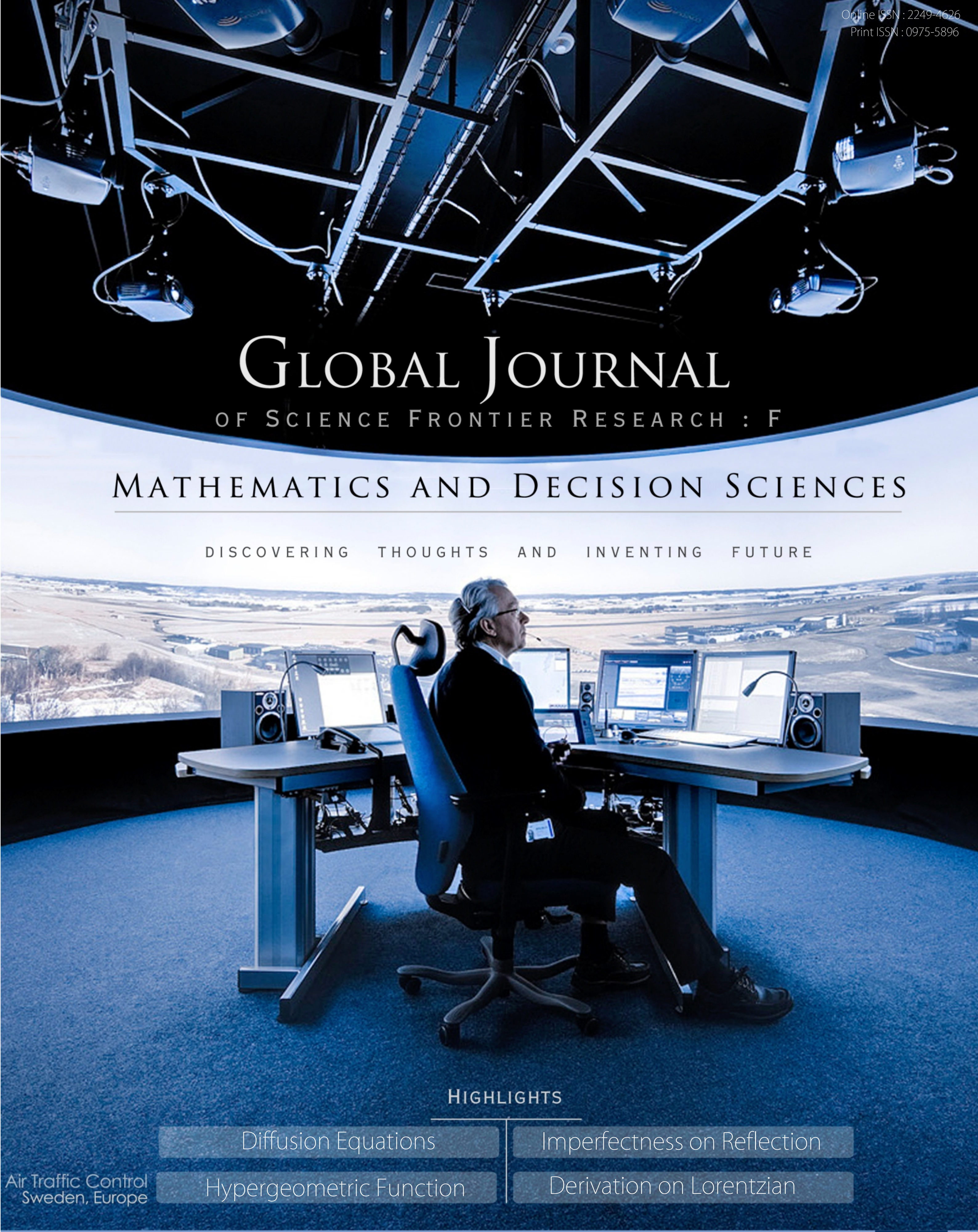
Online ISSN : 2249-4626
Print ISSN : 0975-5896



Global Journal

of Science Frontier Research : F

Mathematics and Decision Sciences

Discovering Thoughts and Inventing Future

Highlights

- Diffusion Equations
- Imperfectness on Reflection
- Hypergeometric Function
- Derivation on Lorentzian

Air Traffic Control
Sweden, Europe

Volume 12 | Issue 2 | Version 1.0

ENG

© 2001-2012 by Global Journal of Science Frontier Research, USA

Global Journal of Science Frontier Research: F
Mathematics & Decision sciences

Global Journal of Science Frontier Research: F
Mathematics & Decision Sciences

Volume 12 Issue 2 (Ver. 1.0)

Open Association of Research Society

© Global Journal of Science Frontier Research .2012 .

All rights reserved.

This is a special issue published in version 1.0 of "Global Journal of Science Frontier Research." By Global Journals Inc.

All articles are open access articles distributed under "Global Journal of Science Frontier Research"

Reading License, which permits restricted use. Entire contents are copyright by of "Global Journal of Science Frontier Research" unless otherwise noted on specific articles.

No part of this publication may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopy, recording, or any information storage and retrieval system, without written permission.

The opinions and statements made in this book are those of the authors concerned. Ultraculture has not verified and neither confirms nor denies any of the foregoing and no warranty or fitness is implied.

Engage with the contents herein at your own risk.

The use of this journal, and the terms and conditions for our providing information, is governed by our Disclaimer, Terms and Conditions and Privacy Policy given on our website http://globaljournals.us/terms-and-condition/menu-id-1463/

By referring / using / reading / any type of association / referencing this journal, this signifies and you acknowledge that you have read them and that you accept and will be bound by the terms thereof.

All information, journals, this journal, activities undertaken, materials, services and our website, terms and conditions, privacy policy, and this journal is subject to change anytime without any prior notice.

Incorporation No.: 0423089
License No.: 42125/022010/1186
Registration No.: 430374
Import-Export Code: 1109007027
Employer Identification Number (EIN):
USA Tax ID: 98-0673427

Global Journals Inc.

*(A Delaware USA Incorporation with "Good Standing"; **Reg. Number: 0423089**)*

Sponsors: *Open Association of Research Society*
Open Scientific Standards

Publisher's Headquarters office

Global Journals Inc., Headquarters Corporate Office, Cambridge Office Center, II Canal Park, Floor No. 5th, ***Cambridge (Massachusetts)***, Pin: MA 02141
United States
USA Toll Free: +001-888-839-7392
USA Toll Free Fax: +001-888-839-7392

Offset Typesetting

Open Association of Research Society, Marsh Road, Rainham, Essex, London RM13 8EU
United Kingdom.

Packaging & Continental Dispatching

Global Journals, India

Find a correspondence nodal officer near you

To find nodal officer of your country, please email us at *local@globaljournals.org*

eContacts

Press Inquiries: *press@globaljournals.org*
Investor Inquiries: *investers@globaljournals.org*
Technical Support: *technology@globaljournals.org*
Media & Releases: *media@globaljournals.org*

Pricing (Including by Air Parcel Charges):

For Authors:

22 USD (B/W) & 50 USD (Color)

Yearly Subscription (Personal & Institutional):

200 USD (B/W) & 250 USD (Color)

Editorial Board Members (HON.)

John A. Hamilton,"Drew" Jr.,
Ph.D., Professor, Management
Computer Science and Software Engineering
Director, Information Assurance Laboratory
Auburn University

Dr. Henry Hexmoor
IEEE senior member since 2004
Ph.D. Computer Science, University at Buffalo
Department of Computer Science
Southern Illinois University at Carbondale

Dr. Osman Balci, Professor
Department of Computer Science
Virginia Tech, Virginia University
Ph.D.and M.S.Syracuse University, Syracuse, New York
M.S. and B.S. Bogazici University, Istanbul, Turkey

Yogita Bajpai
M.Sc. (Computer Science), FICCT
U.S.A.Email:
yogita@computerresearch.org

Dr. T. David A. Forbes
Associate Professor and Range Nutritionist
Ph.D. Edinburgh University - Animal Nutrition
M.S. Aberdeen University - Animal Nutrition
B.A. University of Dublin- Zoology

Dr. Wenying Feng
Professor, Department of Computing & Information Systems
Department of Mathematics
Trent University, Peterborough,
ON Canada K9J 7B8

Dr. Thomas Wischgoll
Computer Science and Engineering,
Wright State University, Dayton, Ohio
B.S., M.S., Ph.D.
(University of Kaiserslautern)

Dr. Abdurrahman Arslanyilmaz
Computer Science & Information Systems Department
Youngstown State University
Ph.D., Texas A&M University
University of Missouri, Columbia
Gazi University, Turkey

Dr. Xiaohong He
Professor of International Business
University of Quinnipiac
BS, Jilin Institute of Technology; MA, MS, PhD,. (University of Texas-Dallas)

Burcin Becerik-Gerber
University of Southern California
Ph.D. in Civil Engineering
DDes from Harvard University
M.S. from University of California, Berkeley & Istanbul University

Dr. Bart Lambrecht
Director of Research in Accounting and FinanceProfessor of Finance
Lancaster University Management School
BA (Antwerp); MPhil, MA, PhD (Cambridge)

Dr. Carlos García Pont
Associate Professor of Marketing
IESE Business School, University of Navarra
Doctor of Philosophy (Management), Massachusetts Institute of Technology (MIT)
Master in Business Administration, IESE, University of Navarra
Degree in Industrial Engineering, Universitat Politècnica de Catalunya

Dr. Fotini Labropulu
Mathematics - Luther College
University of ReginaPh.D., M.Sc. in Mathematics
B.A. (Honors) in Mathematics
University of Windso

Dr. Lynn Lim
Reader in Business and Marketing
Roehampton University, London
BCom, PGDip, MBA (Distinction), PhD, FHEA

Dr. Mihaly Mezei
ASSOCIATE PROFESSOR
Department of Structural and Chemical Biology, Mount Sinai School of Medical Center
Ph.D., Etvs Lornd University
Postdoctoral Training,
New York University

Dr. Söhnke M. Bartram
Department of Accounting and FinanceLancaster University Management SchoolPh.D. (WHU Koblenz)
MBA/BBA (University of Saarbrücken)

Dr. Miguel Angel Ariño
Professor of Decision Sciences
IESE Business School
Barcelona, Spain (Universidad de Navarra)
CEIBS (China Europe International Business School).
Beijing, Shanghai and Shenzhen
Ph.D. in Mathematics
University of Barcelona
BA in Mathematics (Licenciatura)
University of Barcelona

Philip G. Moscoso
Technology and Operations Management
IESE Business School, University of Navarra
Ph.D in Industrial Engineering and Management, ETH Zurich
M.Sc. in Chemical Engineering, ETH Zurich

Dr. Sanjay Dixit, M.D.
Director, EP Laboratories, Philadelphia VA Medical Center
Cardiovascular Medicine - Cardiac Arrhythmia
Univ of Penn School of Medicine

Dr. Han-Xiang Deng
MD., Ph.D
Associate Professor and Research Department Division of Neuromuscular Medicine
Davee Department of Neurology and Clinical NeuroscienceNorthwestern University
Feinberg School of Medicine

Dr. Pina C. Sanelli
Associate Professor of Public Health
Weill Cornell Medical College
Associate Attending Radiologist
NewYork-Presbyterian Hospital
MRI, MRA, CT, and CTA
Neuroradiology and Diagnostic
Radiology
M.D., State University of New York at
Buffalo,School of Medicine and
Biomedical Sciences

Dr. Roberto Sanchez
Associate Professor
Department of Structural and Chemical
Biology
Mount Sinai School of Medicine
Ph.D., The Rockefeller University

Dr. Wen-Yih Sun
Professor of Earth and Atmospheric
SciencesPurdue University Director
National Center for Typhoon and
Flooding Research, Taiwan
University Chair Professor
Department of Atmospheric Sciences,
National Central University, Chung-Li,
TaiwanUniversity Chair Professor
Institute of Environmental Engineering,
National Chiao Tung University, Hsin-
chu, Taiwan.Ph.D., MS The University of
Chicago, Geophysical Sciences
BS National Taiwan University,
Atmospheric Sciences
Associate Professor of Radiology

Dr. Michael R. Rudnick
M.D., FACP
Associate Professor of Medicine
Chief, Renal Electrolyte and
Hypertension Division (PMC)
Penn Medicine, University of
Pennsylvania
Presbyterian Medical Center,
Philadelphia
Nephrology and Internal Medicine
Certified by the American Board of
Internal Medicine

Dr. Bassey Benjamin Esu
B.Sc. Marketing; MBA Marketing; Ph.D
Marketing
Lecturer, Department of Marketing,
University of Calabar
Tourism Consultant, Cross River State
Tourism Development Department
Co-ordinator , Sustainable Tourism
Initiative, Calabar, Nigeria

Dr. Aziz M. Barbar, Ph.D.
IEEE Senior Member
Chairperson, Department of Computer
Science
AUST - American University of Science &
Technology
Alfred Naccash Avenue – Ashrafieh

President Editor (HON.)

Dr. George Perry, (Neuroscientist)
Dean and Professor, College of Sciences
Denham Harman Research Award (American Aging Association)
ISI Highly Cited Researcher, Iberoamerican Molecular Biology Organization
AAAS Fellow, Correspondent Member of Spanish Royal Academy of Sciences
University of Texas at San Antonio
Postdoctoral Fellow (Department of Cell Biology)
Baylor College of Medicine
Houston, Texas, United States

Chief Author (HON.)

Dr. R.K. Dixit
M.Sc., Ph.D., FICCT
Chief Author, India
Email: authorind@computerresearch.org

Dean & Editor-in-Chief (HON.)

Vivek Dubey(HON.)
MS (Industrial Engineering),
MS (Mechanical Engineering)
University of Wisconsin, FICCT
Editor-in-Chief, USA
editorusa@computerresearch.org

Sangita Dixit
M.Sc., FICCT
Dean & Chancellor (Asia Pacific)
deanind@computerresearch.org

Luis Galárraga
J!Research Project Leader
Saarbrücken, Germany

Er. Suyog Dixit
(M. Tech), BE (HONS. in CSE), FICCT
SAP Certified Consultant
CEO at IOSRD, GAOR & OSS
Technical Dean, Global Journals Inc. (US)
Website: www.suyogdixit.com
Email:suyog@suyogdixit.com

Pritesh Rajvaidya
(MS) Computer Science Department
California State University
BE (Computer Science), FICCT
Technical Dean, USA
Email: pritesh@computerresearch.org

Contents of the Volume

Global Journal of Science Frontier Research
Mathematics & Decision Sciences
Volume 12 Issue 2 Version 1.0 February 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Existence positive periodic solution of functional differential equation

By Xuanlong Fan
Qingdao Qiushi College,China

Abstract - The paper is concerned with functional differential equation

$$x'(t) = a(t)g(x(h_1(t)))x(t) - f\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x(t),$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T$, $g(x(h_1(t))) = diag\big(g_1(x_1(h_{11}(t))), \ldots, g_n(x_n(h_{1n}(t)))\big)$, $a(t) = diag(a_1(t), \ldots, a_n(t))$, $f\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big) = diag\Big(f_1(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v), \ldots, f_n(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v)\Big)^T$ *are periodic functions.*

Keywords : *Periodic solution; Functional differential equation; Fixed point; Cone.*

GJSFR-F Classication : *FOR Code: 010109.*

Strictly as per the compliance and regulations of :

© 2012. Xuanlong Fan.This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Ref.
[1] A. Wan, D. Jiang, Existence of positive periodic solutions for functional differential equations, Kyushu J.Math. 56(1)(2002) 193-202.
[5] Wang. H, Positive periodic solutions of functional differential equations, J. Differential Equation 202(2004) 354-366.

Existence positive periodic solution of functional differential equation

Xuanlong Fan

Abstract - The paper is concerned with functional differential equation

$$\begin{aligned} x'(t) &= a(t)g(x(h_1(t)))x(t) - f\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \\ &\quad \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x(t), \end{aligned}$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T$, $g(x(h_1(t))) = diag\big(g_1(x_1(h_{11}(t))), \ldots, g_n(x_n(h_{1n}(t)))\big)$, $a(t) = diag(a_1(t), \ldots, a_n(t))$, $f\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big) = diag\Big(f_1(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v), \ldots, f_n(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v)\Big)^T$ are periodic functions.

Keywords : *Periodic solution; Functional differential equation; Fixed point; Cone.*

I. Introduction

The theory of differential systems have developed by mathematicians (see [1-5]). In this paper, we consider the following system

$$\begin{aligned} x'(t) &= a(t)g(x(h_1(t)))x(t) - f\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \\ &\quad \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x(t), \end{aligned} \tag{1.1}$$

where

(H_1) $a_i\,(i = 1, \ldots, n) \in C(\mathbb{R}, [0, +\infty))$ are T-periodic and there exists $t_1 \in (0, T)$ such that $a_i(t_1) > 0$;

(H_2) $h_{1i}\,(i = 1, \ldots, n) \in C(\mathbb{R}, \mathbb{R})$ are p_1T-periodic, $h_{2i}\,(i = 1, \ldots, n) \in C(\mathbb{R}, \mathbb{R})$ are p_2T-periodic and $h_{3i}\,(i = 1, \ldots, n) \in C(\mathbb{R}, \mathbb{R})$ are p_3T-periodic;

(H_3) $g_i \in C([0, \infty), [0, \infty))$ are continuous, $0 < l_i \leq g_i(u_i) < L_i < \infty$ for all $u_i > 0$, l_i, L_i are two positive constants. There exist positive constant $\mathbb{L}_i$ such that $\big|g_i(u_i) - g_i(v_i)\big| \leq \mathbb{L}_i|u_i - v_i|$.

Author : *Department of Mathematics Qingdao Qiushi College Qingdao, Shandong 266111People's Republic of China.*
E-mail : *fanxuanlong@126.com*

© 2012 Global Journals Inc. (US)

(H_4) $f_i \in C(\mathbb{R} \times [0,\infty) \times [0,\infty) \times [0,\infty) \times \mathbb{R} \times \mathbb{R}, [0,\infty))$ are continuous functions. There exist positive functions $\alpha_{ij}(t) < +\infty$, $\beta_{ij}(t) < +\infty$, such that

$$\begin{aligned} & f_i\Big(t, u, \int_{-\varsigma}^{0} k(v)u(t-v)\mathrm{d}v, u', \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)u'(t-v)\mathrm{d}v\Big) \\ & -f_i\Big(t, v, \int_{-\varsigma}^{0} k(v)v(t-v)\mathrm{d}v, v', \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)v'(t-v)\mathrm{d}v\Big) \\ \leq\ & \sum_{j=1}^{n} \alpha_{ij}(t)|u_i - v_i| + \sum_{j=1}^{n} \beta_{ij}(t)|u_i' - v_i'|. \end{aligned}$$

Throughout this paper, a function is called ω-periodic($\omega > 0$)meaning ω is the least positive periodic of the function. Since p is the least positive rational number such that $\frac{p}{p_0}$, $\frac{p}{p_1}$, $\frac{p}{p_2}$ and $\frac{p}{p_3}$ are the positive integers, $pT = \omega$ is the least positive period of the periodic solutions of Eq.(1.1). System (1.1) contains many mathematical population models of delay differential equations [see(1-3,5,8-12)].

II. PRELIMINARIES

In order to obtain the existence of a periodic solution of system (1.1), we then make the following preparations:

Let $\mathbb{E}$ be a Banach space and K be a cone in $\mathbb{E}$. The semi-order induced by the cone K is denoted by "$\leq$". That is, $x \leq y$ if and only if $y - x \in K$.

Let $\mathbb{E}$, $\mathbb{F}$ be two Banach spaces and $D \subset \mathbb{E}$, a continuous and bounded map $\Phi : \bar{\Omega} \to \mathbb{F}$ is called k-set contractive if for any bounded set $S \subset D$ we have

$$\alpha_{\mathbb{F}}(\Phi(S)) \leq k\alpha_{\mathbb{E}}(S).$$

Φ is called strict-set-contractive if it is k-set-contractive for some $0 \leq k < 1$.

The following lemma cited from Ref. [10,11] which is useful for the proof of our main results of this paper.

Lemma 2.1. [6, 7] *Let K be a cone of the real Banach space X and $K_{r,R} = \{x \in K | r \leq ||x|| \leq R\}$ with $R > r > 0$. Suppose that $\Phi : K_{r,R} \to K$ is strict-set-contractive such that one of the following two conditions is satisfied:*

(i) $\Phi x \not\leq x, \quad \forall x \in K, ||x|| = r$ *and* $\Phi x \not\geq x, \quad \forall x \in K, ||x|| = R.$

(ii) $\Phi x \not\geq x, \quad \forall x \in K, ||x|| = r$ *and* $\Phi x \not\leq x, \quad \forall x \in K, ||x|| = R.$

Then Φ has at least one fixed point in $K_{r,R}$.

Remark 2.1. *Completely continuous operators are 0-set-contractive.*

In order to apply Lemma 2.1 to system (1.1), we consider the Banach space

$$C_\omega^0 = \{x(t) = (x_1(t), \ldots, x_n(t)) | x(t) \in C^0(\mathbb{R}, \mathbb{R}^n), x(t+\omega) = x(t),\ t \in \mathbb{R}\}$$

with the norm defined by $\|x\| = \sum_{i=1}^{n} |x_i|_0$, where $|x_i|_0 = \max_{t\in[0,\omega]}\{x_i(t)\}$, $i = 1, \ldots, n$ and

$$C_\omega^1 = \{x(t) = (x_1(t), \ldots, x_n(t)) | x(t) \in C^1(\mathbb{R}, \mathbb{R}^n), x(t+\omega) = x(t),\ t \in \mathbb{R}\}$$

Ref.

[3] D. Jiang, B. Zhang, Positive periodic solutions of functional differential equations and population models, Electron. J. Differential Equation 71(2002) 1-13.

[8] Y.K. Li, On a periodic neutral delay Lotka-Volterra system, Nonlinear Anal. 39 (2000) 767-778.

© 2012 Global Journals Inc. (US)

with the norm defined by $\|x\|_1 = \sum_{i=1}^{n} |x_i|_1,$, where $|x_i|_1 = \max\{|x_i|_0, |x_i'|_0\}, i = 1, \ldots, n$. Then C_ω^0, C_ω^1 are all Banach space.

Let the map $\Phi = (\Phi_1, \ldots, \Phi_n)$ be defined by

$$\begin{aligned}(\Phi x)(t) &= \int_t^{t+\omega} G(t,s) f\Big(s, x(h_2(s)), \int_{-\varsigma}^{0} k(v)x(s-v)\mathrm{d}v, x'(h_3(s)), \\ &\quad \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(s-v)\mathrm{d}v\Big)x(s)\,\mathrm{d}s \qquad (2.1)\end{aligned}$$

Notes

for $x \in C_\omega^1, t \in \mathbb{R}$, where

$$G(t,s) = diag\big(G_1(t,s), \ldots, G_n(t,s)\big),$$
$$G_i(t,s) = \frac{e^{-\int_t^s a_i(\theta)g_i(x_i(h_{1i}(\theta)))\,\mathrm{d}\theta}}{1 - e^{-\int_0^\omega a_i(\theta)g_i(x_i(h_{1i}(\theta)))\,\mathrm{d}\theta}},\ s \in [t, t+\omega],\ i = 1, \ldots, n.$$

It is easy to see that $G(t+\omega, s+\omega) = G(t,s)$ and

$$\begin{aligned}&\frac{\partial G(t,s)}{\partial t} = a(t)g(x(h_1(t)))G(t,s),\\ &G(t+\omega, s+\omega) = G(t,s),\\ &G(t, t+\omega) - G(t,t) = -I,\\ &\frac{\sigma_i^{L_i}}{1-\sigma_i^{L_i}} \le G_i(t,s) \le \frac{1}{1-\sigma_i^{l_i}},\ \ s \in [t, t+\omega],\end{aligned}$$

where $\sigma_i = e^{-\int_0^\omega a_i(\theta)\,\mathrm{d}\theta}$.

Define the cone K in X by

$$K = \Big\{x \Big| x \in C_\omega^1,\ x_i(t) \ge \delta_i |x_i|_1,\ t \in [0,\omega],\ i = 1, \ldots, n\Big\},$$

where $0 < \delta < I$, $\delta = diag(\delta_1, \ldots, \delta_n)$, $\delta_i = \dfrac{\sigma_i^{L_i}(1-\sigma_i^{l_i})}{1-\sigma_i^{L_i}}$.

Let

$$\xi_{1i} = \min\left\{\inf_{t\in R}\Big\{(\sigma_i^{l_i} - 1) + a_i(t)g_i(x_i(h_{1i}(t)))\Big\},\ \inf_{t\in R}\Big\{\frac{1-\sigma_i^{L_i}}{\sigma_i^{L_i}} - a_i(t)g_i(x_i(h_{1i}(t)))\Big\}\right\};$$

$$\xi_{2i} = \max\left\{\sup_{t\in R}\Big\{(\sigma_i^{l_i} - 1) + a_i(t)g_i(x_i(h_{1i}(t)))\Big\},\ \sup_{t\in R}\Big\{\frac{1-\sigma_i^{L_i}}{\sigma_i^{L_i}} - a_i(t)g_i(x_i(h_{1i}(t)))\Big\}\right\}.$$

(H_5) $0 < \xi_{1i} \le \xi_{2i} \le 1,\ i = 1, \ldots, n.$

Lemma 2.2. *Assume that $(H_1) - (H_5)$ hold, then Φ maps K into K.*

Proof. For any $x \in K$, it is clear that $\Phi x \in C(R,R)$, we have

$$\begin{aligned}(\Phi x)(t+\omega) &= \int_t^{t+\omega} G(t,s) f\Big(s, x(h_2(s)), \int_{-\varsigma}^{0} k(v)x(s-v)\mathrm{d}v, x'(h_3(s)), \\ &\quad \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(s-v)\mathrm{d}v\Big)x(s)\,\mathrm{d}s\end{aligned}$$

Global Journal of Science Frontier Research (F) Volume XII Issue II Version I February 2012

© 2012 Global Journals Inc. (US)

$$
\begin{aligned}
&= \int_t^{t+\omega} G(t+\omega,u+\omega) f\Big(u+\omega, x(h_2(u+\omega)), \int_{-\varsigma}^{0} k(v)x(u+\omega-v)\mathrm{d}v, \\
&\quad x'(h_3(u+\omega)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(u+\omega-v)\mathrm{d}v\Big)x(u+\omega)\,\mathrm{d}s \\
&= \int_t^{t+\omega} G(t,u) f\Big(u, x(h_2(u)), \int_{-\varsigma}^{0} k(v)x(u-v)\mathrm{d}v, \\
&\quad x'(h_3(u)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(u-v)\mathrm{d}v\Big)x(u)\,\mathrm{d}s \\
&= (\Phi x)(t).
\end{aligned}
$$

Thus, $(\Phi x)(t+\omega) = (\Phi x)(t), t \in R$. So $\Phi x \in X$. For $x \in K, t \in [0,\omega]$, we have

$$
\begin{aligned}
|\Phi_i x_i|_0 \;\leq\; & \frac{1}{1-\sigma_i^{l_i}}\Big(\int_t^{t+\omega} f_i\Big(s, x(h_2(s)), \int_{-\varsigma}^{0} k(v)x(s-v)\mathrm{d}v, x'(h_3(s)), \\
& \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(s-v)\mathrm{d}v\Big)x_i(s)\mathrm{d}s\Big),\ i = 1,\ldots,n
\end{aligned}
$$

and

$$
\begin{aligned}
(\Phi_i x_i)(t) \;\geq\; & \frac{\sigma_i^{L_i}}{1-\sigma_i^{L_i}}\Big(\int_t^{t+\omega} f_i\Big(s, x(h_2(s)), \int_{-\varsigma}^{0} k(v)x(s-v)\mathrm{d}v, x'(h_3(s)), \\
& \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(s-v)\mathrm{d}v\Big)x_i(s)\mathrm{d}s\Big),\ i = 1,\ldots,n.
\end{aligned}
$$

So we have $(\Phi_i x_i)(t) \geq \delta_i |\Phi_i x_i|_0$.

If $(\Phi_i x_i)'(t) \geq 0$, then

$$
\begin{aligned}
(\Phi_i x_i)'(t) &= G_i(t,t+\omega) f_i\Big(t+\omega, x(h_2(t+\omega)), \int_{-\varsigma}^{0} k(v)x(t+\omega-v)\mathrm{d}v, \\
&\quad x'(h_3(t+\omega)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t+\omega-v)\mathrm{d}v\Big)x_i(t+\omega) - G_i(t,t) f_i\Big(t, x(h_2(t)), \\
&\quad \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x_i(t) \\
&\quad + a_i(t)g_i(x_i(h_{1i}(t)))(\Phi_i x_i)(t) \\
&= -f_i\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x_i(t) \\
&\quad + a_i(t)g_i(x_i(h_{1i}(t)))(\Phi_i x_i)(t) \\
&\leq \Big((\sigma_i^{l_i}-1) + a_i(t)g_i(x_i(h_{1i}(t)))\Big)(\Phi x_i)(t) \leq (\Phi_i x_i)(t),\ i = 1,\ldots,n. \qquad (2.2)
\end{aligned}
$$

On the other hand, from (2.2), if $(\Phi_i x)'(t) < 0$, then

$$
\begin{aligned}
-(\Phi_i x_i)'(t) &= f_i\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x_i(t) \\
&\quad - a_i(t)g_i(x_i(h_{1i}(t)))(\Phi_i x_i)(t) \\
&\leq \Big(\frac{1-\sigma_i^{L_i}}{\sigma_i^{L_i}} - a_i(t)g_i(x_i(h_{1i}(t)))\Big)(\Phi_i x_i)(t) \leq (\Phi_i x_i)(t),\ i = 1,\ldots,n. \qquad (2.3)
\end{aligned}
$$

Hence, $\Phi x \in K$. The proof of Lemma 2.2 is complete.

Notes

© 2012 Global Journals Inc. (US)

For convenience in the following discussion, we introduce the following notations:

$$\max_{t\in[0,\omega]}\{a_i(t)\} := a_i^M,$$
$$\max_{t\in[0,\omega]\, u\in B(0,R)} f_i\Big(s,u,\int_{-\varsigma}^{0} k(v)u(s-v)\mathrm{d}v, u', \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)u'(s-v)\mathrm{d}v\Big) := \theta_i.$$

Notes

Lemma 2.3. *Assume that* $(H_1)-(H_5)$, *and* $\Big(R\max_{t\in[0,\omega]}\Big\{\sum_{i=1}^{n}\beta_i(t)\Big\}\Big) < 1$ *hold, then* $\Phi : K\bigcap\bar{\Omega}_R \to K$ *is strict-set-contractive, where* $\Omega_R = \{x\in C_\omega^1 : |x|_1 < R\}$.

Proof. It is easy to see that Φ is continuous and bounded. Now we prove that $\alpha_{C_\omega^1}(\Phi(S)) \le \Big(R\max_{t\in[0,\omega]}\Big\{\sum_{i=1}^{n}\beta_i(t)\Big\}\Big)\alpha_{C_\omega^1}(S)$ for any bounded set $S\subset\bar{\Omega}_R$. Let $\eta = \alpha_{C_\omega^1}(S)$. Then, for any positive number $\varepsilon < \Big(R\max_{t\in[0,\omega]}\Big\{\sum_{i=1}^{n}\beta_i(t)\Big\}\Big)\eta$, there is a finite family of subsets $\{S_i\}$ satisfying $S=\bigcup_i S_i$ with $\mathrm{diam}(S_i)\le \eta+\varepsilon$. Therefore

$$\|x_i - y\|_1 \le \eta+\varepsilon \quad \text{for any } x,y\in S_i. \tag{2.4}$$

As S and S_i are precompact in C_ω^0, it follows that there is a finite family of subsets $\{S_{ij}\}$ of S_i such that $S_i=\bigcup_j S_{ij}$ and

$$\|x-y\| \le \varepsilon \quad \text{for any } x,y\in S_{ij}. \tag{2.5}$$

Let $S\subset K$ be an arbitrary open bounded set in K, then there exists a number $R>0$ such that $\|x\|<R$ for any $x=(x_1,\dots,x_n)^T\in S$. In fact, for any $x\in S$ and $t\in[0,\omega]$, we have

$$\begin{aligned}
|(\Phi_i x_i)(t)| &= \Big|\int_t^{t+\omega} G_i(t,s)x_i(s)f_i\Big(s,x(h_2(s)),\int_{-\varsigma}^{0}k(v)x(s-v)\mathrm{d}v, x'(h_3(s)),\\
&\qquad \int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)x'(s-v)\mathrm{d}v\Big)\mathrm{d}s\Big|\\
&\le \frac{1}{1-\sigma_i^{l_i}}\int_t^{t+\omega} x_i(s)f_i\Big(s,x(h_2(s)),\int_{-\varsigma}^{0}k(v)x(s-v)\mathrm{d}v, x'(h_3(s)),\\
&\qquad \int_{\widehat{\varsigma}}\widehat{k}(v)x'(s-v)\mathrm{d}v\Big)\mathrm{d}s\\
&\le \frac{R\omega}{1-\sigma_i^{l_i}}\max_{t\in[0,\omega]\, u_i\in B(0,R)} f_i\Big(s,u,\int_{-\varsigma}^{0}k(v)u(s-v)\mathrm{d}v, u',\\
&\qquad \int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)u'(s-v)\mathrm{d}v\Big)\Big) = \frac{\theta_i R\omega}{1-\sigma_i^{l_i}},\ i=1,\dots,n
\end{aligned}$$

and

$$|(\Phi_i x_i)'(t)| = \Big| - f_i\Big(t,x(h_2(t)),\int_{-\varsigma}^{0}k(v)x(t-v)\mathrm{d}v, x'(h_3(t)),\int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x_i(t)$$

© 2012 Global Journals Inc. (US)

$$+a_i(t)g_i(x_i(h_{1i}(t)))(\Phi_i x_i)(t)\Big|$$
$$\leq \xi_{2i}\frac{\theta_i R\omega}{1-\sigma_i^{l_i}},\ i=1,\ldots,n.$$

Hence

$$\|\Phi x\| \leq \sum_{i=1}^{n} H_i$$

and

$$\|(\Phi x)'\| \leq \sum_{i=1}^{n} \xi_{2i} H_i.$$

Applying the Arzela-Ascoli Theorem, we know that $\Phi(S)$ is precompact in C_ω^0. Then, there is a finite family of subsets $\{S_{ijk}\}$ of S_{ij} such that $S_{ij} = \bigcup_k S_{ijk}$ and

$$|(\Phi x) - (\Phi y)|_0 \leq \varepsilon \ \text{ for any } x, y \in S_{ijk}. \tag{2.6}$$

From (2.4), (2.5), (2.6), (H_3) and (H_4), for any $x, y \in S_{ijk}$, we obtain

$$\begin{aligned}
&|(\Phi_i x_i)' - (\Phi_i y_i)'|_0 \\
&= \max_{t\in[0,\omega]}\Big\{\Big|a_i(t)g_i(x_i(h_{1i}(t)))(\Phi_i x_i)(t) - a_i(t)g_i(y_i(h_{1i}(t)))(\Phi_i y_i)(t) \\
&\quad + f_i\Big(t, y(h_2(t)), \int_{-\varsigma}^{0} k(v)y(t-v)\mathrm{d}v, y'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)y'(t-v)\mathrm{d}v\Big)y(t) \\
&\quad - f_i\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)x_i(t)\Big|\Big\} \\
&\leq \max_{t\in[0,\omega]}\{a_i(t)L|(\Phi_i x_i)(t) - (\Phi_i y_i)(t)| + a_i(t)\mathbb{L}_i|(\Phi_i x_i(t)||x_i(t) - y_i(t)|\} \\
&\quad + \max_{t\in[0,\omega]}\Big\{\Big|x_i(t)\Big[f_i\Big(t, y(h_2(t)), \int_{-\varsigma}^{0} k(v)y(t-v)\mathrm{d}v, y'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)y'(t-v)\mathrm{d}v\Big) \\
&\quad - f_i\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(t-v)\mathrm{d}v\Big)\Big]\Big| \\
&\quad + |x_i(t) - y_i(t)|f_i\Big(t, y(h_2(t)), \int_{-\varsigma}^{0} k(v)y(t-v)\mathrm{d}v, y'(h_3(t)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)y'(t-v)\mathrm{d}v\Big)\Big\} \\
&\leq a_i^M L|(\Phi_i x_i) - (\Phi_i y_i)|_0 + a_i^M \mathbb{L}_i\frac{\theta_i|x_i|_0\omega}{1-\sigma_i^{l_i}}|x_i(t) - y_i(t)| + \max_{t\in[0,\omega]}\Big\{\theta_i|x_i(t) - y_i(t)|\Big\} \\
&\quad + |x_i|_0\Big(\sum_{i=1}^{n}\alpha_i(t)|x_i - y_i|_0 + \sum_{i=1}^{n}\beta_i(t)|x_i' - y_i'|_1\Big) \\
&\leq \Big(a_i^M L + a_i^M\mathbb{L}_i\frac{\theta_i|x_i|_0\omega}{1-\sigma_i^{l_i}} + \theta_i + |x_i|_0\alpha_i(t)\Big)\varepsilon + |x_i|_0\beta_i(t)\Big(\eta + \varepsilon\Big)
\end{aligned}$$

Notes

© 2012 Global Journals Inc. (US)

$$\leq \left(a_i^M L + a_i^M \mathbb{L}_i \frac{\theta_i |x_i|_0 \omega}{1-\sigma_i^{l_i}} + \theta_i + |x_i|_0 \alpha_i(t) + |x_i|_0 \beta_i(t)\right)\varepsilon + |x_i|_0 \beta_i(t)\eta,\ i = 1, \dots, n. \quad (2.7)$$

From (2.6) and (2.7), for any $x, y \in S_{ijk}$, we have

$$\begin{aligned} &\|\Phi x - \Phi y\|_1 \\ &\leq \left(a_i^M L + \max_{1\leq i\leq n}\left\{\sum_{i=1}^n a_i^M \mathbb{L}_i \frac{\theta_1 \omega}{1-\sigma_i^{l_i}}\right\} + \sum_{i=1}^n \theta_i + R \max_{t\in[0,\omega]}\left\{\sum_{i=1}^n \alpha_i(t)\right\}\right. \\ &\left. + R \max_{t\in[0,\omega]}\left\{\sum_{i=1}^n \beta_i(t)\right\}\right)\varepsilon + R \max_{t\in[0,\omega]}\left\{\sum_{i=1}^n \beta_i(t)\right\}\eta. \end{aligned}$$

As ε is arbitrary small, it follows that

$$\alpha_{C_\omega^1}(\Phi(S)) \leq \left(R \max_{t\in[0,\omega]}\left\{\sum_{i=1}^n \beta_i(t)\right\}\right)\alpha_{C_\omega^1}(S).$$

Therefore, Φ is strict-set-contractive. The proof of Lemma 2.3 is complete.

For convenience in the following discussion, we introduce the following notations:

$$\begin{cases} \lim_{u\to 0} \sup \max_{t\in[0,\omega]} \frac{f_i\left(t,u,\int_{-\varsigma}^0 k(v)u(t-v)\mathrm{d}v,u',\int_{-\widehat{\varsigma}}^0 \widehat{k}(v)u'(t-v)\mathrm{d}v\right)}{\sum_{i=1}^n u_i + \sum_{i=1}^n u_i'} = f_i^0, \\ \lim_{u\to\infty} \inf \min_{t\in[0,\omega]} \frac{f_i\left(t,u,\int_{-\varsigma}^0 k(v)u(t-v)\mathrm{d}v,u',\int_{-\widehat{\varsigma}}^0 \widehat{k}(v)u'(t-v)\mathrm{d}v\right)}{\sum_{i=1}^n u_i + \sum_{i=1}^n u_i'} = f_i^\infty. \end{cases} \quad (2.8)$$

III. Main Result

Our main result of this paper is as follows:

Theorem 3.1. *Assume that $(H_1)-(H_5)$ hold, then system (1.3) has at least one positive ω-periodic solution.*

Proof. According (2.8), for any

$$0 < \varepsilon < \min\left\{\frac{1}{2}, \frac{1}{4}\min_{1\leq i\leq n} f_i^\infty\right\},$$

there exist positive numbers $r_0 < R_0$ such that for $i = 1, \dots, n$,

$$\begin{aligned} &f_i\left(t, u, \int_{-\varsigma}^0 k(v)u(t-v)\mathrm{d}v, u', \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)u'(t-v)\mathrm{d}v\right) \\ &< (f_i^0 + \varepsilon)\left(\sum_{i=1}^n u_i + \sum_{i=1}^n u_i'\right) \quad \text{for} \quad 0 < \sum_{i=1}^n |u_i|_1 < r_0 \end{aligned}$$

and

$$f_i\left(t, u, \int_{-\varsigma}^0 k(v)u(t-v)\mathrm{d}v, u', \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)u'(t-v)\mathrm{d}v\right)$$

Notes

© 2012 Global Journals Inc. (US)

$$> \left(f_i^\infty - \varepsilon\right)\left(\sum_{i=1}^n u_i + \sum_{i=1}^n u_i'\right) \quad \text{for} \quad \sum_{i=1}^n |u_i|_1 > R_0$$

Let

$$R = \max\left\{\left(\min_{1\le i\le n}\left\{\frac{2\sigma_i^{L_i}\omega\left(f_i^\infty - \varepsilon\right)}{1-\sigma_i^{L_i}}\delta_i^2\right\}\right)^{-1}, \min_{1\le i\le n}\left\{\delta_i^{-1}\right\}R_0\right\}$$

and

$$0 < r < \min_{1\le i\le n}\left\{\frac{1-\sigma_i^{l_i}}{2\omega\left(f_i^0+\varepsilon\right)}\delta_i,\, r_0\right\}.$$

Then we have $0 < r < R$. From Lemmas 2.2 and 2.3, we know that Φ is strict-set-contractive on $K_{r,R}$. In view of Lemma 2.1, we see that if there exists $x^* \in K$ such that $\Phi x^* = x^*$, then x^* is one positive ω-periodic solution of system (1.1).

