+3} and N_{+3}^{*} are are closed, convex and compact sets in $\mathcal{C}([0,A] \times [2B,3B])$.

For $u \in N_{+3}^{*}$ we define the operators

$$K_{+3}(u)(t,x) = (1+\epsilon)u(t,x),$$

$$L_{+3}(u)(t,x) = -\epsilon u(t,x) + \int_{2B}^{x}\int_{2B}^{\sigma} u(t,y)dyd\sigma - \int_{2B}^{x}\int_{2B}^{\sigma}\Big(u_0(y) + tu_1(y)\Big)dyd\sigma$$

$$-\int_0^t\int_0^\tau\Big(u(s,x) - u^{+2}(s,2B) - (x-2B)u^{+2}(s,2B)\Big)dsd\tau - \int_0^t\int_0^\tau\int_{2B}^{x}\int_{2B}^{\sigma}|u|^{l}(s,y)dyd\sigma dsd\tau,$$

$$P_{+3}(u)(t,x) = K_{+3}(u)(t,x) + L_{+3}(u)(t,x).$$

And etc.

The function

$$u^+ = \begin{cases} u^{+1} & t \in [0,A], x \in [0,B], \\ u^{+2} & t \in [0,A], x \in [B,2B], \\ u^{+3} & t \in [0,A], x \in [2B,3B], \\ \dots \end{cases}$$

is a solution to the Cauchy problem

$$u_{tt} - u_{xx} = |u|^l, \quad t \in [0,A], x \in [0,\infty),$$
$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), \quad x \in [0,\infty),$$

which belongs to the space $\mathcal{C}^2([0,A], \mathcal{C}^2([0,\infty)))$.

IV. Proof of Theorem 1.3

Let A and B are the same constants as in the Section 2. Now consider the Cauchy problem

$$\begin{aligned} & u_{tt} - u_{xx} = |u|^l, \quad t \in [0,A], \quad x \in [-B,0], \\ & u(0,x) = u_0(x), u_t(0,x) = u_1(x), \quad x \in [-B,0]. \end{aligned} \tag{4.1}$$

We define the sets

$$N_{-1} = \Big\{ u \in \mathcal{C}^2([0,A], \mathcal{C}^2([-B,0])) : |u(t,x)| \le D, |u_t(t,x)| \le D, |u_x(t,x)| \le D$$
$$\forall t \in [0,A], \quad \forall x \in [-B,0] \Big\},$$
$$N_{-1}^* = \Big\{ u \in \mathcal{C}^2([0,A], \mathcal{C}^2([-B,0])) : |u(t,x)| \le (1+\epsilon)D, |u_t(t,x)| \le (1+\epsilon)D,$$
$$|u_x(t,x)| \le (1+\epsilon)D \quad \forall t \in [0,A], \quad \forall x \in [-B,0] \Big\},$$

in these sets we define a norm as follows

$$||u|| = \max_{t\in[0,A], x\in[-B,0]} |u(t,x)|,$$

in this way the sets N_{-1} and N_{-1}^* are are closed, convex and compact sets in $\mathcal{C}([0,A] \times [-B,0])$.

For $u \in N_{-1}^*$ we define the operators

$$K_{-1}(u)(t,x) = (1+\epsilon)u(t,x),$$

$$L_{-1}(u)(t,x) = -\epsilon u(t,x) + \int_x^0 \int_\sigma^0 u(t,y)dyd\sigma - \int_x^0 \int_\sigma^0 \Big(u_0(y) + tu_1(y) \Big) dyd\sigma$$
$$- \int_0^t \int_0^\tau \Big(u(s,x) - u^+(s,0) - xu_x^+(s,0) \Big) dsd\tau - \int_0^t \int_0^\tau \int_x^0 \int_\sigma^0 |u|^l(s,y)dyd\sigma dsd\tau,$$

Notes

$$P_{-1}(u)(t,x) = K_{-1}(u)(t,x) + L_{-1}(u)(t,x).$$

As in the Section 2 and in the Section 3 we prove that the Cauchy problem (4.1) has a solution $u^{-1} \in \mathcal{C}^2([0,A],\mathcal{C}^2([-B,0]))$. And etc.
The function

Notes

$$u^- = \begin{cases} u^{-1} & t \in [0,A], x \in [-B,0], \\ u^{-2} & t \in [0,A], x \in [-2B,-B], \\ \dots \end{cases}$$

is a solution to the Cauchy problem

$$u_{tt} - u_{xx} = |u|^l, \quad t \in [0,A], x \in (-\infty,0],$$

$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), \quad x \in (-\infty,0],$$

which belongs to the space $\mathcal{C}^2([0,A],\mathcal{C}^2(-\infty,0])$, and the function

$$u = \begin{cases} u^+ & t \in [0,A], x \in [0,\infty), \\ u^- & t \in [0,A], x \in (-\infty,0], \end{cases}$$

is a solution to the Cauchy problem (1.1), (1.2) which belongs to the space $\mathcal{C}^2([0,A],\mathcal{C}^2(\mathbb{R}))$.

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Notes

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 11 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On Certain Class of Difference Sequence Spaces

By Khalid Ebadullah
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Abstract - In this article we define the sequence spaces $c_0(u, \triangle_v^m, F, p), c(u, \triangle_v^m, F, p)$ and $l_\infty(u, \triangle_v^m, F, p)$ for $F = (f_k)$ a sequence of moduli, *p = (pk)* sequence of positive reals, *v = (vk)* is any fixed sequence of zero complex numbers, $m \in \mathbb{N}$ is a fixed number, and $u \in U$t the set of all sequences and establish some inclusion relations.

Keywords and phrases : *Paranorm, sequence of moduli, difference sequence spaces.*

GJSFR-F Classification : *MSC 2010: 46A45*

ON CERTAIN CLASS OF DIFFERENCE SEQUENCE SPACES

Strictly as per the compliance and regulations of :

Ref.

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Global Journal of Science Frontier Research (F) Volume XII Issue XI Version I Year 2012

On Certain Class of Difference Sequence Spaces

Khalid Ebadullah

Abstract - In this article we define the sequence spaces $c_0(u,\triangle_v^m,F,p),c(u,\triangle_v^m,F,p)$ and $l_\infty(u,\triangle_v^m,F,p)$ for $F=(f_k)$ a sequence of moduli, $p=(p_k)$ sequence of positive reals, $v=(v_k)$ is any fixed sequence of non zero complex numbers, $m\in\mathbb{N}$ is a fixed number, and $u\in U$ the set of all sequences and establish some inclusion relations.

Keywords and phrases : *Paranorm, sequence of moduli, difference sequence spaces.*

I. Introduction

Let $\mathbb{N}$,$\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural,real and complex numbers respectively.
We write

$$\omega=\{x=(x_k):x_k\in\mathbb{R}\ or\ \mathbb{C}\},$$

the space of all real or complex sequences. Let l_∞,c and c_0 denote the Banach spaces of bounded,convergent and null sequences respectively.
The following subspaces of ω were first introduced and discussed by Maddox [13-15].

$$l(p):=\{x\in\omega:\sum_k|x_k|^{p_k}<\infty\},$$
$$l_\infty(p):=\{x\in\omega:\sup_k|x_k|^{p_k}<\infty\},$$
$$c(p):=\{x\in\omega:\lim_k|x_k-l|^{p_k}=0,\quad\text{for}\quad\text{some } l\in\mathbb{C}\},$$
$$c_0(p):=\{x\in\omega:\lim_k|x_k|^{p_k}=0\},$$

where $p=(p_k)$ is a sequence of striclty positive real numbers.
The idea of Difference sequence sets

$$X_\triangle=\{x=(x_k)\in\omega:\triangle x=(x_k-x_{k+1})\in X\},$$

where $X=l_\infty,\ c$ or c_0 was introduced by Kizmaz [9].
In 1981 Kizmaz [9] defined the following sequence spaces,

$$l_\infty(\triangle)=\{x=(x_k)\in\omega:(\triangle x_k)\in l_\infty\},$$

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$$c(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c\},$$

$$c_0(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c_0\},$$

where $\triangle x = (x_k - x_{k+1})$. These are Banach spaces with the norm

$$||x||_\triangle = |x_1| + ||\triangle x||_\infty.$$

After then Et[3] defined the sequence spaces

$$l_\infty(\triangle^2) = \{x = (x_k) \in \omega : (\triangle^2 x_k) \in l_\infty\}$$

$$c(\triangle^2) = \{x = (x_k) \in \omega : (\triangle^2 x_k) \in c\}$$

$$c_0(\triangle^2) = \{x = (x_k) \in \omega : (\triangle^2 x_k) \in c_0\}$$

Where $(\triangle^2 x) = (\triangle^2 x_k) = (\triangle x_k - \triangle x_{k+1})$.
The sequence spaces $l_\infty(\triangle^2), c(\triangle^2)$ and $c_0(\triangle^2)$ are Banach spaces with the norm

$$||x||_\triangle = |x_1| + |x_2| + ||\triangle^2 x||_\infty.$$

After then R. Colak and M. Et [4] defined the sequence spaces

$$l_\infty(\triangle^m) = \{x = (x_k) \in \omega : (\triangle^m x_k) \in l_\infty\},$$

$$c(\triangle^m) = \{x = (x_k) \in \omega : (\triangle^m x_k) \in c\},$$

$$c_0(\triangle^m) = \{x = (x_k) \in \omega : (\triangle^m x_k) \in c_0\},$$

where $m \in \mathbb{N}$,

$$\triangle^0 x = (x_k),$$

$$\triangle x = (x_k - x_{k+1}),$$

$$\triangle^m x = (\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1}),$$

and so that

$$\triangle^m x_k = \sum_{i=0}^{m} (-1)^i \begin{bmatrix} m \\ i \end{bmatrix} \quad x_{k+i}.$$

and showed that these are Banach spaces with the norm

$$||x||_\triangle = \sum_{i=1}^{m} |x_i| + ||\triangle^m x||_\infty.$$

Let U be the set of all sequences $u = (u_k)$ such that $u_k \neq 0 (k = 1, 2, 3....)$.
Malkowsky[16] defined the following sequence spaces

Ref.

[3] Et,M..On some difference sequence spaces,$Dogra-Tr.J.Math.$,17,(1993)18-24.

$$l_\infty(u,\triangle)=\{x=(x_k)\in\omega:(u_k\triangle x_k)\in l_\infty\},$$

$$c(u,\triangle)=\{x=(x_k)\in\omega:(u_k\triangle x_k)\in c\},$$

$$c_0(u,\triangle)=\{x=(x_k)\in\omega:(u_k\triangle x_k)\in c_0\},$$

where $u\in U$.

The concept of paranorm(See[15]) is closely related to linear metric spaces.It is a generalization of that of absolute value.

Let X be a linear space. A function $g:X\longrightarrow R$ is called paranorm, if for all $x,y\in X$,

(PI) $g(x)=0\ if\ x=\theta$,

(P2) $g(-x)=g(x)$,

(P3) $g(x+y)\leq g(x)+g(y)$,

(P4) If (λ_n) is a sequence of scalars with $\lambda_n\rightarrow\lambda\ (n\rightarrow\infty)$ and $x_n,a\in X$ with $x_n\rightarrow a\ \ (n\rightarrow\infty)$, in the sense that $g(x_n-a)\rightarrow 0\ \ (n\rightarrow\infty)$, in the sense that $g(\lambda_n x_n-\lambda a)\rightarrow 0\ \ (n\rightarrow\infty)$.

A paranorm g for which $g(x)=0$ implies $x=\theta$ is called a total paranorm on X, and the pair (X,g) is called a totally paranormed space.

The idea of modulus was structured in 1953 by Nakano.(See[17]).

A function $f:[0,\infty)\longrightarrow[0,\infty)$ is called a modulus if

(P1)$f(t)=0$ if and only if $t=0$,

(P2) $f(t+u)\leq f(t)+f(u)$ for all $t,u\geq 0$,

(P3) f is increasing, and

(P4) f is continuous from the right at zero.

Ruckle [18-20] used the idea of a modulus function f to construct the sequence space

$$X(f)=\{x=(x_k):\sum_{k=1}^{\infty}f(|x_k|)<\infty\}$$

This space is an FK space ,and Ruckle[18-20] proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences.

The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x)=x$ for all real $x\geq 0$.Thus Ruckle[18-20] proved that, for any modulus f.

$$X(f)\subset l_1 \text{ and } X(f)^\alpha=l_\infty$$

The space $X(f)$ is a Banach space with respect to the norm

$$||x||=\sum_{k=1}^{\infty}f(|x_k|)<\infty.\text{(See[18-20])}.$$

Ref.

[15] Maddox,I.J.Some properties of paranormed sequence spaces.,*J.London. Math.Soc.*1(1969),316-322.

Spaces of the type $X(f)$ are a special case of the spaces structured by B.Gramsch in[8].From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Symmetric sequence spaces, which are locally convex have been frequently studied by D.J.H Garling[6-7],G.Köthe[12]and W.H.Ruckle[18-20].

After then E.Kolk [10-11] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ and defined the sequence space

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.\text{(See[10-11])}.$$

After then Vakeel.A.Khan and Lohani[21] defined the following sequence spaces

$$l_\infty(u, \triangle, F) = \{x = (x_k) \in \omega : \sup_{k\geq 0} f_k(|u_k \triangle x_k|) < \infty\},$$

$$c(u, \triangle, F) = \{x = (x_k) \in \omega : \lim_{k\to\infty} f_k(|u_k \triangle x_k - l|) = 0, l \in \mathbb{C}\},$$

$$c_0(u, \triangle, F) = \{x = (x_k) \in \omega : \lim_{k\to\infty} f_k(|u_k \triangle x_k|) = 0\},$$

where $u \in U$.

If we take x_k instead of $\triangle x$,then we have the following sequence spaces

$$l_\infty(u, F) = \{x = (x_k) \in \omega : \sup_{k\geq 0} f_k(|u_k x_k|) < \infty\},$$

$$c(u, F) = \{x = (x_k) \in \omega : \lim_{k\to\infty} f_k(|u_k x_k - l|) = 0, l \in \mathbb{C}\},$$

$$c_0(u, F) = \{x = (x_k) \in \omega : \lim_{k\to\infty} f_k(|u_k x_k|) = 0\},$$

where $u \in U$.

After then C.Asma and R.Colak[1] defined the following sequence spaces

$$l_\infty(u, \triangle, p) = \{x = (x_k) \in \omega : (|u_k \triangle x_k|) \in l_\infty(p)\},$$

$$c(u, \triangle, p) = \{x = (x_k) \in \omega : (|u_k \triangle x_k|) \in c(p)\},$$

$$c_0(u, \triangle, p) = \{x = (x_k) \in \omega : (|u_k \triangle x_k|) \in c_0(p)\},$$

where $u \in U$,$p = (p_k)$ be any sequence of positive reals.

After then again Vakeel.A.Khan and Lohani[21] defined the following sequence spaces

$$l_\infty(u, \triangle, F, p) = \{x = (x_k) \in \omega : \sup_{k\geq 0}(f_k(|u_k \triangle x_k|))^{p_k} < \infty\},$$

$$c(u, \triangle, F, p) = \{x = (x_k) \in \omega : \lim_{k\to\infty}(f_k(|u_k \triangle x_k - l|))^{p_k} = 0, l \in \mathbb{C}\},$$

$$c_0(u, \triangle, F, p) = \{x = (x_k) \in \omega : \lim_{k\to\infty}(f_k(|u_k \triangle x_k|))^{p_k} = 0\},$$

Ref.

[6] Garling,D.J.H. *On Symmetric Sequence Spaces*,Proc.London. Math.Soc.16(1966), 85-106.

Global Journal of Science Frontier Research (F) Volume XII Issue XI Version I Year 2012

which are paranormed spaces paranormed with

$$Q(x) = \sup_{k\geq 0}(f_k(|u_k\triangle x_k|))^{p_k})^{\frac{1}{H}} \leq a$$

where $H = max(1, \sup_{k\geq 0} p_k)$ and $a = f_k(l)$,$l = \sup_{k\geq 0}(|u_k\triangle x_k|)$.

Ref.

[2] Esi.A.,Isik,M. Some generalized difference sequence spaces,$Thai.J.Math.$,3(2),(2005)241-247.

Esi and Isik[2] defined the sequence spaces

$$l_\infty(\triangle_v^m, s, p) = \{x = (x_k) \in \omega : \sup \lim_k k^{-s}|\triangle_v^m x_k|^{p_k} < \infty, s \geq 0\},$$

$$c(\triangle_v^m, s, p) = \{x = (x_k) \in \omega : k^{-s}|\triangle_v^m x_k - L|^{p_k} \to 0 (k \to \infty), s \geq 0, \text{forsome L}\},$$

$$c_0(\triangle_v^m, s, p) = \{x = (x_k) \in \omega : k^{-s}|\triangle_v^m x_k|^{p_k} \to 0 (k \to \infty), s \geq 0\},$$

where $v = (v_k)$ is any fixed sequence of non zero complex numbers, $m \in \mathbb{N}$ is a fixed number,

$$\triangle_v^0 x_k = (v_k x_k), \quad \triangle_v x_k = (v_k x_k - v_{k+1} x_{k+1})$$

and

$$\triangle_v^m x_k = (\triangle_v^{m-1} x_k - \triangle_v^{m-1} x_{k+1})$$

and so that

$$\triangle_v^m x_k = \sum_{i=0}^{m} (-1)^i \begin{bmatrix} m \\ i \end{bmatrix} \quad v_{k+i} x_{k+i}.$$

When s=0, m=1, v=(1,1,1,.......) and $p_k = 1$ for all $k \in \mathbb{N}$, they are just $l_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$ defined by Kizmaz[9].

When s=0 and $p_k = 1$ for all $k \in \mathbb{N}$, they are the following sequence spaces defined by Et and Esi[5]

$$l_\infty(\triangle_v^m) = \{x = (x_k) \in \omega : (\triangle_v^m x_k) \in l_\infty\},$$

$$c(\triangle_v^m) = \{x = (x_k) \in \omega : (\triangle_v^m x_k) \in c\},$$

$$c_0(\triangle_v^m) = \{x = (x_k) \in \omega : (\triangle_v^m x_k) \in c_0\}.$$

II. Main Results

In this article we introduce the following classes of sequence spaces.

$$l_\infty(u, \triangle_v^m, F, p) = \{x = (x_k) \in \omega : \sup_{k\geq 0}(f_k(|u_k\triangle_v^m x_k|))^{p_k} < \infty\},$$

$$c(u, \triangle_v^m, F, p) = \{x = (x_k) \in \omega : \lim_{k\to\infty}(f_k(|u_k\triangle_v^m x_k - l|))^{p_k} = 0, l \in \mathbb{C}\},$$

$$c_0(u, \triangle_v^m, F, p) = \{x = (x_k) \in \omega : \lim_{k\to\infty}(f_k(|u_k\triangle_v^m x_k|))^{p_k} = 0\},$$

Theorem 2.1. $l_\infty(u, \triangle_v^m, F)$ is a Banach space with norm

$$||x||_{(\triangle_v^m)_u} = \sup_{k\geq 0}(f_k(|u_k\triangle_v^m x_k|)) \leq \alpha,$$

where $\alpha = f_k(l)$ and $l = \sup_{k\geq 0}(|u_k\triangle_v^m x_k|)$.

Proof. Let (x^i) be a cauchy sequence in $l_\infty(u, \triangle_v^m, F)$ for each $i \in I\!N$.
Let r, x_0 be fixed. Then for each $\frac{\epsilon}{rx_0} > 0$ there exists a positive integer $I\!N$ such that

$$||x^i - x^j||_{(\triangle_v^m)_u} < \frac{\epsilon}{rx_0} \quad \text{for all i,j} \geq I\!N$$

Using the definition of norm, we get

$$\sup_{k\geq 0} f_k\left(\frac{|u_k(\triangle_v^m x_k^i - \triangle_v^m x_k^j)|}{||x^i - x^j||_{(\triangle_v^m)_u}}\right) \leq \alpha, \quad \text{for all i,j} \geq I\!N$$

ie,

$$f_k\left(\frac{|u_k(\triangle_v^m x_k^i - \triangle_v^m x_k^j)|}{||x^i - x^j||_{(\triangle_v^m)_u}}\right) \leq \alpha, \quad \text{for all i,j} \geq I\!N$$

Hence we can find $r > 0$ with $f_k(\frac{rx_0}{2}) \geq \alpha$ such that

$$f_k\left(\frac{|u_k(\triangle_v^m x_k^i - \triangle_v^m x_k^j)|}{||x^i - x^j||_{(\triangle_v^m)_u}}\right) \leq f_k\left(\frac{rx_0}{2}\right)$$

$$\frac{|u_k(\triangle_v^m x_k^i - \triangle_v^m x_k^j)|}{||x^i - x^j||_{(\triangle_v^m)_u}} \leq \frac{rx_0}{2}$$

This implies that

$$|u_k(\triangle_v^m x_k^i - \triangle_v^m x_k^j)| \leq \frac{rx_0}{2}\frac{\epsilon}{rx_0} = \frac{\epsilon}{2}$$

Since $u_k \neq 0$ for all k, we have

$$|\triangle_v^m x_k^i - \triangle_v^m x_k^j| \leq \frac{\epsilon}{2} \quad \text{for all i,j} \geq I\!N$$

Hence $(\triangle_v^m x_k^i)$ is a cauchy sequence in $\mathbb{C}$
For each $\epsilon > 0$ there exists a positive integer $I\!N$ such that $|\triangle_v^m x_k^i - \triangle_v^m x_k| < \epsilon$ for all $i > I\!N$.
Using the continuity of $F = (f_k)$ we can show that

$$\sup_{k\geq I\!N} f_k(|u_k(\triangle_v^m x_k^i - \lim_{j\to\infty} \triangle_v^m x_k^j)|) \leq \alpha,$$

Thus

$$\sup_{k\geq I\!N} f_k(|u_k(\triangle_v^m x_k^i - \triangle_v^m x_k)|) \leq \alpha,$$

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since $(x^i) \in l_\infty(u, \triangle_v^m, F)$ and $F = (f_k)$ is continuous it follows that $x \in l_\infty(u, \triangle_v^m, F)$
Thus $l_\infty(u, \triangle_v^m, F)$ is complete.

Theorem 2.2. $l_\infty(u, \triangle_v^m, F, p)$ is a complete paranormed space with

$$Q_u(x) = \sup_{k\geq 0}(f_k(|u_k \triangle_v^m x_k|)^{p_k})^{\frac{1}{H}} \leq \alpha$$

where $H = max(1, \sup_{k\geq 0} p_k)$ and $\alpha = f_k(l)$, $l = \sup_{k\geq 0}(|u_k \triangle_v^m x_k|)$.

Proof. Let (x^i) be a cauchy sequence in $l_\infty(u, \triangle_v^m, F, p)$ for each $i \in \mathbb{N}$.
Let $r > 0, x_0$ be fixed.Then for each $\frac{\epsilon}{rx_0} > 0$ there exists a positive integer $\mathbb{N}$ such that

$$Q_u(x^i - x^j)_{(\triangle_v^m)_u} < \frac{\epsilon}{rx_0} \quad \text{for all i,j} \geq \mathbb{N}$$

Using the definition of paranorm, we get

$$\sup_{k\geq 0} f_k\left(\frac{|u_k(\triangle_v^m x_k^i - \triangle_v^m x_k^j)|}{Q_u(x^i - x^j)_{(\triangle_v^m)_u}}\right)^{\frac{p_k}{H}} \leq \alpha, \quad \text{for all i,j} \geq \mathbb{N}$$

ie,

$$f_k\left(\frac{|u_k(\triangle_v^m x_k^i - \triangle_v^m x_k^j)|}{Q_u(x^i - x^j)_{(\triangle_v^m)_u}}\right)^{p_k} \leq \alpha, \quad \text{for all i,j} \geq \mathbb{N}$$

Hence we can find $r > 0$ with $f_k(\frac{rx_0}{2}) \geq \alpha$ such that

$$f_k\left(\frac{|u_k(\triangle_v^m x_k^i - \triangle_v^m x_k^j)|}{Q_u(x^i - x^j)_{(\triangle_v^m)_u}}\right) \leq f_k\left(\frac{rx_0}{2}\right)$$

$$\frac{|u_k(\triangle_v^m x_k^i - \triangle_v^m x_k^j)|}{Q_u(x^i - x^j)_{(\triangle_v^m)_u}} \leq \frac{rx_0}{2}$$

This implies that

$$|u_k(\triangle_v^m x_k^i - \triangle_v^m x_k^j)| \leq \frac{rx_0}{2}\frac{\epsilon}{rx_0} = \frac{\epsilon}{2}$$

Since $u_k \neq 0$ for all k, we have

$$|\triangle_v^m x_k^i - \triangle_v^m x_k^j| \leq \frac{\epsilon}{2} \quad \text{for all i,j} \geq \mathbb{N}$$

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Using the continuity of $F = (f_k)$ we can show that

$$\sup_{k\geq \mathbb{N}} f_k(|u_k(\triangle_v^m x_k^i - \lim_{j\to\infty} \triangle_v^m x_k^j)|)^{\frac{p_k}{H}} \leq \alpha,$$

Thus

$$\sup_{k\geq N} f_k(|u_k(\triangle_v^m x_k^i - \triangle_v^m x_k)|)^{\frac{p_k}{H}} \leq \alpha,$$

since $(x^i) \in l_\infty(u, \triangle_v^m, F, p)$ and $F = (f_k)$ is continuous it follows that $x \in l_\infty(u, \triangle_v^m, F, p)$
Thus $l_\infty(u, \triangle_v^m, F, p)$ is complete.

Theorem 2.3. Let $0 < p_k \leq q_k < \infty$ for each k. Then we have

$$c_0(u, \triangle_v^m, F, p) \subseteq c_0(u, \triangle_v^m, F, q)$$

Proof. Let $x \in c_0(u, \triangle_v^m, F, p)$ that is

$$\lim_{k\to\infty} (f_k(|u_k(\triangle_v^m x_k)|))^{p_k} = 0$$

This implies that

$$f_k(|u_k(\triangle_v^m x_k)|) \leq 1$$

for sufficiently large k, since modulus function is non decreasing.
Hence we get

$$\lim_{k\to\infty} (f_k(|u_k(\triangle_v^m x_k)|))^{q_k} \leq \lim_{k\to\infty} (f_k(|u_k(\triangle_v^m x_k)|))^{p_k} = 0$$

Therefore $x \in c_0(u, \triangle_v^m, F, q)$.