First, we prove that $\Phi x \not\geq x$, $\forall x \in K$, $\|x\|_1 = r$. Otherwise, there exists $x \in K$, $\|x\|_1 = r$ such that $\Phi x \geq x$. So $\|x\| > 0$ and $\Phi x - x \in K$, which implies that

$$(\Phi_i x_i)(t) - x_i(t) \geq \delta_i|\Phi_i x_i - x_i|_1 \geq 0 \quad \text{for any } t\in[0,\omega]. \tag{3.1}$$

Moreover, for $t \in [0,\omega]$, we have

$$\begin{aligned}(\Phi_i x_i)(t) &= \int_t^{t+\omega} G_i(t,s)x_i(s)f_i\Big(s, x(h_2(s)), \int_{-\varsigma}^0 k(v)x(s-v)\mathrm{d}v,\\ &\qquad x'(h_3(s)), \int_{-\widehat{\varsigma}}^0 \widehat{k}(v)x'(s-v)\mathrm{d}v\Big)\mathrm{d}s\\ &\leq \frac{1}{1-\sigma_i^{l_i}}|x_i|_0\left[2\omega\left(f_i^0+\varepsilon\right)\sum_{i=1}^n |x_i|_1\right]\\ &= \frac{2\omega\left(f_i^0+\varepsilon\right)}{1-\sigma_i^{l_i}}|x_i|_0 r\\ &< \delta_i|x_i|_0, \quad i=1,\dots,n.\end{aligned} \tag{3.2}$$

In view of (3.1) and (3.2), we have

$$\|x\| \leq \|\Phi x\| = \sum_{i=1}^n (\Phi_i x_i)|_0 < \max_{1\le i\le n}\{\delta_i\}\|x\| < \|x\|,$$

which is a contradiction. Finally, we prove that $\Phi x \not\leq x$, $\forall x\in K$, $\|x\|_1 = R$ also holds. For this case, we only need to prove that

$$\Phi x \not< x \quad x\in K,\ \|x\|_1 = R.$$

Suppose, for the sake of contradiction, that there exists $x \in K$ and $\|x\|_1 = R$ such that $\Phi x < x$. Thus $x - \Phi x \in K\setminus\{0\}$. Furthermore, for any $t\in[0,\omega]$, we have

$$x(t) - (\Phi x)(t) \geq \delta|x - \Phi x|_1 > 0. \tag{3.3}$$

In addition, for any $t \in [0,\omega]$, we find

$$(\Phi_i x_i)(t) = \int_t^{t+\omega} G_i(t,s)x_i(s)f_i\Big(s, x(h_2(s)), \int_{-\varsigma}^0 k(v)x(s-v)\mathrm{d}v,$$

Notes

© 2012 Global Journals Inc. (US)

Notes

$$x'(h_3(s)), \int_{-\widehat{\varsigma}}^{0} \widehat{k}(v)x'(s-v)\mathrm{d}v\Big)\,\mathrm{d}s$$
$$\geq \frac{\sigma_i^{L_i}}{1-\sigma_i^{L_i}}\delta_i|x_i|_1\left[2\omega\Big(f_i^{\infty}-\varepsilon\Big)\sum_{i=1}^{n}\delta_i|x_i|_1\right]$$
$$\geq \frac{2\sigma_i^{L_i}\omega\Big(f_i^{\infty}-\varepsilon\Big)}{1-\sigma_i^{L_i}}\delta_i|x_i|_1 \min_{1\leq i\leq n}\{\delta_i\}\sum_{i=1}^{n}|x_i|_1, \quad i=1,\ldots,n. \tag{3.4}$$

Thus,

$$\begin{aligned}\|\Phi x\|_0 &= \sum_{i=1}^{n}|(\Phi_i x_i)|_0 \geq \frac{2\sigma_i^{L_i}\omega\Big(f_i^{\infty}-\varepsilon\Big)}{1-\sigma_i^{L_i}}\delta_i|x_i|_1\sum_{i=1}^{n}|x_i|_1\\ &\geq \min_{1\leq i\leq n}\left\{\frac{2\sigma_i^{L_i}\omega\Big(f_i^{\infty}-\varepsilon\Big)}{1-\sigma_i^{L_i}}\delta_i^2\right\}\sum_{i=1}^{n}|x_i|_1\sum_{i=1}^{n}|x_i|_1\\ &\geq \min_{1\leq i\leq n}\left\{\frac{2\sigma_i^{L_i}\omega\Big(f_i^{\infty}-\varepsilon\Big)}{1-\sigma_i^{L_i}}\delta_i^2\right\}R^2 = R. \end{aligned}\tag{3.5}$$

From (3.3) – (3.5), we obtain

$$\|x\| > \|\Phi x\| \geq R,$$

which is a contradiction. Therefore, conditions (i) and (ii) hold. By Lemma 2.2, we see that Φ has at least one nonzero fixed point in K. Therefore, system (1.3) has at least one positive ω-periodic solution. The proof of Theorem 3.1 is complete.

IV. EXAMPLES

Consider the following system [8]

$$x_i'(t) = x_i(t)\left[a_i(t) - \sum_{j=1}^{n}\alpha_{ij}(t)x_j(t-\tau_{ij}) - \sum_{j=1}^{n}\beta_{ij}(t)x_j'(t-\sigma_{ij})\right], \tag{4.1}$$

where a_i, α_{ij}, β_{ij} $(i=1,\ldots,n,\ j=1,\ldots,n) \in \big(\mathbb{R},(0,+\infty)\big)$ are functions with periodic ω, τ_{ij}, σ_{ij} $(i=1,\ldots,n,\ j=1,\ldots,n)\in[0,+\infty)$ are constants.

Corollary 4.1. *Assumed* $(H_1)-(H_5)$ *and* $\max_{1\leq i\leq n}\left\{R\sum_{j=1}^{n}\beta_{ij}(t)\right\} < 1$ *hold, Eq.(4.1) has at least one ω-periodic solution.*

Proof. In this case

$$\begin{aligned} &f_i\Big(t, x(h_2(t)), \int_{-\varsigma}^{0} k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)x'(t-v)\mathrm{d}v\Big)\\ = &\sum_{j=1}^{n}\alpha_{ij}(t)x_j(t-\tau_{ij}) + \sum_{j=1}^{n}\beta_{ij}(t)x_j'(t-\sigma_{ij}),\\ &g_i(x_i(h_1(t))) = 1,\\ &f_i^0 \leq \max_{1\leq i\leq n}\left\{\max_{t\in[0,\omega]}\Big\{\alpha_{ij}(t)\Big\} + \max_{t\in[0,\omega]}\Big\{\beta_{ij}(t)\Big\}\right\} < \infty,\ i=1,\ldots,n\end{aligned}$$

© 2012 Global Journals Inc. (US)

and

$$f_i^{\infty} \le \min_{1\le i\le n}\left\{\min_{t\in[0,\omega]}\left\{\alpha_{ij}(t)\right\} + \min_{t\in[0,\omega]}\left\{\beta_{ij}(t)\right\}\right\} > 0,\ i=1,\dots,n.$$

It follows from Theorem 3.1 that system (4.1) has at least one positive periodic solution. The proof of Theorem 4.1 is complete.

Consider the following system [9]

$$x_i'(t) = x_i(t)\left[a_i(t) - \sum_{j=1}^{n} b_{ij}(t)\int_{-T_{ij}}^{0} K_{ij}(\theta)x_j(t+\theta)\mathrm{d}\theta - \sum_{j=1}^{n} c_{ij}(t)\int_{-\widehat{T}_{ij}}^{0}\widehat{K}_{ij}(\theta)x_j'(t+\theta)\mathrm{d}\theta\right],\quad(4.2)$$

where $a_i,\ b_{ij},\ c_{ij}\ (i=1,\dots,n,\ j=1,\dots,n)\in\big(\mathbb{R},(0,+\infty)\big)$ are functions with periodic ω, $T_{ij},\ \widehat{T}_{ij}\ (i=1,\dots,n,\ j=1,\dots,n)\in[0,+\infty)$, $K_{ij},\ \widehat{K}_{ij}\in(\mathbb{R},\mathbb{R}^+)$ satisfying $\int_{-T_{ij}}^{0}K_{ij}(\theta)x_j(t+\theta)\mathrm{d}\theta=1,\ \int_{-\widehat{T}_{ij}}^{0}\widehat{K}_{ij}(\theta)\mathrm{d}\theta=1,\ i,j=1,\dots,n.$

Corollary 4.2. *Assumed* $(H_1)-(H_5)$ *and* $\max_{1\le i\le n}\left\{R\sum_{j=1}^{n}c_{ij}(t)\right\}<1$ *hold, Eq.(4.1) has at least one* ω*-periodic solution.*

Proof. In this case

$$\begin{aligned}
&f_i\Big(t, x(h_2(t)), \int_{-\varsigma}^{0}k(v)x(t-v)\mathrm{d}v, x'(h_3(t)), \int_{-\widehat{\varsigma}}^{0}\widehat{k}(v)x'(t-v)\mathrm{d}v\Big)\\
=&\sum_{j=1}^{n} b_{ij}(t)\int_{-T_{ij}}^{0}K_{ij}(\theta)x_j(t+\theta)\mathrm{d}\theta + \sum_{j=1}^{n} c_{ij}(t)\int_{-\widehat{T}_{ij}}^{0}\widehat{K}_{ij}(\theta)x_j'(t+\theta)\mathrm{d}\theta,\\
&g_i(x_i(h_1(t)))=1,\\
&f_i^{0}\le \max_{1\le i\le n}\left\{\max_{t\in[0,\omega]}\left\{b_{ij}(t)\right\}+\max_{t\in[0,\omega]}\left\{c_{ij}(t)\right\}\right\}<\infty,\ i=1,\dots,n
\end{aligned}$$

and

$$f_i^{\infty}\le \min_{1\le i\le n}\left\{\min_{t\in[0,\omega]}\left\{b_{ij}(t)\right\}+\min_{t\in[0,\omega]}\left\{c_{ij}(t)\right\}\right\}>0,\ i=1,\dots,n.$$

It follows from Theorem 3.1 that system (4.1) has at least one positive periodic solution. The proof of Theorem 4.1 is complete.

References Références Referencias

[1] A. Wan, D. Jiang, Existence of positive periodic solutions for functional differential equations, Kyushu J.Math. 56(1)(2002) 193-202.

[2] A. Wan, D. Jiang, X. Xu, A new existence theory for positive periodic solutions to functional differential equations, Comput. Math. Appl. 47(2004) 1257-1262.

[3] D. Jiang, B. Zhang, Positive periodic solutions of functional differential equations and population models, Electron. J. Differential Equation 71(2002) 1-13.

[4] Y. Li, Existence and global attractivity of a positive periodic solution fo a class of delay differential equation, Sci. China Ser. A 41(1998) 273-284.

Notes

© 2012 Global Journals Inc. (US)

[5] Wang. H, Positive periodic solutions of functional differential equations, J. Differential Equation 202(2004) 354-366.

[6] N.P. Các, J.A Gatica, Fixed point theorems for mappings in ordered Banach space, J. Math. Anal. Appl. 71(1979) 547-557.

[7] D. Guo, Positive solutions of nonlinear operator equations and its applications to nonlinear integral equations, Adv. Mtath. 13 (1984) 294-310(in chinese).

[8] Y.K. Li, On a periodic neutral delay Lotka-Volterra system, Nonlinear Anal. 39 (2000) 767-778.

[9] Y. Li, Positive periodic solutions of periodic neutral Lotka-Volterra system with distributed delays, Chaos, Solitons and Fractals. 2006, IN PRESS.

[10] Z. Yang, J. Cao, Positive periodic solutions of neutral Lotka-Volterra system with periodic delays, Comput. Math. Appl. 149 (2004) 661-687.

[11] G. Liu, J. Yan, Positive periodic solutions for a neutral delay Lotka-Volterra system. Nonlinear Anal. 67 (2007) 2642-2653.

[12] Z. Liu, L. Chen, Periodic solution of neutral LotkaCVolterra system with periodic delays, J. Math. Anal.Appl. 324(2006) 435-451.

[13] Y. Li, Positive periodic solutions of periodic neutral Lotka-Volterra system with state dependent delays, J. Math. Anal.Appl. 330 (2007) 1347-1362.

[14] Y. Li, Positive periodic solutions of discrete Lotka-Volterra competition systems with state dependent and distributed delays, Mathe. Compu,190(2007) 526-531.

Notes

© 2012 Global Journals Inc. (US)

Notes

This page is intentionally left blank

© 2012 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics & Decision Sciences
Volume 12 Issue 2 Version 1.0 February 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

The Existence of Solution in $H^1(R^N)$ for Nonclassical Diffusion Equations

By LIU Yong-feng & MA Qiao-zhen
Northwest Normal University, Lanzhou, China

Abstract - In this paper, we prove the existence of weak solution for a nonclassical diffusion equations in $H^1(R^N)$. The result in this part are new.

Keywords : *Nonclassical diffusion equations; Weak solution; Absorbing set.*

GJSFR – F Classification : *FOR Code: 010299*

THE EXISTENCE OF SOLUTION IN H1RN FOR NONCLASSICAL DIFFUSION EQUATIONS

Strictly as per the compliance and regulations of :

© 2012 . LIU Yong-feng & MA Qiao-zhen.This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Ref.

[1] Aifantis. E. C, *On the problem of diffusion in solids*, Acta Mech, **37** (1980):265-296.

[2] Lions. J. L, Magenes.E, *Non-homogeneous boundary value problems and applications*, Spring-verlag,Berlin, (1972).

The Existence of Solution in $H^1(R^N)$ for Nonclassical Diffusion Equations

LIU Yong-feng[α] & MA Qiao-zhen[σ]

Abstract - In this paper, we prove the existence of weak solution for a nonclassical diffusion equations in $H^1(R^N)$. The result in this part are new.

Keywords : *Nonclassical diffusion equations; Weak solution; Absorbing set.*

I. Introduction

In this paper, we investigate the following nonclassical diffusion equations

$$u_t - \Delta u_t - \Delta u + f(x,u) = g(x), \quad x \in R^N, \tag{1.1}$$

with the initial data

$$u(x,0) = u_0, \quad x \in R^N. \tag{1.2}$$

This equation is a special form of the nonclassical diffusion equation used in fluid mechanics, solid mechanics and heat conduction theory(see [1, 2]). On bounded domains, the long-time behavior have been discussed by many authors in [3-11].

To our best knowledge, the existence of weak solution in R^N for the nonclassical diffusion equation have not been considered by predecessors.

In this paper, we consider the existence of weak solution in $H^1(R^N)$ if $g(x) \in L^2(R^N)$, and the nonlinearity $f(x,u) = f_1(u) + a(x)f_2(u)$ satisfies:

(F_1) $\alpha_1 \mid u \mid^p - \beta_1 \mid u \mid^2 \leq f_1(u)(u) \leq \gamma_1 \mid u \mid^p + \delta_1 \mid u \mid^2, f_1(u)u \geq 0, p \geq 2,$ *and* $f_1'(u) \geq -c$;

(F_2) $\alpha_2 \mid u \mid^p - \beta_2 \leq f_2(u)(u) \leq \gamma_2 \mid u \mid^p + \delta_2, p \geq 2,$ *and* $f_2'(u) \geq -c$;

and

(A) $a \in L^1(R^N) \cap L^\infty(R^N),\ a(x) > 0.$

where $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i,\ i = 1, 2$, and c are all positive constants.

Author α : College of Mathematics and Information Science, Northwest Normal University, Lanzhou, Gansu 730070, China
E-mail : liuyongfeng1982@126.com

Foundation term: This paper is supported in part by the Natural Sciences Foundation of Gansu province (3ZS061-A25-016), in part by the Education Department Foundation of Gansu Province(0801-02) and NWNU-KJCXGC-03-40.

© 2012 Global Journals Inc. (US)

II. Unique Weak Solution

Lemma 2.1 *([11]) Let $X \subset\subset H \subset Y$ be Banach spaces, with X reflexive. Suppose that u_n is a sequence that is uniformly bounded in $L^2(0,T;X)$, and $\mathrm{d}u_n/\mathrm{dt}$ is uniformly bounded in $L^p(0,T;Y)$, for some $p > 1$. Then there is a subsequence that converges strongly in $L^2(0,T;H)$.*

Theorem 2.1 *Assume (F_1), (F_2) and (A) are satisfied. Then for any $T > 0$ and $u_0 \in H^1(R^N)$, there is a unique solution u of $(1.1)-(1.2)$ such that*

$$u \in \mathcal{C}^1([0,T];H^1(R^N)) \cap L^p(0,T;L^p(R^N)).$$

Moreover, the solution continuously depends on the initial data.

Proof We divide into three steps:

Step 1 For any $n \in N$, we consider the existence of the weak solution for the following problem in $B(0,n) \triangleq B_n \subset R^N$,

$$u_t - \Delta u_t - \Delta u + f(x,u) = g(x), \quad x \in B_n, \tag{2.1}$$

$$u(x,0) = u_0 \in H^1(B_n). \tag{2.2}$$

$$u\mid_{\partial\Omega} = 0. \tag{2.3}$$

Choose a smooth function $\chi_n(x)$ satisfy

$$\chi_n(x) = \begin{cases} 1, & x \in B_{n-1}, \\ 0, & x \notin B_n. \end{cases} \tag{2.4}$$

Since B_n is a bounded domain, so the existence and uniqueness of solutions can be obtained by the standard Faedo-Galerkin methods, see [3,5,8,11], we have the unique weak solution

$$u_n \in \mathcal{C}^1([0,T];H^1(B_n)) \cap L^p(0,T;L^p(B_n)) \ \ and \ u_n(x,0) = \chi_n(x)u_0(x).$$

Step 2 According to Step 1, and we denote $\frac{\mathrm{d}}{\mathrm{dt}}u_n = u_{nt}$, then u_n satisfy

$$u_{nt} - \Delta u_{nt} - \Delta u_n + f(x,u_n) = g(x), \quad x \in B_n, \tag{2.5}$$

$$u_n(x,0) = \chi_n(x)u_0(x), \tag{2.6}$$

$$u_n\mid_{\partial B_n} = 0. \tag{2.7}$$

For the mathematical setting of the problem, we denote $\|\cdot\|_{L^2(B_n)} \triangleq \|\cdot\|_{B_n}$, $\|\cdot\|_{L^1(R^N)} \triangleq \|\cdot\|_1$, $\|\cdot\|_{L^2(R^N)} \triangleq \|\cdot\|$, $\|\cdot\|_{L^\infty(R^N)} \triangleq \|\cdot\|_\infty$.

Multiply (2.5) by u_n in B_n, using $f_1(u)u \geq 0$, (F_2) and (A), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\| \nabla u_n \|^2_{B_n} + \|u_n\|^2_{B_n}) + \| \nabla u_n \|^2_{B_n} \quad &\leq \int_{B_n} a(x)(\beta_2 - \alpha_2 \mid u \mid^p)\mathrm{d}x + \int_{B_n} g u_n \mathrm{d}x \\ &\leq \beta_2 \| a(x) \|_1 - \int_{B_n} \alpha_2 a(x) \mid u \mid^p \mathrm{d}x + \frac{\| g \|^2}{2\lambda} + \frac{\lambda}{2} \| u_n \|^2_{B_n} \end{aligned}$$

By the Poincaré inequality, for some $\nu > 0$, we have

$$\frac{1}{2}\frac{d}{dt}(\| \nabla u_n \|^2_{B_n} + \|u_n\|^2_{B_n}) + \nu(\| \nabla u_n \|^2_{B_n} + \|u_n\|^2_{B_n}) \quad + \int_{B_n} \alpha_2 a(x) \mid u \mid^p \mathrm{d}x$$

Ref.

[3] V. K. Kalantarov, *On the attractors for some non-linear problems of mathematical physics*, Zap. Nauch. Sem. LOMI **152** (1986):50-54.

[11] C.Robinson,*Infinite-dimensional Dynamincal Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global attractors*, Cambridge Universlty Press(2001).

© 2012 Global Journals Inc. (US)

$$\leq \beta_2 \parallel a(x) \parallel_1 + \frac{\parallel g \parallel^2}{2\lambda}. \tag{2.8}$$

Hence, we have

$$\parallel \nabla u_n(T) \parallel^2_{B_n} + \|u_n(T)\|^2_{B_n} + 2\nu \int_0^T (\parallel \nabla u_n(T) \parallel^2_{B_n} + \|u_n(T)\|^2_{B_n}) + 2\int_0^T \int_{B_n} \alpha_2 a(x) \mid u \mid^p \mathrm{d}x$$
$$\leq (2\beta_2 \parallel a(x) \parallel_1 + \frac{\parallel g \parallel^2}{\lambda})T. \tag{2.9}$$

Notes

We get the following estimate:

$$\sup_{t\in[0,T]} \parallel \nabla u_n(t) \parallel^2_{B_n} + \|u_n(t)\|^2_{B_n} \leq C, \tag{2.10}$$

$$\int_0^T (\parallel \nabla u_n(t) \parallel^2_{B_n} + \|u_n(t)\|^2_{B_n}) \leq C, \tag{2.11}$$

$$\int_0^T \int_{B_n} \alpha_2 a(x) \mid u(t) \mid^p \mathrm{d}x \leq C, \tag{2.12}$$

Similar to (2.8), using (F_1), (F_2) and (A), we have

$$\int_0^T \int_{B_n} \mid u(t) \mid^p \mathrm{d}x \leq C. \tag{2.13}$$

where C is independent on n.

According to (F_1) and (F_2), we have

$$\mid f_1(u_n) \mid \leq C(\mid u_n \mid^{p-1} + \mid u_n \mid). \tag{2.14}$$

$$\mid f_2(u_n) \mid \leq C(\mid u_n \mid^{p-1} + 1). \tag{2.15}$$

We choose q such that $\frac{1}{p} + \frac{1}{q} = 1$, then $(p-1)q = p$. Noting that $p \geq 2$, then $1 < q \leq 2$, and we have the embedding $L^p(B_n) \hookrightarrow L^q(B_n)$. According to (2.13) – (2.15), we get

$$\begin{aligned}
\int_0^T \int_{B_n} \mid f_1(u) \mid^q &\leq C \int_0^T \int_{B_n} (\mid u_n \mid^{p-1} + \mid u_n \mid)^q \mathrm{d}x\mathrm{d}t \\
&\leq C \int_0^T \int_{B_n} \mid u_n \mid^{(p-1)q} \mathrm{d}x\mathrm{d}t + C \int_0^T \int_{B_n} \mid u_n \mid^q \mathrm{d}x\mathrm{d}t \\
&\leq C \int_0^T \int_{B_n} \mid u_n \mid^p + C \int_0^T \int_{B_n} \mid u_n \mid^p \mathrm{d}x\mathrm{d}t \\
&\leq C.
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\int_0^T \int_{B_n} \mid f_2(u) \mid^q &\leq C \int_0^T \int_{B_n} \mid a(x) \mid^q (\mid u_n \mid^{p-1} + 1)^q \mathrm{d}x\mathrm{d}t \\
&\leq C \mid a(x) \mid^{q-1}_\infty \int_0^T \int_{B_n} a(x)(\mid u_n \mid^{(p-1)q} + 1)\mathrm{d}x\mathrm{d}t \\
&\leq C \mid a(x) \mid^{q-1}_\infty (C \mid a(x) \mid_1 + \int_0^T \int_{B_n} a(x) \mid u_n \mid^p \mathrm{d}x\mathrm{d}t) \\
&\leq C.
\end{aligned} \tag{2.17}$$

© 2012 Global Journals Inc. (US)

where C is independent on n.

Thanks to (2.16)−(2.17), $f_1(u_n)$ is bounded in $L^p(0,T;L^q(B_n))$, and $af_2(u_n)$ is bounded in $L^p(0,T;L^q(B_n))$.

For $\forall\ v \in L^2(0,T;H_0^1(B_n))$,

$$\begin{aligned}\int_0^T\int_{B_n} -\Delta u_n v &= \int_0^T\int_{B_n} \nabla u_n \nabla v \\ &\leq (\int_0^T \| \nabla u_n \|_{B_n}^2)^{\frac{1}{2}}(\int_0^T \| \nabla v \|_{B_n}^2)^{\frac{1}{2}} \\ &\leq (\int_0^T \| \nabla u_n \|^2)^{\frac{1}{2}}(\int_0^T \| \nabla v \|_{B_n}^2)^{\frac{1}{2}} \\ &\leq C \| \nabla v \|_{H_0^1(B_n)} . \end{aligned} \tag{2.18}$$

where C is independent on n. We can obtain $-\Delta u_n$ is bounded in $L^2(0,T;H^{-1}(B_n))$.

Since $g(x) \in L^2(R^N)$, so

$$g(x) \in L^2(0,T;R^N). \tag{2.19}$$

Hence there exists $s>0$, such that $L^2(0,T;H^{-1}(B_n))$, $L^2(0,T;H_0^1(B_n))$, $L^q(0,T;L^q(B_n))$,$L^2(0,T;L^2(B_n))$ are continuous embedding to $L^q(0,T;H^{-s}(B_n))$.

According to (2.5), (2.16) − (2.19), we obtain

$$u_{nt} - \Delta u_{nt} \in L^q(0,T;H^{-s}(B_n)). \tag{2.20}$$

Hence u_n has a subsequence (we also denote u_n) weak* convergence to u in $L^2(0,T;H^{-1}(B_n))$ and $L^2(0,T;L^2(B_n))$, $u_{nt} - \Delta u_{nt}$ has a subsequence (we also denote $u_{nt} - \Delta u_{nt}$) weak* convergence to $u_t - \Delta u_t$. Let $u_n = 0$ outside of B_n, we can extend u_n to R^N.

As introduced in [3,11], $C_c^\infty(R^N)$ is dense in the dual space of $H^{-1}(B_n))$, $L^2(B_n)$, $L^q(B_n)$ and $H^{-s}(B_n)$, so we can choose $\forall\ \phi \in L^2(0,T;C_c^\infty(R^N)) \cap L^q(0,T;C_c^\infty(R^N))$ as a test function such that

$$\langle \Delta u_n, \phi \rangle \to \langle \Delta u, \phi \rangle \tag{2.21}$$

$$\langle u_{nt} - \Delta u_{nt}, \phi \rangle \to \langle u_t - \Delta u_t, \phi \rangle \tag{2.22}$$

Since $\forall\ \phi \in C_c^\infty(R^N)$, there exists bounded domain $\Omega \subset R^N$ such that $\phi = 0,\ x \notin \Omega$. Hence u_n is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$, and $u_{nt} - \Delta u_{nt} \in L^q(0,T;H^{-s}(\Omega))$. Since $H_0^1(\Omega) \subset\subset L^2(\Omega) \subset H^{-s}(\Omega)$, according to lemma 2.1, there is a subsequence u_n (we also denote u_n) that converges strongly to u in $L^2(0,T;L^2(\Omega))$.

Using the continuity of f_1 and f_2, we have

$$\langle f_1(u_n), \phi \rangle \to \langle f_1(u), \phi \rangle \tag{2.23}$$

$$\langle a(x) f_2(u_n), \phi \rangle \to \langle a(x) f_2(u), \phi \rangle \tag{2.24}$$

According to (2.21) − (2.24), and let $n \to \infty$, we get : $\forall\ \phi \in L^2(0,T;C_c^\infty(R^N)) \cap L^q(0,T;C_c^\infty(R^N))$,

$$\langle u_t - \Delta u_t - \Delta u + f_1(u) + a(x) f_2(u), \phi \rangle = \langle g(x), \phi \rangle \tag{2.25}$$

Ref.

[3] V. K. Kalantarov, *On the attractors for some non-linear problems of mathematical physics*, Zap. Nauch. Sem. LOMI **152** (1986):50-54.

[11] C.Robinson,*Infinite-dimensional Dynamincal Systems: An Introduction to Disspative Parabolic PDEs and the Theory of Global attractors*, Cambridge Universtiy Press(2001).

© 2012 Global Journals Inc. (US)

Hence u is the weak solution of (2.1) − (2.3) and satisfy

$$u \in \mathcal{C}^1([0,T]; H^1(R^N)) \cap L^p(0,T; L^p(R^N)).$$

Step 3 Uniqueness and continuous dependence.

Let u_0, v_0 be in $H^1(R^N)$, and setting $w(t) = u(t) - v(t)$, we see that $w(t)$ satisfies

$$w_t - \Delta w_t - \Delta w + f_1(u) - f_1(v) + a(x)(f_2(u) - f_2(v)) = 0, \quad x \in R^N. \tag{2.26}$$

Notes

Taking the inner product with w of (2.26), using (F_1), (F_2) and (A), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\| \nabla w \|^2 + \|w\|^2) + \| \nabla w \|^2 &\leq | \int (f_1(u) - f_1(v)) w \mathrm{d}x | + | \int a(x)(f_2(u) - f_2(v)) w \mathrm{d}x | \\ &\leq C(1 + \| a \|_\infty) \| w \|^2 \end{aligned} \tag{2.27}$$

By the Gronwall Lemma, we get

$$\| \nabla w(t) \|^2 + \|w(t)\|^2 \leq e^{Ct}(\| \nabla w(0) \|^2 + \|w(0)\|^2). \tag{2.28}$$

This is uniqueness and is continuous dependence on initial conditions.

References Références Referencias

[1] Aifantis. E. C, *On the problem of diffusion in solids*, Acta Mech, **37** (1980):265-296.

[2] Lions. J. L, Magenes.E, *Non-homogeneous boundary value problems and applications*, Spring-verlag,Berlin, (1972).

[3] V. K. Kalantarov, *On the attractors for some non-linear problems of mathematical physics*, Zap. Nauch. Sem. LOMI **152** (1986):50-54.

[4] Y. L. Xiao, *Attractors for a nonclassical diffusion equation*, Acta Math Appl Sinica. **18** (2002): 273-276.

[5] C. Y. Sun, S. Y. Wang, C. K. Zhong, *Global attractors for a nonclassical diffusion equation*, Acta math sinica, English series.**23**(2007):1271-1280.

[6] Q. Z. Ma, C. K. Zhong, *Global attractors of strong solutions to nonclassical diffusion equations*, Journal of Lanzhou University. **40**(2004):7-9.

[7] S. Y. Wang, D. S. Li, C. K. Zhong, *On the dynamics of a class of nonclassical parabolic equations*, J. Math. Anal. Appl. **317**(2006):565-582.

[8] R.Temam, *Infinite-dimensional dynamical systems in mechanics and physic*, Springer, New York,(1997).

[9] C. Sun, M. Yang, *Dynamics of the nonclassical diffusion equations*, Asymptot. Anal . **59**(2008):51-81.

[10] Y.F. Liu, Q.Z. MA, *Exponential Attractors for a Nonclassical Diffusion Equation*, Electronic Journal of Differential Equations .

[11] C.Robinson,*Infinite-dimensional Dynamincal Systems: An Introduction to Disspative Parabolic PDEs and the Theory of Global attractors*, Cambridge Universlty Press(2001).

© 2012 Global Journals Inc. (US)

Notes

This page is intentionally left blank

© 2012 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics & Decision Sciences
Volume 12 Issue 2 Version 1.0 February 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On A Sturm - Liouville Like Four Point Boundary Value Problem

By Svetlin Georgiev Georgiev
University of Sofia, Bulgaria

Abstract - In this article we propose new approach for investigating of Sturm - Liouville like four point boundary value problem. It gives new results.

Keywords and phrases : Sturm - Liouville problem, existence.

GJSFR – F Classification : MSC 2010: 34B24, 34B15.

ON A STURM - LIOUVILLE LIKE FOUR POINT BOUNDARY VALUE PROBLEM

Strictly as per the compliance and regulations of :

OPEN ASSOCIATION OF RESEARCH SOCIETY, USA
RESEARCH | DIVERSITY | ETHICS

© 2012 . Svetlin Georgiev Georgiev.This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Ref.

[1] Zhao, J., F. Geng, J. Zhao, W. Ge. Positive solutions for a new kind Sturm - Liouville like foour point boundary value problem, Applied Mathematics and Computations, 2010, pp.811-819.

On a Sturm - Liouville like four point boundary value problem

Svetlin Georgiev Georgiev

Abstract - In this article we propose new approach for investigating of Sturm - Liouville like four point boundary value problem. It gives new results.

Keywords and phrases : Sturm - Liouville problem, existence.

I. Introduction

In this article we consider the problem

$$\begin{aligned} & x''(t) + h(t)f(t, x(t), x'(t)) = 0, \quad t \in [0,1], \\ & x'(0) - \alpha_1 x(\xi) = 0, \quad x'(1) + \alpha_2 x(\eta) = 0, \end{aligned} \tag{1.1}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\xi \in (0,1)$, $\eta \in (0,1)$, $\xi \neq \eta$, $h(t) \in \mathcal{C}(\mathbb{R})$, $f(\cdot,\cdot,\cdot) \in \mathcal{C}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$ are fixed, $x(t)$ is unknown..

Our aim is to investigate the problem (1.1) for existence of solutions. For this purpose we propose new approach for investigation. This approach gives new results.

Our main result is as follows.

Theorem 1.1. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\xi \in (0,1)$, $\eta \in (0,1)$, $\xi \neq \eta$, $h(t) \in \mathcal{C}(\mathbb{R})$, $f(\cdot,\cdot,\cdot) \in \mathcal{C}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$ be fixed. Then*

1) *the problem* (1.1) *has a bounded solution $x(t) \in \mathcal{C}^2([0,1])$;*

2) *if for the bounded solution $x(t)$ of* 1) *we have*

$$\int_\eta^t \int_0^s h(\tau) f(\tau, x(\tau), x'(\tau)) d\tau ds \neq 0 \quad \text{some} \quad t \in [0,1],$$

then it doesn't coincide with zero on $[0,1]$,

3) *if for the bounded solution $x(t)$ of* 1) *we have*

$$h(t) f(t, x(t), x'(t)) \neq 0 \quad \text{for} \quad \text{some} \quad t \in [0,1],$$

then it doesn't coincide with a constant.

We will compare our result with well known result.

In [1] the problem (1.1) is considered under conditions $0 < \alpha_1 < \frac{1}{\xi}$, $0 < \alpha_2 < \frac{1}{1-\eta}$, $0 < \xi < \eta < 1$, $\alpha_1\alpha_2\eta - \alpha_1\alpha_2\xi + \alpha_1 + \alpha_2 > 0$, $h(t) : [0,1] \longrightarrow [0,\infty)$ is a continuous function,

Author : University of Sofia, Faculty of Mathematics and Informatics, Department of Differential Equations, Blvd \Tzar Osvoboditel" 15, Sofia 1000, Bulgaria. E-mail : sgg2000bg@yahoo.com

© 2012 Global Journals Inc. (US)

a.e. $t \in [0,1]$, $f(t,x,y) \le a(t) + b(t)x + c(t)y$ for suitable functions $a, b, c \in L^1([0,1])$ and it is proved that (1.1) has a nontrivial solution. Evidently our result is better than the result in [1].

II. PROOF OF MAIN RESULT

1) Let D_1 be fixed positive constant and let also

$$M_1 = \max\Big\{\max_{t\in[0,1]} |h(t)|, \max_{[0,1]\times[-D_1,D_1]\times[-D_1,D_1]} |f(\cdot,\cdot,\cdot)|\Big\}.$$

Let $a_1 \in (0,1)$ is enough closed to 1 and $\epsilon \in (0,1)$ are chosen so that $a_1 + \epsilon_1 > 1$ and

$$\begin{aligned} &\epsilon_1 D_1 + (1-a_1)\tfrac{D_1}{|\alpha_2|} + (1-a_1)|\alpha_1|D_1 + (1-a_1)M_1 \le D_1, \\ &\epsilon_1 D_1 + (1-a_1)|\alpha_1|D_1 + (1-a_1)M_1 \le D_1. \end{aligned} \tag{2.1}$$

We define the sets

$$N_1 = \Big\{x(t) \in \mathcal{C}^1([0,1]) : |x(t)| \le D_1, |x'(t)| \le D_1 \quad \forall t \in [0,1]\Big\},$$

$$N_1^* = \Big\{x(t) \in \mathcal{C}^1([0,1]) : |x(t)| \le (a_1+\epsilon_1)D_1, |x'(t)| \le (a_1+\epsilon_1)D_1 \quad \forall t \in [0,1]\Big\}.$$

In these sets we define a norm as follows $||x|| = \max\{\max_{t\in[0,1]} |x(t)|, \max_{t\in[0,1]} |x'(t)|\}$. With this norm the sets N_1 and N_1^* are completely normed spaces. Also since for $x \in N_1$ we have $|x(t)| \le D_1$, $|x'(t)| \le D_1$ for every $t \in [0,1]$ we have that N_1 is a compact subset and closed convex subset of N_1^*.

Under these sets we define the operators

$P_1(x) = (a_1 + \epsilon_1)x,$

$K_1(x) = -\epsilon_1 x - (1-a_1)\frac{x'(1)}{\alpha_2} + (1-a_1)\alpha_1(t-\eta)x(\xi) - (1-a_1)\int_\eta^t \int_0^s h(\tau)f(\tau, x(\tau).x'(\tau))d\tau ds,$

$L_1(x) = P_1(x) + K_1(x).$

Our first observation is

Lemma 2.1. *Let $x(t)$ be a fixed point of the operator L_1. Then $x(t)$ is a solution to the problem* (1.1).

Proof. Since $x(t)$ is a fixed point of the operator L_1 then

$$\begin{aligned} x(t) &= L_1(x) = P_1(x) + K_1(x) \\ &= (a_1+\epsilon_1)x(t) - \epsilon_1 x(t) - (1-a_1)\tfrac{x'(1)}{\alpha_2} + (1-a_1)\alpha_1(t-\eta)x(\xi) \\ &\quad -(1-a_1)\textstyle\int_\eta^t \int_0^s h(\tau)f(\tau,x(\tau),x'(\tau))d\tau \\ &= a_1 x(t) - (1-a_1)\tfrac{x'(1)}{\alpha_2} + (1-a_1)\alpha_1(t-\eta)x(\xi) \\ &\quad -(1-a_1)\textstyle\int_\eta^t \int_0^s h(\tau)f(\tau,x(\tau),x'(\tau))d\tau, \end{aligned}$$

from here

$$(1-a_1)x(t) = -(1-a_1)\frac{x'(1)}{\alpha_2} + (1-a_1)\alpha_1(t-\eta)x(\xi) - (1-a_1)\int_\eta^t \int_0^s h(\tau)f(\tau,x(\tau),x'(\tau))d\tau$$

Ref.

[1] Zhao, J., F. Geng, J. Zhao, W. Ge. Positive solutions for a new kind Sturm - Liouville like foour point boundary value problem, Applied Mathematics and Computations, 2010, pp.811-819.

© 2012 Global Journals Inc. (US)

and

$$x(t) = -\frac{x'(1)}{\alpha_2} + \alpha_1(t-\eta)x(\xi) - \int_\eta^t \int_0^s h(\tau)f(\tau, x(\tau), x'(\tau))d\tau \qquad (2.2)$$

Now we differentiate the last equality with respect t and we obtain

$$x'(t) = \alpha_1 x(\xi) - \int_0^t h(\tau)f(\tau, x(\tau), x'(\tau))d\tau, \qquad (2.3)$$

again we differentiate the last equality with respect t and we have

$$x''(t) = -h(t)f(t, x(t), x'(t)).$$

We put $t = 0$ in (2.3) and we obtain

$$x'(0) = \alpha_1 x(\xi),$$

after we put $t = \eta$ in (2.2) we get

$$x(\eta) = -\frac{x'(1)}{\alpha_2},$$

therefore $x(t)$ satisfies the problem (1.1).

The above Lemma motivate us to search fixed points of the operator L_1. For this purpose we will use the following fixed point theorem.

Theorem 2.2. *(see [2], Corrolary 2.4, pp. 3231) Let X be a nonempty closed convex subset of a Banach space Y. Suppose that T and S map X into Y such that*
(i) *S is continuous, $S(X)$ resides in a compact subset of Y;*
(ii) *$T : X \longrightarrow Y$ is expansive and onto.*
Then there exists a point $x^\star \in X$ with $Sx^\star + Tx^\star = x^\star$.

Here we will use the following definition for expansive operator.

Definition. (see [2], pp. 3230) Let (X, d) be a metric space and M be a subset of X. The mapping $T : M \longrightarrow X$ is said to be expansive, if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq hd(x, y) \quad \forall x, y \in M.$$

Lemma 2.3. *The operator $P_1 : N_1 \longrightarrow N_1^*$ is an expansive operator and onto.*

Proof. Let $x(t) \in N_1$. Then $x(t) \in \mathcal{C}^1([0,1])$, $|x(t)| \leq D_1$, $|x'(t)| \leq D_1$, from here $P_1(x) \in \mathcal{C}^1([0,1])$ and $|P_1(x)| \leq (a_1+\epsilon_1)D_1$, $\left|\frac{d}{dt}P_1(x)\right| \leq (a_1+\epsilon_1)D_1$, i.e. $P_1(x) \in N_1^*$ and $P_1 : N_1 \longrightarrow N_1^*$.

Let $x, y \in N_1$. Then

$$||P_1(x) - P_1(y)|| = (a_1+\epsilon_1)||x-y||,$$

consequently $P_1 : N_1 \longrightarrow N_1^*$ is an expansive operator with a constant $h = a_1 + \epsilon_1 > 1$.

Let now $y \in N_1^*$, $y \neq 0$. Then $x = \frac{y}{a_1+\epsilon_1} \in N_1$ and $P_1(x) = y$, then $P_1 : N_1 \longrightarrow N_1^*$ is onto.

Lemma 2.4. *The operator $K_1 : N_1 \longrightarrow N_1$ is a continuous operator.*

Proof. Let $x(t) \in N_1$. Then $K_1(x) \in \mathcal{C}^1([0,1])$ and

$$|K_1(x)| \leq \epsilon_1|x| + (1-a_1)\frac{|x'(1)|}{|\alpha_2|} + (1-a_1)|\alpha_1||x(\xi)| + (1-a_1)\int_\eta^t\int_0^s |h(\tau)||f(\tau, x(\tau), x'(\tau)|d\tau$$

Ref.

[2] Xiang, T., Rong Yuan. A class of expansive - type Krasnosel'skii fixed point theorems. Nonlinear Analysis, 71(2009), 3229-3239.

© 2012 Global Journals Inc. (US)

$$\leq \epsilon_1 D_1 + (1-a_1)\frac{D_1}{|\alpha_2|} + (1-a_1)|\alpha_1|D_1 + (1-a_1)M_1 \leq D_1,$$

in the last inequality we use the first inequality of (2.1), also

$$\left|\frac{d}{dt}K_1(x)\right| \leq \epsilon_1|x'(t)| + (1-a_1)|\alpha_1||x(\xi)| + (1-a_1)\int_0^t |h(\tau)||f(\tau, x(\tau), x'(\tau)|d\tau$$

$$\leq \epsilon_1 D_1 + (1-a_1)|\alpha_1|D_1 + (1-a_1)M_1 \leq D_1,$$

in the last inequality we use the second inequality of (2.1). Therefore

$$K_1 : N_1 \longrightarrow N_1.$$

Since h and f are continuous functions from $x_n \longrightarrow_{n\longrightarrow\infty} x$, $x_n, x \in N_1$, in the sense of the topology of the set N_1 we have $K_1(x_n) \longrightarrow_{n\longrightarrow\infty} K_1(x)$ in the sense of the topology of the set N_1, in other words the operator $K_1 : N_1 \longrightarrow N_1$ is a continuous operator.

From Lemma 2.1, Theorem 2.2, Lemma 2.3 and Lemma 2.4 follows that the operator L_1 has a fixed point $x^1 \in N_1^*$ which is a solution to the problem (1.1). From (2.3), since f and h are continuous functions, follows that $x^1(t) \in \mathcal{C}^2([0,1])$.

2) If we suppose that the bounded solution $x^1(t) \equiv 0$. Then, from (2.2), we have

$$\int_\eta^t \int_0^s h(\tau)f(\tau, x(\tau), x^{1'}(\tau))d\tau ds = 0 \quad \forall t \in [0,1],$$

which is a contradiction.

3) If we suppose that the bounded solution $x(t)$ coincides with a constant, then from the equation of the problem (1.1) we conclude that

$$h(t)f(t, x^1(t), x^{1'}(t)) = 0 \quad \forall t \in [0,1],$$

which is a contradiction.

Acknowledgements

This article is supported by the grant DD-VU 02/90, Bulgaria.

References Références Referencias

[1] Zhao, J., F. Geng, J. Zhao, W. Ge. Positive solutions for a new kind Sturm - Liouville like four point boundary value problem, Applied Mathematics and Computations, 2010, pp.811-819.

[2] Xiang, T., Rong Yuan. A class of expansive - type Krasnosel'skii fixed point theorems. Nonlinear Analysis, 71(2009), 3229-3239.

Notes

© 2012 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics & Decision Sciences
Volume 12 Issue 2 Version 1.0 February 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Global Dynamics of Classical Solutions to a Model of Mixing Flow

By Kun Zhao
Ohio State University, Columbus

Abstract - We study the long-time dynamics of classical solutions to an initial-boundary value problem for modeling equations of a two-component mixture. Time asymptotically, it is shown that classical solutions converge exponentially to constant equilibrium states as time goes to infinity for large initial data, due to diffusion and boundary effects.

Keywords and phrases : Mixing Flow, Classical Solution, Large-Time Asymptotic Behavior.

2000 Mathematics Subject Classication : 35Q35, 35B65, 35B40

GLOBAL DYNAMICS OF CLASSICAL SOLUTIONS TO A MODEL OF MIXING FLOW

Strictly as per the compliance and regulations of :

Open Association of Research Society, USA
Research | Diversity | Ethics

© 2012 . Kun Zhao.This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

eJournal
Print Journal
epaper
open standard
3d virtual journal

Notes

Global Dynamics of Classical Solutions to a Model of Mixing Flow

Kun Zhao

Abstract - We study the long-time dynamics of classical solutions to an initial-boundary value problem for modeling equations of a two-component mixture. Time asymptotically, it is shown that classical solutions converge exponentially to constant equilibrium states as time goes to infinity for large initial data, due to diffusion and boundary effects.

Keywords and phrases : Mixing Flow, Classical Solution, Large-Time Asymptotic Behavior.

I. Introduction

As one of the core questions in mathematical fluid dynamics, the large-time asymptotic behavior of solutions to Cauchy problem or initial-boundary value problems for modeling equations is of central interest to researchers. Not only is the question physically important, it is also mathematically challenging. Positive answer to this question will undoubtedly benefit mathematicians, physicists and engineers. As is well known, the Navier-Stokes equations (NSE) have been one of the most important modeling systems in mathematical fluid dynamics for more than one hundred years. The comprehension of quantitative and qualitative behavior of the NSE plays an important role in understanding core problems in fluid mechanics, such as the onset of turbulence.

In this paper, we consider the following system of equations:

$$\text{(MF)}\quad \begin{cases} (\rho U)_t + \nabla\cdot(\rho U\otimes U) + \nabla P = \nabla\cdot\big(\mu\nabla U - \lambda\rho[(\nabla U)+(\nabla U)^{\mathrm{T}}] + \nabla(\lambda\rho U)\big) + \\ \qquad\qquad\qquad\qquad\qquad\qquad \nabla\big(\nabla\cdot(\lambda\rho U)\big) + \rho\vec{f}, \\ \rho_t + \nabla\cdot(\rho U) = \lambda\Delta\rho, \\ \nabla\cdot U = 0, \end{cases}$$

which describes the motion of an incompressible two-component mixture under the influence of external forces, with a diffusive mass exchange among the medium particles of various density accounted for [2]. Here, ρ is the density of the mixture, $U=(u,v)$ is the mean velocity, the constants $\mu>0$ and $\lambda>0$ model viscous dissipation and mass exchange, respectively, and $\vec{f}$ stands for external forces. For classical solutions, using the second and third equations, (MF) can be simplified to

Author : Mathematical Biosciences Institute, Ohio State University, Columbus, OH 43210. E-mail : kzhao@mbi.ohio-state.edu

© 2012 Global Journals Inc. (US)

$$\begin{cases} \rho(U_t + U\cdot\nabla U) + \nabla P = \lambda\big(\nabla\rho\cdot\nabla U + U\cdot\nabla(\nabla\rho)\big) + \mu\Delta U + \rho\vec{f}, \\ \rho_t + U\cdot\nabla\rho = \lambda\Delta\rho, \\ \nabla\cdot U = 0. \end{cases} \tag{1.1}$$

System (1.1) generalizes the standard density-dependent incompressible Navier-Stokes equations for non-homogeneous fluid flows:

$$\begin{cases} \rho(U_t + U\cdot\nabla U) + \nabla P = \mu\Delta U + \rho\vec{f}, \\ \rho_t + U\cdot\nabla\rho = 0, \\ \nabla\cdot U = 0, \end{cases} \tag{NS}$$

which are important in applied fields of fluid dynamics such as oceanology and hydrology, and have been well-studied. We refer the reader to [2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and references therein for details. It should be pointed out that a characteristic mathematical feature of (1.1) is its non-diagonality in its main part, which significantly distinguishes itself from (NS).

In real world, flows often move in bounded domains with constraints from boundaries, where initial-boundary value problems appear. Solutions to initial-boundary value problems usually exhibit different behaviors and much richer phenomena comparing with the Cauchy problem. In this paper, we consider (1.1) on a bounded domain in $\mathbb{R}^2$, and the system is supplemented by the following initial and boundary conditions:

$$\begin{cases} (U,\rho)(\mathbf{x},0) = (U_0,\rho_0)(\mathbf{x}), \quad m \le \rho_0(\mathbf{x}) \le M; \\ U|_{\partial\Omega} = 0, \quad \nabla\rho\cdot\mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathbf{n}$ is the unit outward normal to $\partial\Omega$ and m, M are positive constants.

It is well-known that classical solutions to (1.1)–(1.2) exist globally (locally resp.) in time in 2D (3D resp.) (c.f. [2]). However, to the best of the author's knowledge, the large-time asymptotic behavior of the solutions is not well-understood in the literature. In particular, the dynamics of the higher order modes of the solutions is not known. The purpose of this paper is to show that, under certain conditions on the external forcing term $\vec{f}$, the constant equilibrium state $(\bar{\rho}, \mathbf{0})$ is a global attractor of (1.1)–(1.2), for large initial data. Additionally, it is shown that the total Sobolev norm of the perturbation $(\rho-\bar{\rho}, U-\mathbf{0})$ up to the highest oder of derivatives converges exponentially in time due to the boundary effects. Here, $\bar{\rho}$ is the spatial average of ρ over Ω, which is a constant due to the conservation of total mass. The proof requires intensive applications of classical inequalities (Sobolev, Gagliardo-Nirenberg type) and tremendous amount of accurate energy estimates.

Throughout this paper, $\|\cdot\|_{L^p}$, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{W^{s,p}}$ denote the norms of the usual Lebesgue measurable function spaces L^p $(1 \le p < \infty)$, L^∞ and the usual Sobolev space $W^{s,p}$, respectively. For $p = 2$, we denote the norm $\|\cdot\|_{L^2}$ by $\|\cdot\|$ and $\|\cdot\|_{W^{s,2}}$ by $\|\cdot\|_{H^s}$. For simplicity, we will use the following notation: $\|(f_1, f_2, ..., f_m)\|_X \equiv \sum_{i=1}^m \|f_i\|_X$. The

Ref.

[5] R. Danchin, Density-dependent incompressible fluids in critical spaces. *Proc. Royal Soc. Edinburgh* **133** (2003): 1311–1334.
[6] R. Danchin, Navier-Stokes equations with variable density. *Hyperbolic Problems and Related Topics, International Press, Graduate Series in Analysis* (2003): 121–135.

© 2012 Global Journals Inc. (US)

function spaces under consideration are $C([0,T];H^3(\Omega))$ and $L^2([0,T];H^4(\Omega))$, equipped with norms $\sup_{0\le t\le T}\|\Psi(\cdot,t)\|_{H^3}$ and $\left(\int_0^T\|\Psi(\cdot,t)\|_{H^4}^2dt\right)^{1/2}$, respectively. Unless specified, c_i will denote generic constants which are independent of ρ, U and t, but may depend on $\Omega, \lambda, \mu, M, m, \rho_0$ and U_0.

Our main results are summarized in the following theorem.

Notes

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with smooth boundary and suppose that the constant $\mu_1 = 2\mu-\lambda(M-m)>0$. Suppose that the external force $\vec{f}$ is a potential flow, i.e., $\vec{f}=\nabla\phi$ for some scalar function $\phi:\Omega\times[0,\infty)\to\mathbb{R}$. Furthermore, suppose that there exists a constant $F_1>0$ independent of $t\ge 0$ such that $\|\vec{f}\|^2_{C([0,t];H^1(\Omega))}+\|\vec{f}\|^2_{L^2([0,t];H^2(\Omega))}+\|\vec{f}_t\|^2_{C([0,t];L^2(\Omega))}\le F_1$ for any $t\ge 0$. If the initial data $(\rho_0(\mathbf{x}),U_0(\mathbf{x}))\in H^3(\Omega)$ are compatible with the boundary conditions, then there exists a unique solution (ρ,U) to (1.1)–(1.2) globally in time such that $(\rho,U)(\mathbf{x},t)\in C([0,T);H^3(\Omega))\cap L^2([0,T);H^4(\Omega))$ for any $T\ge 0$. Moreover, there exist positive constants α,β and γ independent of t such that the solution satisfies*

$$\|(\rho-\bar{\rho},U)(\cdot,t)\|^2_{H^3}\le\alpha e^{-\beta t},\quad\text{and}\quad\int_0^t\|(\rho-\bar{\rho},U)(\cdot,\tau)\|^2_{H^4}d\tau\le\gamma,\quad\forall\, t\ge 0;$$
$$m\le\rho(\mathbf{x},t)\le M,\quad\forall\, t\ge 0,\ \mathbf{x}\in\Omega,$$

where m and M are given in (1.2).

Remark 1.1. *The external forcing term $\vec{f}$ includes important applications such as $\vec{f}=\mathbf{e}_2=(0,1)^{\mathrm{T}}$, which stands for the effect of gravitational force. Physically speaking, the results indicate that, when the viscous dissipation dominates the mass exchange rate, as time goes on, the velocity of the flow will slow down and the mixture tends to be homogeneous.*

Remark 1.2. *The condition on the diffusion coefficients and the upper-lower bounds of the density can be roughly understood by looking at the stress tensor in the momentum equation in (MF), where competition between viscous dissipation and mass exchange occurs.*

Remark 1.3. *One can generalize the results by manipulating on various boundary conditions for ρ and U. For example, one can work on the Dirichlet boundary condition $\rho|_{\partial\Omega}=\tilde{\rho}$, for some constant $m\le\tilde{\rho}\le M$. In this case, the lower and upper bounds of ρ are direct consequences of maximum principle for parabolic equations, and the equilibrium state of ρ is $\tilde{\rho}$. On the other hand, the results may also be generalized to the Navier type slip boundary condition $U\cdot\mathbf{n}|_{\partial\Omega}=0$, $\omega|_{\partial\Omega}=0$, where ω is the 2D vorticity. Since the underlying analysis is in the similar fashion, we shall not go through the details in this paper.*

The main difficulties of the proof of Theorem 1.1 come from the estimation of the higher order derivatives of the solution, due to the coupling between the velocity and density equations by convection, diffusion, external force and boundary effects. With the density function and the additional nonlinear terms $\nabla\rho\cdot\nabla U$ and $U\cdot\nabla(\nabla\rho)$ standing in the velocity equation, the decay of the higher order derivatives of U is an substantial barrier to overcome. Great efforts have been made to simplify the proof. Current proof involves intensive applications of fundamental inequalities, together with exhaustive combinations

© 2012 Global Journals Inc. (US)

of energy inequalities. The results on Stokes equation by Temam [17], see lemma 2.1, are important in our energy framework. Roughly speaking, because of the lack of the spatial derivatives of the solution at the boundary, our energy framework proceeds as follows: We first apply the standard energy estimate on the solution and the temporal derivatives of the solution. We then apply Temam's results on Stokes equation to recover the spatial derivatives. Such a process will be repeated up to the third order, and then the carefully coupled estimates will be composed into a desired one leading to global regularity and exponential decay of the solution. The condition $\vec{f} = \nabla\phi$ is crucial in our analysis due to the fact that, by combining $\bar{\rho}\nabla\phi$ with ∇P, the density perturbation on the right hand side of the velocity equation will be dominated by the diffusion in the density equation, by virtue of Poincaré inequality. This enables us to combine various energy estimates which eventually lead to the exponential decay of the solution. The result suggests that the diffusions are strong enough to compensate the effects of external force and nonlinear convection in order to prevent the development of singularity of the system and to force the solution to converge to the equilibrium state.

The rest of this paper is organized as follows. In Section 2, we give some basic facts that will be used in this paper. We then prove Theorem 1.1 in Section 3.

II. Preliminaries

In this section, we will list several facts which will be used in the proof of Theorem 1.1. First we recall some useful results from [17].

Lemma 2.1. *Let Ω be any open bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Consider the Stokes problem*

$$\begin{cases} -\mu\Delta U + \nabla P = F \quad \text{in } \Omega, \\ \nabla\cdot U = 0 \quad \text{in } \Omega, \\ U = 0 \quad \text{on } \partial\Omega. \end{cases}$$

If $F \in W^{m,p}$, then $U \in W^{m+2,p}$, $P \in W^{m+1,p}$ and there exists a constant $c_1 = c_1(\mu, m, p, \Omega)$ such that

$$\|U\|^2_{W^{m+2,p}} + \|P\|^2_{W^{m+1,p}} \le c_1\|F\|^2_{W^{m,p}}$$

for any $p \in (1,\infty)$ and the integer $m \ge -1$.

The next lemma will be used in the estimation of higher order spatial derivatives of ρ (c.f. [3]).

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be any open bounded domain with smooth boundary $\partial\Omega$, and let $G \in W^{s,p}(\Omega)$ be a vector-valued function satisfying $\nabla\times G = 0$ and $G\cdot\mathbf{n}|_{\partial\Omega} = 0$, where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. Then there exists a constant $c_2 = c_2(s,p,\Omega)$ such that*

$$\|G\|^2_{W^{s,p}} \le c_2(\|\nabla\cdot G\|^2_{W^{s-1,p}} + \|G\|^2_{L^p})$$

for any $s \ge 1$ and $p \in (1,\infty)$.

As a consequence of Poincaré inequality and Lemma 2.2 we have

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^2$ be any open bounded domain with smooth boundary $\partial\Omega$. For any function $H^s(\Omega) \ni f : \Omega \to \mathbb{R}$ satisfying $\nabla f\cdot\mathbf{n}|_{\partial\Omega} = 0$, let $\bar{f} = \frac{1}{|\Omega|}\int_\Omega f d\mathbf{x}$, where the*

Ref.

[17] R. Temam, *Navier-Stokes Equations.* North Holland, 1977.

© 2012 Global Journals Inc. (US)

integer $s \geq 2$. *Then there exists a constant* $c_3 = c_3(\Omega, s)$ *such that*

$$\|f - \bar{f}\|_{H^s}^2 \leq c_3 \|\Delta f\|_{H^{s-2}}^2.$$

We also need the following Sobolev and Ladyzhenskaya type inequalities which are well-known and standard (c.f. [1, 4, 16]).

Lemma 2.4. *Let* $\Omega \subset \mathbb{R}^2$ *be any open bounded domain with smooth boundary* $\partial\Omega$. *Then the following embeddings and inequalities hold:*

$$\text{(i) } \|f\|_{L^p}^2 \leq c_4 \|f\|_{H^1}^2, \quad \forall\ 1 < p < \infty;$$

$$\text{(ii) } \|f\|_{L^\infty}^2 \leq c_5 \|f\|_{W^{1,p}}^2, \quad \forall\ 2 < p < \infty;$$

$$\text{(iii) } \|f\|_{L^4}^2 \leq c_6 \|f\| \|\nabla f\|, \quad \forall\ f \in H_0^1(\Omega);$$

$$\text{(iv) } \|f\|_{L^4}^2 \leq c_7 \big(\|f\| \|\nabla f\| + \|f\|^2\big), \ \forall\ f \in H^1(\Omega),$$

for some constants $c_i = c_i(p, \Omega)$, $i = 4, ..., 7$.

III. LARGE-TIME BEHAVIOR

In this section we prove Theorem 1.1. Since the global existence has been established in [2], we only show the large-time behavior of the solution. The proof is based on several steps of careful energy estimates which are stated as a sequence of lemmas. First of all, the L^∞ estimate of ρ is a direct consequence of the maximum principle:

Lemma 3.1. *Under the assumptions of Theorem 1.1, it holds that*

$$m \leq \rho(\mathbf{x}, t) \leq M, \quad \forall\ t \geq 0,\ \mathbf{x} \in \Omega.$$

a) Reformulation

In order to perform the asymptotic analysis, we first reformulate the original problem (1.1)–(1.2) to get a new one for the perturbation $(\rho - \bar{\rho}, U)$. Letting $\theta = \rho - \bar{\rho}$ and $Q = P - \bar{\rho}\phi$ we have

$$\begin{cases} \rho(U_t + U \cdot \nabla U) + \nabla Q = \lambda\big(\nabla\theta \cdot \nabla U + U \cdot \nabla(\nabla\theta)\big) + \mu\Delta U + \vec{f}\theta, \\ \theta_t + U \cdot \nabla\theta = \lambda\Delta\theta, \\ \nabla \cdot U = 0. \end{cases} \tag{3.1}$$

The initial and boundary conditions turn out to be

$$\begin{cases} (U, \theta)(\mathbf{x}, 0) = (U_0, \theta_0)(\mathbf{x}) \equiv (U_0, \rho_0 - \bar{\rho})(\mathbf{x}); \\ U|_{\partial\Omega} = 0, \quad \nabla\theta \cdot \mathbf{n}|_{\partial\Omega} = 0. \end{cases} \tag{3.2}$$

b) Decay of $\|(U, \theta)\|$

Lemma 3.2. *Under the assumptions of Theorem 1.1, there exist positive constants* α_1, β_1 *and* γ_1 *independent of* t *such that for any* $t \geq 0$ *it holds that*

$$\|(U, \theta)(\cdot, t)\|^2 \leq \alpha_1 e^{-\beta_1 t}, \quad \text{and} \quad \int_0^t \|(U, \theta)(\cdot, \tau)\|_{H^1}^2 d\tau \leq \gamma_1.$$

Ref.

[2] S.N. Antontsev, A.V. Kazhikhov and V.N. Monakhov, *Boundary value problems in mechanics of nonhomogeneous fluids*. North-Holland, 1990.

© 2012 Global Journals Inc. (US)

Proof. The lemma is proved through careful exploration of the structure of the system. First of all, by taking L^2 inner product of $(3.1)_1$ with U we have

$$\int_\Omega \rho\Big(\frac{|U|^2}{2}\Big)_t d\mathbf{x} + \int_\Omega \rho U\cdot\nabla\Big(\frac{|U|^2}{2}\Big)d\mathbf{x} + \mu\int_\Omega |\nabla U|^2 d\mathbf{x}$$
$$=\lambda\int_\Omega \nabla\theta\cdot\nabla\Big(\frac{|U|^2}{2}\Big)d\mathbf{x} + \lambda\int_\Omega \big(U\cdot\nabla(\nabla\theta)\big)\cdot U d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U d\mathbf{x}.$$

After integration by parts and using the incompressibility condition we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|U|^2 d\mathbf{x} - \frac{1}{2}\int_\Omega \theta_t|U|^2 d\mathbf{x} - \frac{1}{2}\int_\Omega \nabla\cdot(\theta U)|U|^2 d\mathbf{x} + \mu\int_\Omega |\nabla U|^2 d\mathbf{x}$$
$$=-\frac{\lambda}{2}\int_\Omega \Delta\theta|U|^2 d\mathbf{x} + \lambda\int_\Omega \big(U\cdot\nabla(\nabla\theta)\big)\cdot U d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U d\mathbf{x}.$$

Using $(3.1)_2$ we simplify the above equation as

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|U|^2 d\mathbf{x} + \mu\int_\Omega |\nabla U|^2 d\mathbf{x} = \lambda\int_\Omega \big[U\cdot\nabla(\nabla\theta)\big]\cdot U d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U d\mathbf{x}. \quad (3.3)$$

For the first term on the RHS of (3.3), by direct calculations we have

$$\big[U\cdot\nabla(\nabla\theta)\big]\cdot U = \nabla\cdot\big[U(U\cdot\nabla\theta) - (\theta U\cdot\nabla U)\big] + \theta(u_x^2 + 2u_y v_x + v_y^2). \quad (3.4)$$

Therefore, integrating (3.4) over Ω using the boundary condition we get

$$\int_\Omega \big[U\cdot\nabla(\nabla\theta)\big]\cdot U d\mathbf{x} = \int_\Omega \theta(u_x^2 + 2u_y v_x + v_y^2)d\mathbf{x}. \quad (3.5)$$

Using (3.5) we update (3.3) as

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U\|^2 + \mu\|\nabla U\|^2 = \lambda\int_\Omega \theta(u_x^2 + 2u_y v_x + v_y^2)d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U d\mathbf{x}. \quad (3.6)$$

Since $\nabla\cdot U = 0$, we have

$$u_x^2 + 2u_y v_x + v_y^2 = \nabla\cdot(U\cdot\nabla U) - U\cdot\nabla(\nabla\cdot U) = \nabla\cdot(U\cdot\nabla U),$$

which implies that

$$\int_\Omega (u_x^2 + 2u_y v_x + v_y^2)d\mathbf{x} = 0.$$

Since $\bar{\rho}$ is a constant, it follows from (3.6) and the above identity that

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U\|^2 + \mu\|\nabla U\|^2 = \lambda\int_\Omega \Big(\rho - \frac{M+m}{2}\Big)(u_x^2 + 2u_y v_x + v_y^2)d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U d\mathbf{x}. \quad (3.7)$$

Using Lemma 3.1 we estimate the RHS of (3.7) as follows:

$$\Big|\lambda\int_\Omega \Big(\rho - \frac{M+m}{2}\Big)(u_x^2 + 2u_y v_x + v_y^2)d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U d\mathbf{x}\Big|$$
$$\le\lambda\frac{M-m}{2}\|\nabla U\|^2 + \int_\Omega |\theta\vec{f}\cdot U|d\mathbf{x}.$$

Notes

© 2012 Global Journals Inc. (US)

We remark that the coefficient of $\|\nabla U\|^2$ on the RHS of the above estimate is optimal. So we update (3.7) as

$$\frac{d}{dt}\|\sqrt{\rho}U\|^2 + \mu_1\|\nabla U\|^2 \le 2\int_\Omega |\theta\vec{f}\cdot U|d\mathbf{x}, \tag{3.8}$$

where $\mu_1 = 2\mu - \lambda(M-m) > 0$. Using Cauchy-Schwarz and Poincaré inequalities we estimate the RHS of (3.8) as:

Notes

$$\begin{aligned} 2\int_\Omega |\theta\vec{f}\cdot U|d\mathbf{x} &\le \frac{\mu_1}{2c_0}\|U\|^2 + \frac{2c_0}{\mu_1}\|\vec{f}\theta\|^2 \\ &\le \frac{\mu_1}{2}\|\nabla U\|^2 + \frac{2c_0}{\mu_1}\|\vec{f}\theta\|^2. \end{aligned} \tag{3.9}$$

Since $\|\vec{f}\|^2_{C([0,t];H^1(\Omega))} \le F_1$, by Lemma 2.4 (i) we have

$$\begin{aligned} \frac{c_0}{2\mu_1}\|\vec{f}\theta\|^2 &\le \frac{c_0}{2\mu_1}\|\vec{f}\|^2_{L^4}\|\theta\|^2_{L^4} \\ &\le \frac{c_0c_4^2}{2\mu_1}\|\vec{f}\|^2_{H^1}\|\theta\|^2_{H^1} \\ &\le \frac{c_0c_4^2F_1}{2\mu_1}(1+c_0)\|\nabla\theta\|^2. \end{aligned} \tag{3.10}$$

Let $c_8 = c_0c_4^2F_1(1+c_0)/(2\mu_1)$. Combining (3.8)–(3.10) we have

$$\frac{d}{dt}\|\sqrt{\rho}U\|^2 + \frac{\mu_1}{2}\|\nabla U\|^2 \le c_8\|\nabla\theta\|^2. \tag{3.11}$$

The RHS of (3.11) will be compensated by the diffusion in the temperature equation. Taking L^2 inner product of $(3.1)_2$ with θ we have

$$\frac{d}{dt}\|\theta\|^2 + 2\lambda\|\nabla\theta\|^2 = 0. \tag{3.12}$$

Then the operation $(3.12)\times c_8/\lambda + (3.11)$ yields

$$\frac{d}{dt}\left(\frac{c_8}{\lambda}\|\theta\|^2 + \|\sqrt{\rho}U\|^2\right) + c_8\|\nabla\theta\|^2 + \frac{\mu_1}{2}\|\nabla U\|^2 \le 0. \tag{3.13}$$

Since $\rho \le M$, we have

$$\|\sqrt{\rho}U\|^2 \le M\|U\|^2 \le c_0M\|\nabla U\|^2.$$

It follows from (3.13) that

$$\frac{d}{dt}\left(\frac{c_8}{\lambda}\|\theta\|^2 + \|\sqrt{\rho}U\|^2\right) + \beta_1\left(\frac{c_8}{\lambda}\|\theta\|^2 + \|\sqrt{\rho}U\|^2\right) \le 0, \tag{3.14}$$

where

$$\beta_1 = \min\left\{\frac{\lambda}{c_0}, \frac{\mu_1}{2c_0M}\right\}. \tag{3.15}$$

Solving the differential inequality (3.14) we have

$$\left(\frac{c_8}{\lambda}\|\theta\|^2 + \|\sqrt{\rho}U\|^2\right) \le \left(\frac{c_8}{\lambda}\|\theta_0\|^2 + \|\sqrt{\rho_0}U_0\|^2\right)e^{-\beta_1 t}. \tag{3.16}$$

© 2012 Global Journals Inc. (US)

Since $\rho \geq m$, we get from (3.16) that

$$\|(U,\theta)(\cdot,t)\|^2 \leq \alpha_1 e^{-\beta_1 t}, \quad \forall\, t \geq 0, \tag{3.17}$$

where

$$\alpha_1 = \left(\min\left\{\frac{c_8}{\lambda}, m\right\}\right)^{-1}\left(\frac{c_8}{\lambda}\|\theta_0\|^2 + \|\sqrt{\rho_0}U_0\|^2\right). \tag{3.18}$$

Next, upon integrating (3.13) in time and dropping the positive term from the LHS we have

$$\int_0^t c_8\|\nabla\theta(\cdot,\tau)\|^2 + \frac{\mu_1}{2}\|\nabla U(\cdot,\tau)\|^2 d\tau \leq \frac{c_8}{\lambda}\|\theta_0\|^2 + \|\sqrt{\rho_0}U_0\|^2, \quad \forall\, t \geq 0,$$

which, together with (3.17), yields

$$\int_0^t \|(U,\theta)(\cdot,\tau)\|_{H^1}^2 d\tau \leq \gamma_1, \quad \forall\, t \geq 0, \tag{3.19}$$

where

$$\gamma_1 = \frac{\alpha_1}{\beta_1} + \left(\frac{c_8}{\lambda}\|\theta_0\|^2 + \|\sqrt{\rho_0}U_0\|^2\right)\left(\min\{c_8, \mu_1/2\}\right)^{-1}. \tag{3.20}$$

This completes the proof.

Remark 3.1. *The idea of the above proof will be appplied to prove the exponential decay of higher order derivatives of the solution. From (3.15) we see clearly that, the decay rate β_1 tends to zero, as either λ or $\mu_1 = 2\mu - \lambda(M-m)$ tends to zero. Furthermore, by (3.18) we have $\alpha_1 \to \infty$, as $\lambda \to 0$ or $\mu_1 \to 0$. Therefore, as the value of λ either decreases or approaches the threshold value $\frac{2\mu}{M-m}$, the decay of the solution will slow down. By direct calculation we know that the decay rate reaches its maximum when $\lambda = \frac{2\mu}{3M-m}$.*

Remark 3.2. *In what follows, since tremendous amount of combinations of energy estimates will be involved when we deal with the decay of higher order derivatives of the solution, the expressions of the constants appearing in the proofs will become lengthy and hard to read. Therefore, to simplify the presentation, we shall not specify $c_i, \alpha_i, \beta_i, \gamma_i$ in terms of the other time-independent constants.*

c) Decay of $\|\theta\|_{H^1}$

Lemma 3.3. *Under the assumptions of Theorem 1.1, there exist positive constants α_2, β_2 and γ_2 independent of t such that for any $t \geq 0$ it holds that*

$$\|\nabla\theta(\cdot,t)\|^2 \leq \alpha_2 e^{-\beta_2 t}, \quad \text{and} \quad \int_0^t \|\Delta\theta(\cdot,\tau)\|^2 + \|\theta_t(\cdot,\tau)\|^2 d\tau \leq \gamma_2.$$

Proof. Taking L^2 inner product of $(3.1)_2$ with $\Delta\theta$ we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2 + \lambda\|\Delta\theta\|^2 = \int_\Omega (U\cdot\nabla\theta)\Delta\theta \, d\mathbf{x}. \tag{3.21}$$

Using Cauchy-Schwarz and Hölder inequalities we estimate the RHS of (3.21) as

$$\begin{aligned}\left|\int_\Omega (U\cdot\nabla\theta)\Delta\theta d\mathbf{x}\right| &\leq \frac{1}{\lambda}\|U\cdot\nabla\theta\|^2 + \frac{\lambda}{4}\|\Delta\theta\|^2 \\ &\leq \frac{1}{\lambda}\|U\|_{L^4}^2\|\nabla\theta\|_{L^4}^2 + \frac{\lambda}{4}\|\Delta\theta\|^2.\end{aligned}$$

Notes

© 2012 Global Journals Inc. (US)

So we update (3.21) as

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2 + \frac{3}{4}\lambda\|\Delta\theta\|^2 \le \frac{1}{\lambda}\|U\|_{L^4}^2\|\nabla\theta\|_{L^4}^2. \tag{3.22}$$

Applying Lemma 2.4 (iii) to the RHS of (3.22) we have

$$\frac{1}{\lambda}\|U\|_{L^4}^2\|\nabla\theta\|_{L^4}^2 \le c_9\|U\|\|\nabla U\|\|\nabla\theta\|\|D^2\theta\| + c_9\|U\|\|\nabla U\|\|\nabla\theta\|^2. \tag{3.23}$$

For the first term on the RHS of (3.23), using Lemma 2.3 for $\|D^2\theta\|^2$ and Lemma 3.2 for $\|U\|^2$ we have

$$\begin{aligned} c_9\|U\|\|\nabla U\|\|\nabla\theta\|\|D^2\theta\| &\le c_{10}\|\nabla U\|\|\nabla\theta\|\|\Delta\theta\| \\ &\le c_{11}\|\nabla U\|^2\|\nabla\theta\|^2 + \frac{\lambda}{4}\|\Delta\theta\|^2. \end{aligned} \tag{3.24}$$

Applying Poincaré inequality to the second term on the RHS of (3.23) we have

$$c_9\|U\|\|\nabla U\|\|\nabla\theta\|^2 \le c_{12}\|\nabla U\|^2\|\nabla\theta\|^2. \tag{3.25}$$

Combining (3.23)–(3.25) we have

$$\frac{1}{\lambda}\|U\|_{L^4}^2\|\nabla\theta\|_{L^4}^2 \le c_{13}\|\nabla U\|^2\|\nabla\theta\|^2 + \frac{\lambda}{4}\|\Delta\theta\|^2. \tag{3.26}$$

Plugging (3.26) into (3.22) we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2 + \frac{\lambda}{2}\|\Delta\theta\|^2 \le c_{13}\|\nabla U\|^2\|\nabla\theta\|^2. \tag{3.27}$$

Gronwall inequality and Lemma 3.2 then yield (by dropping $\frac{\lambda}{2}\|\Delta\theta\|^2$)

$$\|\nabla\theta(\cdot,t)\|^2 \le \exp\left\{2c_{13}\int_0^t \|\nabla U\|^2 d\tau\right\}\|\nabla\theta_0\|^2 \le e^{2c_{13}\gamma_1}\|\nabla\theta_0\|^2 \equiv c_{14}. \tag{3.28}$$

Plugging (3.28) into (3.27) we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2 + \frac{\lambda}{2}\|\Delta\theta\|^2 \le c_{15}\|\nabla U\|^2. \tag{3.29}$$

To deal with the RHS of (3.29), we consider the estimate (3.13). The combination (3.13) $\times \frac{4c_{15}}{\mu_1}$ + (3.29) gives

$$\frac{d}{dt}\big(E_1(t)\big) + \frac{4c_8c_{15}}{\mu_1}\|\nabla\theta\|^2 + c_{15}\|\nabla U\|^2 + \frac{\lambda}{2}\|\Delta\theta\|^2 \le 0, \tag{3.30}$$

where

$$E_1(t) = \frac{4c_{15}}{\mu_1}\left(\frac{c_8}{\lambda}\|\theta\|^2 + \|\sqrt{\rho}U\|^2\right) + \frac{1}{2}\|\nabla\theta\|^2. \tag{3.31}$$

Using Poincaré inequality one easily checks that there exists a constant $\beta_2 > 0$ independent of t such that

$$\beta_2 E_1(t) \le \left(\frac{4c_8c_{15}}{\mu_1}\|\nabla\theta\|^2 + c_{15}\|\nabla U\|^2\right), \tag{3.32}$$

Using (3.32) we update (3.30) as

$$\frac{d}{dt}\big(E_1(t)\big) + \beta_2 E_1(t) + \frac{\lambda}{2}\|\Delta\theta\|^2 \le 0, \tag{3.33}$$

Notes

© 2012 Global Journals Inc. (US)

which implies that

$$E_1(t) \le e^{-\beta_2 t} E_1(0), \quad \text{and} \quad \frac{\lambda}{2}\int_0^t \|\Delta\theta(\cdot,\tau)\|^2 d\tau \le E_1(0), \quad \forall\, t \ge 0. \tag{3.34}$$

By (3.31) and (3.34) we see that

$$\|\nabla\theta(\cdot,t)\|^2 \le \alpha_2 e^{-\beta_2 t}, \quad \text{and} \quad \int_0^t \|\Delta\theta(\cdot,\tau)\|^2 d\tau \le 2E_1(0)/\lambda, \quad \forall\, t \ge 0, \tag{3.35}$$

where $\alpha_2 = 2E_1(0)$.

To estimate θ_t, we consider $(3.1)_2$. Using (3.26) and (3.35) we have

$$\begin{aligned} \|\theta_t\|^2 &\le 2\|U\cdot\nabla\theta\|^2 + 2\|\lambda\Delta\theta\|^2 \\ &\le 2\|U\|_{L^4}^2\|\nabla\theta\|_{L^4}^2 + 2\lambda^2\|\Delta\theta\|^2 \\ &\le c_{16}\big(\|\Delta\theta\|^2 + \|\nabla U\|^2\|\nabla\theta\|^2\big) + 2\lambda^2\|\Delta\theta\|^2 \\ &\le c_{17}\big(\|\Delta\theta\|^2 + \|\nabla U\|^2\big). \end{aligned} \tag{3.36}$$

Integrating (3.36) in time over $[0,t]$ and using Lemma 3.2 and (3.35) we have

$$\int_0^t \|\theta_t(\cdot,\tau)\|^2 d\tau \le c_{18}, \quad \forall\, t \ge 0. \tag{3.37}$$

We conclude the proof by combining (3.35) and (3.37).

d) Estimate of $\|U\|_{H^2}$

Now we turn to higher order estimates of the solution. The next lemma gives the control of $\|U\|_{H^2}$ by $\|\nabla U\|$, $\|U_t\|$ and estimates of θ. The proof involves intensive applications of Sobolev and Ladyzhenskaya type inequalities.