Theorem 2.4.(a) Let $0 < \inf p_k \leq p_k \leq 1$. Then we have

$$c_0(u, \triangle_v^m, F, p) \subseteq c_0(u, \triangle_v^m, F).$$

(b) Let $1 \leq p_k \leq \sup_k p_k < \infty$. Then we have

$$c_0(u, \triangle_v^m, F) \subseteq c_0(u, \triangle_v^m, F, p).$$

Proof.(a) Let $x \in c_0(u, \triangle_v^m, F, p)$, that is

$$\lim_{k\to\infty} (f_k(|u_k(\triangle_v^m x_k)|))^{p_k} = 0$$

Since $0 < \inf p_k \leq p_k \leq 1$,

$$\lim_{k\to\infty} (f_k(|u_k(\triangle_v^m x_k)|)) \leq \lim_{k\to\infty} (f_k(|u_k(\triangle_v^m x_k)|))^{p_k} = 0$$

Hence $x \in c_0(u, \triangle_v^m, F)$.

(b)Let $p_k \geq 1$ for each k and $\sup_k p_k < \infty$.
Suppose that $x \in c_0(u, \triangle_v^m, F)$.
Then for each $\epsilon > 0$ there exists a positive integer N such that

$$f_k(|u_k(\triangle_v^m x_k)|) \leq \epsilon \quad \text{for all } k \geq N$$

Notes

Since $1 \leq p_k \leq \sup_k p_k < \infty$, we have

$$\lim_{k\to\infty} (f_k(|u_k(\triangle_v^m x_k)|))^{p_k} \leq \lim_{k\to\infty} (f_k(|u_k(\triangle_v^m x_k)|)) \leq \epsilon < 1$$

Therefore $x \in c_0(u, \triangle_v^m, F, p)$.

Notes

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 11 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Security Issues in Wireless Local Area Networks (WLAN)

By Dr. Gurjeet Singh
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Abstract - This paper deals with this wireless local area security technologies and aims to exhibit their potential for integrity, availability and confidentiality. It provides a thorough analysis of the most WLAN packet data services and technologies, which can reveal the data in a secure manner. The paper outlines its main technical characteristics, discusses its architectural aspects based on security and explains the access protocol, the services provided, in secured way. This paper deals with security techniques for wireless local area networks.

GJSFR-F Classification : MSC 2010: 68U01

SECURITY ISSUES IN WIRELESS LOCAL AREA NETWORKS WLAN

Strictly as per the compliance and regulations of :

Ref.
1. Dr. Gurjeet Singh "Performance and Effectiveness of Secure Routing Protocols in MANET" Global Journal of Computer Science & Technology" Vol 12, Issue 5, 2012.

Security Issues in Wireless Local Area Networks (WLAN)

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Abstract - This paper deals with this wireless local area security technologies and aims to exhibit their potential for integrity, availability and confidentiality. It provides a thorough analysis of the most WLAN packet data services and technologies, which can reveal the data in a secure manner. The paper outlines its main technical characteristics, discusses its architectural aspects based on security and explains the access protocol, the services provided, in secured way. This paper deals with security techniques for wireless local area networks.

I. Introduction

A wireless LAN (WLAN) is analogous to a wired LAN but radio waves being the transport medium instead of traditional wired structures. This allows the users to move around in a limited area while being still connected to the network. Thus, WLANS combine data connectivity with user mobility, and, through simplified configuration, enable movable LANs [1]. In other words WLANS provide all the functionality of wired LANs, but without the physical constraints of the wire itself.

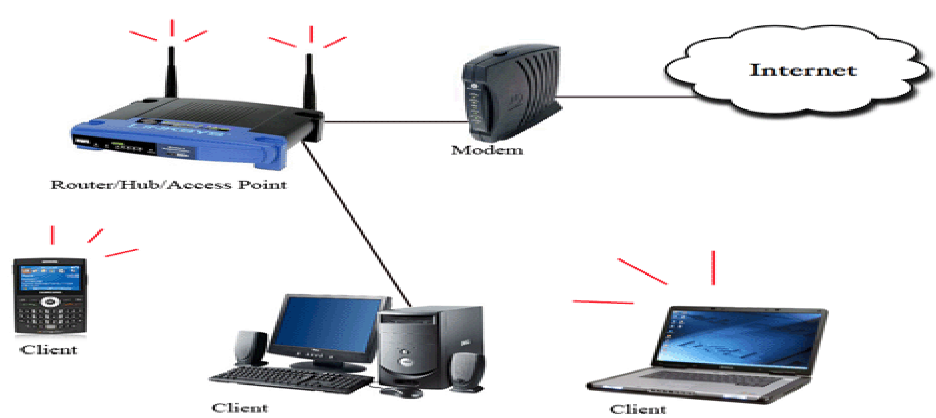

Figure 1.1 : Wireless Local Area Network

Generally a WLAN (in Infrastructure mode, see below) consists of a central connection point called the Access Point (AP). It is analogous to a hub or a switch in traditional star topology based wired local area networks. The Access Point transmits the data between different nodes of a wireless local area network and in most cases serves as the only link between the WLAN and the wired LAN. A typical Access Point can handle

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a handsome amount of users within a radius of about 300 feet. The wireless nodes, also called clients of a WLAN usually consist of Desktop PCs, Laptops or PDAs equipped with wireless interface cards.

II. Types of Wireless Networks

There are three types of wireless networks:

a) Wireless Personal Area Networking (WPAN)

WPAN describes an application of wireless technology that is intended to address usage scenarios that are inherently personal in nature. The emphasis is on instant connectivity between devices that manage personal data or which facilitate data sharing between small groups of individuals. An example might be synchronizing data between a PDA and a desktop computer. Or another example might be spontaneous sharing of a document between two or more individuals. The nature of these types of data sharing scenarios is that they are ad hoc and often spontaneous. Wireless communication adds value for these types of usage models by reducing complexity (i.e. eliminates the need for cables).

b) Wireless Local Area Networking (WLAN)

WLAN on the other is more focused on organizational connectivity not unlike wire based LAN connections. The intent of WLAN technologies is to provide members of workgroups access to corporate network resources be it shared data, shared applications or e-mail but do so in way that does not inhibit a user's mobility. The emphasis is on a permanence of the wireless connection within a defined region like an office building or campus. This implies that there are wireless access points that define a finite region of coverage.

c) Wireless Wide Area Networking (WWAN)

WWAN addresses the need to stay connected while traveling outside this boundary. Today, cellular technologies enable wireless computer connectivity either via a cable to a cellular telephone or through PC Card cellular modems. The need being addressed by WWAN is the need to stay in touch with business critical communications while traveling.

III. IEEE 802.11b Security Features

The security features provided in 802.11b standard [2] are as follows:

a) SSID – Service Set Identifier

SSID acts as a WLAN identifier. Thus all devices trying to connect to a particular WLAN must be configured with the same SSID. It is added to the header of each packet sent over the WLAN (i.e. a BSS) and verified by an Access Point. A client device cannot communicate with an Access Point unless it is configured with the same SSID as the Access Point.

b) WEP - Wired Equivalent Privacy

According to the 802.11 standard, Wired Equivalent Privacy (WEP) was intended to provide "confidentiality that is subjectively equivalent to the confidentiality of a wired local area network (LAN) medium that does not employ cryptographic techniques to enhance privacy" [4].

IEEE specifications for wired LANs do not include data encryption as a requirement. This is because approximately all of these LANs are secured by physical

Ref.

2. Dr. Gurjeet Singh & Dr. Jatinder Singh "Security Issues in Broadband Wireless Networks" Global Journal of Researches in Engineering Electrical and Electronics Engineering, Vol 12, Issue 5, 2012.

means such as walled structures and controlled entrance to building etc. However no such physical boundaries can be provided in case of WLANs thus justifying the need for an encryption mechanism.

WEP provides for Symmetric Encryption using the WEP key. Each node has to be manually configured with the same WEP key. The sending station encrypts the message using the WEP key while the receiving station decrypts the message using the same WEP key. WEP uses the RC4 stream cipher.

Ref.

4. WLAN Association, "Introduction to Wireless LANs", WLAN A Resource Center, 1999, http://www.wlana.com/learn/intro.pdf

c) MAC Address Filters

In this case, the Access Point is configured to accept association and connection requests from only those nodes whose MAC addresses are registered with the Access Point. This scheme provides an additional security layer.

IV. Problem Definition

Ubiquitous network access without wires is the main attraction underlying wireless network deployment. Although this seems as enough attraction, there exists other side of the picture. Before going All-Wireless, organizations should first understand how wireless networks could be vulnerable to several types of intrusion methods.

- *Invasion & Resource Stealing:* Resources of a network can be various devices like printers and Internet access etc. First the attacker will try to determine the access parameters for that particular network. For example if network uses MAC Address based filtering of clients, all an intruder has to do is to determine MAC address and assigned IP address for a particular client. The intruder will wait till that valid client goes off the network and then he starts using the network and its resources while appearing as a valid user.
- *Traffic Redirection:* An intruder can change the route of the traffic and thus packets destined for a particular computer can be redirected to the attacking station. For example ARP tables (which contain MAC Address to IP Address Mapping) in switches of a wired network can be manipulated in such a way that packets for a particular wired station can be re-routed to the attacking station.
- *Denial of Service (DOS):* Two types of DOS attacks against a WLAN can exist. In the first case, the intruder tries to bring the network to its knees by causing excessive interference. An example could be excessive radio interference caused by 2.4 GHz cordless phones or other wireless devices operating at 2.4GHz frequency. A more focused DOS attack would be when an attacking station sends 802.11 dissociate message or an 802.1x EAPOL-logoff message (captured previously) to the target station and effectively disconnects it.
- *Rouge Access Point:* A rogue Access Point is one that is installed by an attacker (usually in public areas like shared office space, airports etc) to accept traffic from wireless clients to whom it appears as a valid Authenticator. Packets thus captured can be used to extract sensitive information or can be used for further attacks before finally being re-inserted into the proper network

These concerns relate to wireless networks in general. The security concerns raised specifically against IEEE 802.11b networks [4] are as following.

- *MAC ADDRESS AUTHENTICATION:* Such sort of authentication establishes the identity of the physical machine, not its human user. Thus an attacker who manages to steal a laptop with a registered MAC address will appear to the network as a legitimate user.

- *ONE-WAY AUTHENTICATION:* WEP authentication is client centered or one-way only. This means that the client has to prove its identity to the Access Point but not vice versa. Thus a rogue Access Point will successfully authenticate the client station and then subsequently will be able to capture all the packets send by that station through it.
- *STATIC WEP KEYS:* There is no concept of dynamic or per-session WEP keys in 802.11b specification. Moreover the same WEP key has to be manually entered at all the stations in the WLAN.
- *SSID:* Since SSID is usually provided in the message header and is transmitted in clear text format, it provides very little security. It is more of a network identifier than a security feature.
- *WEP KEY VULNERABILITY:* WEP key based encryption was included to provide same level of data confidentiality in wireless networks as exists in typical wired networks. However a lot of concerns were raised later regarding the usefulness of WEP. The IEEE 802.11 design community blames 40-bit RC4 keys for this and recommends using 104- or 128-bit RC4 keys instead. Although using larger key size does increase the work of an intruder, it does not provide completely secure solution. Many recent research results have proved this notion [5]. According to these research publications the vulnerability of WEP roots from its initialization vector and not from its smaller key size

V. Virtual Private Network (vpn)

A Virtual Private Network (VPN) is a network technology that creates a secure network connection over a public network such as the Internet or a private network owned by a service provider. Large corporations, educational institutions, and government agencies use VPN technology to enable remote users to securely connect to a private network.

A VPN can connect multiple sites over a large distance just like a Wide Area Network (WAN). VPNs are often used to extend intranets worldwide to disseminate information and news to a wide user base. Educational institutions use VPNs to connect campuses that can be distributed across the country or around the world.

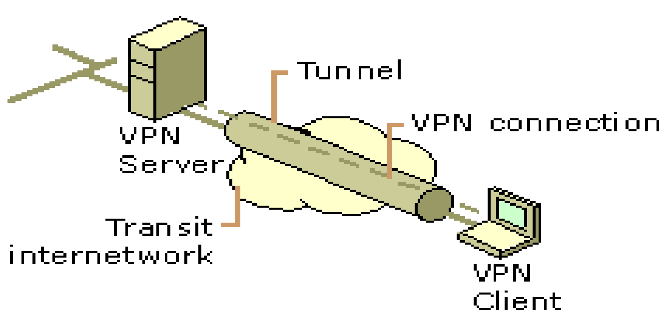

Figure 1.2 : Virtual private network

VPN technology provides three levels of security [7]:

- *Authentication:* A VPN Server should authorize every user logged on at a particular wireless station and trying to connect to WLAN using VPN Client. Thus authentication is user based instead of machine based.

Ref.

5. John Vollbrecht, David Rago, and Robert Moskowitz "Wireless LAN Access Control and Authentication", White Papers at Interlink Networks Resource Library, 2001. http://www.interlinknetworks.com/images/ resource/WLAN_Access_Control.pdf

- *Encryption:* VPN provides a secure tunnel on top of inherently un-secure medium like the Internet. To provide another level of data confidentiality, the traffic passing through the tunnel is also encrypted. Thus even if an intruder manages to get into the tunnel and intercepts the data, that intruder will have to go through a lot of effort and time decoding it (if he is able to decode it).
- *Data authentication:* It guarantees that all traffic is from authenticated devices thus implying data integrity.

Notes

Common Uses of VPNs

The next few subsections describe the more common VPN configurations in more detail.

Remote Access Over the Internet

VPNs provide remote access to corporate resources over the public Internet, while maintaining privacy of information. Figure 2 shows a VPN connection used to connect a remote user to a corporate intranet.