Lemma 3.4. *Under the assumptions of Theorem 1.1, for any positive numbers ε and δ, there exists a constant $d(\varepsilon,\delta)$ independent of t and dependent on ε and δ such that*

$$\|U\|_{H^2}^2 \le \delta\|\nabla\theta_t\|^2 + \varepsilon\|U\|_{H^2}^2 + d(\varepsilon,\delta)\big(\|\nabla U\|^2\|\theta\|_{H^2}^2 + \|\nabla U\|^4 + \|U_t\|^2 + \|\theta\|_{H^1}^2\big).$$

Proof. We rewrite the velocity equation $(3.1)_1$ as the 2D Stokes equation:

$$-\mu\Delta U + \nabla P = \vec{F},$$

where

$$\vec{F} = -\rho U_t - \rho U\cdot\nabla U + \lambda\nabla\theta\cdot\nabla U + \lambda U\cdot\nabla(\nabla\theta) + \vec{f}\theta \equiv \sum_{i=1}^{5} F_i.$$

Since $U|_{\partial\Omega} = 0$, it follows from Lemma 2.1 that

$$\|U\|_{H^2}^2 \le 16c_1 \sum_{i=1}^{5}\|F_i\|^2. \tag{3.38}$$

Now we estimate the summand on the RHS of (3.38) as follows: Using Lemma 3.1 we have

$$\|F_1\|^2 = \|\rho U_t\|^2 \le M^2\|U_t\|^2. \tag{3.39}$$

Notes

© 2012 Global Journals Inc. (US)

Notes

Using Lemma 2.4 (iii), Lemma 3.1 and Lemma 3.3, we have, for any $\varepsilon > 0$:

$$\begin{aligned} \|F_2\|^2 &= \|\rho U \cdot \nabla U\|^2 \\ &\le M^2 \|U\|_{L^4}^2 \|\nabla U\|_{L^4}^2 \\ &\le c_{19} \|U\| \|\nabla U\| \big(\|\nabla U\| \|D^2 U\| + \|\nabla U\|^2 \big) \\ &\le c_{20} \|\nabla U\|^2 \|U\|_{H^2} \\ &\le \frac{c_{21}}{\varepsilon} \|\nabla U\|^4 + \frac{\varepsilon}{48 c_1} \|U\|_{H^2}^2. \end{aligned} \tag{3.40}$$

For F_3, it follows from Lemma 3.3 that

$$\begin{aligned} \|F_3\|^2 &= \lambda^2 \|\nabla\theta \cdot \nabla U\|^2 \\ &\le c_{22} \big(\|\nabla\theta\| \|D^2\theta\| + \|\nabla\theta\|^2 \big) \big(\|\nabla U\| \|D^2 U\| + \|\nabla U\|^2 \big) \\ &\le c_{23} \big(\|D^2\theta\| + \|\nabla\theta\| \big) \big(\|D^2 U\| + \|\nabla U\| \big) \|\nabla U\| \\ &\le \frac{c_{24}}{\varepsilon} \|\theta\|_{H^2}^2 \|\nabla U\|^2 + \frac{\varepsilon}{48 c_1} \|U\|_{H^2}^2. \end{aligned} \tag{3.41}$$

For the estimate of F_4, by Lemma 2.3 and Lemma 2.4 we have

$$\begin{aligned} \|F_4\|^2 &= \lambda^2 \|U \cdot \nabla(\nabla\theta)\|^2 \\ &\le c_{25} \|U\| \|\nabla U\| \|D^2\theta\| \big(\|D^3\theta\| + \|D^2\theta\| \big). \end{aligned} \tag{3.42}$$

To estimate $\|D^3\theta\|$, by Lemma 2.2 we have

$$\begin{aligned} \|D^3\theta\| &\le \sqrt{c_3} \|\Delta\theta\|_{H^1} \\ &\le c_{26} \big(\|\nabla\theta_t\| + \|\nabla(U \cdot \nabla\theta)\| + \|\Delta\theta\| \big) \\ &\le c_{27} \big(\|\nabla\theta_t\| + \|\nabla U \cdot (\nabla\theta)^T\| + \|U \cdot \nabla(\nabla\theta)\| + \|\Delta\theta\| \big). \end{aligned} \tag{3.43}$$

Plugging (3.43) into (3.42) we have

$$\begin{aligned} &\lambda^2 \|U \cdot \nabla(\nabla\theta)\|^2 \\ \le & c_{28} \|U\| \|\nabla U\| \|D^2\theta\| \big(\|\nabla\theta_t\| + \|\nabla U \cdot (\nabla\theta)^T\| + \|U \cdot \nabla(\nabla\theta)\| + \|\theta\|_{H^2} \big). \end{aligned} \tag{3.44}$$

Using Lemma 3.2 and Poincaré inequality we estimate the RHS of (3.44) as follows:

$$\begin{aligned} & c_{28} \|U\| \|\nabla U\| \|D^2\theta\| \big(\|\nabla\theta_t\| + \|\nabla U \cdot (\nabla\theta)^T\| + \|U \cdot \nabla(\nabla\theta)\| + \|\theta\|_{H^2} \big) \\ \le & c_{29} \|\nabla U\| \|\theta\|_{H^2} \big(\|\nabla\theta_t\| + \|\nabla U \cdot (\nabla\theta)^T\| + \|U \cdot \nabla(\nabla\theta)\| \big) + c_{30} \|\nabla U\|^2 \|\theta\|_{H^2}^2 \\ \le & \frac{\delta}{32 c_1} \|\nabla\theta_t\|^2 + \frac{\lambda^2}{2} \|U \cdot \nabla(\nabla\theta)\|^2 + \frac{c_{31}(\delta)}{2\delta} \|\nabla U\|^2 \|\theta\|_{H^2}^2 + \frac{1}{2} \|\nabla U \cdot (\nabla\theta)^T\|^2. \end{aligned}$$

Combining the preceding estimate with (3.44) we have

$$\|F_4\|^2 \le \frac{\delta}{16 c_1} \|\nabla\theta_t\|^2 + \frac{c_{31}(\delta)}{\delta} \|\nabla U\|^2 \|\theta\|_{H^2}^2 + \|\nabla U \cdot (\nabla\theta)^T\|^2. \tag{3.45}$$

In a similar fashion as deriving (3.41) we have

$$\|\nabla U \cdot (\nabla\theta)^T\|^2 \le \frac{c_{32}}{\varepsilon} \|\nabla U\|^2 \|\theta\|_{H^2}^2 + \frac{\varepsilon}{48 c_1} \|U\|_{H^2}^2,$$

© 2012 Global Journals Inc. (US)

which, together with (3.45), yields

$$\|F_4\|^2 \leq \frac{\delta}{16c_1}\|\nabla\theta_t\|^2 + \frac{\varepsilon}{48c_1}\|U\|_{H^2}^2 + \left(\frac{c_{31}(\delta)}{\delta} + \frac{c_{32}}{\varepsilon}\right)\|\nabla U\|^2\|\theta\|_{H^2}^2. \quad (3.46)$$

Finally, using Lemma 2.4 (i) and the condition on $\vec{f}$ we have

$$\|F_5\|^2 = \|\vec{f}\theta\|^2 \leq \|\vec{f}\|_{L^4}^2\|\theta\|_{L^4}^2 \leq c_4^2 F_1\|\theta\|_{H^1}^2. \quad (3.47)$$

Collecting (3.39)–(3.41) and (3.46)–(3.47) and using (3.38) we complete the proof.

Notes

e) Decay of $\|U\|_{H^1}$

With the help of Lemma 3.4 we show the decay of $\|\nabla U\|$ and $\|\theta_t\|$.

Lemma 3.5. *Under the assumptions of Theorem 1.1, there exist positive constants* α_3, β_3 *and* γ_3 *independent of* t *such that for any* $t \geq 0$ *it holds that*

$$\|(\nabla U, \theta_t)(\cdot, t)\|^2 \leq \alpha_3 e^{-\beta_3 t}, \quad \text{and} \quad \int_0^t \|(\nabla\theta_t, U_t)(\cdot, \tau)\|^2 d\tau \leq \gamma_3.$$

Proof. Taking L^2 inner product of $(3.1)_1$ with U_t we have

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla U\|^2 + \int_\Omega \rho|U_t|^2 d\mathbf{x} = -\int_\Omega \rho(U\cdot\nabla U)U_t d\mathbf{x} + \lambda\int_\Omega \big[\nabla\theta\cdot\nabla U + U\cdot\nabla(\nabla\theta)\big]U_t d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U_t d\mathbf{x}. \quad (3.48)$$

We estimate the RHS of (3.48) as follows: By Cauchy-Schwarz inequality we have

$$\left| -\int_\Omega \rho(U\cdot\nabla U)U_t d\mathbf{x} + \lambda\int_\Omega \big[\nabla\theta\cdot\nabla U + U\cdot\nabla(\nabla\theta)\big]U_t d\mathbf{x} + \int_\Omega \theta\vec{f}\cdot U_t d\mathbf{x}\right|$$
$$\leq \frac{m}{8}\|U_t\|^2 + \frac{2}{m}\big\|\big(\rho U\cdot\nabla U + \lambda\nabla\theta\cdot\nabla U + \lambda U\cdot\nabla(\nabla\theta) + \vec{f}\theta\big)\big\|^2. \quad (3.49)$$

For the second term on the RHS of (3.49), it follows from the proof of Lemma 3.4 that

$$\frac{2}{m}\big\|\big(\rho U\cdot\nabla U + \lambda\nabla\theta\cdot\nabla U + \lambda U\cdot\nabla(\nabla\theta) + \vec{f}\theta\big)\big\|^2$$
$$\leq \frac{\lambda}{8}\|\nabla\theta_t\|^2 + \varepsilon_1\|U\|_{H^2}^2 + c_{33}(\varepsilon_1)\big(\|\nabla U\|^2\|\theta\|_{H^2}^2 + \|\nabla U\|^4 + \|\theta\|_{H^1}^2\big),$$

where $\varepsilon_1 > 0$ is a constant to be determined. So we update (3.48) as

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla U\|^2 + \int_\Omega \rho|U_t|^2 d\mathbf{x} \leq \frac{m}{8}\|U_t\|^2 + \frac{\lambda}{8}\|\nabla\theta_t\|^2 + \varepsilon_1\|U\|_{H^2}^2 + c_{33}(\varepsilon_1)\big(\|\nabla U\|^2\|\theta\|_{H^2}^2 + \|\nabla U\|^4 + \|\theta\|_{H^1}^2\big). \quad (3.50)$$

Letting $\varepsilon = 1/2$ and $\delta = 1$ in Lemma 3.4 we have

$$\|U\|_{H^2}^2 \leq c_{34}\big(\|\nabla U\|^2\|\theta\|_{H^2}^2 + \|\nabla U\|^4 + \|U_t\|^2 + \|\theta\|_{H^1}^2 + \|\nabla\theta_t\|^2\big). \quad (3.51)$$

© 2012 Global Journals Inc. (US)

Notes

Plugging (3.51) into (3.50) we have

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla U\|^2+\int_\Omega \rho|U_t|^2 d\mathbf{x} \le \frac{m}{8}\|U_t\|^2+\frac{\lambda}{8}\|\nabla\theta_t\|^2+\varepsilon_1 c_{34}\big(\|U_t\|^2+\|\nabla\theta_t\|^2\big)$$
$$+\big(c_{33}(\varepsilon_1)+c_{34}\big)\big(\|\nabla U\|^2\|\theta\|_{H^2}^2+\|\nabla U\|^4+\|\theta\|_{H^1}^2\big).$$

Choosing $\varepsilon_1=\min\{m/(8c_{34}),\lambda/(8c_{34})\}$ and using the fact that $\rho\ge m$ we have

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla U\|^2+\frac{3m}{4}\|U_t\|^2 \le \frac{\lambda}{4}\|\nabla\theta_t\|^2+c_{35}\big(\|\nabla U\|^2\|\theta\|_{H^2}^2+\|\nabla U\|^4+\|\theta\|_{H^1}^2\big). \quad (3.52)$$

Next, by taking the temporal derivative of $(3.1)_2$ we have

$$\theta_{tt}+U_t\cdot\nabla\theta+U\cdot\nabla\theta_t=\lambda\Delta\theta_t. \quad (3.53)$$

Taking L^2 inner product of (3.53) with θ_t we have

$$\frac{1}{2}\frac{d}{dt}\|\theta_t\|^2+\lambda\|\nabla\theta_t\|^2=-\int_\Omega (U_t\cdot\nabla\theta)\theta_t d\mathbf{x}. \quad (3.54)$$

Using Cauchy-Schwarz inequality we have

$$\left|-\int_\Omega (U_t\cdot\nabla\theta)\theta_t d\mathbf{x}\right| \le \frac{m}{4}\|U_t\|^2+\frac{1}{m}\|(\nabla\theta)\theta_t\|^2$$
$$\le \frac{m}{4}\|U_t\|^2+\frac{1}{m}\|\nabla\theta\|_{L^4}^2\|\theta_t\|_{L^4}^2. \quad (3.55)$$

For the RHS of (3.55), by Lemma 2.4 (iii) and Lemma 3.3 we have

$$\frac{1}{m}\|\nabla\theta\|_{L^4}^2\|\theta_t\|_{L^4}^2 \le c_{36}(\|\nabla\theta\|\|D^2\theta\|+\|\nabla\theta\|^2)(\|\theta_t\|\|\nabla\theta_t\|+\|\theta_t\|^2)$$
$$\le c_{37}\big(\|D^2\theta\|+\|\nabla\theta\|\big)\|\theta_t\|\|\nabla\theta_t\|+c_{36}\|\theta\|_{H^2}^2\|\theta_t\|^2 \quad (3.56)$$
$$\le \frac{\lambda}{4}\|\nabla\theta_t\|^2+c_{38}\|\theta\|_{H^2}^2\|\theta_t\|^2.$$

Combining (3.54)–(3.56) we have

$$\frac{1}{2}\frac{d}{dt}\|\theta_t\|^2+\frac{3\lambda}{4}\|\nabla\theta_t\|^2 \le \frac{m}{4}\|U_t\|^2++c_{38}\|\theta\|_{H^2}^2\|\theta_t\|^2. \quad (3.57)$$

Coupling (3.52) and (3.57) we have

$$\frac{d}{dt}\big(\mu\|\nabla U\|^2+\|\theta_t\|^2\big)+m\|U_t\|^2+\lambda\|\nabla\theta_t\|^2$$
$$\le c_{39}\big(\|\nabla U\|^2\|\theta\|_{H^2}^2+\|\nabla U\|^4+\|\theta\|_{H^1}^2+\|\theta\|_{H^2}^2\|\theta_t\|^2\big) \quad (3.58)$$
$$\le c_{40}\big(\|\theta\|_{H^2}^2+\|\nabla U\|^2\big)\big(\mu\|\nabla U\|^2+\|\theta_t\|^2\big)+c_{39}\|\theta\|_{H^1}^2.$$

Applying Gronwall inequality to (3.58) and using Lemma 3.2 and Lemma 3.3 we have

$$\mu\|\nabla U\|^2+\|\theta_t\|^2\le c_{41}, \quad \text{and} \quad \int_0^t m\|U_t\|^2+\lambda\|\nabla\theta_t\|^2 d\tau \le c_{42}. \quad (3.59)$$

Plugging the first part of (3.59) into (3.58) we have

$$\frac{d}{dt}\big(\mu\|\nabla U\|^2+\|\theta_t\|^2\big)+m\|U_t\|^2+\lambda\|\nabla\theta_t\|^2 \le c_{43}\big(\|\theta\|_{H^2}^2+\|\nabla U\|^2\big). \quad (3.60)$$

© 2012 Global Journals Inc. (US)

To show the exponential decay of $\|\nabla U\|$ and $\|\theta_t\|$, we consider the estimate (3.30). By absorbing the RHS of (3.60) into the LHS of (3.30) we have

$$\frac{d}{dt}\big(E_2(t)\big) + c_{44}D_2(t) \le 0, \tag{3.61}$$

for some constant $c_{44} > 0$ independent of t, where, by virtue of Poincaré inequality,

$$E_2(t) \cong \|(U,\theta)(\cdot,t)\|_{H^1}^2 + \|\theta_t(\cdot,t)\|^2,$$
$$D_2(t) \cong \|(U,\theta_t)(\cdot,t)\|_{H^1}^2 + \|\theta(\cdot,t)\|_{H^2}^2 + \|U_t(\cdot,t)\|^2.$$

Here $\cong$ denotes the equivalence of quantities. Then the lemma follows immediately from (3.61) and (3.59). This completes the proof.

f) Decay of $\|\theta\|_{H^2}$

Lemma 3.6. *Under the assumptions of Theorem 1.1, there exist constants $\alpha_4, \beta_4, \gamma_4 > 0$ independent of t such that for any $t \ge 0$ it holds that*

$$\|\theta(\cdot,t)\|_{H^2}^2 \le \alpha_4 e^{-\beta_4 t}, \quad \text{and} \quad \int_0^t \|U(\cdot,\tau)\|_{H^2}^2 d\tau \le \gamma_4.$$

Proof. We note that, by Lemma 2.3, Lemma 2.4 and Lemma 3.5 it holds that

$$\begin{aligned}\|\theta\|_{H^2}^2 \le c_3\|\Delta\theta\|^2 &\le c_{45}\big(\|\theta_t\|^2 + \|U\cdot\nabla\theta\|^2\big)\\ &\le c_{46}\big(\|\theta_t\|^2 + \|U\|_{H^1}^2(\|\nabla\theta\|\|\theta\|_{H^2} + \|\nabla\theta\|^2)\big)\\ &\le c_{47}\big(\|\theta_t\|^2 + \|\nabla\theta\|^2\big) + \frac{1}{2}\|\theta\|_{H^2}^2,\end{aligned}$$

which implies that

$$\|\theta\|_{H^2}^2 \le c_{48}\big(\|\theta_t\|^2 + \|\nabla\theta\|^2\big). \tag{3.62}$$

Then the exponential decay of $\|\theta\|_{H^2}^2$ follows from Lemma 3.3 and Lemma 3.5.

Next, by (3.51) and Lemma 3.5 we have

$$\begin{aligned}\|U\|_{H^2}^2 &\le c_{34}\big(\|\nabla U\|^2\|\theta\|_{H^2}^2 + \|\nabla U\|^4 + \|U_t\|^2 + \|\theta\|_{H^1}^2 + \|\nabla\theta_t\|^2\big)\\ &\le c_{49}\big(\|\theta\|_{H^2}^2 + \|\nabla U\|^2 + \|U_t\|^2 + \|\nabla\theta_t\|^2\big),\end{aligned} \tag{3.63}$$

which, together with Lemmas 3.2, 3.3 and 3.5, implies that

$$\int_0^t \|U(\cdot,\tau)\|_{H^2}^2 d\tau \le c_{50}.$$

This completes the proof.

g) Decay of $\|\theta\|_{H^3}$ and $\|U\|_{H^2}$

Lemma 3.7. *Under the assumptions of Theorem 1.1, there exist positive constants α_5, β_5 and γ_5 independent of t such that for any $t \ge 0$ it holds that*

$$\|U(\cdot,t)\|_{H^2}^2 + \|(\nabla\theta_t, U_t)(\cdot,t)\|^2 \le \alpha_5 e^{-\beta_5 t}, \quad \text{and} \quad \int_0^t \|(\nabla U_t, \Delta\theta_t)(\cdot,\tau)\|_{H^2}^2 d\tau \le \gamma_5.$$

Proof. Taking the temporal derivative of $(3.1)_1$ we have

$$\theta_t(U_t + U\cdot\nabla U) + \rho(U_{tt} + U_t\cdot\nabla U + U\cdot\nabla U_t) + \nabla P_t \tag{3.64}$$

Notes

© 2012 Global Journals Inc. (US)

$$=\mu\Delta U_t+\lambda\big(\nabla\theta_t\cdot\nabla U+\nabla\theta\cdot\nabla U_t+U_t\cdot\nabla(\nabla\theta)+U\cdot\nabla(\nabla\theta_t)\big)+\vec{f}\theta_t+\vec{f}_t\theta.$$

Taking L^2 inner product of (3.64) with U_t, after integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U_t\|^2+\mu\|\nabla U_t\|^2+\frac{1}{2}\int_\Omega(\theta_t-U\cdot\nabla\theta)|U_t|^2d\mathbf{x}=\sum_{i=1}^{7}R_i+\lambda\int_\Omega(\nabla\theta\cdot\nabla U_t)\cdot U_t d\mathbf{x},$$

where

$$R_1=-\int_\Omega(\theta_t U\cdot\nabla U)\cdot U_t d\mathbf{x},\quad R_2=-\int_\Omega(\rho U_t\cdot\nabla U)\cdot U_t d\mathbf{x};$$

$$R_3=\lambda\int_\Omega(\nabla\theta_t\cdot\nabla U)\cdot U_t d\mathbf{x},\quad R_4=\lambda\int_\Omega(U_t\cdot\nabla(\nabla\theta))\cdot U_t d\mathbf{x},$$

$$R_5=-\lambda\int_\Omega\nabla\theta_t\cdot(U\cdot\nabla U_t)d\mathbf{x};$$

$$R_6=\lambda\int_\Omega\theta_t\vec{f}\cdot U_t d\mathbf{x},\quad R_7=\lambda\int_\Omega\theta\vec{f}_t\cdot U_t d\mathbf{x}.$$

Using the boundary condition we have

$$\lambda\int_\Omega(\nabla\theta\cdot\nabla U_t)\cdot U_t d\mathbf{x}=-\frac{\lambda}{2}\int_\Omega\Delta\theta|U_t|^2d\mathbf{x}.$$

Moreover, since $\theta_t=\lambda\Delta\theta-U\cdot\nabla\theta$, we have

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U_t\|^2+\mu\|\nabla U_t\|^2=\sum_{i=1}^{9}R_i,\tag{3.65}$$

where

$$R_8=\int_\Omega(U\cdot\nabla\theta)|U_t|^2d\mathbf{x},\quad R_9=-\lambda\int_\Omega\Delta\theta|U_t|^2d\mathbf{x}.$$

We estimate $R_i, i=1,...,9$ as follows: By Lemma 2.4, Lemma 3.5 and Poincaré inequality we have

$$\begin{aligned}|R_1|&\le\|\theta_t\|_{L^4}\|U\|_{L^4}\|\nabla U\|_{L^4}\|U_t\|_{L^4}\\&\le c_{51}\|\theta_t\|_{H^1}\|\nabla U\|_{H^1}\|U_t\|_{H^1}\\&\le c_{52}\|\theta_t\|_{H^1}\|U\|_{H^2}\|\nabla U_t\|\\&\le\varepsilon\|\nabla U_t\|^2+\frac{c_{53}}{\varepsilon}\big(\|\theta_t\|^2+\|\nabla\theta_t\|^2\big)\|U\|_{H^2}^2\\&\le\varepsilon\|\nabla U_t\|^2+\frac{c_{54}}{\varepsilon}\|U\|_{H^2}^2+\frac{c_{53}}{\varepsilon}\|\nabla\theta_t\|^2\|U\|_{H^2}^2,\end{aligned}$$

where $\varepsilon>0$ is a constant to be determined. Similarly, we have

$$\begin{aligned}|R_2|&\le\|\rho\|_{L^\infty}\|\nabla U\|\|U_t\|_{L^4}^2\\&\le c_{55}\|U_t\|\|\nabla U_t\|\\&\le\varepsilon\|\nabla U_t\|^2+\frac{c_{56}}{\varepsilon}\|U_t\|^2.\end{aligned}$$

Using Lemma 3.1 and Lemma 3.5 we have

$$|R_3|\le\frac{\lambda}{2}\|\nabla\theta_t\|^2+\frac{\lambda}{2}\|\nabla U\|_{L^4}^2\|U_t\|_{L^4}^2$$

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue II Version I February 2012

© 2012 Global Journals Inc. (US)

$$\leq \frac{\lambda}{2}\|\nabla\theta_t\|^2 + c_{57}\big(\|\nabla U\|\|\nabla^2 U\| + \|\nabla U\|^2\big)\|U_t\|\|\nabla U_t\|$$
$$\leq \frac{\lambda}{2}\|\nabla\theta_t\|^2 + c_{58}\|\nabla^2 U\|\|U_t\|\|\nabla U_t\| + c_{59}\|U_t\|\|\nabla U_t\|$$
$$\leq \varepsilon\|\nabla U_t\|^2 + c_{60}(\varepsilon)\big(\|U\|_{H^2}^2\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2 + \|U_t\|^2\big);$$

and

$$|R_4| \leq \lambda\|\theta\|_{H^2}\|U_t\|_{L^4}^2$$
$$\leq c_{61}\|\theta\|_{H^2}\|U_t\|\|\nabla U_t\|$$
$$\leq \varepsilon\|\nabla U_t\|^2 + \frac{c_{62}}{\varepsilon}\|\theta\|_{H^2}^2\|\sqrt{\rho}U_t\|^2.$$

By Sobolev embedding we have

$$|R_5| \leq \varepsilon\|\nabla U_t\|^2 + \frac{c_{63}}{\varepsilon}\|U\|_{L^\infty}^2\|\nabla\theta_t\|^2$$
$$\leq \varepsilon\|\nabla U_t\|^2 + \frac{c_{64}}{\varepsilon}\|U\|_{H^2}^2\|\nabla\theta_t\|^2.$$

Since $\|\vec{f}_t\|_{C([0,t];H^1(\Omega))}^2 + \|\vec{f}_t\|_{C([0,t];L^2(\Omega))}^2 \leq F_1$, using Poincaré inequality we have

$$|R_6| \leq \frac{\varepsilon}{c_0}\|U_t\|^2 + \frac{c_{65}}{\varepsilon}\|\vec{f}\|_{L^4}^2\|\theta_t\|_{L^4}^2$$
$$\leq \varepsilon\|\nabla U_t\|^2 + \frac{c_{66}}{\varepsilon}\|\theta_t\|_{H^1}^2,$$

and

$$|R_7| \leq \frac{\varepsilon}{c_0}\|U_t\|^2 + \frac{c_{67}}{\varepsilon}\|\vec{f}_t\|^2\|\theta\|_{L^\infty}^2$$
$$\leq \varepsilon\|\nabla U_t\|^2 + \frac{c_{68}}{\varepsilon}\|\theta\|_{H^2}^2.$$

The last two terms are treated as

$$|R_8| \leq \|U\cdot\nabla\theta\|\|U_t\|_{L^4}^2$$
$$\leq c_{69}\|U\|_{L^4}\|\nabla\theta\|_{L^4}\|U_t\|\|\nabla U_t\|$$
$$\leq c_{70}\|\theta\|_{H^2}\|U_t\|\|\nabla U_t\|$$
$$\leq \varepsilon\|\nabla U_t\|^2 + \frac{c_{71}}{\varepsilon}\|\theta\|_{H^2}^2\|\sqrt{\rho}U_t\|^2;$$

and

$$|R_9| \leq \lambda\|\Delta\theta\|\|U_t\|_{L^4}^2$$

$$\leq c_{72}\|\theta\|_{H^2}\|U_t\|\|\nabla U_t\|$$
$$\leq \varepsilon\|\nabla U_t\|^2 + \frac{c_{73}}{\varepsilon}\|\theta\|_{H^2}^2\|\sqrt{\rho}U_t\|^2.$$

Plugging above estimates into (3.65) we have

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U_t\|^2 + \mu\|\nabla U_t\|^2 \leq 9\varepsilon\|\nabla U_t\|^2 + K(t)(\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2) + Z(t), \qquad (3.66)$$

where

$$K(t) = c_{74}(\varepsilon)\big(\|U\|_{H^2}^2 + \|\theta\|_{H^2}^2\big),$$

Notes

© 2012 Global Journals Inc. (US)

$$Z(t) = c_{75}(\varepsilon)\big(\|U_t\|^2 + \|U\|_{H^2}^2 + \|\theta_t\|_{H^1}^2 + \|\theta\|_{H^2}^2\big).$$

Next, taking L^2 inner product of (3.53) with $\Delta\theta_t$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\theta_t\|^2 + \lambda\|\Delta\theta_t\|^2 &= \int_\Omega (U_t\cdot\nabla\theta + U\cdot\nabla\theta_t)\Delta\theta_t d\mathbf{x}\\ &\le \frac{\lambda}{2}\|\Delta\theta_t\|^2 + \lambda\big(\|U_t\cdot\nabla\theta\|^2 + \|U\cdot\nabla\theta_t\|^2\big).\end{aligned} \tag{3.67}$$

The second term on the RHS of (3.67) is estimated as

$$\begin{aligned}&\lambda\big(\|U_t\cdot\nabla\theta\|^2 + \|U\cdot\nabla\theta_t\|^2\big)\\ &\le c_{76}\|U_t\|_{L^4}^2\big(\|\nabla\theta\|\|D^2\theta\| + \|\nabla\theta\|^2\big) + \lambda\|U\|_{L^\infty}^2\|\nabla\theta_t\|^2\\ &\le c_{77}\|U_t\|\|\nabla U_t\|\|\theta\|_{H^2} + c_{78}\|U\|_{H^2}^2\|\nabla\theta_t\|^2\\ &\le \varepsilon\|\nabla U_t\|^2 + \frac{c_{79}}{\varepsilon}\|\theta\|_{H^2}^2\|\sqrt{\rho}U_t\|^2 + c_{78}\|U\|_{H^2}^2\|\nabla\theta_t\|^2.\end{aligned}$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta_t\|^2 + \frac{\lambda}{2}\|\Delta\theta_t\|^2 \le \varepsilon\|\nabla U_t\|^2 + \frac{c_{79}}{\varepsilon}\|\theta\|_{H^2}^2\|\sqrt{\rho}U_t\|^2 + c_{78}\|U\|_{H^2}^2\|\nabla\theta_t\|^2. \tag{3.68}$$

Combining (3.66) and (3.68) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2\big) + \mu\|\nabla U_t\|^2 + \frac{\lambda}{2}\|\Delta\theta_t\|^2\\ &\le 10\varepsilon\|\nabla U_t\|^2 + \tilde{K}(t)\big(\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2\big) + \tilde{Z}(t),\end{aligned} \tag{3.69}$$

where $\tilde{K}(t)$ and $\tilde{Z}(t)$ are equivalent to $K(t)$ and $Z(t)$ respectively. Choosing $\varepsilon = \mu/20$ in (3.69) we have

$$\frac{d}{dt}\big(\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2\big) + \mu\|\nabla U_t\|^2 + \lambda\|\Delta\theta_t\|^2 \le 2\tilde{K}(t)(\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2) + 2\tilde{Z}(t). \tag{3.70}$$

By virtue of Lemmas 3.5–3.6 we know that $\tilde{K}(t), \tilde{Z}(t)$ are uniformly integrable in time for any $t \ge 0$. Applying Gronwall inequality to (3.70) we have

$$\|(\sqrt{\rho}U_t, \nabla\theta_t)(\cdot,t)\|^2 \le c_{79}, \quad \text{and} \quad \int_0^t \|(\nabla U_t, \Delta\theta_t)(\cdot,\tau)\|^2 d\tau \le c_{80}, \quad \forall\, t \ge 0. \tag{3.71}$$

Plugging the first part of (3.71) into (3.70) we have

$$\frac{d}{dt}\big(\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2\big) + \mu\|\nabla U_t\|^2 + \lambda\|\Delta\theta_t\|^2 \le c_{81}Y(t), \tag{3.72}$$

where

$$Y(t) = \|U_t\|^2 + \|U\|_{H^2}^2 + \|\theta_t\|_{H^1}^2 + \|\theta\|_{H^2}^2.$$

By virtue of (3.63), Poincaré inequality and Lemma 2.3 we have

$$Y(t) \le c_{82}\big(\|U_t\|^2 + \|\nabla U\|^2 + \|\nabla\theta_t\|^2 + \|\Delta\theta\|^2\big). \tag{3.73}$$

Plugging (3.73) into (3.72) we have

$$\frac{d}{dt}\big(\|\sqrt{\rho}U_t\|^2 + \|\nabla\theta_t\|^2\big) + \mu\|\nabla U_t\|^2 + \lambda\|\Delta\theta_t\|^2 \le c_{83}\|(U_t, \nabla U, \nabla\theta_t, \Delta\theta)\|^2. \tag{3.74}$$

© 2012 Global Journals Inc. (US)

where

$$E_3(t) \cong \|(\theta, \theta_t, U)\|_{H^1}^2 + \|U_t\|^2,$$
$$D_3(t) \cong \|(\theta, \theta_t)\|_{H^2}^2 + \|(U, U_t)\|_{H^1}^2.$$

Then the lemma follows directly from (3.63), (3.71), (3.75) and Lemma 3.6. This completes the proof.

As a consequence of Lemma 3.7 we have

Lemma 3.8. *Under the assumptions of Theorem 1.1, there exist positive constants α_6 and β_6 independent of t such that for any $t \geq 0$ it holds that*

$$\|\theta(\cdot, t)\|_{H^3}^2 \leq \alpha_6 e^{-\beta_6 t}.$$

Proof. By virtue of Lemma 2.3 we have

$$\begin{aligned}\|\theta\|_{H^3}^2 \leq c_3\|\Delta\theta\|_{H^1}^2 &\leq c_{85}\big(\|\Delta\theta\|^2 + \|\nabla\theta_t\|^2 + \|\nabla(U\cdot\nabla\theta)\|^2\big)\\ &\leq c_{86}\big(\|\Delta\theta\|^2 + \|\nabla\theta_t\|^2 + \|U\|_{H^2}^2\|\theta\|_{H^2}^2\big).\end{aligned}$$

Then the lemma follows from Lemma 3.6 and Lemma 3.7. This completes the proof.

h) Decay of $\|U\|_{H^3}$

Lemma 3.9. *Under the assumptions of Theorem 1.1, there exist positive constants α_7, β_7 and γ_6 independent of t such that for any $t \geq 0$ it holds that*

$$\|U(\cdot, t)\|_{H^3}^2 \leq \alpha_7 e^{-\beta_7 t}, \quad \text{and} \int_0^t \big(\|\theta_t(\cdot,\tau)\|_{H^2}^2 + \|U_{tt}(\cdot,\tau)\|^2\big)d\tau \leq \gamma_6.$$

Proof. Taking L^2 inner product of (3.64) with U_{tt} we have

$$\begin{aligned}&\frac{\mu}{2}\frac{d}{dt}\|\nabla U_t\|^2 + \|\sqrt{\rho}U_{tt}\|^2\\ &= \int_\Omega \big[-\rho_t U_t - \rho_t U\cdot\nabla U - \rho U_t\cdot\nabla U - \rho U\cdot\nabla U_t\\ &+\lambda\big(\nabla\rho_t\cdot\nabla U + \nabla\rho\cdot\nabla U_t + U_t\cdot\nabla(\nabla\rho) + U\cdot\nabla(\nabla\rho_t)\big) + \vec{f}\rho_t + \vec{f}_t\rho\big]\cdot U_{tt}d\mathbf{x}.\end{aligned} \tag{3.76}$$

Using previously established estimates and Lemma 2.4, we can show that (since there is no essential difficulties, we omit the details)

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla U_t\|^2 + \frac{1}{2}\|\sqrt{\rho}U_{tt}\|^2 \leq c_{87}\big(\|\nabla U_t\|^2 + \|\theta_t\|_{H^2}^2 + \|\theta\|_{H^2}^2\big). \tag{3.77}$$

By absorbing the RHS of (3.77) into the LHS of (3.75) we have

$$\frac{d}{dt}E_4(t) + c_{88}D_4(t) \leq 0, \quad \forall\, t \geq 0, \tag{3.78}$$

where

$$E_4(t) \cong \|(U, U_t, \theta, \theta_t)\|_{H^1}^2,$$
$$D_4(t) \cong \|(\theta, \theta_t)\|_{H^2}^2 + \|(U, U_t)\|_{H^1}^2 + \|U_{tt}\|^2.$$

So that, for any $t \geq 0$ it holds that

$$\|U_t(\cdot, t)\|_{H^1}^2 \leq c_{89}e^{-c_{90}t}, \quad \text{and} \int_0^t \big(\|\theta_t(\cdot,\tau)\|_{H^2}^2 + \|U_{tt}(\cdot,\tau)\|^2\big)d\tau \leq c_{91}. \tag{3.79}$$

Notes

© 2012 Global Journals Inc. (US)

With the help of previous estimates and Lemma 2.1, by direct calculations, we have

$$\|U\|_{H^3}^2 \leq c_{92}\big(\|U\|_{H^2}^2 + \|\theta\|_{H^3}^2 + \|U_t\|_{H^1}^2\big).$$

Then the lemma follows from Lemma 3.7, Lemma 3.8 and (3.79). This completes the proof.

i) Uniform estimate of $\|(\theta, U)\|_{H^4}$

We now prove the uniform estimates of $\|(\theta, U)\|_{H^4}$ in order to complete the proof of Theorem 1.1.

Lemma 3.10. *Under the assumptions of Theorem 1.1, there exists a positive constant* γ_7 *independent of* t *such that for any* $t \geq 0$ *it holds that*

$$\int_0^t \|(U,\theta)(\cdot,t)\|_{H^4}^2 d\tau \leq \gamma_7, \quad \forall\ t \geq 0.$$

Proof. By Lemma 2.3, Lemma 2.1 and Lemma 3.9, it is straightforward to show that

$$\begin{aligned} \|\theta\|_{H^4}^2 &\leq c_{93}\big(\|\theta_t\|_{H^2}^2 + \|\theta\|_{H^3}^2\big), \\ \|U\|_{H^4}^2 &\leq c_{94}\big(\|U_t\|_{H^2}^2 + \|\theta\|_{H^4}^2\big). \end{aligned} \tag{3.80}$$

Since $U_t|_{\partial\Omega} = 0$, by Lemma 2.1 and (3.64) we have

$$\|U_t\|_{H^2}^2 \leq c_{95}\big(\|U_{tt}\|^2 + \|\rho_t\|_{H^2}^2 + \|U\|_{H^3}^2\|\rho\|_{H^3}^2\big). \tag{3.81}$$

Then the lemma follows from Lemma 3.9, (3.80) and (3.81). This completes the proof.

Lemmas 3.8–3.10 conclude our main result, Theorem 1.1.

Acknowledgement

The research of K. Zhao was partially supported by the National Science Foundation through grant DMS 0807406. The author would also like to thank the generous support from the National Science Foundation under agreement no. 0635561.

References Références Referencias

[1] R. Adams, *Sobolev Spaces.* Academic, New York, 1975.

[2] S.N. Antontsev, A.V. Kazhikhov and V.N. Monakhov, *Boundary value problems in mechanics of nonhomogeneous fluids.* North-Holland, 1990.

[3] J.P. Bourguignon and H. Brezis, Remarks on the Euler equation. *J. Funct. Anal.* **15** (1975): 341–363.

[4] L.C. Evans, *Partial Differential Equations.* AMS, 1998.

[5] R. Danchin, Density-dependent incompressible fluids in critical spaces. *Proc. Royal Soc. Edinburgh* **133** (2003): 1311–1334.

[6] R. Danchin, Navier-Stokes equations with variable density. *Hyperbolic Problems and Related Topics, International Press, Graduate Series in Analysis* (2003): 121–135.

[7] R. Danchin, Fluides incompressibles densit variable. *Sminaire quations aux Drives Partielles*, 2002–2003, **Exp. No. XI**, Palaiseau, 2002–2003.

[8] R. Danchin, The inviscid limit for density-dependent incompressible fluids. *Ann. Fac. Sci. Toulouse Math.* **15** (2006): 637-688.

[9] R. Danchin, Local and global well-posedness results for flows of inhomogeneous viscous fluids. *Advances in Differential Equations* **9** (2004): 353–386.

© 2012 Global Journals Inc. (US)

[10] R. Danchin, Density dependent incompressible fluids in bounded domains. *J. Math. Fluid Mech.* **8** (2006): 333–381.

[11] B. Desjardins, Global existence results for the incompressible density-dependent Navier-Stokes equations in the whole space. *Differential and Integral Equations* **10** (1997): 587–598.

[12] S. Itoh and A. Tani, Solvability of nonstationary problems for nonhomogeneous incompressible fluids and the convergence with vanishing viscosity. *Tokyo J. Math.* **22** (1999): 17–42.

[13] O.A. Ladyzhenskaya and V.A. Solonnikov, The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids. *J. Soviet Math.* **9** (1978): 697–749.

[14] J. Leray, Sur le mouvement dun liquide visqueux remplissant lespace. *Acta Mathematica* **63** (1934): 193–248.

[15] P.L. Lions, *Mathematical Topics in Fluid Mechanics, Vol. I.* Oxford University Press, New York, 1996.

[16] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Uraltseva, *Linear and Quasi-linear Equations of Parabolic Type*. AMS, 1968.

[17] R. Temam, *Navier-Stokes Equations*. North Holland, 1977.

Notes

© 2012 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics & Decision Sciences
Volume 12 Issue 2 Version 1.0 February 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On Some Classes of Analytic Functions Defined by Subordination

By A. El -Sayed Ahmed
Taif University , Saudi Arabia

Abstract - In this paper , we define some general classes of analytic functions by subordination . Our new results extend and improve a lot of known results (see [6]).

Keywords and phrases : Convex functions, univalent functions, subordination.

GJSFR – F Classification : 30C45

ON SOME CLASSES OF ANALYTIC FUNCTIONS DEFINED BY SUBORDINATION

Strictly as per the compliance and regulations of :

RESEARCH | DIVERSITY | ETHICS

© 2012 . A. El -Sayed Ahmed.This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

eJournal · Print Journal · epaper · open standard · 3d virtual journal

On Some Classes of Analytic Functions Defined by Subordination

A. El-Sayed Ahmed

Abstract - In this paper , we define some general classes of analytic functions by subordination . Our new results extend and improve a lot of known results (see [6]).

Keywords and phrases : *Convex functions, univalent functions, subordination.*

I. Introduction

Let A be the class of functions f which are analytic in the unit disk $\Delta = \{z : |z| < 1\}$ and are given by

$$f(z) = 1 + \sum_{n=k}^{\infty} a_k z^k, \qquad n \in N. \tag{1.1}$$

A function f analytic in Δ is said to be univalent in a domain D if

$$f(z_1) = f(z_2) \Longrightarrow z_1 = z_2 \qquad z_1, z_2 \in D.$$

The class of all univalent functions f in Δ and have form (1.1) will be denoted by S.

A domain D is called convex if for every pair of points w_1 and w_2 in the interior of D, the line-segment joining w_1 to w_2 lies wholly in D. A function f which maps Δ onto a convex domain is called a convex function.The necessary and sufficient condition for $f \in S$ to be convex in Δ is that $\mathrm{Re}\frac{(zf'(z))'}{f'(z)} > 0, \quad z \in \Delta$. The class off all functions convex and univalent in Δ is denoted by C.

A domain D is said to be starlike with respect to $w = 0$ if the linear segment joining $w = 0$ to any other point of D lies wholly in D. If a function f maps Δ onto a starlike domain with respect to $w = 0$, then f is said to be starlike. The necessary and sufficient condition for $f \in S$ to be starlike is that

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0, \quad z \in \Delta.$$

This class is denoted by S^*, and it was studied first by Alexander [3].

Let $f(z)$ and $g(z)$ be analytic in Δ. We say that $f(z)$ is subordinate to $g(z)$ if there exists a function $\phi(z)$ analytic (not necessarily univalent) in Δ satisfying $\phi(0) = 0$ and $|\phi(z)| < 1$ such that

Author : *Taif University , Faculty of Science , Mathematics Department, Saudi Arabia. E-mail : ahsayed80@hotmail.com*

Ref.

[6] Z.M.G. Kishka, On some Classes of analytic functions, Bulletin de la Societe' Royale des Sciences de Lie'ge, 62(5-6)(1993), 313-360.

[3] J. Alexander , Functions with map the interior of the unit circle upon simple regions, Ann. Math. 17(1915-1916),12-22.

© 2012 Global Journals Inc. (US)

$$f(z) = g(\phi(z)) \quad (|z| < 1). \tag{1.2}$$

Subordination is denoted by $f(z) \prec g(z)$. For more details on univalent functions by subordination, we refer to [1,2,5,7-16].

Let B be the class of functions, analytic in Δ and of the form

$$w(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad n \in N, \tag{1.3}$$

and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Delta$. Based on the class B Janowski [4] defined the class $P[A, B]$, as follows:

Let p be analytic function in Δ, given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{1.4}$$

Then $p(z)$ is said to be in the class $P[A, B]$; $-1 \leq B < A \leq 1$; if and only if, for $z \in \Delta$

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \qquad ; w \in B. \tag{1.5}$$

Concerning the class $P[A, B]$ Janowski [4] proved the following lemma:

Lemma 1.1 [4]. Let $p \in P[A, B]$, and given by (1.4). Then

(i) $- |p_n| \leq A - B$,

$(ii) - \frac{1-Ar}{1-Br} \leq \text{Re p(z)} \leq \frac{1+Ar}{1+Br}$,

$(iii) - |\arg \text{p(z)}| \leq \sin^{-1} \frac{(A-B)r}{1-ABr^2}$

These results are sharp.

Let N and D be analytic in Δ, D maps Δ onto a many -sheeted starlike region, $N(0) = D(0)$, and

$$\frac{N'(z)}{D'(z)} \in P[A, B], \quad \text{then} \quad \frac{N(z)}{D(z)} \in P[A, B].$$

In [14], Ravichandran et.al defined the class $P_n[A, B]$ as follows:

For $-1 \leq B < A \leq 1$ and

$$p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + ..., \quad n \in N,$$

we say that $p \in P_n[A, B]$ if

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \quad z \in \Delta.$$

The class with the property that $\frac{zf'(z)}{f(z)} \in P_n[A, B]$ is denoted by $ST_n[A, B]$. If $n = 1$, we drop the subscript. Also, Ravichandran et.al [14] obtained the following lemma:

Lemma 1.2 [14]. If $p \in P_n[A, B]$, then

$$\left| p(z) - \frac{1 - ABr^{2n}}{1 - B^2 r^{2n}} \right| \leq \frac{(A - B)r^n}{1 - B^2 r^{2n}}, \quad |z| = r < 1. \tag{1.6}$$

For the special case $p \in P_n(\alpha) = P_n[1 - 2\alpha, -1]$, we get

Ref.

[1] F. M. Al-Oboudi and K. A. Al-Amoudi, Subordination results for classes of analytic functions related to conic domains defined by a fractional operator, J. Math. Anal. Appl. 354(2)(2009), 412-420..

[2] R. Aghalary, J.M. Jahangiri and S.R. Karni, Starlikeness and convexity conditions for classes of functions defined by subordination, J. Inequal. Pure Appl. Math. 5, No.2, Paper No.31, 11 p. (2004).

© 2012 Global Journals Inc. (US)

$$\left| p(z) - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}} \right| \le \frac{2(1-\alpha)r^n}{1-r^{2n}}, \quad |z| = r < 1.$$

In this paper, we define the classes:

$$\mathbf{P} = P^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, \ldots, A_N, B_N],$$

$$\mathbf{P}_n = P'^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, \ldots, A_N, B_N];$$

Notes

of analytic functions of the single complex variable z in the unit disk $\Delta = \{z : |z| < 1\}$. Moreover we study some of their basic properties. Besides we study the behavior of functions of these classes under some differential and integral operators. Concerning the class:

$$P^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[A_1, B_1, A_2, B_2, A_3, B_3, \ldots, A_N, B_N],$$

which denotes the class of functions q that are analytic in Δ and are represented by

$$q(z) = \sum_{j=1}^{N} \alpha_j \left[\frac{k_j+2}{4} p_j(z) - \frac{k_j-2}{4} u_j(z)\right],$$

where $p_j, u_j \in P[A_j, B_j]$, α_j are non-negative real numbers ; $\sum_{j=1}^{\infty} \alpha_j = 1$; $-1 \le B_j < A_j \le 1$, $k_j \ge 2$ and $j = 1, 2, 3, \ldots, N$.

The following lemma is useful in the sequel.

Lemma 1.3 [6]**.** If $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$ is regular in Δ, $\phi_1(z)$ and $h(z)$ are convex univalent in Δ such that $\psi(z) \prec \phi_1(z)$, then $\psi(z) * h(z) \prec \phi_1(z) * h(z)$, $z \in \Delta$, where

$$\phi_1(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \psi(z) * \phi_1(z) = \sum_{n=0}^{\infty} b_n a_n z^n.$$

II. The Class P

Suppose that

$$\mathbf{P} = P^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, \ldots, A_N, B_N]$$

denotes the class of functions q_n that are analytic in Δ and are represented by

$$q(z) = \sum_{j=1}^{N} \alpha_j \left[\frac{k_j+2}{4} p_j(z) - \frac{k_j-2}{4} u_j(z)\right], \tag{2.1}$$

where $p_j, u_j \in P_n[A_j, B_j]$, α_j are non-negative real numbers ; $\sum_{j=1}^{\infty} \alpha_j = 1$; $-1 \le B_j < A_j \le 1$, $k_j \ge 2$ and $j = 1, 2, 3, \ldots, N$. Since, for

$$p(z) = 1 + \sum_{k=n}^{\infty} a_k z^k, \quad n \in N,$$

we say that $p \in P_n[A_j, B_j]$ if $p(z) \prec \frac{1+A_j z}{1+B_j z}$, $z \in \Delta$.

Lemma 2.1. The class $\mathbf{P}$ is a convex set.

Proof. We want to prove that for $\alpha, \beta \ge 0$, $\alpha + \beta = 1$ and for

$$q_1, q_2 \in P^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, \ldots, A_N, B_N],$$

that $q(z) = \frac{1}{\alpha+\beta}[\alpha q_1(z) + \beta q_2(z)]$, belongs to the class

Global Journal of Science Frontier Research (F) Volume XII Issue II Version I February 2012

© 2012 Global Journals Inc. (US)

we say that $p \in P_n[A_j, B_j]$ if $p(z) \prec \frac{1+A_j z}{1+B_j z}$, $z \in \Delta$.

Lemma 2.1. The class **P** is a convex set.

Proof. We want to prove that for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ and for

$$q_1, q_2 \in P^{\alpha_1,\alpha_2,\alpha_3,...,\alpha_N}_{k_1,k_2,k_3,...,k_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, ..., A_N, B_N],$$

that $q(z) = \frac{1}{\alpha+\beta}[\alpha q_1(z) + \beta q_2(z)]$, belongs to the class

$$P^{\alpha_1,\alpha_2,\alpha_3,...,\alpha_N}_{k_1,k_2,k_3,...,k_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, ..., A_N, B_N].$$

This can simply seen by letting

$$q_1(z) = \sum_{j=1}^{N} \alpha_j \Big[\frac{k_j + 2}{4} f_j(z) - \frac{k_j - 2}{4} f_j^*(z)\Big],$$

where $f_j, f_j^* \in P_n[A_j, B_j]$, α_j are non-negative real numbers ; $\sum_{j=1}^{N} \alpha_j = 1$; $-1 \leq B_j < A_j \leq 1$, $k_j \geq 2$.

Also, let

$$q_2(z) = \sum_{j=1}^{N} \alpha_j \Big[\frac{k_j + 2}{4} g_j(z) - \frac{k_j - 2}{4} g_j^*(z)\Big],$$

where $g_j, g_j^* \in P_n[A_j, B_j]$. Then, we see that

$$\frac{1}{\alpha + \beta}[\alpha q_1(z) + \beta q_2(z)] = \sum_{j=1}^{N} \frac{\alpha_j}{\alpha + \beta} \Big[\frac{k_j + 2}{4}[\alpha f_j + \beta g_j] - \frac{k_j - 2}{4}[\alpha f_j^* + \beta g_j^*]\Big]$$

$$= \sum_{j=1}^{N} \alpha_j \Big[\frac{k_j + 2}{4} p_j(z) - \frac{k_j - 2}{4} u_j(z)\Big].$$

Then we arrive at the proof of our Lemma, since the class $P_n[A_j, B_j]$ is convex.

Lemma 2.2. Let

$$q \in P^{\alpha_1,\alpha_2,\alpha_3,...,\alpha_N}_{k_1,k_2,k_3,...,k_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, ..., A_N, B_N].$$

Then for

$$p(z) = 1 + \sum_{k=n}^{\infty} a_k z^k;$$

we have

(*i*) $|a_n| \leq \sum_{j=1}^{N} \frac{\alpha_j k_j}{2}(A_j - B_j)$ for all n.

$$(ii)\Big\{\sum_{s=1}^{N} \alpha_s \prod_{\substack{j=1;\\ j\neq s}}^{N} (1 - B_j^2 r^{2n})\big(1 - \frac{k_s}{2}(A_s - B_s)r^n - A_s B_s r^{2n}\big)\Big\} \Big/ \prod_{\substack{j=1;\\ j\neq s}}^{N} (1 - B_j^2 r^{2n})$$

$$\leq Rep(z)$$

$$\leq \Big\{\sum_{s=1}^{N} \alpha_s \prod_{\substack{j=1;\\ j\neq s}}^{N} (1 - B_j^2 r^{2n})\big(1 + \frac{k_s}{2}(A_s - B_s)r^n - A_s B_s r^{2n}\big)\Big\} \Big/ \prod_{\substack{j=1;\\ j\neq s}}^{N} (1 - B_j^2 r^{2n})$$

Notes

© 2012 Global Journals Inc. (US)

(iii) $q \in P_n$ for $|z| < r_0$, where r_0 is the least positive root of the equation

$$(2.2) \qquad \sum_{s=1}^{N} \alpha_s \prod_{\substack{j=1;\\ j\neq s}}^{N} (1 - B_j^2 r^{2n})\big(1 - \frac{k_s}{2}(A_s - B_s)r^n - A_s B_s r^{2n}\big) = 0,$$

and $P_n = P_n[1, -1]$ is the class of functions of positive real part. These results are sharp.

Notes

Proof. The proof of the assertion (i) is very similar to the proof of the assertion (i) of Lemma 1.1 [4]. To prove assertion (ii) of Lemma 2.2, let $p_j, u_j \in P_n[A_j, B_j]$; $-1 \leq B_j < A_j \leq 1$, $n \in N$ and $j = 1, 2, 3, ..., N$. Now, let

$$p(z) = 1 + \sum_{k=n}^{\infty} a_k z^k \prec \frac{1 + A_j z}{1 + B_j z}.$$

Then, we can write $p(z) = \frac{1+A_j\phi(z)}{1+B_j\phi(z)}$, where $\phi(z)$ is analytic in Δ, $\phi(0) = 0$ and $|\phi(z)| < 1$. Expressing $\phi(z)$ in terms of $p(z)$, we get that $\phi(z) = \frac{p(z)-1}{A_j - B_j p(z)} = \frac{a_n}{A_j - B_j} z^n + ... = z^n \Psi(z)$, where $|\Psi(z)| \leq 1$. Therefore $|\phi(z)| \leq z^n$, and hence from the subordination principle, we have that $\left|\frac{1-A_j\phi(z)}{1-B_j\phi(z)}\right| \leq Rep(z) \leq |p(z)| \leq \frac{1+A_j\phi(z)}{1+B_j\phi(z)}$, which implies that,

$$(2.3) \qquad \left|\frac{1 - A_j r^n}{1 - B_j r^n}\right| \leq Rep(z) \leq |p(z)| \leq \frac{1 + A_j r^n}{1 + B_j r^n}.$$

Moreover the double inequality (2.3) will be also satisfied for the functions $u_j(z)$. Now, since

$$q \in P_{k_1,k_2,k_3,...,k_N}^{\alpha_1,\alpha_2,\alpha_3,...,\alpha_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, ..., A_N, B_N],$$

then using relation (2.1), it follows that

$$(2.4) \qquad \sum_{j=1}^{N} \alpha_j \big[\frac{k_j + 2}{4} \min Rep_j(z) - \frac{k_j - 2}{4} \max u_j(z)\big] \leq Req(z)$$
$$\leq \sum_{j=1}^{N} \alpha_j \big[\frac{k_j + 2}{4} \max Rep_j(z) - \frac{k_j - 2}{4} \min u_j(z)\big].$$

Introducing the double inequality (2.3) in the double inequality (2.4), we obtain the following double inequality

$$\sum_{j=1}^{N} \alpha_j \Big\{\frac{k_j + 2}{4}\big(\frac{1 - A_j r^n}{1 - B_j r^n}\big) - \frac{k_j - 2}{4}\big[\frac{1 + A_j r^n}{1 + B_j r^n}\big]\Big\} \leq Req(z)$$
$$\leq \sum_{j=1}^{N} \alpha_j \big\{\frac{k_j + 2}{4}\big(\frac{1 + A_j r^n}{1 + B_j r^n}\big) - \frac{k_j - 2}{4}\big[\frac{1 - A_j r^n}{1 - B_j r^n}\big]\big\},$$

which yields, after simplification the required double inequality. The result of part (iii) of Lemma 2.2 follows easily from part (ii) of the same Lemma ; since

© 2012 Global Journals Inc. (US)

$$Req(z) \geq \left\{\sum_{s=1}^{N} \alpha_s \prod_{\substack{j=1;\\ j\neq s}}^{N} (1-B_j^2 r^{2n})\left(1-\frac{k_s}{2}(A_s-B_s)r^n - A_s B_s r^{2n}\right)\right\} \Big/ \prod_{\substack{j=1;\\ j\neq s}}^{N} (1-B_j^2 r^{2n}),$$

thus Re $q(z) > 0$, for $|z| = r_0$, where r_0 is the least positive root of the equation

$$\sum_{s=1}^{N} \alpha_s \prod_{\substack{j=1;\\ j\neq s}}^{N} (1-B_j^2 r^{2n})\left(1-\frac{k_s}{2}(A_s-B_s)r^n - A_s B_s r^{2n}\right) = 0.$$

The function

$$q(z) = \sum_{j=1}^{N} \alpha_j \left\{ \frac{(1-\frac{k_j}{2}(A_j-B_j)z^n - A_j B_j z^{2n})}{(1-B_j^2 z^{2n})} \right\},$$

shows that the results of part (ii) and (iii) of Lemma 2.2 are sharp.

Lemma 2.3. Let

$$q \in P_{k_1,k_2,k_3,\ldots,k_N}^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}[n; A_1, B_1, A_2, B_2, A_3, B_3, ..., A_N, B_N].$$

Then

$$(i) \frac{1}{2\pi}\int_0^{2\pi} \left|q(re^{i\theta})\right|^2 d\theta \leq 1 + \frac{\left[\sum_{j=1}^{N} \frac{\alpha_j k_j}{2}(A_j-B_j)\right]^2 r^{2n}}{1-r^2},$$

$$(ii) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left|q(re^{i\theta})\right|^2 d\theta \leq \sum_{j=1}^{N} \frac{\alpha_j k_j}{2}\left[\frac{A_j-B_j}{1-B_j^2 r^{2n}}\right].$$

Proof. Let

$$q(z) = 1 + \sum_{k=n}^{\infty} a_k z^k.$$

Then by using Parseval's identity and the result of (i) given in Lemma 2.2, we get

$$\frac{1}{2\pi}\int_0^{2\pi} \left|q(re^{i\theta})\right|^2 d\theta = \sum_{k=0}^{\infty} |a_k|^2 r^{2k} \leq 1 + \sum_{k=n}^{\infty}\left[\sum_{j=1}^{N} \frac{\alpha_j k_j}{2}(A_j-B_j)\right]^2 r^{2k}$$

$$= 1 + \frac{\left[\sum_{j=1}^{N} \frac{\alpha_j k_j}{2}(A_j-B_j)\right]^2}{(1-r^2)} r^{2n}$$

Now, using relation (2.1), we get that

$$q'(z) = \sum_{j=1}^{N} \alpha_j \left[\frac{k_j+2}{4} Rep_j'(z) - \frac{k_j-2}{4} Reu_j'(z)\right]. \tag{2.5}$$

Moreover, for $p_j' \in P_n[A_j, B_j]$; we have

$$p_j'(z) = \frac{(A_j-B_j)\phi'_j(z)}{[1+B_j\phi_j(z)]^2},$$

© 2012 Global Journals Inc. (US)

Notes

then

$$(2.6) \qquad \frac{1}{2\pi}\int_0^{2\pi}\left|q(re^{i\theta})\right|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{\left|(A_j-B_j)\right|\times\left|w_j'(re^{i\theta})\right|d\theta}{\left|1+B_jw_j(re^{i\theta})\right|} \le \frac{A_j-B_j}{1-B_j^2r^{2n}}.$$

Applying (2.6) in (2.5), it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|q'(re^{i\theta})\right|^2 d\theta \le \sum_{j=1}^{N}\frac{\alpha_j}{2\pi}\int_0^{2\pi}\left[\frac{k_j+2}{4}\left|p_j'(re^{i\theta})\right| + \frac{k_j-2}{4}\left|u_j'(re^{i\theta})\right|\right]d\theta$$
$$\le \sum_{j=1}^{N}\frac{\alpha_jk_j}{2}\left[\frac{A_j-B_j}{1-B_j^2r^{2n}}\right].$$

III. The Class P_N

A function f analytic in Δ is said to belong to the class

$$P'^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n; A_1,B_1,A_2,B_2,A_3,B_3,\ldots,A_N,B_N],$$

if and only if,

$$f' \in P^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N}_{k_1,k_2,k_3,\ldots,k_N}[n; A_1,B_1,A_2,B_2,A_3,B_3,\ldots,A_N,B_N].$$

Lemma 3.1. The class $\mathbf{P}_n$ is a convex set.

Proof. The proof of this Lemma is very similar to the proof of Lemma 1.4 (see [6]). Now, we give the following theorem:

Theorem 3.1. Let $f \in \mathbf{P}_n$. Then f is univalent for $|z| < r_0$; where r_0 is the least positive root of the equation (2.2). This result is sharp.

Proof. Let $f \in \mathbf{P}_n$, hence it follows from Lemma 1.2.2 assertion (iii) that $Ref(z) > 0$, $|z| < r_0$; where r_0 is the least positive root of the equation (2.2).

The sharpness follows from the function $f_1(z)$ defined by

$$f_1(z) =$$
$$\left\{\int_0^z \sum_{s=1}^{N}\alpha_s \prod_{j=1;j\neq s}^{N}(1-B_j^2\zeta^{2n})\Big(1-\frac{k_s}{2}(A_s-B_s)\zeta^n - A_sB_s\zeta^{2n}\Big)d\zeta\right\}\Big/\prod_{\substack{j=1;\\ j\neq s}}^{N}(1-B_j^2\zeta^{2n})$$

Theorem 3.2. Let $f \in \mathbf{P}_n$. Then f maps $|z| < r_1 = (\sqrt{2}-1)r_0^n$ onto a convex domain, where r_0 is the least positive root of the equation (2.2). This result is sharp.

Proof. Let $f \in \mathbf{P}_n$. Hence it follows from Lemma 2.2 assertion (iii) that $Ref(z) > 0$, $|z| < r_0$; where r_0 is the least positive root of the equation (2.2). Let w be any complex number such that $|w| < r_0$. Then the function

$$G(z) = P\Big(\frac{r_0^{2n}(z+w)}{r_0^{2n}+z\bar{w}}\Big) = P(w) + P'(w)\left[1-\frac{|w|^2}{r_0^{2n}}\right]z + \ldots$$

Ref.

[6] Z.M.G. Kishka, On some Classes of analytic functions, Bulletin de la Societe' Royale des Sciences de Lie'ge, 62(5-6)(1993), 313-360.

© 2012 Global Journals Inc. (US)

is analytic in $|z| < r_0$ and $Re\ G(z) > 0$ for $|z| < r_0$. Hence

$$\left|P'(w)(1 - \frac{|w|^2}{r_0^{2n}})\right| \leq \frac{2P(w)}{r_0^n},$$

which implies that,

$$\left|\frac{P'(w)}{P(w)}\right| \leq \frac{2r_0^n}{r_0^{2n} - |w|^2}.$$

Since w is any complex number with $|w| < r_0$, we can write the above inequality as

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{2r_0^n|z|}{r_0^{2n} - |z|^2},$$

which implies that,

$$Re(1 + \frac{zf''(z)}{f'(z)}) \geq 1 - \frac{2r_0^n|z|}{r_0^{2n} - |z|^2} = \frac{r_0^{2n} - 2r_0^n|z| - |z|^2}{r_0^{2n} - |z|^2} > 0,$$

for all $|z| < r_1 = (\sqrt{2} - 1)r_0^n$, where r_0 is the least positive root of the equation (2.2).

The function

$$f(z) = \int_0^z \frac{1 + \zeta^n}{1 - \zeta^n} d\zeta$$

shows that $(\sqrt{2} - 1)$ can not replaced by a smaller constant.

Theorem 3.3. Let $f \in \mathbf{P}_n$. Then for $z = re^{i\theta}$, we have

$$\begin{aligned}(3.1)\quad |f(z)| \geq \sum_{j=1}^{N} \alpha_j \Bigg\{ & r\left[1 - \frac{A_j k_j r^n}{2(n+1)}\right]\gamma(B_j) + \frac{A_j \Phi(B_j)}{(B_j + \gamma(B_j))} r \\ & + \left[1 - \frac{A_j \Phi(B_j)}{(B_j + \gamma(B_j))}\right]\left[\sum_{S=0}^{\infty} \beta_j^{2s} \frac{r^{2ns+1}}{2ns+1}\right] \\ & - \frac{k_j}{2}(A_j - B_j)\Phi(B_j)\left[\sum_{S=0}^{\infty} \beta_j^{2s} \frac{r^{2ns+n+1}}{2ns+n+1}\right]\Bigg\},\end{aligned}$$

where,

$$\gamma(B_j) = \begin{cases} 1, & B_j = 0, \\ 0, & B_j \neq 0 \end{cases}$$

and

$$\Phi(B_j) = \begin{cases} 0, & B_j = 0, \\ 1, & B_j \neq 0. \end{cases}$$

This result is sharp for the function

$$f_0(z) = \sum_{j=1}^{N} \alpha_j \left\{ z\left[1 - \frac{A_j k_j z^n}{2(n+1)}\right]\gamma(B_j) + \frac{A_j \Phi(B_j)}{(B_j + \gamma(B_j))} z\right.$$

Notes

© 2012 Global Journals Inc. (US)

$$+\left[1-\frac{A_j\Phi(B_j)}{(B_j+\gamma(B_j))}\right]\left[\sum_{s=0}^{\infty}\beta_j^{2s}\frac{z^{2ns+1}}{2ns+1}\right]$$
$$\left.-\frac{k_j}{2}(A_j-B_j)\Phi(B_j)\left[\sum_{s=0}^{\infty}\beta_j^{2s}\frac{z^{2ns+n+1}}{2ns+n+1}\right]\right\}.$$

Proof. Since,

$$|f(z)|\geq\int_0^r Re(f'(te^{i\theta}))dt$$

Using part (ii) of Lemma 2.2, for $f'(z)=p(z)$;

$$p(z)\in P'^{\alpha_1,\alpha_2,\alpha_3,...,\alpha_N}_{k_1,k_2,k_3,...,k_N}[n;A_1,B_1,A_2,B_2,A_3,B_3,...,A_N,B_N],$$

we get that,

(3.2) $$|f(z)|\geq\int_0^r\sum_{j=1}^{N}\alpha_j\left\{\frac{1-\frac{k_j}{2}(A_j-B_j)t^n-A_jB_jt^{2n}}{1-B_j^2t^{2n}}\right\}dt.$$

But

$$\frac{1-\frac{k_j}{2}(A_j-B_j)t^n-A_jB_jt^{2n}}{1-B_j^2t^{2n}}=\begin{cases}[1-\frac{k_jA_j}{2}t^n]; & B_j=0\\ \frac{A_j}{B_j}+\frac{[1-\frac{A_j}{B_j}]-\frac{k_j}{2}(A_j-B_j)t^n}{1-B_j^2t^{2n}}; & \\ & B_j\neq 0.\end{cases}$$

Thus

$$I=\int_0^r\frac{1-\frac{k_j}{2}(A_j-B_j)t^n-A_jB_jt^{2n}}{1-B_j^2t^{2n}}dt$$

$$=\begin{cases}[1-\frac{k_jA_j}{2(n+1)}r^n]r; & B_j=0\\ \frac{A_j}{B_j}r+\left\{(1-\frac{A_j}{B_j})\left[\sum_{s=0}^{\infty}\beta_j^{2s}\frac{r^{2ns+1}}{2ns+1}\right]-\frac{k_j}{2}(A_j-B_j)\left[\sum_{s=0}^{\infty}\beta_j^{2s}\frac{r^{2ns+n+1}}{2ns+n+1}\right]\right\}, & \\ s=1,2,...,N;\ B_j\neq 0,\end{cases}$$

which implies that,

(3.3) $$I=\left[1-\frac{k_jA_j}{2(n+1)}r^n\right]r\gamma(B_j)+\frac{A_j\Phi(B_j)}{(B_j+\gamma(B_j))}r$$

$$+\left[1-\frac{A_j\Phi(B_j)}{(B_j+\gamma(B_j))}\right]\left[\sum_{s=0}^{\infty}\beta_j^{2s}\frac{r^{2ns+1}}{2ns+1}\right]-\frac{k_j}{2}(A_j-B_j)\Phi(B_j)\left[\sum_{s=0}^{\infty}\beta_j^{2s}\frac{r^{2ns+n+1}}{2ns+n+1}\right].$$

Introducing (3.3) in the right hand side of inequality (3.2), we obtain inequality (3.1).

Remark 3.1. If we put $n=1$ in Theorems 3.1, 3.2 and 3.3, we obtain the corresponding results in [6].

References Références Referencias

[1] F. M. Al-Oboudi and K. A. Al-Amoudi, Subordination results for classes of analytic functions related to conic domains defined by a fractional operator, J. Math. Anal. Appl. 354(2)(2009), 412-420..

[2] R. Aghalary, J.M. Jahangiri and S.R. Karni, Starlikeness and convexity conditions for classes of functions defined by subordination, J. Inequal. Pure Appl. Math. 5, No.2, Paper No.31, 11 p. (2004).

Ref.

[6] Z.M.G. Kishka, On some Classes of analytic functions, Bulletin de la Societe' Royale des Sciences de Lie'ge, 62(5-6)(1993), 313-360.

[3] J. Alexander , Functions with map the interior of the unit circle upon simple regions, Ann. Math. 17(1915-1916),12-22.

[4] A. Janowski, Some extermal problems for certain families of analytic functions, Ann.Polon. Math. 28(1973),297-326.

[5] D.J. Hallenbeck, and K.T. Hallenbeck, Extreme points and support points of subordination families. J. Math. Anal. Appl. 251, No.1(2000), 157-166.

[6] Z.M.G. Kishka, On some Classes of analytic functions, Bulletin de la Societe' Royale des Sciences de Lie'ge, 62(5-6)(1993), 313-360.

[7] J. Kaczmarski , Extermal properties of close-to-convex functions, Mathematiyka Fizyka. Z. 3(1985), 19-39.

[8] O.S. Kwon and S. Owa, The subordination theorem for λ-spirallike functions of order α, Int. J. Appl. Math. 11 No.2(2002), 113-119.

[9] R. Goel and B. Mehoke, Some invariance properties of a subclass of close-to-convex functions, Indian J. Pur. and Appl.Math., 12 (1981), 1210-1249.

[10] A. O. Mostafa, Applications of differential subordination to certain subclasses of p-valent meromorphic functions involving a certain operator, Math. Comput. Modelling 54(5-6)(2011), 1486-1498..

[11] S. Owa and H.M. Srivastava, Some subordination theorems involving a certain family of integral operators, Integral Transforms Spec. Funct. 15 No.5(2004), 445-454.

[12] R. Parvatham and T. Shanmugan, On analytic functions with reference to an integral operators, Bull. Austral. Math. So., 28(1983),207-21.

[13] N.N. Pascu and S. Owa, Subordination chains and univalence criteria, Bull. Korean Math. Soc. 40, No.4(2003), 671-675.

[14] V. Ravichandran, F. Ronning and T.N. Shanmugam, Radius of convexity and radius of starlikeness for some classes of analytic functions , Complex Variables Theory Appl. 33, No.1-4(1997), 265-280.

[15] R. Singh and S. Singh, Subordination by convex functions, Int. J. Math. Math. Sci. 24, No.8(2000), 563-568.

[16] H.M. Srivastava and A.A. Attiya, Some subordination results associated with certain subclasses of analytic functions, J. Inequal. Pure Appl. Math. 5 No.4, No.82(2004).

Notes

© 2012 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics & Decision Sciences
Volume 12 Issue 2 Version 1.0 February 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Generalization of Ramanujan's identities in terms of q-products and continued fractions

By M.P. Chaudhary
International Scientific Research and Welfare Organization, New Delhi, India

Abstract - In this paper, we generalized seven Ramanujan's identities in terms of q-products and continued fractions, using properties of Jacobi's triple product identities. Findings are new and not available in the literature of special functions.

Keywords and phrases : Jacobi's triple product identities, q-products and continued fraction.

GJSFR-F Classication : Primary 05A17, 11A15; Secondary 11P81, 11P83.

Strictly as per the compliance and regulations of :

© 2012. M.P. Chaudhary.This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

eJournal · Print Journal (physical existance) · epaper · open standard · 3d virtual journal

Generalization of Ramanujan's identities in terms of q-products and continued fractions

M.P. Chaudhary

Abstract - In this paper, we generalized seven Ramanujan's identities in terms of q-products and continued fractions, using properties of Jacobi's triple product identities. Findings are new and not available in the literature of special functions.

Keywords : *Jacobi's triple product identities, q-products and continued fraction.*

I. Introduction

For $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) \tag{1.1}$$

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{(n-1)}) \tag{1.2}$$

$$(a_1,a_2,a_3,...,a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty...(a_k;q)_\infty \tag{1.3}$$

Ramanujan [2, p.1(1.2)] has defined general theta function, as

$$f(a,b) = \sum_{-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \quad ; \quad |ab| < 1, \tag{1.4}$$

Jacobi's triple product identity [3,p.35] is given, as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{1.5}$$

Special cases of Jacobi's triple products identity are given, as

$$\phi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty \tag{1.6}$$

$$(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.7}$$

Ref.

2. B.C. Berndt; *What is a q-series?*, preprint.
3. B.C. Berndt; *Ramanujan's notebook Part III*, Springer-Verlag, New York, 1991.

Author : *International Scientific Research and Welfare Organization, New Delhi, India. E-mail : mpchaudhary_2000@yahoo.com*

© 2012 Global Journals Inc. (US)

$$f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.8}$$

Equation (1.8) is known as Euler's pentagonal number theorem. Euler's another well known identity is as

$$(q;q^2)_\infty^{-1} = (-q;q)_\infty \tag{1.9}$$

Throughout this paper we use the following representations

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty \cdots (q^t;q^n)_\infty = (q^a,q^b,q^c \cdots q^t;q^n)_\infty \tag{1.10}$$

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty \cdots (q^t;q^n)_\infty = (q^a,q^b,q^c \cdots q^t;q^n)_\infty \tag{1.11}$$

$$(-q^a;q^n)_\infty(-q^b;q^n)_\infty(q^c;q^n)_\infty \cdots (q^t;q^n)_\infty = (-q^a,-q^b,q^c \cdots q^t;q^n)_\infty \tag{1.12}$$

Computation of q-product identities:

Now we can have following q-products identities, as

$$(q^2;q^2)_\infty = \prod_{n=0}^{\infty}(1-q^{2n+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{2(4n)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+1)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+2)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+3)+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{8n+2}) \times \prod_{n=0}^{\infty}(1-q^{8n+4}) \times \prod_{n=0}^{\infty}(1-q^{8n+6}) \times \prod_{n=0}^{\infty}(1-q^{8n+8})$$

or,

$$\begin{aligned}(q^2;q^2)_\infty &= (q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty \\ &= (q^2,q^4,q^6,q^8;q^8)_\infty\end{aligned} \tag{1.13}$$

also we can compute

$$(q^2;q^2)_\infty = (q^2;q^4)_\infty(q^4;q^4)_\infty \tag{1.14}$$

$$(q^4;q^4)_\infty = \prod_{n=0}^{\infty}(1-q^{4n+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{4(3n)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+1)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+2)+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{12n+4}) \times \prod_{n=0}^{\infty}(1-q^{12n+8}) \times \prod_{n=0}^{\infty}(1-q^{12n+12})$$

or,

$$\begin{aligned}(q^4;q^4)_\infty &= (q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty \\ &= (q^4,q^8,q^{12};q^{12})_\infty\end{aligned} \tag{1.15}$$

$$(q^4;q^{12})_\infty = \prod_{n=0}^{\infty}(1-q^{12n+4}) = \prod_{n=0}^{\infty}(1-q^{12(5n)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+1)+4}) \times$$

$$\times \prod_{n=0}^{\infty}(1-q^{12(5n+2)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+3)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+4)+4})$$

Notes

© 2012 Global Journals Inc. (US)

$$= \prod_{n=0}^{\infty}(1 - q^{60n+4}) \times \prod_{n=0}^{\infty}(1 - q^{60n+16}) \times \prod_{n=0}^{\infty}(1 - q^{60n+28}) \times$$

$$\times \prod_{n=0}^{\infty}(1 - q^{60n+40}) \times \prod_{n=0}^{\infty}(1 - q^{60n+52})$$

or,

$$(q^4; q^{12})_\infty = (q^4; q^{60})_\infty (q^{16}; q^{60})_\infty (q^{28}; q^{60})_\infty (q^{40}; q^{60})_\infty (q^{52}; q^{60})_\infty$$
$$= (q^4, q^{16}, q^{28}, q^{40}, q^{52}; q^{60})_\infty \qquad (1.16)$$

Similarly we can compute following as

$$(q^5; q^5)_\infty = (q^5; q^{15})_\infty (q^{10}; q^{15})_\infty (q^{15}; q^{15})_\infty$$
$$= (q^5, q^{10}, q^{15}; q^{15})_\infty \qquad (1.17)$$

$$(q^6; q^6)_\infty = (q^6; q^{24})_\infty (q^{12}; q^{24})_\infty (q^{18}; q^{24})_\infty (q^{24}; q^{24})_\infty$$
$$= (q^6, q^{12}, q^{18}, q^{24}; q^{24})_\infty \qquad (1.18)$$

$$(q^6; q^{12})_\infty = (q^6; q^{60})_\infty (q^{18}; q^{60})_\infty (q^{30}; q^{60})_\infty (q^{42}; q^{60})_\infty (q^{54}; q^{60})_\infty$$
$$= (q^6, q^{18}, q^{30}, q^{42}, q^{54}; q^{60})_\infty \qquad (1.19)$$

The outline of this paper is as follows. In sections 2, some results on continued fraction [5-8], and also some well known results recorded by Ramanujan [9], are listed, those are useful to the rest of the paper. In section 3, we established seven new results by generalizing Rmanujan's identities in terms of q-products and continued fractions, using the properties Jacobi's triple product identities. Findings are new and not available in the literature of special functions. In section 4, we provide the proofs for newly established results.

II. Preliminaries

In [9, p. 224], Ramanujan recorded following identities

Entry(i):

$$\frac{\ (q^7)\ \ (q^9) - \ (-q^7)\ \ (-q^9)}{\ (q)\ \ (q^{63}) - \ (-q)\ \ (-q^{63})} = q^6 \qquad (2.1)$$

Entry(ii):

$$\frac{\ (q^5)\ \ (q^{11}) - \ (-q^5)\ \ (-q^{11})}{\ (q)\ \ (q^{55}) - \ (-q)\ \ (-q^{55})} = q^5 \qquad (2.2)$$

Entry(iii):

$$\frac{\ (q^3)\ \ (q^{13}) - \ (-q^3)\ \ (-q^{13})}{\ (q)\ \ (q^{39}) - \ (-q)\ \ (-q^{39})} = q^3 \qquad (2.3)$$

In [9, p. 230], Ramanujan recorded following identities

Entry(vii):

$$\ (q)\ \ (q^{11}) - \ (-q)\ \ (-q^{11}) = 2qf(q^2, q^{10})f(q^{44}, q^{88}) + 2q^{15}\phi(q^6)\ \ (q^{132}) \qquad (2.4)$$

Ref.

5. G.E. Andrews; *An introduction to Ramanujan's Lost notebook*, Amer. Math. Monthly. 86(2)(1979), 89-98.

9. S. Ramanujan; *Notebooks (Volume I)*, Tata Institute of Fundamental Research, Bombay, 1957.

© 2012 Global Journals Inc. (US)

In [9, p. 299], Ramanujan recorded following identities

Entry(ii):

$$\phi(q)\phi(q^{27}) - \phi(-q)\phi(-q^{27}) = 4qf(-q^6)f(-q^{18}) + 4q^7 \quad (q^2) \quad (q^{54}) \tag{2.5}$$

Entry(iii):

$$\phi(q)\phi(q^{35}) - \phi(-q)\phi(-q^{35}) = 4qf(-q^{10})f(-q^{14}) + 4q^9 \quad (q^2) \quad (q^{70}) \tag{2.6}$$

Entry(iv):

$$\phi(q^5)\phi(q^7) - \phi(-q^5)\phi(-q^7) = 4q^3 \quad (q^{10}) \quad (q^{14}) - 2q^3f(-q^2)f(-q^{70}) \tag{2.7}$$

In [7], following continued fractional identities is given

$$(q^2;q^2)_\infty(-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1+\vdots}}}}}}} \tag{2.8}$$

Following Rogers-Ramanujan continued fraction is one of the most celebrated identities associated with Ramanujan's academic career [8],

$$C(q) = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} = 1 + \cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cfrac{q^5}{1+\vdots}}}}} \tag{2.9}$$

In [5, equation (1.6)], the famous Rogers-Ramanujan continued fraction identity is given

$$\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\vdots}}}}} \tag{2.10}$$

In [6, equation (4.21)], following Ramanujan continued fraction identity is given

$$\frac{(-q^3;q^4)_\infty}{(-q;q^4)_\infty} = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^3+q^2}{1+\cfrac{q^5}{1+\cfrac{q^7+q^4}{1+\cfrac{q^9}{1+\cfrac{q^{11}+q^6}{1+\vdots}}}}}}} \tag{2.11}$$

Ref.