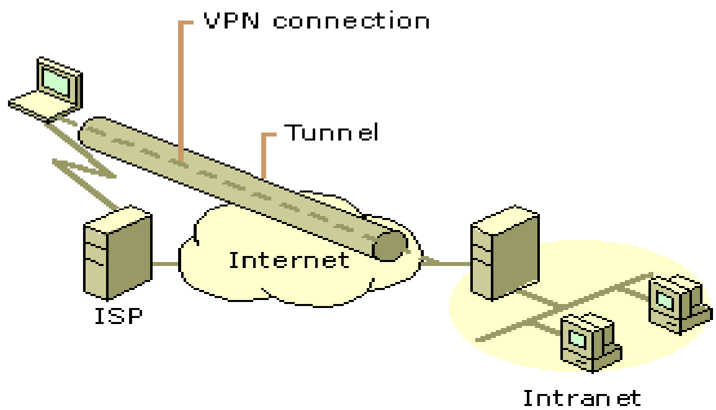

Figure 1.3 : VPN connection to connect a remote client to a private intranet

Rather than making a long distance (or 1-800) call to a corporate or outsourced network access server (NAS), the user calls a local ISP. Using the connection to the local ISP, the VPN software creates a virtual private network between the dial-up user and the corporate VPN server across the Internet.

Connecting Networks Over the Internet

There are two methods for using VPNs to connect local area networks at remote sites:

- **Using dedicated lines to connect a branch office to a corporate LAN**. Rather than using an expensive long-haul dedicated circuit between the branch office and the corporate hub, both the branch office and the corporate hub routers can use a local dedicated circuit and local ISP to connect to the Internet. The VPN software uses the local ISP connections and the Internet to create a virtual private network between the branch office router and corporate hub router.
- **Using a dial-up line to connect a branch office to a corporate LAN**. Rather than having a router at the branch office make a long distance (or 1-800) call to a corporate or outsourced NAS, the router at the branch office can call the local ISP. The VPN software uses the connection to the local ISP to create a VPN between the branch office router and the corporate hub router across the Internet.

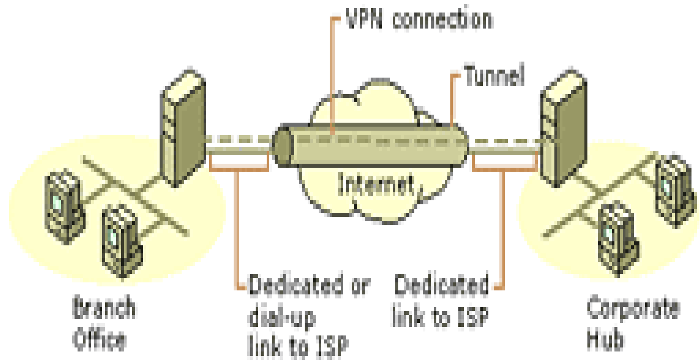

Figure 1.4 : Using a VPN connection to connect two remote sites

In both cases, the facilities that connect the branch office and corporate offices to the Internet are local. The corporate hub router that acts as a VPN server must be connected to a local ISP with a dedicated line. This VPN server must be listening 24 hours a day for incoming VPN traffic.

Connecting Computers over an Intranet

In some corporate internetworks, the departmental data is so sensitive that the department's LAN is physically disconnected from the rest of the corporate internetwork. Although this protects the department's confidential information, it creates information accessibility problems for those users not physically connected to the separate LAN.

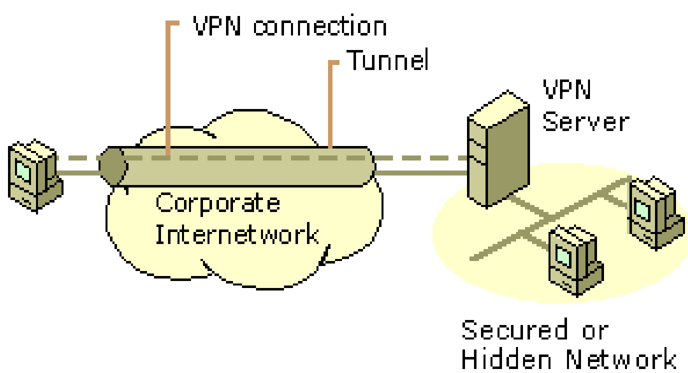

Figure 1.5 : Using a VPN connection to connect to a secured or hidden network

VPNs allow the department's LAN to be physically connected to the corporate internetwork but separated by a VPN server. The VPN server is not acting as a router between the corporate internetwork and the department LAN. A router would connect the two networks, allowing everyone access to the sensitive LAN. By using a VPN, the network administrator can ensure that only those users on the corporate internetwork who have appropriate credentials (based on a need-to-know policy within the company) can establish a VPN with the VPN server and gain access to the protected resources of the department. Additionally, all communication across the VPN can be encrypted for data confidentiality. Those users who do not have the proper credentials cannot view the department LAN.

Notes

VI. Cisco Leap (Light Weight Authentication Protocol)

Cisco LEAP, or EAP Cisco Wireless, is an 802.1X authentication type for wireless LANs that supports strong mutual authentication between the client and a RADIUS server. LEAP is a component of the Cisco Wireless Security Suite. Cisco introduced LEAP in December 2000 as a preliminary way to quickly improve the overall security of wireless LAN authentication. LEAP is a widely deployed, market-proven EAP authentication type.

Cisco's LEAP fills two noteworthy WLAN security holes [4]:

- Mutual Authentication between Client Station and Access Point: We described in Section 2 (Problem Definition) of Rogue Access Points. This was because of the One-Way, Client Centered Authentication between the Client and the Access Point. LEAP requires two-way authentication, i.e., a station can also verify the identity of the Access Point before completing the connection.
- Distribution of WEP Keys on a Per-session Basis: As opposed to the static WEP Keys in 802.11 specifications, LEAP protocol supports the notion of dynamic session keys. Both the Radius Server and Cisco client independently generate this key. Thus the key is not transmitted through the air where it could be intercepted.

VII. Secure Socket Layer (SSL)

Stands for "Secure Sockets Layer." SSL is a secure protocol developed for sending information securely over the Internet. Many websites use SSL for secure areas of their sites, such as user account pages and online checkout. Usually, when you are asked to "log in" on a website, the resulting page is secured by SSL. SSL encrypts the data being transmitted so that a third party cannot "eavesdrop" on the transmission and view the data being transmitted. Only the user's computer and the secure server are able to recognize the data. SSL keeps your name, address, and credit card information between you and merchant to which you are providing it. Without this kind of encryption, online shopping would be far too insecure to be practical. When you visit a Web address starting with "https," the "s" after the "http" indicates the website is secure. These websites often use SSL certificates to verify their authenticity. The below figure 1.6 shows the high level protocols

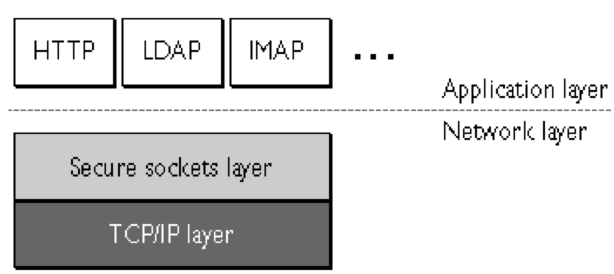

Figure 1.6 : SSL runs above TCP and below High Level Protocols

VIII. Access Point

Wireless **access points** (APs or WAPs) are specially configured nodes on wireless local area networks (WLANs). Access points act as a central transmitter and receiver of WLAN radio signals. Access points used in home or small business networks are generally

Ref.

4. WLAN Association, "Introduction to Wireless LANs", WLAN A Resource Center, 1999, http://www.wlana.com/learn/intro.pdf

small, dedicated hardware devices featuring a built-in network adapter, antenna, and radio transmitter. Access points support Wi-Fi wireless communication standards. Although very small WLANs can function without access points in so-called "ad hoc" or peer-to-peer mode, access points support "infrastructure" mode. This mode bridges WLANs with a wired Ethernet LAN and also scales the network to support more clients. Older and base model access points allowed a maximum of only 10 or 20 clients; many newer access points support up to 255 clients.

Notes

- Model Setup: Cisco Aironet 350 Series
- Data Rates: 1, 2, 5.5, 11 Mbps
- Network Standard: IEEE 802.11b
- Uplink:Auto-Sensing 0/100BaseT Ethernet
- Frequency Band: 2.4 to 2.497 GHz
- Network Architecture: Infrastructure
- Wireless Medium: Direct Sequence Spread Spectrum (DSSS)

IX. Experimental Results

There were four solutions suggested in response to the WEP vulnerability problems. Among those, IEEE 802.1x (i.e. EAP based) and Cisco LEAP will be treated as similar solutions for analysis and testing purposes and thus our test setup will only include Cisco LEAP solution for both cases. WEP based configuration will be implemented in order to emphasize and practically demonstrate the vulnerability in WEP based security. Various test results are discussed and illustrated as follows:

Legends:

------ Represents security control; ···· Represents data flow

→ Represents interception

SP Represents a Java program that exchanges sample data with the client

a) WEP Based Approach

In this approach, WEP keys will be manually configured in both desktops and Access Point to enable WEP Key based encryption. SP will generate sample data. Then the Laptop armed with hacking software would try to break the WEP key.

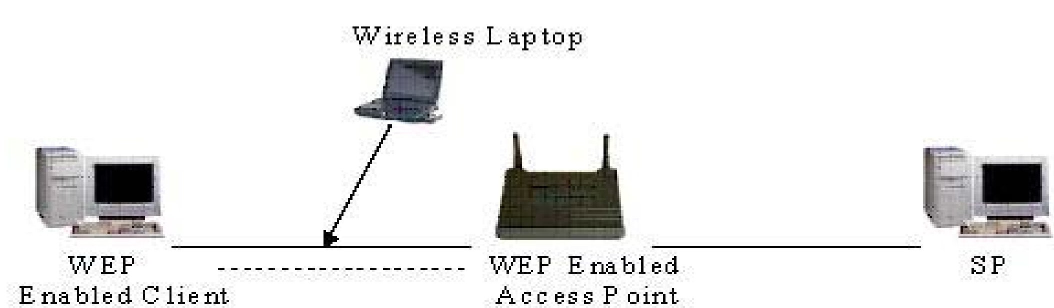

Figure 1.7 : WEP-enabled Set-up

Notes

b) LEAP Based Approach

In this approach one of the desktops will act as RADIUS server, while the client will be configured to use LEAP.

Figure 1.8 : LEAP-enabled Set-up

c) VPN Based Approach

In the VPN approach, the Access Point will be VPN aware; i.e. it will only accept and forward VPN traffic to a desktop computer configured as VPN server (and an optional AAA server). The second desktop computer will be installed with VPN client software.

Figure 1.9 : VPN-enabled Set-up

An alternate approach would be to have the access point act as a VPN server. However this is not the approach most widely used primarily because of performance considerations.

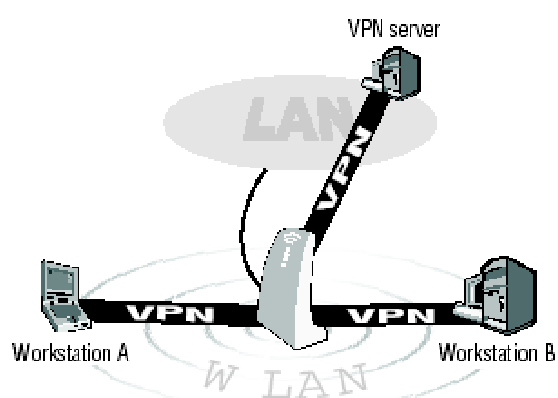

Figure 1.10 : VPN Server

d) SSL Based Approach

One of the desktops will be configured as a server (most probably a web server) implementing SSL. The second desktop will act as a SSL client. Again all traffic has to pass through Access Point.

Figure 1.11 : SSL-enabled Set-up

Notes

X. Conclusion

The wireless local area network provides physical flexibility in that it does not matter where within the space the user is working they are still able to use the network. With a wired network it is necessary to decide where computers will be used and install the ports there. Often the use of space changes with time, and then either the space has to be rewired or long trailing cables are used to get from the computer to the port. With a wireless network the performance of the network will deteriorate as the usage increases but unless there is very high demand all users will be able to access the network. The network can reach places that wired networks cannot, this includes out of doors where up to several hundred metres from buildings the signal can be reached. Also, it is relatively easy to set up an access point linked back to the campus network for use in remote premises.

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Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 11 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On Semi 3-Crossed Module by Using Simplicial Algebra

By Ali Mutlu & Berrin Mutlu
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GJSFR-F Classification : MSC 2010: 18D35, 18G30,18G50, 18G55,55Q05, 55Q20

ON SEMI 3 -CROSSED MODULE BY USING SIMPLICIAL ALGEBRA

Strictly as per the compliance and regulations of :

Global Journal of Science Frontier Research (F) Volume XII Issue XI Version I Year 2012

On Semi 3−Crossed Module by Using Simplicial Algebra

Ali Mutlu[α] & Berrin Mutlu[σ]

Abstract - Using simplicial algebra, semi 3 − crossed module of commutative algebra is defined and some of the examples and results of semi 3 − crossed module are given.

Keywords and Phrases : Crossed Module, 2 − crossed Module, Semi 3 − crossed module, Simplicial Algebra.