7. R.Y. Denis; *On certain q-series and continued fractions*, Math. Students, 44(1-4)(1983), 70-76.

9. S. Ramanujan; *Notebooks (Volume I)*, Tata Institute of Fundamental Research, Bombay, 1957.

© 2012 Global Journals Inc. (US)

III. Main Results

In this section, we established seven new results by using (.) and $\phi(.)$ functions in Ramanujan identities [9], or in more general language we can say that by using the properties of Jacobi's triple product identity, as (.) and $\phi(.)$ functions are special cases of it, and further applying the properties of continued fraction identities. These results are new, and not recorded in the literature of special functions

$$q^6=\left[\frac{(-q^7;q^{14})_\infty(-q^9;q^{18})_\infty-(q^7;q^{14})_\infty(q^9;q^{18})_\infty}{(-q;q^2)_\infty(-q^{63};q^{126})_\infty-(q;q^2)_\infty(q^{63};q^{126})_\infty}\right]\times$$

$$\times\frac{(-q,q;q^2)_\infty(-q^{63},q^{63};q^{126})_\infty}{(q^2;q^2)_\infty(-q^7;q^{14})_\infty(-q^9;q^{18})_\infty(q^{126};q^{126})_\infty}\times$$

$$\times\cfrac{1}{1-\cfrac{q^7}{1+\cfrac{q^7(1-q^7)}{1-\cfrac{q^{21}}{1+\cfrac{q^{14}(1-q^{14})}{1-\cfrac{q^{35}}{1+\cfrac{q^{21}(1-q^{21})}{1+\vdots}}}}}}}\times\cfrac{1}{1-\cfrac{q^9}{1+\cfrac{q^9(1-q^9)}{1-\cfrac{q^{27}}{1+\cfrac{q^{18}(1-q^{18})}{1-\cfrac{q^{45}}{1+\cfrac{q^{27}(1-q^{27})}{1+\vdots}}}}}}} \qquad (3.1)$$

$$q^5=\left[\frac{(-q^5;q^{10})_\infty(-q^{11};q^{22})_\infty-(q^5;q^{10})_\infty(q^{11};q^{22})_\infty}{(-q;q^2)_\infty(-q^{55};q^{110})_\infty-(q;q^2)_\infty(q^{55};q^{110})_\infty}\right]\times$$

$$\times\frac{(-q,q;q^2)_\infty(-q^{55},q^{55};q^{110})_\infty}{(q^2;q^2)_\infty(-q^5;q^{10})_\infty(-q^{11};q^{22})_\infty(q^{110};q^{110})_\infty}\times$$

$$\times\cfrac{1}{1-\cfrac{q^5}{1+\cfrac{q^5(1-q^5)}{1-\cfrac{q^{15}}{1+\cfrac{q^{10}(1-q^{10})}{1-\cfrac{q^{25}}{1+\cfrac{q^{15}(1-q^{15})}{1+\vdots}}}}}}}\times\cfrac{1}{1-\cfrac{q^{11}}{1+\cfrac{q^{11}(1-q^{11})}{1-\cfrac{q^{33}}{1+\cfrac{q^{22}(1-q^{22})}{1-\cfrac{q^{55}}{1+\cfrac{q^{33}(1-q^{33})}{1+\vdots}}}}}}} \qquad (3.2)$$

$$q^3=\left[\frac{(-q^3;q^6)_\infty(-q^{13};q^{26})_\infty-(q^3;q^6)_\infty(q^{13};q^{26})_\infty}{(-q;q^2)_\infty(-q^{39};q^{78})_\infty-(q;q^2)_\infty(q^{39};q^{78})_\infty}\right]\times$$

$$\times\frac{(-q,q;q^2)_\infty(-q^{39},q^{39};q^{78})_\infty}{(q^2;q^2)_\infty(q^{78};q^{78})_\infty(-q^3;q^6)_\infty(-q^{13};q^{26})_\infty}\times$$

Ref.

9. S. Ramanujan; *Notebooks (Volume I)*, Tata Institute of Fundamental Research, Bombay, 1957.

© 2012 Global Journals Inc. (US)

$$\times \cfrac{1}{1-\cfrac{q^3}{1+\cfrac{q^3(1-q^3)}{1-\cfrac{q^9}{1+\cfrac{q^6(1-q^6)}{1-\cfrac{q^{15}}{1+\cfrac{q^9(1-q^9)}{1+\vdots}}}}}}} \times \cfrac{1}{1-\cfrac{q^{13}}{1+\cfrac{q^{13}(1-q^{13})}{1-\cfrac{q^{39}}{1+\cfrac{q^{26}(1-q^{26})}{1-\cfrac{q^{65}}{1+\cfrac{q^{39}(1-q^{39})}{1+\vdots}}}}}}} \qquad (3.3)$$

$$2q(q^{12};q^{12})_\infty\left[(-q^2,-q^{10};q^{12})_\infty(-q^{44},q^{88};q^{132})_\infty+q^{14}(-q^6;q^{12})^2_\infty\frac{(q^{264};q^{264})_\infty}{(q^{132};q^{264})_\infty}\right]$$

$$=\left[\frac{(-q;q^2)_\infty(-q^{11};q^{22})_\infty-(q;q^2)_\infty(q^{11};q^{22})_\infty}{(-q;q^2)_\infty(-q^{11};q^{22})_\infty}\right]\times$$

$$\times \cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1+\vdots}}}}}}} \times \cfrac{1}{1-\cfrac{q^{11}}{1+\cfrac{q^{11}(1-q^{11})}{1-\cfrac{q^{33}}{1+\cfrac{q^{22}(1-q^{22})}{1-\cfrac{q^{55}}{1+\cfrac{q^{33}(1-q^{33})}{1+\vdots}}}}}}} \qquad (3.4)$$

$$(q^2;q^2)_\infty(q^{54};q^{54})_\infty\left[(-q;q^2)^2_\infty(-q^{27};q^{54})^2_\infty-(q;q^2)^2_\infty(q^{27};q^{54})^2_\infty\right]$$

$$=4q(q^6;q^6)_\infty(q^{18};q^{18})_\infty+4q^7\times$$

$$\times \cfrac{1}{1-\cfrac{q^2}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^6}{1+\cfrac{q^4(1-q^4)}{1-\cfrac{q^{10}}{1+\cfrac{q^6(1-q^6)}{1+\vdots}}}}}}} \times \cfrac{1}{1-\cfrac{q^{54}}{1+\cfrac{q^{54}(1-q^{54})}{1-\cfrac{q^{162}}{1+\cfrac{q^{108}(1-q^{108})}{1-\cfrac{q^{270}}{1+\cfrac{q^{162}(1-q^{162})}{1+\vdots}}}}}}} \qquad (3.5)$$

$$(q^2;q^2)_\infty(q^{70};q^{70})_\infty\left[(-q;q^2)^2_\infty(-q^{35};q^{70})^2_\infty-(q;q^2)^2_\infty(q^{35};q^{70})^2_\infty\right]$$

$$=4q(q^{10};q^{10})_\infty(q^{14};q^{14})_\infty+4q^9\times$$

$$\times \cfrac{1}{1-\cfrac{q^2}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^6}{1+\cfrac{q^4(1-q^4)}{1-\cfrac{q^{10}}{1+\cfrac{q^6(1-q^6)}{1+\vdots}}}}}}} \times \cfrac{1}{1-\cfrac{q^{70}}{1+\cfrac{q^{70}(1-q^{70})}{1-\cfrac{q^{210}}{1+\cfrac{q^{140}(1-q^{140})}{1-\cfrac{q^{350}}{1+\cfrac{q^{210}(1-q^{210})}{1+\vdots}}}}}}} \qquad (3.6)$$

Notes

© 2012 Global Journals Inc. (US)

Notes

$$(q^{10};q^{10})_\infty(q^{14};q^{14})_\infty\left[(-q^5;q^{10})_\infty^2(-q^7;q^{14})_\infty^2-(q^5;q^{10})_\infty^2(q^7;q^{14})_\infty^2\right]$$
$$=-2q^3(q^2;q^2)_\infty(q^{70};q^{70})_\infty+4q^3\times$$
$$\times\cfrac{1}{1-\cfrac{q^{10}}{1+\cfrac{q^{10}(1-q^{10})}{1-\cfrac{q^{30}}{1+\cfrac{q^{20}(1-q^{20})}{1-\cfrac{q^{50}}{1+\cfrac{q^{30}(1-q^{30})}{1+\vdots}}}}}}}\times\cfrac{1}{1-\cfrac{q^{14}}{1+\cfrac{q^{14}(1-q^{14})}{1-\cfrac{q^{42}}{1+\cfrac{q^{28}(1-q^{28})}{1-\cfrac{q^{70}}{1+\cfrac{q^{42}(1-q^{42})}{1+\vdots}}}}}}} \quad (3.7)$$

IV. Proofs For Main Results (3.1) to (3.7)

Proof of (3.1): In (1.7), put $q=-q,q^7,-q^7,q^9,-q^9,q^{63},-q^{63}$ respectively, we get

$$(-q)=\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty},\ \psi(q^7)=\frac{(q^{14};q^{14})_\infty}{(q^7;q^{14})_\infty},\ \psi(-q^7)=\frac{(q^{14};q^{14})_\infty}{(-q^7;q^{14})_\infty} \quad (3.1.1)$$

$$(q^9)=\frac{(q^{18};q^{18})_\infty}{(q^9;q^{18})_\infty},\ \psi(-q^9)=\frac{(q^{18};q^{18})_\infty}{(-q^9;q^{18})_\infty} \quad (3.1.2)$$

$$(q^{63})=\frac{(q^{126};q^{126})_\infty}{(q^{63};q^{126})_\infty},\ \psi(-q^{63})=\frac{(q^{126};q^{126})_\infty}{(-q^{63};q^{126})_\infty} \quad (3.1.3)$$

Now, substituting the values from (3.1.1) to (3.1.3), and using (1.7) into (2.1), after simplifications by applying the properties of q-product identities and further using continued fraction (2.8), we get desired result (3.1).

Proofs of (3.2) and (3.3): On similar lines of proof for (3.1), we can easily obtain proofs for (3.2) and (3.3).

Proof of (3.4): In (1.7), put $q=-q,q^{11},-q^{11},q^{132}$, respectively, we get

$$(-q)=\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty},\ \psi(q^{11})=\frac{(q^{22};q^{22})_\infty}{(q^{11};q^{22})_\infty},\ \psi(-q^{11})=\frac{(q^{22};q^{22})_\infty}{(-q^{11};q^{22})_\infty},\ \psi(q^{132})=\frac{(q^{264};q^{264})_\infty}{(q^{132};q^{264})_\infty} \quad (3.4.1)$$

again by putting $q=q^6$ in (1.6), we get

$$\phi(q^6)=(-q^6;q^{12})_\infty^2(q^{12};q^{12})_\infty \quad (3.4.2)$$

also by putting $a=q^2,b=q^{10}$ and $a=q^{44},b=q^{88}$ respectively in (1.5), we get

$$f(q^2,q^{10})=(-q^2;q^{12})_\infty(-q^{10};q^{12})_\infty(q^{12};q^{12})_\infty \quad (3.4.3)$$

$$f(q^{44},q^{88})=(-q^{44};q^{132})_\infty(-q^{88};q^{132})_\infty(q^{132};q^{132})_\infty \quad (3.4.4)$$

© 2012 Global Journals Inc. (US)

Now, substituting the values from (3.4.1) to (3.4.4), and using (1.7) into (2.4), after simplifications by applying the properties of q-product identities and further using continued fraction (2.8), we get desired result (3.4).

Proof of (3.5): In (1.6), put $q = -q, q^{27}, -q^{27}$, respectively, we get

$$\phi(-q) = (q; q^2)^2_\infty (q^2; q^2)_\infty \tag{3.5.1}$$

and

$$\phi(q^{27}) = (-q^{27}; q^{54})^2_\infty (q^{54}; q^{54})_\infty,\ \phi(-q^{27}) = (q^{27}; q^{54})^2_\infty (q^{54}; q^{54})_\infty \tag{3.5.2}$$

by substituting $q = q^2,\ q^{54}$ respectively in (1.7), we get

$$(q^2) = \frac{(q^4; q^4)_\infty}{(q^2; q^4)_\infty},\ \psi(q^{54}) = \frac{(q^{108}; q^{108})_\infty}{(q^{54}; q^{108})_\infty} \tag{3.5.3}$$

again by substituting $q = q^6,\ q^{18}$ respectively in (1.8), we get

$$f(-q^6) = (q^6; q^6)_\infty,\ f(-q^{18}) = (q^{18}; q^{18})_\infty \tag{3.5.4}$$

Now, substituting the values from (3.5.1) to (3.5.4), and using (1.6) into (2.5), after simplifications by applying the properties of q-product identities and further using continued fraction (2.8), we get desired result (3.5).

Proofs of (3.6) and (3.7): On similar lines of proof for (3.5), we can easily obtain proofs for (3.6) and (3.7).

References Références Referencias

1. M.P. Chaudhary; *On q-product identities,* preprint.
2. B.C. Berndt; *What is a q-series?,* preprint.
3. B.C. Berndt; *Ramanujan's notebook Part III,* Springer-Verlag, New York, 1991.
4. B.C. Berndt; *Ramanujan's notebook Part V,* Springer-Verlag, New York, 1998.
5. G.E. Andrews; *An introduction to Ramanujan's Lost notebook,* Amer. Math. Monthly. 86(2)(1979), 89-98.
6. G.E. Andrews; *Ramanujan's Lost notebook III. The Rogers-Ramanujan continued fraction,* Advances in Mathematics 41(2)(1981), 186-208.
7. R.Y. Denis; *On certain q-series and continued fractions,* Math. Students, 44(1-4)(1983), 70-76.
8. R.Y. Denis; *On basic hypergeometric functions and continued fractions,* Math. Students, 15(1-4)(1984), 47-58.
9. S. Ramanujan; *Notebooks (Volume I),* Tata Institute of Fundamental Research, Bombay, 1957.
10. S. Ramanujan; *Notebooks (Volume II),* Tata Institute of Fundamental Research, Bombay, 1957.

Notes

© 2012 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics & Decision Sciences
Volume 12 Issue 2 Version 1.0 February 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On Quivers and Incidence Algebras

By Viji M. & R.S.Chakravarti
Cochin University of Science and Technology, Cochin, Kerala

Abstract - By giving a generalized definition for the quiver algebra we obtain a surjective homomorphism between the quiver algebra of locally finite acyclic quiver and the incidence algebra of corresponding poset.

Keywords and phrases : Incidence algebra, Quiver, Path algebra.

AMS Subject Classication: 16S50.

ON QUIVERS AND INCIDENCE ALGEBRAS

Strictly as per the compliance and regulations of :

Open Association of Research Society, USA
Research | Diversity | Ethics

© 2012. Viji M. & R.S.Chakravarti.This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Global Journal of Science Frontier Research (F) Volume XII Issue II Version I February 2012

On Quivers and Incidence Algebras

Viji M.[α] & R.S.Chakravarti[σ]

Abstract - By giving a generalized definition for the quiver algebra we obtain a surjective homomorphism between the quiver algebra of locally finite acyclic quiver and the incidence algebra of corresponding poset.

Keywords and Phrases : Incidence algebra, Quiver, Path algebra.

I. Introduction

A *quiver* ([2]) $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two set: Q_0(whose elements are called *points*, or *vertices*) and Q_1 (whose elements are called *arrows*) and two maps $s, t : Q_1 \to Q_0$ which associates to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$, respectively. Hereafter we use the notation $Q = (Q_0, Q_1)$ or simply Q to denote a quiver. A *path of length* l in Q is a sequence of arrows $(\alpha_1, \alpha_2, ..., \alpha_l)$ of Q, of length l, such that $s(\alpha_{i+1}) = t(\alpha_i)$. A path of length 0, from a point a to a is denoted by ε_a and it is called *stationary path*.

Let Q be a quiver. The *Path Algebra* KQ, of Q is the $K-$algebra, whose underlying $K-$vector space has as a basis, the set of all paths $(a|\alpha_1, \alpha_2, ..., \alpha_l|b)$ of length ≥ 0. The product of 2 basis elements $(a|\alpha_1, \alpha_2, ..., \alpha_l|b)$ and $(c|\beta_1, \beta_2, ..., \beta_m|d)$ of KQ is defined as,

$$(a|\alpha_1, \alpha_2, ..., \alpha_l|b).(c|\beta_1, \beta_2, ..., \beta_m|d) = \delta_{bc}(a|\alpha_1, ..., \alpha_l, \beta_1, ...\beta_m|d).$$

Let KQ_l be the subspace of KQ generated by the set Q_l of all paths of length l, where $l \geq 0$. It is clear that $(KQ_n).(KQ_m) \subseteq (KQ_{n+m})$ and we have the direct sum decomposition

$$KQ = KQ_0 \oplus KQ_1 \oplus ... \oplus KQ_l \oplus ... .$$

KQ is an associative algebra. It has an identity if and only if Q_0 is finite and acyclic.

Let Q be a quiver. The two sided ideal of the path algebra KQ generated (as an ideal) by the arrows of Q is called the *arrow ideal* of KQ and is denoted by R_Q. So

Author α : Dept. of Mathematics, St.Thomas' College, Thrissur-680001, Kerala. E-mail : vijigeethanjaly@gmail.com
Author σ : Dept. of Mathematics, Cochin University of Science and Technology, Cochin-682022, Kerala.
E-mail : rsc@cusat.ac.in

Ref.

[2] Ibrahim Assem, Daniel Simson and Andrzej Skowronski; *Elements of Representation Theory of Associative algebras, Vol.1 Techniques of Representation Theory*, London Mathematical Society Student Texts- vol. 65, Cambridge University press, 20.

© 2012 Global Journals Inc. (US)

$$R_Q = KQ_1 \oplus KQ_2 \oplus ... \oplus KQ_l \oplus ... .$$

Let R_Q^l denote the ideal of KQ generated, as a $K-$ vectorspace, by the set of all paths of length $\geq l$.

A two-sided ideal I of KQ is said to be *admissible* if there exists $m \geq 2$ such that $R_Q^m \subseteq I \subseteq R_Q^2$. If I is an admissible ideal of KQ, the pair (Q, I) is called *bound quiver* and the quotient algebra KQ/I is called a *bound quiver algebra*.

A Quiver Q is said to be *connected* if the underlying graph is connected. An algebra A is said to be *connected* if A is not a direct product of two algebras, or equivalently, 0 and 1 are the only central idempotents.

A partially ordered set X is said to be *locally finite* if, the subset $X_{yz} = \{x \in X : y \leq x \leq z\}$ is finite for each $y \leq z \in X$. The Incidence algebra $I(X, R)$ of a locally finite partially ordered set X over the commutative ring R with identity is
$I(X, R) = \{f : X \times X \to R \mid f(x, y) = 0 \; if \; x \nleq y\}$
with operations defined by

$$(f + g)(x, y) = f(x, y) + g(x, y),$$

$$(f.g)(x, y) = \sum_{x \leq z \leq y} f(x, z).g(z, y),$$

$$(r.f)(x, y) = r.f(x, y)$$

for all $f, g \in I(X, R)$, $r \in R$ and $x, y, z \in X$.

The identity element of $I(X, R)$ is $\delta(x, y) = \begin{cases} 1 & if \; x = y \\ 0 & Otherwise \end{cases}$

For a finite partially ordered set X, the incidence algebra $I(X, K)$ is a subalgebra of the matrix algebra $M_n(K)$. The following theorem characterize finite dimensional incidence algebras.([1], Theorem 4.2.10)

Theorem 1. *Let K be a field and S be a subalgebra of $M_n(K)$. Then there exists a partially ordered set X of order n such that $I(X, K) \cong S$ if and only if*

(i) S contains n pairwise orthogonal idempotent and

(ii) $S/J(S)$ is commutative.

And, for incidence algebras of lower finite partially ordered sets we have the following characterization: ([3], Theorem 2.)

Theorem 2. *Let V be a $K-$vector space with dimension $|X|$, for a suitable set X. Let S be a subalgebra of $End_K V$. Then there exists a lower finite partial ordering in X such that $S \cong I(X, K)$ if and only if,*

Ref.

[1] E.Spiegel and C.J.O'Donnell ; *Incidence Algebras*, Monographs and Textbooks in Pure and Applied Mathematics, Vol.206, Marcel Dekker, Newyork, 1997.

© 2012 Global Journals Inc. (US)

(1) $1 \in S$

(2) $S/J(S)$ *is commutative*

(3) For each $x \in X$*, there is an* $E_x \in S$ *of rank 1, such that*

$$E_x.E_y = \delta_{xy}E_x \text{ and } \bigoplus_{x\in X} E_x(V) = V$$

(4) $X_y = \{z \in X \mid E_z.S.E_y \neq 0\}$ *is finite for each* $y \in X$

Notes

II. The Partially Ordered Set Corresponding To An Acyclic Quiver

Let Q be an acyclic quiver. Let Q_0 denote the set of all points of Q. We may define an order on Q_0 by $i \leq j$ if and only if there exists a path from i to j. Since $\varepsilon_a \in Q$, $\forall a \in Q_0$, we have $i \leq i$, $\forall i \in Q_0$. If $i \neq j$ and $i < j$, then $j \nleq i$, since Q is acyclic. If there exist a path α from i to j and β from j to k, $\alpha\beta$ is a path from i to k. So $i \leq j$ and $j \leq k$ implies $i \leq k$. So $(Q_0, \leq)$ is a partially ordered set. Clearly $(Q_0, \leq)$ is locally finite for a finite quiver Q.

Proposition 1. Let Q be a finite acyclic quiver such that there exists at most 1 path from i to j, for each pair $i, j \in Q_0$. Then the path algebra KQ is isomorphic to the incidence algebra $I(Q_0, K)$.

Proof. Let $Q = (Q_0, Q_1)$ be a finite acyclic quiver such that, there exists at most 1 path from i to j, for each pair $i, j \in Q_0$. If $i \leq j$, denote the unique path from i to j by α_{ij}. Define $\phi : KQ \to I(Q_0, K)$ such that $\alpha_{ij} \mapsto E_{ij}$ where $E\delta_{ij}$ is the function which assumes the value 1 at (i, j) and zero elsewhere. This is an isomorphism from KQ to $I(Q_0, K)$, since ϕ is a bijective map from basis of KQ $(\{\alpha_{ij}|i, j \in Q_0\})$ to a basis of $I(Q_0, K)$ $(\{\delta_{ij}|i, j \in Q_0\})$ and it preserves addition, multiplication and identity element. Hence the theorem.

Definition 1. If a quiver $Q = (Q_0, Q_1)$ is such that there exists atmost one path from x to y for each pair $x, y \in Q_0$, then we call Q a *unique path quiver*.

Proposition 2. Let K be a field and S be a subalgebra of $M_n(K)$. Then there is a unique path quiver $Q = (Q_0, Q_1)$ with n vertices such that $KQ \cong S$ if and only if

(i) S contains n pairwise orthogonal idempotents and

(ii) $S/J(S)$ is commutative.

Proposition 3. Given a finite acyclic quiver $Q = (Q_0, Q_1)$ there exists a surjective homomorphism from KQ onto the associated incidence algebra $I(Q_0, K)$ and this becomes an isomorphism if and only if Q is such that, there exists atmost one path from i to j, for each pair $i, j \in Q_0$.

© 2012 Global Journals Inc. (US)

Proof. Since Q is finite and acyclic, KQ is finite dimensional with the set of all paths as its basis. Let $Q_0 = \{x_1, x_2, ..., x_n\}$ and let for each $i, j \in Q_0$, $i \leq j$ whenever there exists a path from i to j. $I(Q_0, K)$ will be isomorphic to a subalgebra S of $T_n(K)$. If E_{ij} denote the $n \times n$ matrix with 1 at the $(i, j)th$ position and zeros elsewhere. It is clear that whenever $i \leq j$, $E_{ij} \in S$. Now define $\varphi : KQ \rightarrow I(Q_0, K)$ such that $\varphi(\alpha) = E_{ij}$ if α is a path from i to j. If $\alpha : i \rightarrow j$ and $\beta : m \rightarrow n$ are two paths Q, $\alpha\beta = 0$ if $j \neq m$ and $\alpha\beta$ is a path from i to n, if $j = m$.

$$\varphi(\alpha\beta) = \begin{cases} E_{in} & \text{if } j = m \\ 0 & \text{Otherwise} \end{cases}$$

$$= E_{ij}.E_{mn}$$

$$= \varphi(\alpha).\varphi(\beta)$$

$\phi(\sum_{i \in Q_0} \varepsilon_i) = I_n$ since $\varepsilon_i \mapsto E_{ii}$. Hence ϕ is a surjective map from the basis of KQ onto a basis of $I(Q_0, K)$, which is compatible with the addition, multiplication and scalar multiplication. Hence ϕ is a surjective homomorphism from KQ onto $I(Q_0, K)$. Clearly if there exists at most one path from i to j for each pair $i, j \in Q_0$, then $dim(KQ) = dim(I(Q_0, K))$. So $KQ \cong I(Q_0, K)$.

Remark 1. Under the above defined surjective homomorphism ϕ, we can reach at the following results.

(1) If Q is a finite acyclic quiver then the Jacobson radical of KQ will be mapped on to the Jacobson Radical of $I(Q_0, K)$

(2) R_Q^l will be mapped on to the two sided ideal J_l of $I(Q_0, K)$, where $J_l = \{f \in I(Q_0, K) | f(x, y) = 0 \ \textit{if the length of the longest chain from } x \textit{ to } y \textit{ is } \leq l\}$

Definition 2. Let $Q = (Q_0, Q_1)$ be an acyclic quiver. Then Q is said to be a *locally finite quiver*, if for each pair $i, j \in Q_0$, there exists only finitely many paths from i to j and is said to be *lower finite* if for each $x \in Q_0$ there exist only finitely many paths that ends at x.

Note that if $Q = (Q_0, Q_1)$ is an acyclic locally finite quiver, then the associated partial order set is also locally finite.

Proposition 4. If Q is an acyclic locally finite quiver and $(Q_0, \leq)$ is the associated locally finite partially ordered set, then there exists a homomorphism $\phi : KQ \rightarrow I(Q_0, K)$ and this homomorphism is injective if and only if Q is such that, for each pair $i, j \in Q_0$ there exists at most one path from i to j.

Notes

© 2012 Global Journals Inc. (US)

Notes

Proof. Let V be a $K-$vector space of dimension $\mid Q_0 \mid$. Let $\{v_i \mid i \in Q_0\}$ be a basis of V. For each pair $i, j \in Q_0$ there exists $E_{ij} \in End_K V$ such that $E_{ij}(v_k) = \delta_{jk} v_i$. Let $S = span\{E_{ij} \mid i, j \in Q_0, i \leq j\}$. This is a subalgebra of $I(Q_0, K)$, since E_{ij} can be mapped to $\delta_{ij} \in I(Q_0, K)$. These δ_{ij}s will span a subalgebra of $I(Q_0, K)$. Denote this subalgebra by A. We have, $S \cong A$. Call this isomorphism by ψ .

Now, consider a basis of KQ, which is the set of all paths in Q. If α is a path from i to j, then define $\phi : KQ \to S$ such that $\alpha \mapsto E_{ij}$.This is a homomorphism from KQ to S.

Now, $\phi \circ \psi : KQ \to I(Q_0, K)$ is a homomorphism. It is clear that this becomes injective if and only if there exists at most one path from i to j for each pair $i, j \in Q_0$.

Remark 2. KQ has an identity if and only if Q is finite and acyclic. But $I(Q_0, K)$ always has an identity. So that $\phi \circ \psi$ can not be surjective in general.

Remark 3. Associated to a finite acyclic quiver we get a unique partially ordered set. But the converse is not true. For example, corresponding to $X = \{1, 2\}$ together with the usual ordering we get countably many quivers with n arrows between 1 and 2 for any natural number $n \in \mathbb{N}$.

III. Path Algebra: A Generalized Definition

Definition 3. Let Q be a quiver, and let P be the set of all paths in Q. A *Path Algebra* of Q is defined as $\left\{\sum_{\alpha \in P} c_\alpha \alpha \mid c_\alpha \in K, \alpha \in P\right\}$. We define addition and scalar multiplication componentwise. If $(a \mid \alpha_1, \alpha_2, ..., \alpha_l \mid b)$ and $(c \mid \beta_1, \beta_2, ..., \beta_m \mid d)$ are any paths in Q, we define their product as,

$(a \mid \alpha_1, \alpha_2, ..., \alpha_l \mid b).(c \mid \beta_1, \beta_2, ..., \beta_m \mid d) = \delta_{bc}(a \mid \alpha_1, ..., \alpha_l, \beta_1, ...\beta_m \mid d)$. The product of two arbitrary elements of KQ can be defined by assuming distributivity of multiplication of paths over arbitrary summation.

$$\therefore \left(\sum_{\alpha \in P} c_\alpha \alpha\right)\left(\sum_{\beta \in P} d_\beta \beta\right) = \sum_{\alpha, \beta \in P} c_\alpha d_\beta \alpha\beta$$

This is well defined since $\alpha\beta = 0$ if $t(\alpha) \neq s(\beta)$ and since $\alpha\beta$ is a path, it is of finite length and so it can be expressed as a product of 2 paths only in finitely many ways.

$$\text{Define } KQ_l = \left\{\sum_{\alpha \in P} c_\alpha \alpha \mid c_\alpha = 0 \text{ if length of } \alpha \neq l\right\}.$$

© 2012 Global Journals Inc. (US)

KQ can be expressed as a direct product of KQ_l for $l \geq 0$. i.e.,

$$KQ = KQ_0 \times KQ_1 \times ... \times KQ_l \times ...$$

Clearly $(KQ_n).(KQ_m) \subseteq KQ_{n+m}\ \forall n, m \geq 0$.

Note that if Q is a finite acyclic quiver, then our generalized definition and old definition of path algebra coincides. So the results we obtained in the previous section for finite acyclic quiver holds, even when we use the generalized definition of path algebra. For a finite acyclic quiver Q, the set of all its paths P, will serve as a basis for KQ. Here after we use the generalized definition of path algebra.

Notes

Proposition 5. Let Q be a quiver and KQ be the corresponding path algebra.Then,

(a) KQ is an associative algebra.

(b) The element $\sum\limits_{a\in Q_0} \varepsilon_a$ is the identity in KQ.

(c) KQ is finite dimensional if and only if Q is finite and acyclic.

Proof. (a) The fact that KQ is an associative algebra, follows directly from the definition of multiplication, because, the product of paths is the composition of paths and hence it is associative. Any element in KQ is an arbitrary linear combination of paths. So associativity holds in general, since we have distributivity of multiplication over arbitrary summation.

(b) Let $\sum\limits_{\alpha\in P} c_\alpha \alpha \in KQ$ be arbitrary.

$$\begin{aligned}\left(\sum_{a\in Q_0} \varepsilon_a\right) \cdot \left(\sum_{\alpha\in P} c_\alpha \alpha\right) &= \sum_{\alpha\in P} c_\alpha \left[\left(\sum_{a\in Q_0} \varepsilon_a\right).\alpha\right] \\ &= \sum_{\alpha\in P} c_\alpha \left(\sum_{a\in Q_0} \varepsilon_a.\alpha\right), \\ &= \sum_{\alpha\in P} c_\alpha \alpha\end{aligned}$$

since $\varepsilon_a.\alpha = \begin{cases} \alpha, \text{ if } s(\alpha) = a \\ 0, \text{otherwise} \end{cases}$

Similarly since $\alpha.\varepsilon_a = \begin{cases} \alpha, \text{ if } t(\alpha) = a \\ 0, \text{otherwise} \end{cases}$,

we get $\left(\sum\limits_{\alpha\in P} c_\alpha \alpha\right) \cdot \left(\sum\limits_{a\in Q_0} \varepsilon_a\right) = \left(\sum\limits_{\alpha\in P} c_\alpha \alpha\right)$

Therefore, $\sum\limits_{a\in Q_0} \varepsilon_a$ serves as the identity of KQ.

© 2012 Global Journals Inc. (US)

Notes

(c) If Q is infinite, so is the set P. Span$(P) \subseteq KQ$ and P is linearly independent. So that KQ is infinite dimensional.

Now if Q is cyclic, then there is atleast one cycle, say ω in Q.
Then $\omega^l \in P\ \forall\, l \geq 1$, which implies P is infinite and hence KQ is also infinite dimensional.

Conversely, if Q is finite and acyclic, then $\mid P \mid$ is finite and in this case P serves as a basis for KQ. Hence KQ is finite dimensional.

Proposition 6. Let $Q = (Q_0, Q_1)$ be a unique path quiver then an element $a \in KQ$ is a unit if and only if the coefficient a_{xx} of the stationary path ε_x is nonzero for all $x \in Q_0$.

Proof. Let $a = \sum_{x,y \in Q_0} a_{xy}\alpha_{xy}$ be a unit element of KQ, where α_{xy} is the unique path from x to y, if there is one. Then there exists a $b = \sum_{x,y \in Q_0} b_{xy}\alpha_{xy}$ in KQ such that $ab = \sum_{x \in Q_0} \varepsilon_x$. That is

$$\sum_{x,y,z,u \in Q_0} a_{xy} b_{zu} \alpha_{xy} \alpha_{zu} = \sum_{x \in Q_0} \varepsilon_x$$

$$\Rightarrow \sum_{x,u \in Q_0} \quad \sum_{y \in Q_0} a_{xy} b_{yu} \Bigg) \alpha_{xu} = \sum_{x \in Q_0} \varepsilon_x$$

Equating coefficients on both sides we may conclude that the coefficients of each stationary path should be nonzero.

Conversely, suppose that $a = \sum_{x,y \in Q_0} a_{xy}\alpha_{xy}$ is such that $a_{xx} \neq 0$ for all $x \in Q_0$. Then there is an element $b \in KQ$ such that

$$b_{xy} = 1/a_{xx},\ if\ x = y$$

$$= \tfrac{-1}{a_{xx}} \sum_{z \in Q_0 - \{x\}} a_{xz} b_{zy},\ if\ x \neq y$$

So that if $x = y$ coefficient of $\varepsilon_x = a_{xx}.b_{xx} = 1$ and

if $x \neq y$, coefficient of α_{xy} in the product $a.b = \sum_{z \in Q_0} a_{xz} b_{zy}$

But,

$$\sum_{z \in Q_0} a_{xz} b_{zy} = \qquad a_{xx} b_{xy} + \sum_{z \in Q_0 - \{x\}} a_{xz} b_{zy}$$

$$= a_{xx} \tfrac{-1}{a_{xx}} \sum_{z \in Q_0 - \{x\}} a_{xz} b_{zy} + \sum_{z \in Q_0 - \{x\}} a_{xz} b_{zy}$$

$$= 0$$

Hence $a.b = \sum_{x \in Q_0} \varepsilon_x$ which implies that a is a unit.

© 2012 Global Journals Inc. (US)

Remark 4. $\{\varepsilon_a \mid a \in Q_0\}$ of all stationary paths in Q is a set of primitive orthogonal idempotents for KQ such that $\sum_{a\in Q_0} \varepsilon_a = 1 \in KQ$.

Proposition 7. Let Q be a quiver and KQ be its path algebra. Then KQ is connected if and only if Q is connected.

Proof. To prove this, we first prove that KQ is connected if and only if there does not exist a nontrivial partition $I \dot{\cup} J$ of Q_0 such that if $i \in I$ and $j \in J$ then, $\varepsilon_i(KQ)\varepsilon_j = 0 = \varepsilon_j(KQ)\varepsilon_i$. Assume that there exists such a partition for Q_0. Let $c = \sum_{j\in J} \varepsilon_j$. Since the partition is nontrivial $c \neq 0$ or 1. Since ε_j's are primitive orthogonal idempotents and multiplication in KQ is distributive over arbitrary sum, we can conclude that c is an idempotent. Also,

$c.\varepsilon_i = 0 = \varepsilon_i.c, \ \forall i \in I$ and

$c.\varepsilon_j = 0 = \varepsilon_j.c, \ \forall j \in J$.

According to our hypothesis $\varepsilon_i.a.\varepsilon_j = 0 = \varepsilon_j.a.\varepsilon_i$, $\forall i \in I$ and $\forall j \in J$ and $\forall a \in KQ$.

Therefore,

$$\begin{aligned} c.a &= \left.\sum_{j\in J} \varepsilon_j \right) .a \\ &= \left.\sum_{j\in J} \varepsilon_j.a \right) .1 \\ &= \left.\sum_{j\in J} \varepsilon_j.a \right) . \left(\sum_{i\in I} \varepsilon_i + \sum_{k\in J} \varepsilon_k\right) \\ &= \sum_{k,j\in J} \varepsilon_j a \varepsilon_k \\ &= \left.\sum_{j\in J} \varepsilon_j + \sum_{i\in I} \varepsilon_i \right) a \left(\sum_{k\in J} \varepsilon_k\right) \\ &= a.c \end{aligned}$$

which implies c is a nontrivial central idempotent. Hence KQ is not connected. Conversely, if KQ is not connected, it contains a nontrivial central idempotent, say c.

Therefore,

$$\begin{aligned} c &= 1.c.1 \\ &= \left.\sum_{i\in Q_0} \varepsilon_i \right) .c. \left.\sum_{j\in Q_0} \varepsilon_j \right) \\ &= \sum_{i,j\in Q_0} \varepsilon_i c \varepsilon_j \end{aligned}$$

Notes

© 2012 Global Journals Inc. (US)

$= \sum_{i \in Q_0} \varepsilon_i c \varepsilon_i$, since c is central

Now let $c_i = \varepsilon_i c = c\varepsilon_i = \varepsilon_i c \varepsilon_i \in \varepsilon_i (KQ) \varepsilon_i$

So that, $c_i^2 = (\varepsilon_i c \varepsilon_i)(\varepsilon_i c \varepsilon_i) = \varepsilon_i c^2 \varepsilon_i = \varepsilon_i c \varepsilon_i = c_i$,

hence c_i is an idempotent.

But ε_i's are primitive, so that either $c_i = 0$ or $c_i = 1$, since

Notes

$\varepsilon_i = \varepsilon_i (1 - c_i + c_i)$

$= \varepsilon_i (1 - c_i) + \varepsilon_i c_i$

So, $\varepsilon_i = \varepsilon_i c_i$ or $\varepsilon_i = \varepsilon_i (1 - c_i)$.

Let $I = \{i \in Q_0 / c_i = 0\}$ and $J = \{j \in Q_0 / c_j = 1\}$. Since $c \neq 0, 1$, this is a nontrivial partition of Q_0. And if $i \in I$ then, $\varepsilon_i c = c\varepsilon_i = 0$ and if $j \in J$ then, $\varepsilon_j c = c\varepsilon_j = \varepsilon_j$.

Therefore if $i \in I$ and $j \in J$, $\varepsilon_i(KQ)\varepsilon_j = \varepsilon_i(KQ)c\varepsilon_j = \varepsilon_i c(KQ)\varepsilon_j = 0$.

Similarly, $\varepsilon_j(KQ)\varepsilon_i = 0$.

Now assume that KQ is not connected. Let Q' be a connected component of Q. Let Q'' be the full subquiver of Q having the set of points $Q_0'' = Q_0 \backslash Q_0'$. Since Q is not connected, both Q_0' and Q_0'' are nonempty. Let $a \in Q_0'$ and $b \in Q_0''$. Since Q is not connected, then if α is any path in Q, either α is entirely contained in Q' or α is entirely contained in Q''

If α is contained in Q' then, $\alpha.\varepsilon_b = 0$ and so $\varepsilon_a.\alpha.\varepsilon_b = 0$.

If α is contained in Q'' then, $\varepsilon_a.\alpha = 0$ and so $\varepsilon_a.\alpha.\varepsilon_b = 0$.

Therefore, $\varepsilon_a (KQ) \varepsilon_b = 0$. Similarly, $\varepsilon_b (KQ) \varepsilon_a = 0$

This implies KQ is not connected.

Now assume that Q is connected but KQ is not. We have a nontrivial disjoint union of Q_0 such that $Q_0 = Q_0' \dot\cup Q_0''$ and if $a \in Q'$ and $b \in Q''$ then, $\varepsilon_a (KQ) \varepsilon_b = 0 = \varepsilon_b (KQ) \varepsilon_a$.

Since Q is connected, there exists some $a_0 \in Q_0'$ and some $b_0 \in Q_0''$ such that they are neighbors. Without loss of generality, suppose that there exists an arrow $\alpha : a_0 \to b_0$. Therefore, $\alpha = \varepsilon_{a_0}.\alpha.\varepsilon_{b_0} \in \varepsilon_{a_0} (KQ) \varepsilon_{b_0} = 0$, which is a contradiction. Hence KQ is connected.

Definition 4. Let Q be a quiver and KQ be its path algebra. The two-sided ideal of KQ, is called *arrow ideal* and is denoted by R_Q if it is defined by,

$$R_Q = \left\{ \sum_{\alpha \in P} c_\alpha \alpha | \ c_\alpha = 0, \text{ if } \alpha \text{ is a stationary path} \right\}$$

© 2012 Global Journals Inc. (US)

Let R_Q^l denote the two-sided ideal of KQ generated by the paths of length $\geq l$. So that

$$R_Q^l = \left\{ \sum_{\alpha \in P} c_\alpha \alpha \mid c_\alpha = 0, \text{ if } \alpha \text{ is a path of length less than } l \right\}.$$

Therefore $\frac{R_Q^l}{R_Q^{l+1}} \cong KQ_l$

Definition 5. A two-sided ideal I of KQ is said to be *admissible* if there exists $m \geq 2$ such that

$$R_Q^m \subseteq I \subseteq R_Q^2.$$

If I is an admissible ideal of KQ, the pair (Q, I) is called *bound quiver* and the quotient algebra KQ/I is called a *bound quiver algebra*.