I. Introduction

Simplicial algebras play an important role in homological algebras homotopy theory and algebraic $K-$theory. In each theory the internal structures has been studied relatively. The present article intends to study the $4-$types of a simplicial algebra.

Crossed module was initially defined by J.H.C. Whitehead in [10] as a model for $2-$types (homotopy) and used it in various contexts, especially in his investigation into the algebraic structure of second relative homotopy groups. We use the definition and elementary theory of crossed module of a commutative algebra given by [9].

Higher dimensional analogues of crossed modules of groups and commutative algebras have been defined respectively: [4] has defined a $2-$crossed module of groups as model for $2-$types. A $2-$crossed module of algebras was given by [5].

In this paper, we extend the crossed module to $4-$types by using simplicial method. We also give the description of semi $3-$crossed module of commutative algebras and present some applications of Peiffer elements on Moore complex of a simplicial algebra. In particular we investigate Moore complex sequence for $i \geq k \in \{0, 1, \ldots, n+2\}$. Let $\mathbf{A}$ be a simplicial algebra and $NA_i = 0$ where $NA_i = \bigcap_{i=0}^{n-1} Kerd_i$ is a Moore complex of $\mathbf{A}$. We examine the simplicial long sequence and the Moore long sequence as follows respectively.

Ref.

[5] Grandjeán A.R. & Vale, M.J., 2-Modulos cruzados en la cohomologia de André-Quillen, *Memorias de la Real Academia de Ciencias* **22**

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$$\cdots A_n \underset{s_{n-1},\ldots s_0}{\overset{d_n,\ldots,d_0}{\rightleftarrows}} A_{n-1} \cdots A_2 \underset{s_0,s_1}{\overset{d_0,d_1,d_2}{\rightleftarrows}} A_1 \underset{s_0}{\overset{d_0,d_1}{\rightleftarrows}} A_0$$

and

$$\cdots 0 \longrightarrow 0 \longrightarrow NA_n \longrightarrow \cdots \longrightarrow NA_2 \longrightarrow NA_1 \longrightarrow NA_0 \, .$$

Also we iterate relation between the Moore long exact sequence consists of crossed complex, $2-$crossed module, square complex, $2-$crossed complex which is defined in [6]. We describe semi $3-$crossed module of a commutative algebra, using by $C_{\alpha,\beta}$ Peiffer elements are defined in [2]. Our aim is given relation between algebraic topology constructions in this article. Observe that Moore complex is the relation between structure of algebraic topology and a simplicial algebra.

II. Construction of Semi 3-Crossed Module

Before giving definition of semi $3-$crossed module it will be helpful to have notion of a pre-crossed module and introduce description of pre $2-$crossed module.
Throughout this article we denote an action of $c_0 \in C_0$ on $c_1 \in C_1$ by $c_0 \cdot c_1$.

Definition 2.1 *Let C_0 be a $k-$algebra with identity. A* pre-crossed module *of commutative algebras is a C_0-algebra, C_1 together with a C_0-algebra morphism.*

$$\partial : C_1 \longrightarrow C_0,$$

such that for all $c_1 \in C_1,\ c_0 \in C_0\ \partial(c_0 \cdot c_1) = c_0 \partial(c_1)$.

Now we may describe the definition of a pre $2-$crossed module and semi $3-$crossed module of commutative algebras.

Definition 2.2 *A* pre$-2-$crossed module *of $k-$algebras consists of complex of C_0-algebra*

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

with $\partial_2,\ \partial_1$ morphisms of C_0-algebra, where the algebra C_0 acts on itself by multiplication such that

$$C_2 \xrightarrow{\partial_2} C_1$$

is pre-crossed module in which C_1 acts on C_2, (we require that for all $x \in C_2,\ \ y \in C_1$ and $z \in C_0$ $(xy)z = x(yz)$) further, there is a C_0-bilinear function giving

$$\{\otimes\} : C_1 \otimes C_1 \to C_2$$

Ref.

[2] Arvasi Z. , Applications in Commutative Algebra of The Moore Complex of a Simplicial Algebra, *Ph.D. Thesis*, University of Wales, BANGOR, (1994).

Notes

called Peiffer lifting, which satisfies the following axioms:

$$\begin{array}{lrcl} 2CM1_p & \partial_2\{y_0 \otimes y_1\} & = & y_0y_1 - y_0\partial_1(y_1) \\ 2CM2_p & \{y_0 \otimes y_1y_2\} & = & \{y_0y_1 \otimes y_2\} + \partial_1(y_2)\ \{y_0 \otimes y_1\} \\ 2CM3_p & \{y_0 \otimes y_1\} \cdot z & = & \{y_0 \otimes y_1 \cdot z\} \end{array}$$

for all $x, x_1, x_2 \in C_2, \quad y, y_0, y_1, y_2 \in C_1$ *and,* $z \in C_0$.

Let **A** be a simplicial algebra with the Moore complex **NA**. Then the complex of algebras

$$NA_2 \xrightarrow{\partial_2} NA_1 \xrightarrow{\partial_1} NA_0$$

is a pre 2−crossed module of algebras, where the Peiffer map is defined as follows:

$$\begin{array}{rcl} \{\ \otimes\ \}: \ NA_1 \otimes NA_1 & \longrightarrow & NA_2 \\ (x_0 \otimes x_1) & \longmapsto & s_1(x_0)(s_1(x_1) - s_0(x_1)). \end{array}$$

It is obvious that the pre-crossed module condition is obviously satisfied. Indeed it is sufficient to show that $\partial_2,\ \partial_1$ are pre-crossed modules and pre 2−crossed module axioms are verified. That is NA_0 acts on NA_1 via s_0 and NA_1 acts on NA_2 and also s_1 and NA_0 acts on NA_2 via s_1s_0. Thus

$$\partial_1(x_0 \cdot x_1) = \partial_1(s_0(x_0)x_1) = x_0\partial_1(x_1)$$

$$\partial_2(x_1 \cdot x_2) = \partial_2(s_1(x_1)x_2) = x_1\partial_2(x_2)$$

2CM1$_p$:

$$\begin{array}{rcl} \partial_2\{x_0 \otimes\ x_1\} & = & \partial_2(s_1(x_0)(s_1(x_1) - s_0(x_1))), \\ & = & x_0x_1 - x_0\partial_1(x_1). \end{array}$$

Other two conditions are clear where $\partial_1,\ \partial_2$ are restrictions of $d_1,\ d_2$ respectively.

Now we can give the definition of a semi 3−crossed module of commutative algebras.

Definition 2.3 *A* semi 3−crossed module *of k−algebras consists of a complex* C_0*−algebra*

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

with $\partial_3, \partial_2, \partial_1$ *are morphisms of* C_0*−algebra, where the algebra* C_0 *acts on itself by multiplication, such that*

$$C_3 \xrightarrow{\partial_3} C_2$$

is a crossed module and

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

is a pre 2*−crossed module. Thus* C_2 *acts on* C_3 *and we require that for all* $w \in C_3, \quad x \in C_2, \quad y \in C_1$ *and* $z \in C_0$ *that*

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$$(wx)(yz) = \Big(w(x(yz))\Big).$$

Furthermore there is also a C_0-equivalent function defined as

$$\{\ \otimes\ \}: C_2 \otimes C_2 \to C_3$$

Mutlu-Arvasi mapping may be defined as follows

$$\{x_2 \otimes x_2'\} = H(x_2 \otimes x_2') = s_1(x_2)s_0(x_2') - s_1(x_2)s_1(x_2') + s_2(x_2)s_2(x_2')$$

if the following conditions are verified.

3CM1$_s$ ∂_2, ∂_1 *are pre-crossed modules,* ∂_3 *is a crossed module*

3CM2$_s$ $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ *is a pre* $2-$*crossed module*

3CM3$_s$ $\partial_3 H(x_2 \otimes x_2') = s_1d_2(x_2)s_0d_2(x_2') - s_1d_2(x_2)s_1d_2(x_2') + x_2x_2'$

3CM4$_s$ (a) $H(x_2 \otimes \partial_3(y_3)) = s_2(x_2)y_3$

(b) $H(\partial_3(y_3) \otimes x_2) = s_2(x_2)y_3$

3CM5$_s$ $H(x_2 \otimes \partial_3(y_3))H(\partial_3(y_3) \otimes x_2) = 0$

3CM6$_s$ $H(\partial_3(y_3) \otimes \partial_3(y_3')) = y_3y_3'$

where $x_2, x_2' \in C_2$ *and* $y_3, y_3' \in C_3$.

Theorem 2.4 ***(a)*** *If $NA_i = 0$ for $\forall i \geq 1$ in the Moore long sequence, then the Moore long sequence become only an algebra i.e., A_0 be an algebra.*

(b) *If $NA_i = 0$ for $\forall i \geq 2$ in the Moore long sequence, then the Moore long sequence be a crossed module i.e., $\cdots 0 \to 0 \to NA_1 \to NA_0$ is a crossed module.*

(c) *If $NA_i = 0$ for $\forall i \geq 3$ in the Moore long sequence, then the Moore long sequence become a $2-$crossed module i.e., $\cdots 0 \to 0 \to NA_2 \to NA_1 \to NA_0$ is a $2-$crossed module.*

(d) *If $NA_i = 0$ for $\forall i \geq 4$ in the Moore long sequence, then the Moore long sequence be semi $3-$crossed module i.e., $\cdots 0 \to 0 \to NA_3 \to NA_2 \to NA_1 \to NA_0$ is a $3-$semi crossed module.*

(e) *If $NA_i = 0$ for $\forall i \geq n+1$ in the Moore long sequence, then the Moore long sequence become an $n-$crossed complex i.e., $\cdots 0 \to 0 \to NA_n \to NA_{n-1} \to \cdots \to NA_3 \to NA_2 \to NA_1 \to NA_0$ is an $n-$crossed complex.*

(f) *If $NA_i = 0$ for $\forall i \geq n+2$ in the Moore long sequence, then the Moore long sequence be a $T-$complex.*

(g) *If $C_{\alpha,\beta}(x_\alpha, y_\beta) = 0$ hypercrossed complex pairings are described in [2] and [3], then the Moore long sequence be a crossed complex.*

Notes

Proof: **(a)** Suppose that $NA_i = 0$ for $\forall i \geq 1$ and so the Moore long sequence obtains as follows. $\cdots 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to NA_0 = A_0$. Therefore $NA_1 = \mathrm{Ker} d_0^1$ is an ideal of A_0.
On the other hand, if $a \in \mathrm{Ker} d_0^1$, then $NA_1 = 0$ since $d_0(a) = 0$.
(b) If $NA_i = 0$, for $\forall i \geq 2$ $(1 \leq i \leq n+2)$, then $\cdots 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to NA_1 \to NA_0 = A_0$ be a crossed module (see [2, 9]).
On other word, recall that $C_{\alpha,\beta}(x_\alpha \otimes y_\beta) = 0$ in [2], then for $\alpha = (1)$, $\beta = (0)$

$$\begin{aligned} C_{(1),(0)}(x_1 \otimes y_1) &= NA_1 \times NA_1 \to NA_2 \\ C_{(1),(0)}(x_1 \otimes y_1) &= s_1(x_1)(s_1(y_1) - s_0(y_1)) = 0 \end{aligned}$$

since $NA_1 \to NA_0$ be crossed module i.e, NA_0 acts on NA_1 together with $x_1 \cdot y_1 = s_1(x_1)s_0(y_1)$ and so crossed axioms are verified indeed,

$$\partial_1(x_1)y_1 = x_1\partial_1(y_1)$$

and

$\partial_1(x_1)y_1 = x_1 s_0 d_1(y_1) = x_1 d_1(y_1) = x_1 y_1$ (∂_1 is defined by restriction d_1). (see [2]) Also $1-$truncated hypercrossed complex, $1-$hypercrossed complex and $1-$crossed complex (see [3]).
(c) If $NA_i = 0$ for $\forall i \geq 3$, then the Moore long sequence $\cdots 0 \to 0 \to NA_2 \to NA_1 \to NA_0$ is a $2-$crossed module and $C^{(3)}_{\alpha,\beta}(x_\alpha \otimes y_\beta) = 0$ for $\alpha, \beta \in P(3)$. (see [2]) So the Peiffer lifting is defined as follows.

$$\{\otimes\} : NA_1 \otimes NA_1 \to NA_2$$

$$\{x \otimes y\} \mapsto s_1(x_1)(s_1(y_1) - s_0(y_1)) = 0$$

and $2-$crossed module conditions are also satisfied. On the other hand, 2CM2, 2CM4 (a) and (b) of $2-$crossed module axioms give us $C^{(3)}_{\alpha,\beta}(x_\alpha \otimes y_\beta) = 0$, which implies $NA_3 = 0$. (see [2])
(d) Let $\mathbf{A}$ be a simplicial algebra with the Moore complex $N\mathbf{A}$. Then the complex of algebras

$$NA_3/\partial_4(NA_4 \cap I_4) \xrightarrow{\partial_3} NA_2 \xrightarrow{\partial_2} NA_1 \xrightarrow{\partial_1} NA_0$$

is a semi $3-$crossed module of algebras, where and also I_4 is the ideal generated by the degenerate elements. Now we can define Mutlu-Arvasi map as follows:

$$\begin{aligned} \{\ \otimes\ \} :\ NA_2 \otimes NA_2 &\longrightarrow NA_3/\partial_4(NA_4 \cap I_4) \\ (x_2 \otimes y_2) &\longmapsto s_1(x_2)s_0(y_2) - s_1(x_2)s_1(y_2) + s_2(x_2)s_2(y_2) \end{aligned}$$

here the right hand side denotes an ideal in $NA_3/\partial_4(NA_4 \cap I_4)$ represented by the corresponding element in NA_3.
3CM1$_\mathbf{s}$ (a) ∂_2, ∂_1 are pre-crossed modules that is NA_1 acts on NA_2 via s_1 and NA_0 acts on NA_1 via s_0. Thus $\partial_1(x_0 \cdot y_1) = \partial_1(s_0(x_0)y_1) = x_0\partial_1(y_1)$ and $\partial_2(y_1 \cdot y_2) = \partial_2(s_1(y_1)y_2) = y_1\partial_2(y_2) = y_1\partial_2(y_2)$.

Ref.
[2] Arvasi Z. , Applications in Commutative Algebra of The Moore Complex of a Simplicial Algebra, *Ph.D. Thesis*, University of Wales, BANGOR, (1994).

(b) It is readily checked that the morphism $\partial_3 : NA_3/\partial_4(NA_4 \cap D_4) \to NA_2$ is a crossed module i.e., NA_2 acts on $NA_3/\partial_4(NA_4 \cap D_4)$ via s_2 and we have $\partial_4 C_{(3)(2)}(x_3 \otimes y_3) = d_4(s_3 x_3(s_2 y_3 - s_3 y_3) = 0$ via mod $\partial_4(NA_4 \cap D_4)$ from [2]. Thus $\partial_4 C_{(3)(2)}(x_3 \otimes y_3) = x_3(s_2\partial_2(y_3) - y_3)$ mod $\partial_4(NA_4 \cap D_4)$ so $\partial_3(x_3 \cdot y_3) = \partial_3(s_3(x_3))y_3 = x_3\partial_3(y_3)$ and $(\partial_3(x_3))y_3 = x_3 s_2 \partial_3(y_3) = x_3 y_3$ is obtained

3CM2$_s$ $NA_2 \to NA_1 \to NA_0$ is a pre 2−crossed module, where Peiffer map is defined as follows:
$\{x_0 \otimes \ x_1\} \longmapsto s_1(x_0)(s_1(x_1) - s_0(x_1))$.

3CM3$_s$

$$\begin{aligned}\partial_4 H(x_2 \otimes y_2) &= s_1 d_2(x_2) s_0 d_2(y_2) - s_1 d_2(x_2) s_1 d_2(y_2) + x_2 y_2 \\ &= s_1 d_2(x_2)(s_0 d_2(y_2) - s_1 d_2(x_2)) + x_2 y_2\end{aligned}$$

3CM4$_s$ (a) Using the hypercrossed complex parings are defined in [2] and then

$$0 \equiv \partial_4 C^{(4)}_{(3,1)(0)}(x_2 \otimes y_3) = s_1(x_2)s_0 d_3(y_3) - s_1(x_2)s_2 d_3(y_3) + s_2(x_2)s_2 d_3(y_3) - s_2(x_2)y_3 \mod \partial_4(NA_4 \cap I_4).$$

is calculated. Thus, we have

$$H(x_2 \otimes \partial_3(y_3)) = s_1(x_2)s_0 d_3(y_3) - s_1(x_2)s_2 d_3(y_3) + s_2(x_2)s_2 d_3(y_3) \mod \partial_4(NA_4 \cap I_4)$$

and therefore we obtain

$$H(x_2 \otimes \partial_3(y_3)) = s_2(x_2)y_3 \mod \partial_4(NA_4 \cap I_4).$$

(b) Again using the hypercrossed complex parings in [2] then

$$\begin{aligned}0 \equiv \partial_4 C^{(4)}_{(1)(0,3)}(y_3 \otimes x_2) = \ & s_1 d_3(y_3)s_0(x_2) - s_1 d_3(y_3)s_1(x_2) + \\ & s_1 d_3(y_3)s_0(x_2) - s_1 d_3(y_3)s_1(x_2) + y_3 s_2(x_2) \\ & \mod \partial_4(NA_4 \cap I_4).\end{aligned}$$

is found. This equality also holds

$$H(\partial_3(y_3) \otimes x_2) = s_1 d_3(y_3)s_0(x_2) - s_1 d_3(y_3)s_1(x_2) + s_2 d_3(y_3)s_2(x_2) \mod \partial_4(NA_4 \cap I_4).$$

and so we acquire

$$H(\partial_3(y_3) \otimes x_2) = -s_2(x_2)y_3 \mod \partial_4(NA_4 \cap I_4)$$

is commutated. Thus, the result of is given (a) and (b) of $3CM4_s$ as above.

3CM5$_s$

$$H(x_2 \otimes \partial_3(y_3)) + H(\partial_3(y_3) \otimes x_2) = s_2(x_2)y_3 - s_2(x_2)y_3 = 0.$$

Ref.

[2] Arvasi Z. , Applications in Commutative Algebra of The Moore Complex of a Simplicial Algebra, *Ph.D. Thesis*, University of Wales, BANGOR, (1994).

3CM6$_\mathbf{s}$ By [2] we may also be written this equation as.

$$\begin{aligned} 0 \equiv \partial_4 C^{(4)}_{(1)(0)}(y_3 \otimes y_3') &= s_1d_3(y_3)s_0d_3(y_3') - s_1d_3(y_3)s_1d_3(y_3') \\ &+ s_2d_3(y_3)s_2d_3(y_3') - y_3'y_3 \mod \partial_4(NA_4 \cap I_4). \end{aligned}$$

Using the equation is obtained as

$$\begin{aligned} H(\partial_3(y_3) \otimes \partial_3(y_3')) &= s_1d_3(y_3)s_0d_3(y_3') - s_1d_3(y_3)s_1d_3(y_3') \\ &+ s_2d_3(y_3)s_2d_3(y_3') \mod \partial_4(NA_4 \cap I_4). \end{aligned}$$

Hence, we yield

$$H(\partial_3(y_3) \otimes \partial_3(y_3')) \equiv y_3y_3' \mod \partial_4(NA_4 \cap I_4)$$

(e) If $NA_i = 0$ for $\forall i \geq n+1$ in the Moore long sequence, then the Moore long sequence be an $n-$crossed complex with $C^{(n+1)}_{(\alpha)(\beta)}(x \otimes y) = 0$. Recall that from [2] we have the trivial map as follows.

$$C^{(n+1)}_{(\alpha)(\beta)}(x \otimes y) = NA_{(n+1))-\#\alpha} \otimes NA_{(n+1))-\#\beta} \to NA_{n+1}.$$

And so this NA_n also be a commutative algebra for $n \geq 2$ since

$$\begin{aligned} 0 &= \partial_{n+1} C^{(n+1)}_{(n-1),(n)}(x \otimes y) \\ &= s_{n-1}d_n(x)y - xy \\ &= (\phi^{(n+1)}_{n-1} d_n(x))y - xy \\ &= yx. \end{aligned}$$

Here NA is a simplicial chain complex where NA_n is commutative for $n \geq 2$, $\phi^{(n+1)}_{n-1}$ is an action of NA_0 on NA_n for each $n \geq 1$ and ∂_n is NA_0-algebra morphism defined as

$$\cdots \longrightarrow NA_n/\partial_{n+1}K_{n+1} \longrightarrow NA_{n-1}/\partial_n K_n \longrightarrow \cdots \longrightarrow$$

$$NA_2/\partial_3K_3 \longrightarrow NA_1/\partial_2K_2 \longrightarrow NA_0$$

this is obviusly a crossed complex, where $K_i = NA_i \cap I_i$.

To prove the opposite of it let $NA_n/\partial_{n+1}K_{n+1}$ for $n \geq 2$, then

Thus $C^{(n-1)}_{(\alpha)(\beta)}(x, y) = 0$ implies that $NA_{n+1} = 0$. This is also an n-truncated complex. (see [3]).

(f) If $NA_i = 0$ for $\forall i \geq n+2$, then the Moore long sequence be a $T-$complex. To proof see [1] and [3].

(g) If $C^{(n-1)}_{(\alpha)(\beta)}(x_\alpha, y_\beta) = 0$, then the Moore long sequence become a crossed complex.

Therefore, we have the following results.

Ref.

[2] Arvasi Z. , Applications in Commutative Algebra of The Moore Complex of a Simplicial Algebra, *Ph.D. Thesis*, University of Wales, BANGOR, (1994).

Corollary 2.5 *If $NA_3/\partial_4(NA_4\cap I_4)=0$, then $NA_2\rightarrow NA_1\rightarrow NA_0$ corresponds a 2−crossed module. (see [2])*

Proof: Let **A** be a simplicial algebra with the Moore complex **NA.** Then the complex of algebras

$$NA_2\xrightarrow{\partial_2}NA_1\xrightarrow{\partial_1}NA_0$$

is a 2−crossed module of algebras, where the Peiffer map is defined as follows:

$$\begin{array}{rccl}\{\ \otimes\ \}: & NA_1\otimes NA_1 & \longrightarrow & NA_2/\partial_3(NA_3\cap I_3)\\ & (x_1\otimes y_1) & \longmapsto & s_1x_1(s_1y_1-s_0y_1).\end{array}$$

Here the right hand side denotes a coset in $NA_2/\partial_3(NA_3\cap I_3)$ represented by an element in NA_2 and $\partial_3(NA_3\cap I_3)=0$. □

Corollary 2.6 *If $NA_0=0$, then $NA_3/\partial_4(NA_4\cap I_4)\xrightarrow{\partial_3}NA_2\xrightarrow{\partial_2}NA_1$ is a 2−crossed module with defined Peiffer map as*

$$\begin{array}{rccl}\{\ \otimes\ \}: & NA_2\times NA_2 & \longrightarrow & NA_3/\partial_4(NA_4\cap I_4)\\ & (x_2\otimes\ y_2) & \longmapsto & s_2(x_2)s_2(y_2)-s_2(x_2)s_2(y_2).\end{array}$$

Proof: Indeed the function is satisfied 2−crossed module axioms.

2CM1: $\partial_3\{x_2\otimes y_2\}=x_2s_1\partial_2(y_2)-x_2y_2$.

2CM2: $\{\partial_3(x_2)\otimes\partial_3(y_2)\}=x_2y_2$ since $\partial_4C^{(4)}_{(2)(1)}(x_2\otimes y_2)=s_2d_3(x_3)(s_1d_3(y_3)-s_2d_3(y_3))+x_3y_3=0 \mod \partial_4(NA_4\cap I_4)$. (see [2])

2CM3: $\{x_2\otimes y_2y_2'\}=\{x_2y_2\otimes y_2'\}+\partial_2y_2'\{x_2\otimes y_2\}$

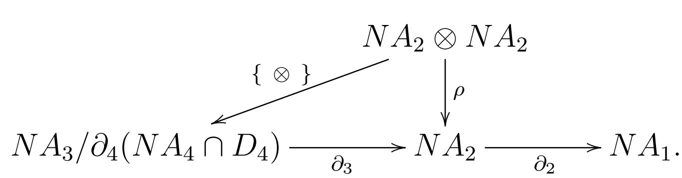
2CM4: (a) Let $\partial_4C^{(4)}_{(3,2)(1)}(x_2\otimes y_3)=s_2(x_2)(s_1d_3(y_3)-s_2d_3(y_3)+y_3)$. So $\partial_4C^{(4)}_{(3,2)(1)}(x_2\otimes y_3)=0 \mod \partial_4(NA_4\cap I_4)$. (see [2]) Then $\{x_2\otimes\partial_3(y_3)\}=s_2(x_2)y_3$ is obtained by the definition of action.

(b) Let $\partial_4C^{(4)}_{(2)(3,1)}(y_3\otimes x_2)=(s_2d_3(y_3)-y_3)(s_1(x_2)-s_2(x_2))$. So $\partial_4C^{(4)}_{(2)(3,1)}(y_3\otimes x_2)=0 \mod \partial_4(NA_4\cap I_4)$. (see [2]) Then $\{\partial_3(y_3)\otimes\ x_2\}=y_3x_2-\partial_2(y_3)x_2 \mod \partial_4(NA_4\cap I_4)$ is found.

2CM5

$$\begin{aligned}\{x_2\otimes y_2\}\cdot z &= s_2(x_2)(s_2(y_2)-s_0(y_2))\cdot z\\ &= s_2s_0(z)s_2(x_2)(s_2(y_2)-s_0(y_2))\\ &= s_2(s_0(zx_2))(s_2(y_2)-s_0(y_2))\\ &= s_2(x_2z)(s_2(y_2)-s_0(y_2))\\ &= \{x_2\cdot z\otimes y_2\}.\end{aligned}$$

Now we can consider the following diagram of morphism



Ref.
[2] Arvasi Z. , Applications in Commutative Algebra of The Moore Complex of a Simplicial Algebra, *Ph.D. Thesis*, University of Wales, BANGOR, (1994).