Proposition 8. Let Q be a quiver and I be an admissible ideal of KQ. The set $\{e_a = \varepsilon_a + I \mid a \in Q_0\}$ is a set of primitive orthogonal idempotents of the bound quiver algebra KQ/I and $\sum_{a \in Q_0} e_a = 1_{KQ/I}$

Proof. Since e_a is the image of ε_a under the canonical homomorphism from $KQ \to KQ/I$, and $\sum_{a \in Q_0} \varepsilon_a = 1$, it is clear that $\{e_a = \varepsilon_a + I \mid a \in Q_0\}$ is a set of orthogonal idempotents such that $\sum_{a \in Q_0} e_a = 1_{KQ/I}$. Now we have to prove that each e_a is primitive. That is only idempotents of $e_a (KQ/I) e_a$ are zero and e_a. Any idempotent of $e_a (KQ/I) e_a$ can be written in the form $e = \lambda \varepsilon_a + \omega + I$, $\lambda \in K$ and ω is a linear combination of cycles of length ≥ 1. Therefore, since e is an idempotent,

$$(\lambda \varepsilon_a + \omega)^2 + I = (\lambda \varepsilon_a + \omega) + I$$

$$i.e., \ (\lambda \varepsilon_a + \omega)^2 - (\lambda \varepsilon_a + \omega) \in I$$

$$i.e., \ (\lambda^2 - \lambda) \varepsilon_a + (2\lambda - 1) \omega + \omega^2 \in I$$

Since $I \subseteq R_Q^2$, $(\lambda^2 - \lambda) = 0$ which implies $\lambda = 0 \, or \, 1$

If $\lambda = 0$, $e = \omega + I$ and then, ω is an idempotent modulo I.Since $R_Q^m \subseteq I$ for some $m \geq 2$, $\omega^m \in I$ and so $\omega \in I$. So that $e = 0 \in KQ/I$.

If $\lambda = 1$, then $e = \varepsilon_a + \omega + I$ and $e_a - e = -\omega + I$ is an idempotent in $e_a (KQ/I) e_a$. So that ω is an idempotent modulo I, which implies $\omega^m \in I$ which in turn implies that $\omega \in I$. Hence $e_a - e \in I$ and $e_a = e$ modulo I.

Proposition 9. Let Q be a quiver and I be an admissible ideal of KQ. The bound quiver algebra KQ/I is connected if and only if Q is a connected quiver.

Notes

© 2012 Global Journals Inc. (US)

Proof. Let Q be not connected. By Proposition 5, we have KQ is not connected. And this implies that there exists a nontrivial central idempotent γ (*neither* 0 *nor* 1) which can be chosen as a sum of paths of stationary paths. Then $c = \gamma + I \neq I$. If $c = 1 + I$ then, $1 - \gamma \in I$, which is not possible, since $I \subseteq R_Q^2$. Hence c is a nontrivial central idempotent of KQ/I and so KQ/I is not connected as an algebra.

Notes

Conversely, assume that Q is a connected quiver, but KQ/I is not a connected algebra. Then, there exists a nontrivial partition $Q_0 = Q_0' \dot{\cup} Q_0''$ such that whenever $x \in Q_0'$ and $y \in Q_0''$, then $e_x (KQ/I) e_y = 0 = e_y (KQ/I) e_x$. Since Q is a connected quiver, There is some $a \in Q_0'$ and $b \in Q_0''$ that are neighbors. With out loss of generality we may assume that there exists an arrow from a to b. Then, $\alpha = \varepsilon_a \alpha \varepsilon_b$ and so, $\overline{\alpha} = \alpha + I$ satisfies $\overline{\alpha} = e_a \overline{\alpha} e_b \in e_a (KQ/I) e_b = 0$. As $\overline{\alpha} \neq I \left(\because I \subseteq R_Q^2\right)$, This is a contradiction. Hence KQ/I is connected.

IV. The Relation Between The Path Algebra of an Acyclic Quiver and the Incidence Algebra of the Associated Poset

In the second section, we discussed some homomorphism between Path algebras of finite and acyclic quivers and Incidence algebras of associated partially ordered sets. Now we discuss the same for infinite dimensional algebras.

Proposition 10. Let Q be a unique path quiver. Then $KQ \cong I(Q_0, K)$

Proof. Given that there exists atmost one path from x to y for each pair $x, y \in Q_0$. Denote this path by α_{xy}. An arbitrary element $a \in KQ$ can be written as

$$a = \sum_{\alpha_{xy} \in P} a_{xy} \alpha_{xy}$$

Define $\Phi : KQ \to I(Q_0, K)$ by $\Phi(a) = f_a$ where,

$$f_a(x, y) = a_{xy}$$

If $x \nleq y$, there is no path from x to y, so that the coefficient of α_{xy} in $a = a_{xy} = 0$.

So that $f_a(x, y) = 0$. Hence $f_a \in I(Q_0, K)$.

Now let $a = \sum_{\alpha_{xy} \in P} a_{xy} \alpha_{xy}$ and $b = \sum_{\alpha_{xy} \in P} b_{xy} \alpha_{xy}$

Then,

$$f_{a+b} = \Phi(a + b) = \Phi \left(\sum_{\alpha_{xy} \in P} a_{xy} \alpha_{xy} + \sum_{\alpha_{xy} \in P} b_{xy} \alpha_{xy} \right)$$

© 2012 Global Journals Inc. (US)

$$= \Phi\left(\sum_{\alpha_{xy}\in P}(a_{xy}+b_{xy})\alpha_{xy}\right)$$

So that $\Phi(a+b)=\Phi(a)+\Phi(b)$.

Let $\Phi(ab) \ = f_{ab}$. Then,

$$\begin{aligned}\Phi(ab) &= \Phi\left(\left(\sum_{\alpha_{xy}\in P} a_{xy}\alpha_{xy}\right)\ \sum_{\alpha_{uv}\in P} b_{uv}\alpha_{uv}\right)\right) \\ &= \Phi\left(\sum_{\alpha_{xy},\alpha_{uv}\in P} a_{xy}b_{uv}\left(\alpha_{xy}\alpha_{uv}\right)\right) \\ &= \Phi\ \sum_{\alpha_{xv}\in P}\ \sum_{x\leqslant y\leqslant v} a_{xy}b_{yv}\right)\alpha_{xv}\right)\end{aligned}$$

Therefore, $f_{ab}(x,y) \ = \ \sum_{x\leqslant z\leqslant y} a_{xz}b_{zy}\right) = (f_a.f_b)(x,y)$, which implies $\Phi(ab)=\Phi(a).\Phi(b)$

$$\Phi\ \sum_{a\in Q_0}\varepsilon_a\right) \ = \delta = \text{identity in}\ \ I\left(Q_0,K\right)$$

$\Phi\left(c.a\right) \quad = c.\Phi(a),\ \text{ for }\ c\in K,\ \ a\in KQ$

So that Φ is a homomorphism from KQ to $I(Q_0,K)$.

Now let $f\in I(Q_0,K)$, then there exists an $a=\sum_{\alpha_{xy}\in P} f(x,y)\alpha_{xy}\in KQ$ such that $\Phi(a)=f$. Hence Φ is onto.

If $a,b\in KQ$ *such that* $\Phi(a)=\Phi(b)$ then,

$$\begin{aligned}&\Phi(a)(x,y)=\Phi(b)(x,y)\ \ \forall x,y\in Q_0\\ &i.e.\quad a_{xy}=b_{xy}\ \ \forall x,y\in Q_0\\ &i.e.\quad a=b\end{aligned}$$

So Φ is one-one and hence it is an isomorphism.

Combining theorem 2 and proposition 10 we can reach at the following result

Proposition 11. Let K be a field and V be a $K-$vectorspace. Let S be a subalgebra of $End_K(V)$. Then there exists a lower finite unique path quiver $Q=(Q_0,Q_1)$ with $\mid Q_0\mid = dim(V)$ such that $KQ\cong S$ if and only if

(i) $1\in S$

Notes

© 2012 Global Journals Inc. (US)

(ii) $S/J(S)$ is commutative.

(iii) For each $x \in Q_0$, there is $E_x \in S$ of rank 1 such that $E_x.E_y = \delta_{xy} E_x$ where δ_{xy} is the Kronecker's delta and $\sum_{x \in Q_o} E_x(V) = V$

(iv) $X_y = \{z \in Q_0 : E_z.S.E_y \neq 0\}$ is finite for each $y \in Q_0$

Notes

Proposition 12. Let Q be a locally finite acyclic quiver. Then there exists a surjective homomorphism from KQ to $I(Q_0, K)$.

Proof. Let Q be a locally finite acyclic quiver and P be the set of all paths in Q. Since Q is locally finite, there exists only finitely many paths from x to y for each pair $x, y \in Q_0$. Let n_{xy} denote the number of paths from x to y in Q, and let $\alpha_{xy}^{(1)}, \alpha_{xy}^{(2)}, ..., \alpha_{xy}^{(n_{xy})}$ denote the n_{xy} paths from x to y in Q. Let $a \in KQ$ be arbitrary. So that a can be written as $a = \sum_{\alpha \in P} a_\alpha \alpha$. Let a_{xy} denote the sum of coefficients of all paths from x to y that comes in a. Define $\Phi : KQ \to I(Q_0, K)$ by $\Phi(a) = f_a$, where $f_a(x, y) = a_{xy}$ As in the previous proposition, it is easy to verify that $f_a \in I(Q_0, K)$, Φ preserves addition and scalar multiplication, Φ maps identity of KQ to identity of $I(Q_0, K)$. Now we prove that Φ preserves multiplication. Let $a = \sum_{\alpha \in P} a_\alpha \alpha$ and $b = \sum_{\beta \in P} b_\beta \beta$. So that $ab = \sum_{\alpha, \beta \in P} a_\alpha b_\beta \alpha\beta$. Let us denote the sum of coefficients of all paths from x to y that comes in ab by $(ab)_{xy}$. Note that $\alpha\beta$ is a path from x to y if and only if $s(\alpha) = x$ and $t(\beta) = y$ and $t(\alpha) = s(\beta)$. So,

$$\begin{aligned}
(ab)_{xy} &= \sum_{\substack{x \leq z \leq y \\ 1 \leq m \leq n_{xz} \\ 1 \leq n \leq n_{zy}}} a_{\alpha_{xz}^{(m)}} b_{\beta_{zy}^{(n)}} \\
&= \sum_{x \leq z \leq y} \left(\sum_{1 \leq m \leq n_{xz}} a_{\alpha_{xz}^{(m)}} \right) \left(\sum_{1 \leq n \leq n_{zy}} b_{\beta_{zy}^{(n)}} \right) \\
&= \sum_{x \leq z \leq y} a_{xz} b_{zy} \\
&= \sum_{x \leq z \leq y} f_a(x, z) f_b(z, y) \\
&= (f_a.f_b)(x, y)
\end{aligned}$$

So that Φ is a homomorphism from KQ to $I(Q_0, K)$. Now, let $f \in I(Q_0, K)$ and denote any fixed path from x to y by α_{xy}. So that there exists some $a = \sum_{x, y \in Q_0} f(x, y) \alpha_{xy} \in KQ$ such that $\Phi(a) = f$.

Hence Φ is a surjective homomorphism.

References Références Referencias

[1] E.Spiegel and C.J.O'Donnell ; *Incidence Algebras*, Monographs and Textbooks in Pure and Applied Mathematics, Vol.206, Marcel Dekker, Newyork, 1997.

© 2012 Global Journals Inc. (US)

[2] Ibrahim Assem, Daniel Simson and Andrzej Skowronski; *Elements of Representation Theory of Associative algebras, Vol.1 Techniques of Representation Theory*, London Mathematical Society Student Texts- vol. 65, Cambridge University press, 20.

[3] Viji M. and R.S.Chakravarti; *When is an algebra of endomorphisms an incidence algebra?* Global Journal of Science Frontier Research, volume 10(4),September 2010, 67-68.

Notes

© 2012 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics & Decision Sciences
Volume 12 Issue 2 Version 1.0 February 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Two Summation Formulae Relating Hypergeometric Function

By Salahuddin
P.D.M College of Engineering,Haryana,India

Abstract - The aim of the present paper is to obtain two summation formulae associated to Hypergeometric function.The results derived in this paper are of general character and are believed to be new.

Keywords and phrases : Contiguous relation,Recurrence relation, Gauss second summation theorem.

A.M.S. Subject Classification (2010): 33C05 , 33C20 ,33D50 , 33D60.

TWO SUMMATION FORMULAE RELATING HYPERGEOMETRIC FUNCTION

Strictly as per the compliance and regulations of :

Open Association of Research Society, USA
Research | Diversity | Ethics

© 2012. Salahuddin.This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Global Journal of Science Frontier Research (F) Volume XII Issue II Version I February 2012

Ref.

6. Rainville, E. D.; The contiguous function relations for ${}_pF_q$ with applications to Bateman's $J_n^{u,v}$ and Rice's $H_n\ (\zeta, p, \nu)$, *Bull. Amer. Math. Soc.*, 51(1945), 714-723.

Two Summation Formulae Relating Hypergeometric Function

Salahuddin

Abstract - The aim of the present paper is to obtain two summation formulae associated to Hypergeometric function.The results derived in this paper are of general character and are believed to be new.

Keywords and Phrases : Contiguous relation,Recurrence relation, Gauss second summation theorem.

I. Introduction

The special function is one of the central branches of Mathematical sciences initiated by LEuler .But systematic study of the Hypergeometric functions were initiated by C.F Gauss, an imminent German Mathematician in 1812 by defining the Hypergeometric series and he had also proposed notation for Hypergeometric functions. Since about 250 years several talented brains and promising Scholars have been contributed to this area. Some of them are C.F Gauss, G.H Hardy , S. Ramanujan ,A.P Prudnikov , W.W Bell , Yu. A Brychkov and G.E Andrews.

Generalized Gaussian Hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{matrix} a_1, a_2, \cdots, a_A & ; \\ & & z \\ b_1, b_2, \cdots, b_B & ; \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} $$

or

$$ {}_AF_B\left[\begin{matrix} (a_A) & ; \\ & & z \\ (b_B) & ; \end{matrix}\right] \equiv {}_AF_B\left[\begin{matrix} (a_j)_{j=1}^{A} & ; \\ & & z \\ (b_j)_{j=1}^{B} & ; \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \qquad (1) $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers. The series converges for all finite z if $A \leq B$, converges for$|\ z\ |< 1$ if A=B+1, diverges for all z, $z \neq 0$ if$A > B + 1$.

Contiguous Relation is defined by

Following Eq. (10), p-51 of ref [6], we write

$$ (a-b)\ {}_2F_1\left[\begin{matrix} a,\ b\ ; & \\ c\ \ ; & z \end{matrix}\right] = a\ {}_2F_1\left[\begin{matrix} a+1,\ b\ ; & \\ c\ \ ; & z \end{matrix}\right] - b\ {}_2F_1\left[\begin{matrix} a,\ b+1\ ; & \\ c\ ; & z \end{matrix}\right] \qquad (2) $$

Author : P.D.M College of Engineering,Bahadurgarh ,Haryana,India. E-mails: sludn@yahoo.com; vsludn@gmail.com

© 2012 Global Journals Inc. (US)

Recurrence relation is defined by

$$\Gamma(z+1) = z\,\Gamma(z) \tag{3}$$

Gauss second summation theorem is defined by [Prudnikov., 491(7.3.7.3)]

$${}_2F_1\left[\begin{matrix} a,\ b\ ; \\ \frac{a+b+1}{2}\ ; \end{matrix}\ \ \frac{1}{2}\right] = \frac{\Gamma(\frac{a+b+1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})} \tag{4}$$

$$= \frac{2^{(b-1)}\,\Gamma(\frac{b}{2})\,\Gamma(\frac{a+b+1}{2})}{\Gamma(b)\,\Gamma(\frac{a+1}{2})} \tag{5}$$

II. Main Results of Summation Formulae

$${}_2F_1\left[\begin{matrix} a,\ b\ ; \\ \frac{a+b+23}{2}\ ; \end{matrix}\ \ \frac{1}{2}\right] = \frac{2^b\,\Gamma(\frac{a+b+23}{2})}{(a-b)\,\Gamma(b)}\times$$

$$\times\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{1024a(654729075-1396704420a+1094071221a^2-444647600a^3+107494190a^4)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+\right.\right.$$

$$+\frac{1024a(-16486680a^5+1646778a^6-106800a^7+4335a^8-100a^9+a^{10}-400914000b)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024a(4564470450ab-1410623712a^2b+1263684888a^3b-155769600a^4b+42918540a^5b)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024a(-2331168a^6b+255192a^7b-5040a^8b+210a^9b+2644887945b^2-265793584ab^2)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024a(3183848164a^2b^2-293010704a^3b^2+257688830a^4b^2-11918928a^5b^2+3222324a^6b^2)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024a(-57456a^7b^2+5985a^8b^2+368444608b^3+2290676024ab^3-33209568a^2b^3)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024a(529562376a^3b^3-17364480a^4b^3+14271432a^5b^3-217056a^6b^3+54264a^7b^3)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024a(407004318b^4+126838376ab^4+413414806a^2b^4-904400a^3b^4+26340650a^4b^4)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

Notes

© 2012 Global Journals Inc. (US)

Notes

$$+\frac{1024a(-271320a^5b^4+203490a^6b^4+32111520b^5+117320364ab^5+9767520a^2b^5)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024a(21434280a^3b^5+352716a^5b^5+9231474b^6+4019792ab^6+7533652a^2b^6+180880a^3b^6)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024a(293930a^4b^6+357312b^7+1020984ab^7+93024a^2b^7+116280a^3b^7+38367b^8+14364ab^8)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024a(20349a^2b^8+560b^9+1330ab^9+21b^{10})}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024b(654729075-400914000a+2644887945a^2+368444608a^3+407004318a^4+32111520a^5)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024b(9231474a^6+357312a^7+38367a^8+560a^9+21a^{10}-1396704420b+4564470450ab)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024b(-265793584a^2b+2290676024a^3b+126838376a^4b+117320364a^5b+4019792a^6b)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024b(1020984a^7b+14364a^8b+1330a^9b+1094071221b^2-1410623712ab^2)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024b(3183848164a^2b^2-33209568a^3b^2+413414806a^4b^2+9767520a^5b^2+7533652a^6b^2)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024b(93024a^7b^2+20349a^8b^2-444647600b^3+1263684888ab^3-293010704a^2b^3)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024b(529562376a^3b^3-904400a^4b^3+21434280a^5b^3+180880a^6b^3+116280a^7b^3)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024b(107494190b^4-155769600ab^4+257688830a^2b^4-17364480a^3b^4+26340650a^4b^4)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024b(293930a^6b^4-164866680b^5+42918540ab^5-11918928a^2b^5+14271432a^3b^5)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

© 2012 Global Journals Inc. (US)

$$+\frac{1024b(-271320a^4b^5+352716a^5b^5+1646778b^6-2331168ab^6+3222324a^2b^6)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{(1024b-217056a^3b^6+203490a^4b^6-106800b^7+255192ab^7-57456a^2b^7+54264a^3b^7)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$\left.+\frac{1024b(4335b^8-5040ab^8+5985a^2b^8-100b^9+210ab^9+b^{10})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}\right\}-$$

$$-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{2048(654729075+400914000a+2644887945a^2-368444608a^3+407004318a^4}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+\right.$$

$$+\frac{2048(-32111520a^5+9231474a^6-357312a^7+38367a^8-560a^9+21a^{10}+1396704420b)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{2048(4564470450ab+265793584a^2b+2290676024a^3b-126838376a^4b+117320364a^5b)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{2048(-4019792a^6b+1020984a^7b-14364a^8b+1330a^9b+1094071221b^2+1410623712ab^2)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{2048(3183848164a^2b^2+33209568a^3b^2+413414806a^4b^2-9767520a^5b^2+7533652a^6b^2)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{2048(-93024a^7b^2+20349a^8b^2+444647600b^3+1263684888ab^3+293010704a^2b^3)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{2048(529562376a^3b^3+904400a^4b^3+21434280a^5b^3-180880a^6b^3+116280a^7b^3)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{2048(107494190b^4+155769600ab^4+257688830a^2b^4+17364480a^3b^4+26340650a^4b^4)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{2048(293930a^6b^4+164866680b^5+42918540ab^5+11918928a^2b^5+14271432a^3b^5+271320a^4b^5)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

Notes

© 2012 Global Journals Inc. (US)

Notes

$$+\frac{2048(352716a^5b^5+1646778b^6+2331168ab^6+3222324a^2b^6+217056a^3b^6+203490a^4b^6)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{2048(+106800b^7+255192ab^7+57456a^2b^7+54264a^3b^7+4335b^8+5040ab^8+5985a^2b^8)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{2048(100b^9+210ab^9+b^{10})}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{2048(654729075+1396704420a+1094071221a^2+444647600a^3+107494190a^4+16486680a^5)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{2048(1646778a^6+106800a^7+4335a^8+100a^9+a^{10}+400914000b+4564470450ab)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{2048(1410623712a^2b+1263684888a^3b+155769600a^4b+42918540a^5b+2331168a^6b)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{2048(255192a^7b+5040a^8b+210a^9b+2644887945b^2+265793584ab^2+3183848164a^2b^2)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{2048(293010704a^3b^2+257688830a^4b^2+11918928a^5b^2+3222324a^6b^2+57456a^7b^2)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{2048(5985a^8b^2-3684446608b^3+2290676024ab^3+33209568a^2b^3+529562376a^3b^3)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{2048(17364480a^4b^3+14271432a^5b^3+217056a^6b^3+54264a^7b^3+407004318b^4)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{2048(-126838376ab^4+413414806a^2b^4+904400a^3b^4+26340650a^4b^4+271320a^5b^4)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{2048(203490a^6b^4-321115520b^5+117320364ab^5-9767520a^2b^5+21434280a^3b^5)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{2048(352716a^5b^5+9231474b^6-4019792ab^6+7533652a^2b^6-180880a^3b^6+293930a^4b^6)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

© 2012 Global Journals Inc. (US)

$$+\frac{2048(-357312b^{7}+1020984ab^{7}-93024a^{2}b^{7}+116280a^{3}b^{7}+38367b^{8}-14364ab^{8})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{2048(20349a^{2}b^{8}-560b^{9}+1330ab^{9}+21b^{10})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}\Bigg\}\Bigg] \quad (6)$$

$${}_2F_1\left[\begin{matrix}a, & b & ; & \frac{1}{2}\\ \frac{a+b+24}{2} & & ; & \end{matrix}\right]=\frac{2^{b}\,\Gamma(\frac{a+b+24}{2})}{(a-b)\,\Gamma(b)}\times$$

$$\times\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a}{2})}\Bigg\{\frac{2048(3715891200a-5441863680a^{2}+3264915456a^{3}-1076416000a^{4})}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+\right.$$

$$+\frac{2048(218683520a^{5}-28865760a^{6}+2524368a^{7}-145200a^{8}+5280a^{9}-110a^{10}+a^{11})}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{2048(3715891200b+18690693120a^{2}b-4089046016a^{3}b+3093104256a^{4}b-317412480a^{5}b)}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{2048(75431664a^{6}b-3589344a^{7}b+347424a^{8}b-6160a^{9}b+231a^{10}b+5441863680b^{2})}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{2048(18690693120ab^{2}+9866191104a^{3}b^{2}-699103328a^{4}b^{2}+531899984a^{5}b^{2}-21114016a^{6}b^{2})}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{2048(4975872a^{7}b^{2}-79002a^{8}b^{2}+7315a^{9}b^{2}+3264915456b^{3}+4089046016ab^{3})}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{2048(9866191104a^{2}b^{3}+1327912432a^{4}b^{3}-35814240a^{5}b^{3}+25467904a^{6}b^{3}-341088a^{7}b^{3})}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{2048(74613a^{8}b^{3}+1076416000b^{4}+3093104256ab^{4}+699103328a^{2}b^{4}+1327912432a^{3}b^{4})}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{2048(55711040a^{5}b^{4}-497420a^{6}b^{4}+319770a^{7}b^{4}+218683520b^{5}+317412480ab^{5})}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{2048(531899984a^{2}b^{5}+35814240a^{3}b^{5}+55711040a^{4}b^{5}+646646a^{6}b^{5}+28865760b^{6})}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

Notes

© 2012 Global Journals Inc. (US)

$$+\frac{2048(75431664ab^{6}+21114016a^{2}b^{6}+25467904a^{3}b^{6}+497420a^{4}b^{6}+646646a^{5}b^{6}+2524368b^{7})}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{2048(3589344ab^{7}+4975872a^{2}b^{7}+341088a^{3}b^{7}+319770a^{4}b^{7}+145200b^{8}+347424ab^{8})}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{2048(79002a^{2}b^{8}+74613a^{3}b^{8}+5280b^{9}+6160ab^{9}+7315a^{2}b^{9}+110b^{10}+231ab^{10}+b^{11})}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{4096b(3715891200+1199554560a+4962674688a^{2}+720247296a^{3}+469992064a^{4})}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{4096b(34181280a^{5}+7691376a^{6}+270864a^{7}+24816a^{8}+330a^{9}+11a^{10}-1199554560b)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{4096b(12030259200ab+1008349696a^{2}b+3230041600a^{3}b+198001888a^{4}b+113212512a^{5}b)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{4096b(3702160a^{6}b+743424a^{7}b+9702a^{8}b+770a^{9}b+4962674688b^{2}-1008349696ab^{2})}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{4096b(5777911552a^{2}b^{2}+181722688a^{3}b^{2}+473992848a^{4}b^{2}+12633936a^{5}b^{2}+6273344a^{6}b^{2})}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{4096b(75240a^{7}b^{2}+13167a^{8}b^{2}-720247296b^{3}+3230041600ab^{3}-181722688a^{2}b^{3})}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{4096b(747974976a^{3}b^{3}+9586640a^{4}b^{3}+20837376a^{5}b^{3}+198968a^{6}b^{3}+85272a^{7}b^{3})}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{4096b(469992064b^{4}-198001888ab^{4}+473992848a^{2}b^{4}-9586640a^{3}b^{4}+30749600a^{4}b^{4})}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{4096b(135660a^{5}b^{4}+248710a^{6}b^{4}-34181280b^{5}+113212512ab^{5}-12633936a^{2}b^{5})}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{4096b(20837376a^{3}b^{5}-135660a^{4}b^{5}+352716a^{5}b^{5}+7691376b^{6}-3702160ab^{6})}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

Notes

© 2012 Global Journals Inc. (US)

$$+\frac{4096b(6273344a^2b^6-198968a^3b^6+248710a^4b^6-270864b^7+743424ab^7-75240a^2b^7)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{4096b(+85272a^3b^7+24816b^8-9702ab^8+13167a^2b^8-330b^9+770ab^9+11b^10)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}\Bigg\}-$$

$$-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+1}{2})}\Bigg\{\frac{4096a(3715891200-1199554560a+4962674688a^2-720247296a^3+469992064a^4)}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{4096a(-34181280a^5+7691376a^6-270864a^7+24816a^8-330a^9+11a^{10}+1199554560b)}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{4096a(12030259200ab-1008349696a^2b+3230041600a^3b-198001888a^4b+113212512a^5b)}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{4096a(-3702160a^6b+743424a^7b-9702a^8b+770a^9b+4962674688b^2+1008349696ab^2)}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{4096a(5777911552a^2b^2-181722688a^3b^2+473992848a^4b^2-12633936a^5b^2+6273344a^6b^2)}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{4096a(-75240a^7b^2+13167a^8b^2+720247296b^3+3230041600ab^3+181722688a^2b^3)}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{4096a(747974976a^3b^3-9586640a^4b^3+20837376a^5b^3-198968a^6b^3+85272a^7b^3)}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{4096a(469992064b^4+198001888ab^4+473992848a^2b^4+9586640a^3b^4+30749600a^4b^4)}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{4096a(-135660a^5b^4+248710a^6b^4+34181280b^5+113212512ab^5+12633936a^2b^5)}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{4096a(20837376a^3b^5+135660a^4b^5+352716a^5b^5+7691376b^6+3702160ab^6+6273344a^2b^6)}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{4096a(198968a^3b^6+248710a^4b^6+270864b^7+743424ab^7+75240a^2b^7+85272a^3b^7)}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

Notes

© 2012 Global Journals Inc. (US)

$$+\frac{4096a(24816b^8+9702ab^8+13167a^2b^8+330b^9+770ab^9+11b^{10})}{\left[\prod_{\eta=0}^{10}\{a-b-2\eta\}\right]\left[\prod_{\vartheta=1}^{11}\{a-b+2\vartheta\}\right]}+$$

$$+\frac{2048(3715891200a+5441863680a^2+3264915456a^3+1076416000a^4+218683520a^5)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

Notes

$$+\frac{2048(28865760a^6+2524368a^7+145200a^8+5280a^9+110a^{10}+a^{11}+3715891200b)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{2048(18690693120a^2b+4089046016a^3b+3093104256a^4b+317412480a^5b+75431664a^6b)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{2048(3589344a^7b+347424a^8b+6160a^9b+231a^10b-5441863680b^2+18690693120ab^2)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{2048(9866191104a^3b^2+699103328a^4b^2+531899984a^5b^2+21114016a^6b^2+4975872a^7b^2)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{2048(79002a^8b^2+7315a^9b^2+3264915456b^3-4089046016ab^3+9866191104a^2b^3)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{2048(1327912432a^4b^3+35814240a^5b^3+25467904a^6b^3+341088a^7b^3+74613a^8b^3)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{2048(-1076416000b^4+3093104256ab^4-699103328a^2b^4+1327912432a^3b^4+55711040a^5b^4)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{2048(497420a^6b^4+319770a^7b^4+218683520b^5-317412480ab^5+531899984a^2b^5)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{2048(-35814240a^3b^5+55711040a^4b^5+646646a^6b^5-28865760b^6+75431664ab^6)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{2048(-21114016a^2b^6+25467904a^3b^6-497420a^4b^6+646646a^5b^6+2524368b^7-3589344ab^7)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

$$+\frac{2048(4975872a^2b^7-341088a^3b^7+319770a^4b^7-145200b^8+347424ab^8)}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}+$$

© 2012 Global Journals Inc. (US)

$$+\frac{2048(-79002a^2b^8+74613a^3b^8+5280b^9-6160ab^9+7315a^2b^9-110b^{10}+231ab^{10}+b^{11})}{\left[\prod_{\delta=0}^{11}\{a-b-2\delta\}\right]\left[\prod_{\zeta=1}^{10}\{a-b+2\zeta\}\right]}\Bigg\}\Bigg] \quad (7)$$

III. Derivation of Summation Formula (6)

Substituting $c=\frac{a+b+23}{2}$ and $z=\frac{1}{2}$ in equation (2), we get

$$(a-b)\ {}_2F_1\left[\begin{matrix} a,\ b & ; & \frac{1}{2} \\ \frac{a+b+23}{2} & ; & \end{matrix}\right]=a\ {}_2F_1\left[\begin{matrix} a+1,\ b & ; & \frac{1}{2} \\ \frac{a+b+23}{2} & ; & \end{matrix}\right]-b\ {}_2F_1\left[\begin{matrix} a,\ b+1 & ; & \frac{1}{2} \\ \frac{a+b+23}{2} & ; & \end{matrix}\right]$$

Now applying the formula obtained by Salahuddin [Salahuddin.,p.12(9)], we get

$$L.H.S=a\,\frac{2^b\ \Gamma(\frac{a+b+23}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a+1}{2})}\left\{\frac{1024(654729075-1396704420a+1094071221a^2)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+\right.\right.$$

$$+\frac{1024(-444647600a^3+107494190a^4-16486680a^5+1646778a^6-106800a^7+4335a^8-100a^9)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(a^{10}-400914000b+4564470450ab-1410623712a^2b+1263684888a^3b-155769600a^4b)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(42918540a^5b-2331168a^6b+255192a^7b-5040a^8b+210a^9b+2644887945b^2)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(-265793584ab^2+3183848164a^2b^2-293010704a^3b^2+257688830a^4b^2-11918928a^5b^2)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(3222324a^6b^2-57456a^7b^2+5985a^8b^2+3684444608b^3+2290676024ab^3-33209568a^2b^3)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(529562376a^3b^3-17364480a^4b^3+14271432a^5b^3-217056a^6b^3+54264a^7b^3)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(407004318b^4+126838376ab^4+413414806a^2b^4-904400a^3b^4+26340650a^4b^4)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue II Version I February 2012

© 2012 Global Journals Inc. (US)

Notes

$$+\frac{1024(-271320a^5b^4+203490a^6b^4+32111520b^5+117320364ab^5+9767520a^2b^5)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(21434280a^3b^5+352716a^5b^5+9231474b^6+4019792ab^6+7533652a^2b^6+180880a^3b^6)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(293930a^4b^6+357312b^7+1020984ab^7+93024a^2b^7+116280a^3b^7)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\left.\frac{1024(38367b^8+14364ab^8+20349a^2b^8+560b^9+1330ab^9+21b^{10})}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}\right\}-$$

$$-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+2}{2})}\left\{\frac{1024(654729075+400914000a+2644887945a^2-368444608a^3+407004318a^4)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+\right.$$

$$+\frac{1024(-32111520a^5+9231474a^6-357312a^7+38367a^8-560a^9+21a^{10}+1396704420b)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(4564470450ab+265793584a^2b+2290676024a^3b-126838376a^4b+117320364a^5b)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(-4019792a^6b+1020984a^7b-14364a^8b+1330a^9b+1094071221b^2+1410623712ab^2)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(3183848164a^2b^2+33209568a^3b^2+413414806a^4b^2-9767520a^5b^2+7533652a^6b^2)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(-93024a^7b^2+20349a^8b^2+444647600b^3+1263684888ab^3+293010704a^2b^3)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(529562376a^3b^3+904400a^4b^3+21434280a^5b^3-180880a^6b^3+116280a^7b^3)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(107494190b^4+155769600ab^4+257688830a^2b^4+17364480a^3b^4+26340650a^4b^4)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(293930a^6b^4+164486680b^5+42918540ab^5+11918928a^2b^5+14271432a^3b^5+271320a^4b^5)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

© 2012 Global Journals Inc. (US)

$$+\frac{1024(352716a^5b^5+1646778b^6+2331168ab^6+3222324a^2b^6+217056a^3b^6+203490a^4b^6)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(106800b^7+255192ab^7+57456a^2b^7+54264a^3b^7+4335b^8+5040ab^8+5985a^2b^8)}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}+$$

$$+\frac{1024(100b^9+210ab^9+b^{10})}{\left[\prod_{\varphi=1}^{10}\{a-b-(2\varphi-1)\}\right]\left[\prod_{\omega=1}^{11}\{a-b+(2\omega-1)\}\right]}\Bigg\}\Bigg]-$$

$$-b\,\frac{2^{b+1}\,\Gamma(\frac{a+b+23}{2})}{\Gamma(b+1)}\left[\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})}\left\{\frac{1024(654729075+1396704420a+1094071221a^2+444647600a^3)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+\right.\right.$$

$$+\frac{1024(107494190a^4+16486680a^5+1646778a^6+106800a^7+4335a^8+100a^9+a^{10})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024(400914000b+4564470450ab+1410623712a^2b+1263684888a^3b+155769600a^4b)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024(42918540a^5b+2331168a^6b+255192a^7b+5040a^8b+210a^9b+2644887945b^2)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024(265793584ab^2+3183848164a^2b^2+293010704a^3b^2+257688830a^4b^2+11918928a^5b^2)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024(3222324a^6b^2+57456a^7b^2+5985a^8b^2-368444608b^3+2290676024ab^3+33209568a^2b^3)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024(529562376a^3b^3+17364480a^4b^3+14271432a^5b^3+217056a^6b^3+54264a^7b^3)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024(407004318b^4-126838376ab^4+413414806a^2b^4+904400a^3b^4+26340650a^4b^4)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024(271320a^5b^4+203490a^6b^4-321111520b^5+117320364ab^5-9767520a^2b^5+21434280a^3b^5)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

$$+\frac{1024(352716a^5b^5+9231474b^6-4019792ab^6+7533652a^2b^6-180880a^3b^6+293930a^4b^6)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+$$

Notes

© 2012 Global Journals Inc. (US)

$$
+\frac{1024(-357312b^{7}+1020984ab^{7}-93024a^{2}b^{7}+116280a^{3}b^{7}+38367b^{8}-14364ab^{8}+20349a^{2}b^{8})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+
$$

$$
+\frac{1024(-560b^{9}+1330ab^{9}+21b^{10})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}\Bigg\}-
$$

Notes

$$
-\frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\Bigg\{\frac{1024(654729075-400914000a+2644887945a^{2}+368444608a^{3}+407004318a^{4})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+
$$

$$
+\frac{1024(32111520a^{5}+9231474a^{6}+357312a^{7}+38367a^{8}+560a^{9}+21a^{10}-1396704420b)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+
$$

$$
+\frac{1024(4564470450ab-265793584a^{2}b+2290676024a^{3}b+126838376a^{4}b+117320364a^{5}b)}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+
$$

$$
+\frac{1024(4019792a^{6}b+1020984a^{7}b+14364a^{8}b+1330a^{9}b+1094071221b^{2}-1410623712ab^{2})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+
$$

$$
+\frac{1024(3183848164a^{2}b^{2}-33209568a^{3}b^{2}+413414806a^{4}b^{2}+9767520a^{5}b^{2}+7533652a^{6}b^{2})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+
$$

$$
+\frac{1024(93024a^{7}b^{2}+20349a^{8}b^{2}-444647600b^{3}+1263684888ab^{3}-293010704a^{2}b^{3})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+
$$

$$
+\frac{1024(529562376a^{3}b^{3}-904400a^{4}b^{3}+21434280a^{5}b^{3}+180880a^{6}b^{3}+116280a^{7}b^{3}+107494190b^{4})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+
$$

$$
+\frac{1024(-155769600ab^{4}+257688830a^{2}b^{4}-17364480a^{3}b^{4}+26340650a^{4}b^{4}+293930a^{6}b^{4})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+
$$

$$
+\frac{1024(-16486680b^{5}+42918540ab^{5}-11918928a^{2}b^{5}+14271432a^{3}b^{5}-271320a^{4}b^{5})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+
$$

$$
+\frac{1024(352716a^{5}b^{5}+1646778b^{6}-2331168ab^{6}+3222324a^{2}b^{6}-217056a^{3}b^{6}+203490a^{4}b^{6})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+
$$

$$
+\frac{1024(-106800b^{7}+255192ab^{7}-57456a^{2}b^{7}+54264a^{3}b^{7}+4335b^{8}-5040ab^{8}+5985a^{2}b^{8})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}+
$$

Global Journal of Science Frontier Research (F) Volume XII Issue II Version I February 2012

© 2012 Global Journals Inc. (US)

$$+\frac{1024(-100b^9+210ab^9+b^{10})}{\left[\prod_{\varsigma=1}^{11}\{a-b-(2\varsigma-1)\}\right]\left[\prod_{\tau=1}^{10}\{a-b+(2\tau-1)\}\right]}\Bigg\}\Bigg]$$

On simplification ,we get the result (6).

On the same way, we can prove the result (7).

IV. Conclusion

Notes

In this paper we have derived two summation formulae with the help of contiguous relation . However, the formulae presented herein may be further developed to extend this result .Thus we can only hope that the development presented in this work will stimulate further interest and research in this important area of classical special functions. Just as the mathematical properties of the Gauss hypergeometric function are already of immense and significant utility in mathematical sciences and numerous other areas of pure and applied mathematics, the elucidation and discovery of the formulae of hypergeometric functions considered herein should certainly eventually prove useful to further developments in the broad areas alluded to above.

References Références Referencias

1. Abramowitz, Milton., A and Stegun, Irene , *Handbook of Mathematical Functions with Formulas , Graphs , and Mathematical Tables*(National Bureau of Standards, 1970).
2. Arora, Asish, Singh, Rahul , Salahuddin. ; Development of a family of summation formulae of half argument using Gauss and Bailey theorems *Journal of Rajasthan Academy of Physical Sciences.*, 7(2008), 335-342.
3. Bells, Richard. Wong, Roderick; *Special Functions, A Graduate Text*(Cambridge Studies in Advanced Mathematics, 2010).
4. Luke, Y.L, *Mathematical Functions and their Approximations*(Academic Press Inc, 1975).
5. Prudnikov, A. P., Brychkov, Yu. A. and Marichev, O.I.; *Integrals and Series Vol. 3: More Special Functions.* Nauka, Moscow, 1986. Translated from the Russian by G.G. Gould, Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo, Melbourne, 1990.
6. Rainville, E. D.; The contiguous function relations for ${}_pF_q$ with applications to Bateman's $J_n^{u,v}$ and Rice's $H_n\ (\zeta,p,\nu)$, *Bull. Amer. Math. Soc.*, 51(1945), 714-723.
7. Salahuddin ; Two Summation Formulae Based On Half Argument Associated To Hypergeometric Function, *Global journal of Science Frontier Research*, 10(2010),08-35.