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The algebra NA_2 acts, in two way on the algebra $NA_3/\partial_4(NA_4 \cap I_4)$ by multiplication via s_1 and via s_2 both within A_3. The action via s_1 will also be denoted by $x \cdot y = s_1(x)y$ and the action via s_2 will be denoted by $xy = s_2(x)y$. The action of NA_1 on NA_3 is given as follows: from equality $(s_1(x) - s_2s_1d_2(x))y \equiv 0 \mod NA_3/\partial_4(NA_4 \cap I_4)$, there is a commutative diagram

$$\begin{array}{ccc} NA_3/\partial_4(NA_4 \cap I_4) \otimes NA_2 & \longrightarrow & NA_3/\partial_4(NA_4 \cap I_4) \\ \downarrow & & \downarrow \\ NA_3/\partial_4(NA_4 \cap I_4) \otimes NA_1 & \longrightarrow & NA_3/\partial_4(NA_4 \cap I_4) \end{array}$$

given by

$$\begin{array}{ccc} (y \otimes x) & \longmapsto & x \cdot y = s_1(x)y \\ \downarrow & & \downarrow \\ (x \otimes \partial_2(y)) & \longmapsto & \partial_2(xy) = s_2s_1d_2(x)y \end{array}$$

which gives an equality

$$\partial_2(xy) = s_2s_1d_2(x)y = s_1(x)y.$$

Let us define the map ρ by $\rho(x \otimes x') = \partial_2(x)x' - xx'$ for $x, x' \in NA_2$, that is the Peiffer element in NA_2 which corresponds to $\{x \otimes x'\}$. Thus if the map ρ is the trivial map $\partial_2 : NA_2 \to NA_1$ is a crossed module.

Now if the the Moore long sequence is iterated as follows, so then two results are obtained where $K_i = NA_i \cap I_i$.

$$\cdots 0 \longrightarrow 0 \longrightarrow NA_n/\partial_{n+1}K_{n+1} \longrightarrow NA_{n-1}/\partial_n K_n$$

$$\longrightarrow \cdots \longrightarrow NA_1/\partial_2 K_2 \longrightarrow NA_0$$

Corollary 2.7

$$\cdots 0 \longrightarrow 0 \longrightarrow NA_k/\partial_{k+1}K_{k+1} \xrightarrow{\partial_k} NA_{k-1} \xrightarrow{\partial_{k-1}} NA_{k-2}$$

$$\longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

is a 2−crossed module with defined Peiffer element

$$\{x_{k-1} \otimes \ y_{k-1}\} = C_{(1)(0)}(x_{k-1} \otimes \ y_{k-1}).$$

So the 2−crossed module conditions are clearly verified.

Corollary 2.8

$$\cdots 0 \longrightarrow 0 \longrightarrow NA_k/\partial_{k+1}K_{k+1} \xrightarrow{\partial_k} NA_{k-1} \xrightarrow{\partial_{k-1}} NA_{k-2} \xrightarrow{\partial_{k-2}} NA_{k-3}$$

$$\xrightarrow{\partial_{k-3}} 0 \longrightarrow \cdots \longrightarrow 0$$

Notes

is a semi $3-$*crossed module, where the Mutlu-Arvasi map is defined as follows:*

$$\{x_{k-1} \otimes \ y_{k-1}\} = C^{(k)}_{(0)(1)}(x_{k-1} \otimes y_{k-1})$$

It is clear that semi $3-$*crossed module conditions are satisfied.*

Corollary 2.9 *A* $3-$*truncated complex is a semi* $3-$*crossed module.*

We may follow the same procure as we did in Corolarly 2.8 in order to get to results.

Corollary 2.10 *The category of semi* $3-$*crossed modules is equivalent to the category of simplicial algebras with Moore complex of length 3.*

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Notes

Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 11 Version 1.0 Year 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Solution of Fractional Kinetic Equation with Laplace and Fourier Transform

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Abstract - In earlier paper Saxena et al.(2002,2003)[18],[19] derived the solutions of a number of fractional kinetic equations in terms of generalized Mittage-Leffler functions which extended the work of Haubold and Mathai (2000)[5].The objects of present paper is to investigate the solution of fractional diffusion equation involving Mittag-Leffler functions. The method involves simultaneous application of Laplace and Fourier transforms with time and space variable respectively. The results obtained are in a form of H-function.

Keywords : Mittage-Leffler function, Fractional Kinetic Equation, Laplace Transform, Fourier Transform amd H-functions.

GJSFR-F Classification : MSC 2010: 65T50, 44A10

SOLUTION OF FRACTIONAL KINETIC EQUATION WITH LAPLACE AND FOURIER TRANSFORM

Global Journal of Science Frontier Research (F) Volume XII Issue XI Version I Year 2012

Ref.

6. Hilfer,R(Ed):2000, *Application of fractional calculus in physics, World Scientific,Singapore.*

Solution of Fractional Kinetic Equation with Laplace and Fourier Transform

Satendra Kumar Tripathi [α] & Renu Jain [σ]

Abstract - In earlier paper Saxena et al.(2002,2003)[18],[19] derived the solutions of a number of fractional kinetic equations in terms of generalized Mittage-Leffler functions which extended the work of Haubold and Mathai (2000)[5].The objects of present paper is to investigate the solution of fractional diffusion equation involving Mittag-Leffler functions. The method involves simultaneous application of Laplace and Fourier transforms with time and space variable respectively. The results obtained are in a form of H-function.

Keywords : Mittage-Leffler function, Fractional Kinetic Equation, Laplace Transform, Fourier Transform amd H-functions.

I. Introduction

Fundamental law of physics are written as equations for the time evolution of a quantity X(t), dX(t)/dt=-AX(t), where this could be Maxwell's equation or Schroedinger's equation (If A is limited to linear operators), or it could be Newton's law of motion or Einstein's equations for geodesics (If A may also be a non linear operator). The mathematical solution (for linear operators) is X(t) =X(0)Exp{-At}. The initial value of the quantity at t=0 is given by X(0).

The same exponential behavior referred to above arises if X(t) represents the scalar number density of species at time t that do not interact with each other. If one denote A_p the production rate and A_d the destruction rate, respectively, the number density obey an exponential equation where the coefficient A is equal to the different of A_p-A_d. Subsequently, A_p^{-1} is the average time between production and A_d^{-1} is the average time between destruction. This type of behavior arises frequently in biology, chemistry and physics (Hilfer, 2000; Metzler and Klafter, 2000) [6],[12]. This paper in Section 2 summarizes mathematical result concerning solution of the diffusion equations in section 3 and section 4 respectively, widely distributed in the literature or of very recent origin. These involve the Mittage-Leffler function, H-function and the application of fractional calculus, Fourier transform and Laplace transform to them.

The section 3 and section 4 presented in a closed form solution of a fractional diffusion equation in terms of H-function.

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II. Mathematical Prerequisites

A generalization of the Mittage-Leffler function (Mittage-Leffler, 1903,1905)[9],[10]

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(n\alpha+1)},(\alpha\in C,Re(\alpha)>0) \quad (1)$$

was introduced by wiman(1905)[20] in the general form

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(n\alpha+\beta)},(\alpha,\beta\in C,Re(\alpha)>0) \quad (2)$$

The main result of these functions are available in the handbook of Erdelyi Magnus. Oberhettinger and Tricomi (1955, Section18.1)[4]and the monographs written by Dzherbashyas (1966,1993)[1][2], Prabhakar(1971)[14] introduced a generalization of (2) in the form

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n}z^{n}}{\Gamma(n\alpha+\beta)n!},(\alpha,\beta,\gamma\in C,Re(\alpha)>0) \quad (3)$$

Where

$$(\gamma)_{0}=1,\ (\gamma)_{k}=\gamma(\gamma+1)(\gamma+2)\ldots\ldots\ldots\ldots(\gamma+k-1)\ \ (k=1,2\ldots..)\ \ \gamma\neq 0 \quad (4)$$

For $\gamma=1$

$$E_{\alpha,\beta}^{1}(z)=E_{\alpha,\beta}(z),$$

For $\gamma=1,\beta=1$

$$E_{\alpha,1}^{1}(z)=E_{\alpha}(z) \quad (5)$$

The Mellin-Barnas integral representation for this function follows from the integral

$$E_{\alpha,\beta}^{\gamma}(z)=\frac{1}{\Gamma(\gamma)}\frac{1}{2\pi\omega}\int_{\Omega}\frac{\Gamma(-\xi)\Gamma(\gamma+\xi)(-z)^{\xi}}{\Gamma(\beta+\xi\alpha)}d\xi \quad (6)$$

where $\omega=(-1)^{1/2}$ The contour Ω is straight line parallel to the imaginary axis at a distance 'c' from the origin and separating the poles of $\Gamma(-\xi)$ at the point $\xi=\nu(\nu=0,1,2\ldots.)$ from those of $\Gamma(\gamma+\xi)$ at the points $\xi=-\gamma-\nu(\nu=0,1,2\ldots.)$. If we calculate the residues at the poles of $\Gamma(\gamma+\xi)$ at the points $\xi=-\gamma-\nu(\nu=0,1,2\ldots.)$ then it gives the analytic continuation formula of this function in the form[2]

$$E_{\alpha,\beta}^{\gamma}(z)=\frac{(-z)^{-\gamma}}{\Gamma(\gamma)}\sum_{\nu=0}^{\infty}\frac{\Gamma(\gamma+\nu)}{\Gamma(\beta-\alpha\gamma-\alpha\nu)}\frac{(-z)^{-\nu}}{\nu!},|z|>1 \quad (7)$$

Ref.

9. Mittage-Leffler, G.M.:1903, *Sur la nouvelle fonction $E_{\alpha}(x)$, C.R.Acad.Sci., Paris, (ser.II)*, ***137***, *554-558.*

From (7) it follows that for large z its behavior is given by

$$E^{\gamma}_{\alpha,\beta}(z) \sim O(|z|^{-\gamma}), |z| > 1 \tag{8}$$

The H-function is defined by means of Mellin-Barnes type integral in the following manner (Mathai and Saxena, 1978 p-2)[8]

Ref.

8. Mathai, A.M and Saxena, R.K.;1978, *The H-function with application in Statistics and other disciplines,Halsted Press[John Wiley and Sons],*

$$H^{m,n}_{p,q}(z) = H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right] = H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1, A_1)\dots.(a_p, A_p)\\(b_1, B_1)\dots.(b_q, B_q)\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi i}\int \theta(s) z^{-\xi}\, d\xi \tag{9}$$

$$where\ \theta(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j\xi)\prod_{j=1}^{n}\Gamma(1 - a_j - A_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j\xi)\prod_{j=n+1}^{p}\Gamma(a_j + A_j\xi)} \tag{10}$$

$$m, n, p, q \in N_0\ with\ 1 \le n \le p, 1 \le m \le q, A_j, B_j \in R_+\ a_j, b_j \in R$$

$$(i = 1,2 \dots..p, j = 1,2 \dots\dots q)$$

$$A_i(b_j + k) \ne B_j(a_i - l - 1)\ \ (k, l \in N_0\,; i = 1,2 \dots.n, j = 1,2 \dots m) \tag{11}$$

Where we employ the usual notations $N_0 = (0,1,2\dots)$ $R = (-\infty,\infty)$ $R_+ = (0,\infty)$ and C defines the complex number field. Ω is a suitable contour separating the poles of $\Gamma(b_j + B_j\xi)$ from those of $\Gamma(1 - a_j - A_j\xi)$.

A detailed and comprehensive account of the H-function along with convergence condition is available from Mathai and Saxena (1978)[8]

It follows from (7) that the generalized Mittag-Leffler function

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} H^{1,1}_{1,2}\left[-z\left|\begin{matrix}(1-\gamma, 1)\\(0,1)(1-\beta,\alpha)\end{matrix}\right.\right] (\alpha, \beta, \gamma \in C, Re(\alpha) > 0) \tag{12}$$

Putting $\gamma = 1$ in(12)

$$E_{\alpha,\beta}(z) = H^{1,1}_{1,2}\left[-z\left|\begin{matrix}(0,1)\\(0,1)(1-\beta,\alpha)\end{matrix}\right.\right] \tag{13}$$

If we further take $\beta = 1$ in (13) we get

$$E_{\alpha}(z) = H^{1,1}_{1,2}\left[-z\left|\begin{matrix}(0,1)\\(0,1)(0,\alpha)\end{matrix}\right.\right] \tag{14}$$

From Prudnikov, A.P., Brychkov, Yu.A. and Marichev, O.I (1989,p.355,eq2.25.3.2) [15] and Mathai and Saxena(1978,p.49)[8] it follows that the cosine transform of the H-function is given

$$\int_0^\infty t^{\rho-1}\cos kt\, H_{p,q}^{m,n}\left[at^\mu \left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]dt$$

$$=\frac{\pi}{k^\rho}H_{q+1,p+2}^{n+1,m}\left[\frac{k^\mu}{a}\left|\begin{matrix}(1-b_q,B_q)\left(\frac{1}{2}+\frac{\rho}{2},\frac{\mu}{2}\right)\\(\rho,\mu)(1-a_p,A_p)\left(\frac{1}{2}+\frac{\rho}{2},\frac{\mu}{2}\right)\end{matrix}\right.\right] \tag{15}$$

The Riemann-Liouvile fractional integral of order $v \in C$ is defined by Miller and Ross(1993,p.45;)[11] see also Srivastva and saxena,2001)[17]

$${}_0D_t^{-\nu}f(t)=\frac{1}{\Gamma(\nu)}\int_0^t (t-u)^{\nu-1}f(u)du \tag{16}$$

where $Re(\nu)>0$ following Samko, S.G., Kilbas, A. A. and Marichev, O.I. (1993,p.37)[16] we define the fractional derivative for $\alpha>0$ in the form

$${}_0D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(u)}{(t-u)^{\alpha-n+1}}du,(n=[Re(\alpha)]+1) \tag{17}$$

where $[Re(\alpha)]$means the integral part of $Re(\alpha)$.
In particular, if $0<\alpha<1$

$${}_0D_t^{\alpha}f(t)=\frac{d}{dt}\int_0^t \frac{f(u)du}{(t-u)^{\alpha}} \tag{18}$$

And in $\alpha=n\in N$ then

$${}_0D_t^{\alpha}f(t)=D^n f(t) \tag{19}$$

is the usual derivative of *n*.
From Erdelyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F.G (1954,p.182) [3] we have

$$L\{{}_0D_t^{-\nu}f(t)\}=s^{-\nu}F(s) \tag{20}$$

$$F(s)=L\{f(t);s\}=\int_0^\infty e^{-st}f(t)dt \tag{21}$$

where $Re(s)>0$

The Laplace transform of the fractional derivative is given by Oldham and spanier(1974,p.134,eq 8.1.3;)[13]see also (srivastva and saxena 2001)[17]

$$L\{{}_0D_t^{-\nu}f(t)\}=s^{\alpha}F(s)-\sum_{k=1}^{n}s^{k-1}\,{}_0D_t^{\alpha-k}f(t)|_{t=0} \tag{22}$$

Ref.

11. **Miller, K.S and Ross, B.:1993** An introduction the Fractional Calculus and Fractional differential equation, John Wiley and Sons, Newyork.

In this we present solution of the fractional diffusion equation given by (Metzler and Klafter 2000;Jorgenson and Lang,2001)[12][7]

Theorem 1. Consider the fractional diffusion equation

$$\boldsymbol{N(x,t) - N_0 t^{\mu-1} = -c^{\nu}\ {}_0D_t^{-\nu}\ {}_0D_x^{\nu}\, N(x,t)} \tag{23}$$

with initial condition

$$\boldsymbol{{}_0D_t^{\nu-k}\, N(x,t)|_{t=0} = 0\ \ and\ {}_0D_t^{-\nu-k}\, N(x,t)|_{x=0} = 0, k = 1,2 \dots n} \tag{24}$$

Where $n = [Re(\nu)] + 1; c^{\nu}$ *is diffusion constant then for the solution of (23) is given by*

$$\boldsymbol{N(x,t) = \frac{N_0\Gamma(\mu)}{c^t}\ H_{1,1}^{1,0}\left[\frac{|x|^{\nu}}{(ct)^{\nu}}\middle|\begin{matrix}(\mu+\nu,\nu)\\(1+\nu,\nu)\end{matrix}\right]} \tag{25}$$

Proof- $$N(x,t) - N_0 t^{\mu-1} = -c^{\nu}\ {}_0D_t^{-\nu}\ {}_0D_x^{\nu}\, N(x,t)$$

Apply Laplace and fourier transform with time variable and space vaiable respectively to (23) we get

$$N^*(k,s) - N_0\frac{\Gamma(\mu)}{s^{\mu}} = -c^{\nu}k^{\nu}s^{-\nu}N^*(k,s)$$

$$N^*(k,s)\{1 + ({}^{s}/_{c})^{-\nu}k^{\nu}\} = N_0 s^{-\mu}\Gamma(\mu)$$

$$N^*(k,s) = N_0 s^{-\mu}\Gamma(\mu)\left\{1 + \left({}^{s}/_{kc}\right)^{-\nu}\right\}^{-1}$$

$$= N_0 s^{-\mu}\Gamma(\mu)\sum_{r=0}^{\infty}\frac{(1)_r\left[-\left({}^{s}/_{kc}\right)^{-\nu}\right]^r}{r!}$$

$$= N_0\Gamma(\mu)\sum_{r=0}^{\infty}\frac{(1)_r(kc)^{r\nu}(-1)^r}{r!}s^{-\nu r-\mu}$$

where $N^*(k,s)$ Laplace and Fourier transform of $N(x,t)$
Taking inverse Laplace transform

$$N(k,t) = N_0\Gamma(\mu)\sum_{r=0}^{\infty}(kc)^{r\nu}\,(-1)^r L^{-1}\{s^{-\nu r-\mu}\}$$

Ref.

12. Metzler, R and Klafter,J.:2000 *The random walk's guide to anomalous diffusion: Afractionaldynamics approach, Phys.Rep.339,1-77.*

$$N(k,t) = N_0\Gamma(\mu)\sum_{r=0}^{\infty}(kc)^{r\nu}\,(-1)^r\frac{t^{\mu+r\nu-1}}{\Gamma(r\nu+\mu)}$$

$$= N_0\Gamma(\mu)t^{\mu-1}\,E_{\nu,\mu}(-c^\nu k^\nu t^\nu)$$

which can we expressed in terms of H-function

$$= N_0\Gamma(\mu)t^{\mu-1}H_{1,2}^{1,1}\left[c^\nu k^\nu t^\nu \,\middle|\, \begin{matrix}(0,1)\\(0,1)(1-\mu,\nu)\end{matrix}\right]$$

Now take inverse fourier transformation

$$N(x,t) = \frac{1}{\pi}\int_0^\infty \cos kx\; t^{\mu-1}N_0\Gamma(\mu)H_{1,2}^{1,1}\left[c^\nu k^\nu t^\nu \,\middle|\, \begin{matrix}(0,1)\\(0,1)(1-\mu,\nu)\end{matrix}\right]dk$$

$$= \frac{t^{\mu-1}N_0\Gamma(\mu)}{\pi}\,\frac{\pi}{|x|}H_{3,3}^{2,1}\left[\frac{|x|^\nu}{(ct)^\nu}\,\middle|\, \begin{matrix}(1,1)(\mu,\nu)\left(1,{}^{\nu}/_{2}\right)\\(1,1)(1,\nu)\left(1,{}^{\nu}/_{2}\right)\end{matrix}\right]$$

Applying a result of Mathai and Saxena (1978, p.4.eq1.2.1) the above expression becomes

$$N(x,t) = \frac{N_0\Gamma(\mu)}{|x|}H_{2,2}^{2,0}\left[\frac{|x|^\nu}{(ct)^\nu}\,\middle|\, \begin{matrix}(\mu,\nu)\left(1,{}^{\nu}/_{2}\right)\\(1,\nu)\left(1,{}^{\nu}/_{2}\right)\end{matrix}\right]$$

If we employ the formula Mathai and Saxena (1978,p.4.eq1.2.4)

$$x^\sigma H_{p,q}^{m,n}\left[z\,\middle|\, \begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] = H_{p,q}^{m,n}\left[z\,\middle|\, \begin{matrix}(a_p+\sigma A_p,A_p)\\(b_q+\sigma B_q,B_q)\end{matrix}\right]$$

$$N(x,t) = \frac{N_0\Gamma(\mu)}{ct}H_{2,2}^{2,0}\left[\frac{|x|^\nu}{(ct)^\nu}\,\middle|\, \begin{matrix}(\mu+\nu,\nu)\left(1,{}^{\nu}/_{2}\right)\\(1+\nu,\nu)\left(1,{}^{\nu}/_{2}\right)\end{matrix}\right]$$

$$N(x,t) = \frac{N_0\Gamma(\mu)}{ct}H_{1,1}^{1,0}\left[\frac{|x|^\nu}{(ct)^\nu}\,\middle|\, \begin{matrix}(\mu+\nu,\nu)\\(1+\nu,\nu)\end{matrix}\right]$$

Theorem 2- Consider the fractional diffusion equation (Metzler and Klafter 2000;Jorgenson and Long,2001)[12][7]

$${}_0D_t^\nu\,N(x,t) - E_\nu(-d^\nu t^\nu) = -c^\nu\frac{\partial^2}{\partial x^2}N(x,t) \qquad (26)$$

with initial condition

Ref.

7. **Jorgenson, J. and Lang, S.:2001,** *The ubiquitous heat kernel, in mathematics Unlimited-2001 and Beyond, Eds.B. Engquist and W.Schmid, Springer-Verlag.Berlin and Heidelberg.*

Notes

$$ {}_0D_t^{\nu-k}\, N(x,t)|_{t=0} = 0 \quad k = 1,2..n \qquad (27) $$

Where $n = [Re(\nu)] + 1; c^{\nu}$ *is diffusion constant.*
Then for the solution of (26) is given by

$$ \frac{1}{2d^{\nu/2}} \sin\left(d^{\nu/2}x\right) * \frac{1}{(ct)^{\nu}} H_{1,1}^{1,0}\left[\frac{|x|^2}{(ct)^{\nu}}\middle|\begin{matrix}(1-\nu/2,\nu)\\(0,2)\end{matrix}\right] $$

$$ -\frac{1}{2d^{\nu/2}} \sin\left(d^{\nu/2}x\right) H_{1,2}^{1,1}\left[d^{\nu}t^{\nu}\middle|\begin{matrix}(0,1)\\(0,1)(0,\nu)\end{matrix}\right] \qquad (28) $$

Proof-

$$ {}_0D_t^{\nu}\, N(x,t) - E_{\nu}(-d^{\nu}t^{\nu}) = -c^{\nu}\frac{\partial^2}{\partial x^2}N(x,t) $$

Applying the fourier transform with respect to the space variable *x* and the Laplace transform with respect to the time variable *t.* we get

$$ s^{\nu}N^{*}(k,s) - \frac{s^{\nu-1}}{s^{\nu}+d^{\nu}} = -c^{\nu}k^2N^{*}(k,s) $$

$$ \{s^{\nu}+c^{\nu}k^2\}N^{*}(k,s) = \frac{s^{\nu-1}}{s^{\nu}+d^{\nu}} $$

$$ N^{*}(k,s) = \frac{s^{\nu-1}}{\{s^{\nu}+d^{\nu}\}\{s^{\nu}+c^{\nu}k^2\}} $$

$$ = \frac{s^{\nu-1}}{c^{\nu}k^2-d^{\nu}}\left[\frac{1}{s^{\nu}+d^{\nu}} - \frac{1}{s^{\nu}+c^{\nu}k^2}\right].. \qquad (29) $$

To invert equation(29).It is convenient to first invert the Laplace transformation and fourier transform.Apply inverse Laplace transform we obtain

$$ N(k,t) = \frac{1}{c^{\nu}k^2t^{\nu}}[E_{\nu}(-d^{\nu}t^{\nu}) - E_{\nu}(-c^{\nu}k^2t^{\nu})].. \qquad (30) $$

Which can expressed in terms of H-function

$$ N(k,t) = \frac{1}{c^{\nu}k^2-d^{\nu}}\left\{H_{1,2}^{1,1}\left[d^{\nu}t^{\nu}\middle|\begin{matrix}(0,1)\\(0,1)(0,\nu)\end{matrix}\right] - H_{1,2}^{1,1}\left[c^{\nu}k^{\nu}t^{\nu}\middle|\begin{matrix}(0,1)\\(0,1)(0,\nu)\end{matrix}\right]\right\} \qquad (31) $$

Invert the fourier transform

$$N(x,t) = \frac{1}{\pi}\int_0^\infty coskx \frac{1}{c^\nu k^2 - d^\nu}\left\{H_{1,2}^{1,1}\left[d^\nu t^\nu \left| \begin{matrix} (0,1) \\ (0,1)(0,\nu) \end{matrix} \right.\right] dk\right.$$

$$\left. -\frac{1}{\pi}\int_0^\infty coskx \frac{1}{c^\nu k^2 - d^\nu} H_{1,2}^{1,1}\left[c^\nu k^\nu t^\nu \left| \begin{matrix} (0,1) \\ (0,1)(0,\nu) \end{matrix} \right.\right] dk\right\}$$

$$= -\frac{1}{2d^{\nu/2}}\sin(d^{\nu/2}x)\, H_{1,2}^{1,1}\left[d^\nu t^\nu \left| \begin{matrix} (0,1) \\ (0,1)(0,\nu) \end{matrix} \right.\right] + \frac{1}{2d^{\nu/2}}\sin(d^{\nu/2}x)$$

$$* \frac{1}{|x|} H_{3,3}^{2,1}\left[\frac{|x|^2}{(ct)^\nu} \left| \begin{matrix} (1,1)(1,\nu)(1,1) \\ (1,2)(1,1)(1,1) \end{matrix} \right.\right]$$

$$= -\frac{1}{2d^{\nu/2}}\sin(d^{\nu/2}x)\, H_{1,2}^{1,1}\left[d^\nu t^\nu \left| \begin{matrix} (0,1) \\ (0,1)(0,\nu) \end{matrix} \right.\right]$$

$$+ \frac{1}{2d^{\nu/2}}\sin(d^{\nu/2}x) * \frac{1}{(c^\nu t^\nu)^{1/2}} H_{2,2}^{2,0}\left[\frac{|x|^2}{(ct)^\nu} \left| \begin{matrix} (1-\nu/2,\nu)\left(1/2,1\right) \\ (0,2)\left(1/2,1\right) \end{matrix} \right.\right]$$

$$= \frac{1}{2d^{\nu/2}}\sin(d^{\nu/2}x) * \frac{1}{(ct)^\nu} H_{1,1}^{1,0}\left[\frac{|x|^2}{(ct)^\nu} \left| \begin{matrix} (1-\nu/2,\nu) \\ (0,2) \end{matrix} \right.\right]$$

$$- \frac{1}{2d^{\nu/2}}\sin(d^{\nu/2}x)\, H_{1,2}^{1,1}\left[d^\nu t^\nu \left| \begin{matrix} (0,1) \\ (0,1)(0,\nu) \end{matrix} \right.\right]$$

III. Conclusion

The fractional kinetic equation has been extended to generalized fractional equation (23) and (26). Their respective solutions are given in terms of Mittag-Leffler function and their generalization, which can also be represented as Fox's H-function.

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Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
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