© 2012 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics & Decision Sciences
Volume 12 Issue 2 Version 1.0 February 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Effect of Imperfectness on Reflection and Transmission Coefficients in Swelling Porous Elastic Media at an Imperfect Boundary

By Rajneesh Kumar , Divya & Kuldeep Kumar
Kurukshetra University, Kurukshetra, Haryana, India

Abstract - The present investigation is concerned with the reflection and transmission of plane waves at an imperfect interface between two different swelling porous elastic media. The expression for various amplitude ratios due to the incidence of longitudinal wave in solid (PS), transverse wave in solid (SVS) are obtained for imperfect boundary and are deduced for normal stiffness, transversal stiffness and welded contact. The resulting amplitude ratios are computed and depicted graphically for a specific model. The present investigation has immense application in structural problems, geophysics etc.

Keywords : longitudinal waves, transversal waves, normal stiffness, transversal stiffness, welded contact.

GJSFR-F Classification: FOR Code: 019999.

EFFECT OF IMPERFECTNESS ON REFLECTION AND TRANSMISSION COEFFICIENTS IN SWELLING POROUS ELASTIC MEDIA AT AN IMPERFECT BOUNDARY

Strictly as per the compliance and regulations of :

OPEN ASSOCIATION OF RESEARCH SOCIETY, USA
Research | Diversity | Ethics

© 2012. Rajneesh Kumar , Divya & Kuldeep Kumar.This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Effect of Imperfectness on Reflection and Transmission Coefficients in Swelling Porous Elastic Media at an Imperfect Boundary

Rajneesh Kumar α, Divya σ & Kuldeep Kumar ρ

Abstract - The present investigation is concerned with the reflection and transmission of plane waves at an imperfect interface between two different swelling porous elastic media. The expression for various amplitude ratios due to the incidence of longitudinal wave in solid (PS), transverse wave in solid (SVS) are obtained for imperfect boundary and are deduced for normal stiffness, transversal stiffness and welded contact. The resulting amplitude ratios are computed and depicted graphically for a specific model. The present investigation has immense application in structural problems, geophysics etc.

Keywords : longitudinal waves, transversal waves, normal stiffness, transversal stiffness, welded contact.

I. Introduction

Dynamic analysis of theories of porous media is a subject with application in various branches of geophysics, civil and mechanical engineering. Based on the work of Von Terzaghi [1,2], Biot [3] proposed a general theory of three dimensional deformations of fluid saturated porous elastic solids. Subsequently, Biot [4,5,6,7] presented the models for describing the dynamic behaviour of fluid saturated porous media. He examines both high and low frequency limits and shows the existence of two longitudinal waves and one shear wave, which are dispersive and dissipative. Biot theory was based on the assumption of compressible constituents and till recently, some of his results have been taken as standard references and the basis for subsequent analysis in acoustic, geophysics and other fields. Based on the Fillunger model [8], (which is further based on the concept of volume fractions combined with surface porosity coefficients), Bowen [9], Boer and Ehlers[10,11] and Ehlers[12] develop and use another interesting theory in which all the constituents of a porous medium are assumed as soil; solid constituents are incompressible and liquid constituents which are generally water or oils are also incompressible.

Swelling porous medium (material) is a porous material that swells (shrinks) upon whetting (drying). Eringen [13] point out the importance of theories of mixtures to the applied field of swelling porous elastic soils as a continuum theory of mixtures for porous elastic solids filled with fluid and gas. Bofill and Quintanilla[14] discus the problem of anti-plane shear deformations of swelling porous elastic soils in case of fluid saturation or gas saturation. Gales [15] investigates the spatial behavior of solutions describing harmonic vibrations of right cylinder in the isothermal linear theory of swelling porous elastic soils. Gales [16] investigates some theoretical problems concerning waves and vibrations within the context of isothermal linear theory of swelling porous elastic soils with fluid, or gas saturation. Kleintelter, Park and Cushman[17] study various problems on swelling porous elastic soils.

A perfectly bonded interface is a surface across which both traction and displacement are continuous. Thus when solving harmonic wave problem in the neighborhood of a perfectly bonded interface

Author α : Department of Mathematics, Kurukshetra University, Kurukshetra, Haryana, India 136 119.
E-mail : rajneesh_kukmath@rediffmail.com
Author σ : Department of Mathematics, N.I.T., Kurukshetra, Haryana, India 136119.
E-mail : divyataneja82@yahoo.com
Author ρ : Department of Mathematics, N.I.T., Kurukshetra, Haryana, India 136119.

Ref.
4. Biot M.A., Theory of Propagation of elastic waves in a fluid-saturated Porous solid-I Low frequency range, J. Acoust. Soc. Am. 28, 168-178, (1956).
5. Biot M.A., Theory of Propagation of elastic waves in a fluid-saturated Porous solid-II Higher frequency range. J. Acoust. Soc. Am. 28, 179-191, (1956).

© 2012 Global Journals Inc. (US)

between two different elastic media, wave solution in one medium must be matched with those in the second medium through interface condition. The generalization of the concept is that of an imperfectly bonded interface for which the displacement and temperature distribution across a surface need not be continuous. Debonding and imperfect contact however are known to exist in composites, in the domain of electrical, thermal conduction or elasticity.

Kumar and Singh[18,19] study some problems on propagation of plane waves at an imperfect surface. Kumar et al [20] studied some problems on reflection and transmission of waves at an imperfect boundary.

The exact nature beneath the earth surface is not known. For the purpose of theoretical investigation about the earth interior one has to consider various appropriate model. The problem of waves and their reflection is very useful to understand the internal structure of earth and to explore various useful material in form of rocks buried inside the earth, for example mineral and crystals etc.

The spring like model has been adopted in the present work between two swelling porous elastic half space as has been represented by the boundary conditions in the text. K_n , K_t , K_{nf} , K_{tf} used in the boundary conditions are spring constant type material parameters. $K_n \to \infty$, $K_t \to \infty$; $K_{nf} \to \infty$, $K_{tf} \to \infty$ implies the continuity of displacement components in case of solid and fluid respectively and therefore the two solids are perfectly bonded together or to say that the two solids are in welded contact. Reflection and transmission of plane waves in swelling porous elastic field at the imperfect boundary surface have been studied due to incidence of longitudinal and transversal waves. The amplitude ratios of various reflected and transmitted waves are computed and shown graphically. As such a model may be found in the earth's crust, the results of the problem can be applicable to engineering, seismology and geophysics problem.

II. Basic Equations

Following Eringen [13], the field equations in linear theory of swelling porous elastic soils are

$$\mu u^{s}_{i,jj} + (\lambda + \mu) u^{s}_{j,ji} - \sigma^{f} u^{f}_{j,ji} + \xi^{ff} (\dot{u}^{f}_{i} - \dot{u}^{s}_{i}) + f^{s}_{i} = \rho^{s}_{0} \ddot{u}^{s}_{i}, \tag{1}$$

$$\mu_v \dot{u}^{f}_{i,jj} + (\lambda_v + \mu_v) \dot{u}^{f}_{j,ji} - \sigma^{f} u^{s}_{j,ji} - \sigma^{ff} u^{f}_{j,ji} - \xi^{ff} (\dot{u}^{f}_{i} - \dot{u}^{s}_{i}) + f^{f}_{i} = \rho^{f}_{0} \ddot{u}^{f}_{i}, \tag{2}$$

$$t^{s}_{ij} = (-\sigma^{f} u^{f}_{r,r} + \lambda u^{s}_{r,r}) \delta_{ij} + \mu (u^{s}_{i,j} + u^{s}_{j,i}), \tag{3}$$

$$t^{f}_{ij} = (-\sigma^{f} u^{s}_{r,r} - \sigma^{ff} u^{f}_{r,r} + \lambda_v \dot{u}^{f}_{r,r}) \delta_{ij} + \mu_v (\dot{u}^{f}_{i,j} + \dot{u}^{f}_{j,i}), \tag{4}$$

i,j=1,2,3

where, the superscripts s and f denote respectively, the elastic solid and the fluid; u^s_i and u^f_i are the displacement components of solid and fluid respectively. The functions (f^s_i, f^f_i) are the body forces, ρ^s_0, ρ^f_0 are the densities of each constituent and $\lambda, \mu, \lambda_v, \mu_v, \sigma^f, \sigma^{ff}, \xi^{ff}$ are constitutive constants. Subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, and a superposed dot denotes time differentiation, t^s_{ij} , t^f_{ij} are the partial stress tensors.

III. Formulation of the Problem and Solution

We consider two homogeneous swelling porous elastic half spaces in contact with each other at a plane surface which we designate as the plane z = 0 of a rectangular Cartesian co-ordinate system oxyz. We consider plane waves in the xz - plane with wave front parallel to the yz - plane and all the field variables depend only on x,z and t.

For two dimensional problem, we assume the displacement vector

$$\vec{u}^{i} = (u^{i}_{1}, 0, u^{i}_{3}) \qquad i=s,f \tag{5}$$

We define the non-dimensional quantities as

$$x' = \frac{\omega^*}{c_2} x,\ z' = \frac{\omega^*}{c_2} z,\ u^{i\prime}_{1} = \frac{\omega^*}{c_2} u^{i}_{1},\ u^{i\prime}_{3} = \frac{\omega^*}{c_2} u^{i}_{3},\ t_{ij}{'}^{i} = \frac{t^{i}_{ij}}{\mu},\ \omega^* = \frac{\xi^{ff}}{\rho^{s}_{0}}, c^2_2 = \frac{\mu}{\rho^{s}_{0}},\ t' = \omega^* t, \tag{6}$$

Expressing the displacement components $u^s_1, u^s_3, u^f_1, u^f_3$ by the scalar potential functions $\phi^i(x, z, t)$ and $\psi^i(x, z, t)$ in dimensionless form

Ref.

18. Kumar R., Singh M., Propagation of generalized thermoelastic plane waves at an imperfect interface, Int. J of Applied Mechanics and Engineering, 12(3), 713-732, 2007.
19. Kumar R., Singh M., Reflection and transmission of thermoelastic plane waves at an imperfect boundary, Bull. Cal. Math. Soc., 101(4), 419-436, 2009.

© 2012 Global Journals Inc. (US)

$$u_1^i = \frac{\partial \phi^i}{\partial x} - \frac{\partial \psi^i}{\partial z}, \qquad u_3^i = \frac{\partial \phi^i}{\partial z} + \frac{\partial \psi^i}{\partial x} \tag{7}$$

Using equations (1)-(2), (5)-(7) we obtain two coupled system of equations in absence of body forces

$$\begin{bmatrix} (1+a_1)\nabla^2 - a_3\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2} & -a_2\nabla^2 + a_3\frac{\partial}{\partial t} \\ -h_2\nabla^2 + h_4\frac{\partial}{\partial t} & ((1+h_1)\frac{\partial}{\partial t} - h_3)\nabla^2 - h_4\frac{\partial}{\partial t} - h_5\frac{\partial^2}{\partial t^2} \end{bmatrix} \begin{bmatrix} \phi^s \\ \phi^f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} -\nabla^2 + a_3\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} & -a_3\frac{\partial}{\partial t} \\ -h_4\frac{\partial}{\partial t} & (-\nabla^2 + h_4)\frac{\partial}{\partial t} + h_5\frac{\partial^2}{\partial t^2} \end{bmatrix} \begin{bmatrix} \psi^s \\ \psi^f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9}$$

where, Δ^2 is the Laplacian operator and

$$a_1 = \frac{\lambda+\mu}{\mu}, \quad a_2 = \frac{\sigma^f}{\mu}, \quad a_3 = \frac{\xi^{ff} c_2^2}{\omega^* \mu}, \quad h_1 = \frac{\lambda_v + \mu_v}{\mu_v}, \; h_2 = \frac{\sigma^f}{\mu_v \omega^*}, \; h_3 = \frac{\sigma^{ff}}{\mu_v \omega^*}, \; h_4 = \frac{\xi^{ff} c_2^2}{\omega^{*2} \mu_v}, \; h_5 = \frac{\rho_0^f c_2^2}{\mu_v \omega^*},$$

IV. Reflection and Transmission

We consider a longitudinal wave in solid (PS)/longitudinal wave in fluid (PF) /transverse wave in solid (SVS)/ transverse wave in fluid(SVF) propagating through the medium M_1 which is designated as the region z=0 and incident at the plane z=0, with its direction of propagating with angle θ_0 normal to the surface. Corresponding to each incident wave, we get reflected PS, PF, SVS, SVF waves and transmitted PS, PF, SVS, SVF waves in medium $\overline{M}$ as shown in Fig. 1.

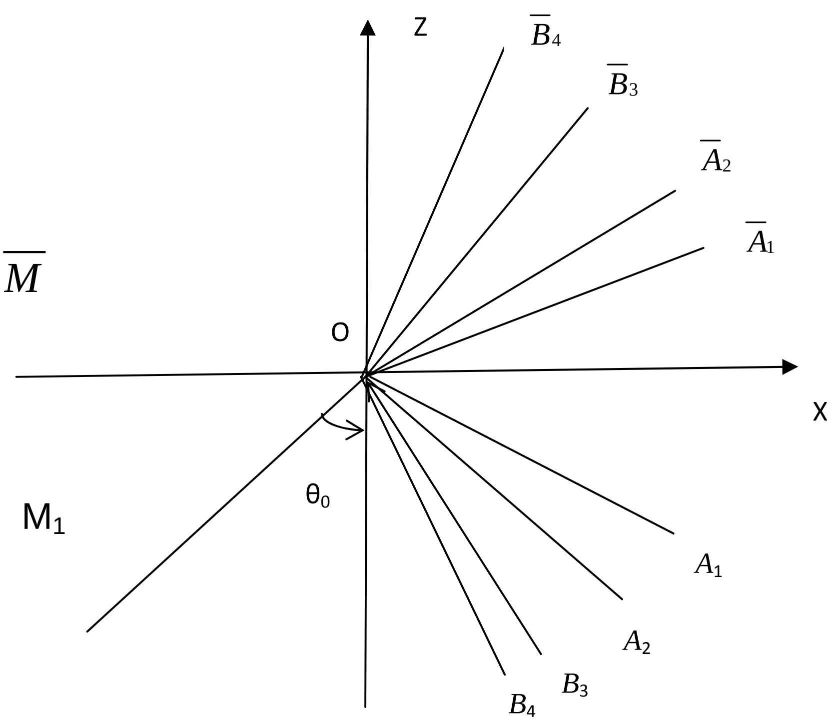


Fig. 1 : Geometry of the problem

© 2012 Global Journals Inc. (US)

We assume the solutions of the system of equations (8)-(9) in the form

$$\left[\phi^s,\phi^f,\psi^s,\psi^f\right]=\left[\phi_1^s,\phi_1^f,\psi_1^s,\psi_1^f\right]e^{i\{k\{x\sin\theta-z\cos\theta\}-\omega t\}} \quad (10)$$

where k is the wave number and ω is the complex circular frequency.

Making use of equation (10) in (8)-(9) we obtain two quadratic equations in V^2 given by

$$AV^4+BV^2+C=0, \qquad A_1V^4+B_1V^2+C_1=0 \quad (11)$$

where $V=\omega/k$ is the velocity of the waves: V_1,V_2 are the velocities of the reflected longitudinal PS and PF waves respectively, given by equation $(11)_1$, and V_3,V_4 are the velocities of transverse SVS and SVF waves respectively given by equation $(11)_2$.

where,

$$A=\frac{i(h_5a_3+h_4)}{\omega}+h_5,\ C=(-i\omega(1+h_1)-h_3)(1+a_1)-h_2a_2,\ B_1=i\left(\omega-\frac{h_4}{\omega}\right)-h_5-a_3,\ C_1=-i\omega,\ \tau_1=(h_5+\frac{ih_4}{\omega}),$$

$$B=(1+h_1)(i\omega-a_3)-(1+a_1)\tau_1+h_3\tau_0+\frac{i}{\omega}(h_4a_2+h_2a_3),A_1=\frac{a_3}{\omega^2}(h_5-h_4)+\frac{i}{\omega}(a_3h_5+h_4)+h_5,\ \tau_0=(1+\frac{ia_3}{\omega}),$$

V. Boundary Conditions

We consider two-bonded swelling porous elastic half-spaces as shown in Fig. 1. Imperfect bonding considered here means that the traction is continuous across the interface but that the small displacement is assumed to depend linearly on the traction vector. If the size and spacing between the imperfections is much smaller than the wave-length at the interface, we can use spring boundary conditions at z=0 [21] as

$$(i)\ \bar{t}_{33}^s=K_n(u_3^s-\bar{u}_3^s)\quad (ii)\ \bar{t}_{33}^f=K_{nf}(u_3^f-\bar{u}_3^f)\quad (iii)\ \bar{t}_{31}^s=K_t(u_1^s-\bar{u}_1^s)\quad (iv)\ \bar{t}_{31}^f=K_{tf}(u_1^f-\bar{u}_1^f)$$

$$(v)\ t_{33}^s=\bar{t}_{33}^s\quad (vi)\ t_{33}^f=\bar{t}_{33}^f\quad (vii)\ t_{31}^s=\bar{t}_{31}^s\quad (viii)\ t_{31}^f=\bar{t}_{31}^f$$

where K_n, K^f_n are normal stiffness in case of solid and fluid respectively and K_t , K^f_t are transversal stiffness in case of solid and fluid respectively.

In view of (10), we assume the values of $\phi^s,\phi^f,\psi^s,\psi^f$ for medium M_1 and $\phi^s,\phi^f,\psi^s,\psi^f$ for medium $\bar{M}$ satisfying the boundary conditions as

$$\{\phi^s,\phi^f\}=\sum_{i=1}^{2}\{1,\eta_i\}[A_{0i}e^{i\{k_i\{x\sin\theta_{0i}-z\cos\theta_{0i}\}-\omega_it\}}+P_i],\ \ \{\bar{\phi}^s,\bar{\phi}^f\}=\sum_{i=1}^{2}\{1,\bar{\eta}_i\}[\bar{A}_ie^{i\{\bar{k}_i\{x\sin\bar{\theta}_i-z\cos\bar{\theta}_i\}-\bar{\omega}_it\}}]$$

$$\{\psi^s,\psi^f\}=\sum_{j=3}^{4}\{1,\eta_j\}[B_{0j}e^{i\{k_j\{x\sin\theta_{0j}-z\cos\theta_{0j}\}-\omega_jt\}}+P_j],\ \{\bar{\psi}^s,\bar{\psi}^f\}=\sum_{j=3}^{4}\{1,\bar{\eta}_j\}[\bar{B}_je^{i\{\bar{k}_j\{x\sin\bar{\theta}_j-z\cos\bar{\theta}_j\}-\bar{\omega}_jt\}}] \quad (12)$$

where,

$$P_i=A_ie^{i\{k_i\{x\sin\theta_i+z\cos\theta_i\}-\omega_it\}},\ P_j=B_je^{i\{k_j\{x\sin\theta_j+z\cos\theta_j\}-\omega_jt\}},\ \eta_i=\frac{\omega(1+a_1)-ia_3V_i-\omega V_i^2}{a_2\omega-ia_3V_i^2},$$

$$\eta_j=\frac{-\omega+(ia_3+\omega)V_j^2}{ia_3V_j^2},\ \bar{\eta}_i=\frac{\omega(1+\bar{a}_1)-i\bar{a}_3\bar{V}_i-\omega\bar{V}_i^2}{\bar{a}_2\omega-i\bar{a}_3\bar{V}_i^2},\ \bar{\eta}_j=\frac{-\omega+(i\bar{a}_3+\omega)\bar{V}_j^2}{i\bar{a}_3\bar{V}_j^2},\ (i=1,2\ \&\ j=3,4)$$

A_{0i} are the amplitudes of the incident PS wave, PF wave and B_{0j} are the amplitudes of the incident SVS wave, SVF wave respectively. A_i are the amplitudes of the reflected PS wave (PSR), PF wave (PFR) and B_j are the amplitudes of the reflected SVS wave (SVSR) and SVF wave (SVFR), A are the amplitudes of the transmitted PS wave (PST), transmitted PF wave (PFT), j A are the amplitudes of transmitted SVS wave (SVST) and transmitted SVF wave (SVFT) respectively.

Notes

© 2012 Global Journals Inc. (US)

In order to satisfy the boundary conditions, the extension of the Snell's law will be

$$\frac{sin\theta_0}{V_0}=\frac{sin\theta_1}{V_1}=\frac{sin\theta_2}{V_2}=\frac{sin\theta_3}{V_3}=\frac{sin\theta_4}{V_4}=\frac{sin\overline{\theta}_1}{\overline{V}_1}=\frac{sin\overline{\theta}_2}{\overline{V}_2}=\frac{sin\overline{\theta}_3}{\overline{V}_3}=\frac{sin\overline{\theta}_4}{\overline{V}_4} \tag{13}$$

$$\text{Where, } k_1V_1=k_2V_2=k_3V_3=k_4V_4=\overline{k}_1\overline{V}_1=\overline{k}_2\overline{V}_2=\overline{k}_3\overline{V}_3=\overline{k}_4\overline{V}_4=\omega \text{ at z=0} \tag{14}$$

Making use of potentials given by (12) in boundary conditions, we obtain a system of eight non-homogeneous equations which can be written as

$$\sum_{i,j=1}^{8} a_{ij}Z_j=Y_i \tag{15}$$

where,

$$a_{1p}=-liK_nk_ps_p,\ a_{1e}=-liK_nk_e\sin\theta_e,\ a_{1r}=(\frac{\overline{\sigma}^f}{\overline{\mu}}\overline{\eta}_p-\frac{\overline{\lambda}}{\overline{\mu}}-2\overline{s}_p^2)\overline{k}_p^2-K_ni\overline{k}_p\overline{s}_p\frac{\mu}{\overline{\mu}},\ a_{1d}$$

$$=(2\overline{k}_e^2\overline{s}_e+K_ni\overline{k}_e\frac{\mu}{\overline{\mu}})\sin\overline{\theta}_e,$$

$$a_{2p}=lK_{nf}\eta_pik_ps_p,\ a_{2e}=-\frac{\mu}{\overline{\mu}}K_{nf}\eta_eik_eSin\theta_e,\ a_{2r}=(\frac{\overline{\sigma}^f}{\overline{\mu}}+(\frac{\overline{\sigma}^{ff}}{\overline{\mu}}+(\frac{\overline{\lambda}_v}{\overline{\mu}}+2\overline{\mu}_v\overline{s}_p^2)i\overline{\omega}^*\overline{\omega}_p)\overline{\eta}_p)$$

$$\overline{k}_p^2-K_{nf}i\overline{k}_p\overline{\eta}_p\overline{s}_pl$$

$$a_{3r}=-(2\overline{k}_p^2\overline{s}_p+K_ti\overline{k}_pl)\sin\overline{\theta}_p,\ a_{2d}=(-2\overline{\mu}_v\overline{\omega}^*\overline{k}_e^2\overline{\omega}_3\overline{s}_e+K_{nf}\overline{k}_el)i\sin\overline{\theta}_e\overline{\eta}_e,\ a_{3p}=K_tik_p\sin\theta_pl,$$

$$,\ a_{3e}=(-1)^eK_tik_es_el,$$

$$a_{3d}=\overline{k}_e^2(-\overline{s}_e^2+\sin^2\overline{\theta}_e)-i\overline{k}_e\overline{s}_eK_tl,\ a_{4p}=-\eta_pK_{tf}ik_p\sin\theta_pl,\ a_{4e}=\eta_eK_{tf}ik_e\,s_e\,l\ ,$$

$$a_{4r}=(-\overline{l}\,\overline{\omega}^*2\overline{\omega}_p\overline{k}_p^2\overline{s}_p+l\overline{k}_pK_{tf})\overline{\eta}_p\sin\overline{\theta}_p,a_{4d}=(\overline{l}\,\overline{\omega}^*\overline{k}_e^2(\overline{s}_e^2-\sin^2\overline{\theta}_e)+il\overline{k}_eK_{tf}\overline{s}_e)\overline{\eta}_e,$$

$$a_{5p}=(m\eta_p-n-2s_p^2)k_p^2,\quad a_{5e}=-2k_e^2\sin\theta_es_3,\ a_{5r}=(-\overline{m}\overline{\eta}_p+\overline{n}+2\overline{s}_p^2)\overline{k}_p^2,\ a_{5d}=-2\overline{k}_e^2\sin\overline{\theta}_e\overline{s}_e,$$

$$a_{61}=(m+v\eta_p+\frac{i\omega^*\omega_p\eta_p}{\mu}(\lambda_v+2\mu_vs_p^2))k_p^2,\ a_{6e}=\frac{2\mu_v\omega^*}{\mu}(ik_e^2\sin\theta_es_e\omega_e\eta_e),$$

$$a_{6r}=(-\overline{m}+(\overline{v}+\overline{n}\overline{\omega}^*i\overline{\omega}_p-2\overline{l}\,\overline{\omega}^*i\overline{\omega}_p\overline{s}_p^2)\overline{\eta}_p)\overline{k}_p^2,a_{6d}=2\overline{l}\,\overline{\omega}^*i\overline{\omega}_e\overline{\eta}_e\sin\overline{\theta}_e\overline{s}_e\overline{k}_e^2,\ a_{7p}=-2k_p^2\sin\theta_ps_p,$$

$$a_{7e}=k_e^2(-\sin^2\theta_e+s_e^2),a_{7r}=-2\overline{k}_p^2\overline{s}_p\sin\overline{\theta}_p,\ a_{7d}=k_e^2(-\sin^2\overline{\theta}_e+\overline{s}_e^2),a_{8p}=2l_v\omega^*ik_p^2\sin\theta_ps_p\omega_p\eta_p,$$

$$a_{8e}=l_v\omega^*i\omega_e\eta_ek_e^2(\sin^2\theta_e-s_e^2),\ a_{8r}=-2\overline{l}\,\overline{\omega}^*i\overline{k}_p^2\sin\overline{\theta}_p\overline{s}_1\overline{\omega}_p\overline{\eta}_p,a_{8d}=-\overline{l}\,\overline{\omega}^*i\overline{k}_e^2(\overline{s}_e^2-\sin^2\overline{\theta}_e)\overline{\omega}_e\overline{\eta}_e,$$

Where,

$$s_d=\left(\frac{V_d}{V_0}\right)\left[\left(\frac{V_0}{V_d}\right)^2-Sin^2\theta_0\right]^{\frac{1}{2}},\overline{s}_d=\left(\frac{\overline{V}_d}{V_0}\right)\left[\left(\frac{V_0}{\overline{V}_d}\right)^2-Sin^2\theta_0\right]^{\frac{1}{2}},\ l=\frac{\mu}{\overline{\mu}},\ \overline{l}=\frac{\overline{\mu}_v}{\overline{\mu}},m=\frac{\sigma^f}{\mu},$$

$$\overline{m}=\frac{\overline{\sigma}^f}{\overline{\mu}},n=\frac{\lambda}{\mu},\overline{n}=\frac{\overline{\lambda}}{\overline{\mu}}$$

where , $d=p,e,r,d$; $p=1,2$; $e=3,4$; $r=5,6$; $d=7,8$

© 2012 Global Journals Inc. (US)

and

$$Z_1=\frac{A_1}{A^*},\ Z_2=\frac{A_2}{A^*},\ Z_3=\frac{B_1}{A^*},\ Z_4=\frac{B_2}{A^*},Z_5=\frac{\bar{A}_1}{A^*},\ Z_6=\frac{\bar{A}_2}{A^*},\ Z_7=\frac{\bar{B}_1}{A^*},\ Z_8=\frac{\bar{B}_2}{A^*},$$

(i) For incident PS –wave:

$$A^*=A_{01},\ A_{02}=B_{03}=B_{04}=0$$

$$Y_1=a_{11},\ \ Y_2=a_{21},\ \ Y_3=-a_{31},\ \ Y_4=-a_{41},Y_5=-a_{51},\ Y_6=-a_{61},\ Y_7=a_{71},\ Y_8=a_{81}$$

(ii) For incident PF–wave:

$$A^*=A_{02},\ A_{01}=B_{03}=B_{04}=0$$

$$Y_1=a_{12},\ \ Y_2=a_{22},\ \ Y_3=-a_{32},\ \ Y_4=-a_{42},\ Y_5=-a_{52},\ \ Y_6=-a_{62},\ \ Y_7=a_{72},\ \ Y_8=a_{82}$$

(iii) For incident SVS –wave:

$$A^*=B_{03},\ A_{01}=A_{02}=B_{04}=0$$

$$Y_1=-a_{13},\ \ Y_2=-a_{23},\ \ Y_3=-a_{33},\ \ Y_4=a_{43},\ \ Y_5=a_{53},\ \ Y_6=a_{63},\ \ Y_7=-a_{73},\ \ Y_8=-a_{83}$$

(iv) For incident SVF –wave:

$$A^*=B_{04},\ A_{01}=A_{02}=B_{03}=0$$

$$Y_1=-a_{14},\ \ Y_2=-a_{24},\ \ Y_3=-a_{34},\ \ Y_4=a_{44}\ ,Y_5=a_{54},\ \ Y_6=a_{64},\ \ Y_7=-a_{74},\ \ Y_8=-a_{84}\ ,$$

where, Z_1, Z_2 , Z_3, Z_4 are the amplitude ratios of reflected PS-, PF-, SVS-, SVF-waves and $\bar{Z}_1$, $\bar{Z}_2$, $\bar{Z}_3$, $\bar{Z}_4$, are the amplitude ratios of transmitted PS-, PF-, SVS-, SVF-waves.

Case -I: Normal Stiffness (NS):

$K_n\neq 0$, $K_{nf}\neq 0$, $K_t\rightarrow\infty$, $K_{tf}\rightarrow\infty$ Correspond to the case of normal stiffness and we obtain a system of eight non-homogeneous equations with the changed values of a_{ij} as

$$a_{3p}=ik_pl\sin\theta_p,\ a_{3e}=(-1)^e ik_els_e,\ a_{3r}=-i\bar{k}_pl\sin\bar{\theta}_p,\ a_{3d}=-i\bar{k}_el\bar{s}_e,\ a_{41}=-\eta_pik_pl\sin\theta_p,\ a_{4e}=l\eta_eik_es_e,$$
$$a_{4r}=(-1)^r\eta_pi\bar{k}_pl\sin\bar{\theta}_p,\ a_{4d}=-l\eta_ei\bar{k}_e\bar{s}_e,$$

Case - II: Transverse Stiffness (TS):

$K_n\rightarrow\infty$, $K_{nf}\rightarrow\infty$, $K_t\neq 0$, $K_{tf}\neq 0$ Correspond to the case of transverse stiffness. We obtain a system of eight non-homogeneous equations with the changed values of a_{ij} as

$$a_{1p}=-ilk_ps_p,\ a_{1e}=-ik_e\sin\theta_e,a_{1r}=-il\bar{k}_p\bar{s}_p,a_{1d}=i\bar{k}_el\sin\bar{\theta}_e,a_{2p}=-\eta_plik_ps_p,\ a_{2e}=-\eta_eilk_e\sin\theta_e,$$
$$a_{2r}=-i\bar{\eta}_pl\bar{k}_p\bar{s}_p,\ a_{2d}=il\bar{\eta}_e\bar{k}_e\sin\bar{\theta}_e$$

Case - III: Welded Contact (WC):

$K_n\rightarrow\infty$, $K_{nf}\rightarrow\infty$, $K_t\rightarrow\infty$, $K_{tf}\rightarrow\infty$ Correspond to the case of transverse stiffness. We obtain a system of eight non-homogeneous equations with the changed values of a_{ij} as

$$a_{1p}=-ilk_ps_p,a_{1e}=-ilk_e\sin\theta_e,a_{1r}=-i\bar{k}_pl\bar{s}_p,a_{1d}=i\bar{k}_el\sin\bar{\theta}_e,a_{2p}=-i\eta_plk_ps_p,\ a_{2e}=-i\eta_elk_e\sin\theta_e,$$
$$a_{2r}=-i\bar{k}_pl\bar{\eta}_p\bar{s}_p,a_{2d}=i\bar{k}_el\bar{\eta}_e\sin\bar{\theta}_e,\ a_{3p}=ik_pl\sin\theta_p,a_{3e}=-ilk_es_e,a_{3r}=-i\bar{k}_pl\sin\bar{\theta}_p,a_{3d}=-il\bar{k}_e\bar{s}_e,$$
$$a_{4p}=-i\eta_plk_p\sin\theta_p,a_{4e}=i\eta_elk_es_e,\ a_{4r}=i\bar{\eta}_p\bar{k}_pl\sin\bar{\theta}_p,a_{4d}=i\bar{\eta}_e\bar{k}_e\ \bar{s}_e$$

© 2012 Global Journals Inc. (US)

VI. Numerical Results and Discussion

In order to illustrate theoretical results obtained in the preceding sections, we now present some numerical results. For numerical computation, the physical data is given below:

$$\lambda = 2.238\times10^{10}\, N/m^2,\ \mu = 2.992\times10^{10}\, N/m^2,\ \lambda_v = 2.05\times10^{10}\, NSec/m^2\ ,\ \mu_v = 2.5\times10^{10}\, NSec/m^2,$$

$$\sigma^f = 1.42\times10^{10}\, N/m^2,\ \sigma^{ff} = 1.75\times10^{10}\, N/m^2,\ \rho_0^s = 2.65\times10^3\, NS\,ec^2/m^4,\ \rho_0^f = 1.92\times10^3\, NS\,ec^2/m^4,$$

$$\xi^{ff} = 1.745\times10^3\, NSec/m^4,\ \bar{\lambda} = 0.91\times10^{10}\, N/m^2,\ \bar{\mu} = 1.11\times10^{10}\, N/m^2,\ \bar{\lambda}_v = 1.15\times10^{10}\, NSec/m^2\ ,$$

$$\bar{\mu}_v = 1.29\times10^{10}\, NSec/m^2,\ \bar{\rho}_0^s = 1.25\times10^3\, NS\,ec^2/m^4,\ \bar{\rho}_0^f = 0.12\times10^3\, NS\,ec^2/m^4,\ \bar{\sigma}^f = 0.7\times10^{10}\, N/m^2,$$

$$\bar{\sigma}^{ff} = 0.5\times10^{10}\, N/m^2,\ \bar{\xi}^{ff} = 0.1\times10^3\, NSec/m^4$$

A computer programme has been developed and amplitude ratios of various reflected and transmitted waves have been computed. The variations of amplitude ratios for swelling porous elastic solid with stiffness (ST), normal stiffness (NS), transversal stiffness (TS), welded contact (WC) with angle of incidence θ_0 of the incident PS wave, incident SVS wave are shown graphically in Figures 2-3

VII. Incident PS-Wave

Fig. 2(a)-2(h) depicts the variation in values of amplitude ratios $|Z_s|$, $s = 1,2,3,4,5,6,7,8$ when PS wave is incident.

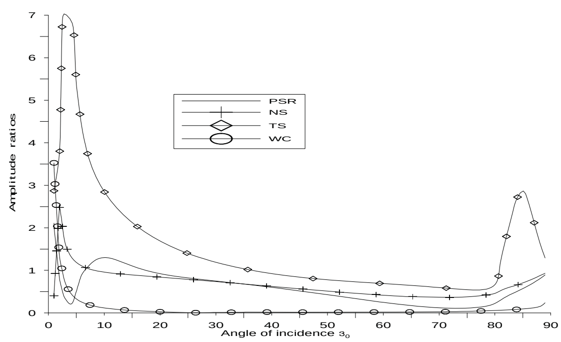


Fig. 2(a) Variation in amplitude ratios of Reflected PS wave when PS wave is incident

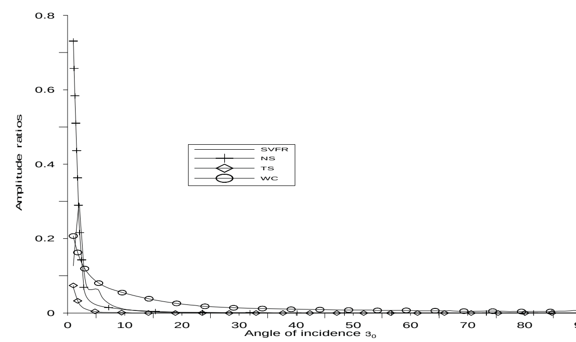


Fig. 2(d) Variation in amplitude ratios of Reflected SVF wave when PS wave is incident

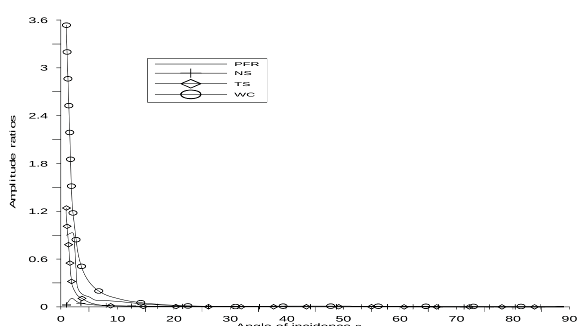


Fig. 2(b) Variation in amplitude ratios of Ref lected PF wave when PS wave is incident

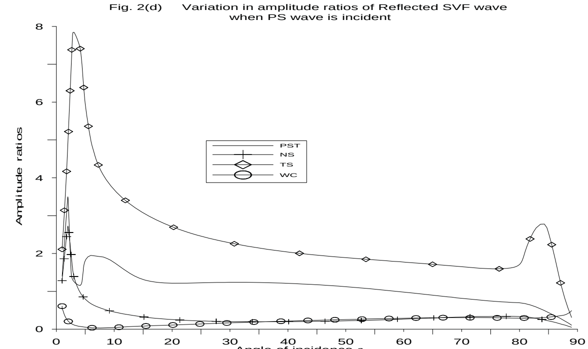


Fig. 2(e) Variation in amplitude ratios of Transmitted PS wave when PS wave is incident

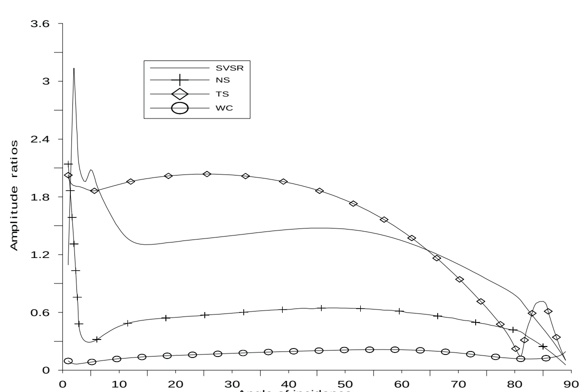


Fig. 2(c) Variation in amplitude ratios of Reflected SVS wave when PS wave is incident

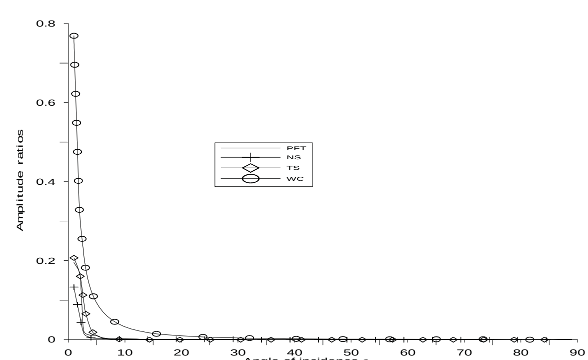


Fig. 2(f) Variation in amplitude ratios of Transmitted PF wave when PS wave is incident

© 2012 Global Journals Inc. (US)

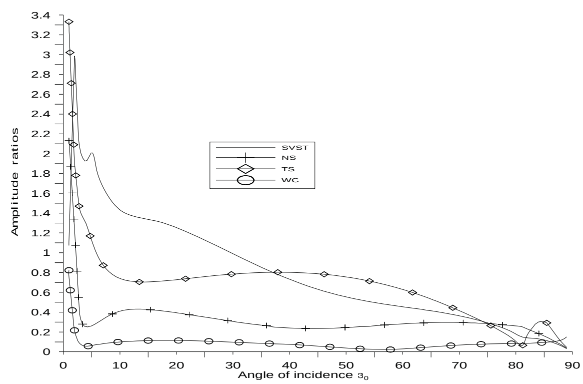


Fig. 2(g) Variation in amplitude ratios of Transmitted SVS wave when PS wave is incident

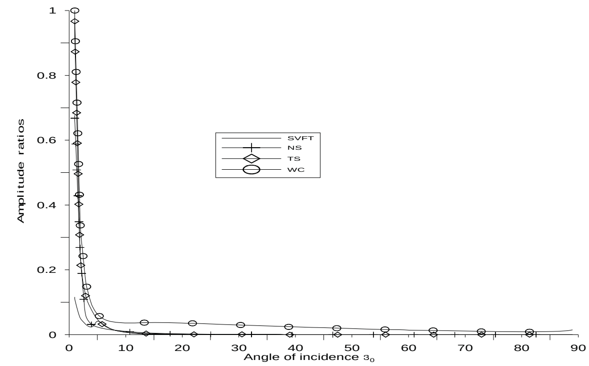


Fig. 2(h) Variation in amplitude ratios of Transmitted SVF wave when PS wave is incident

From Fig. 2(a), we notice that the values of amplitude ratios $|Z_1|$ for PSR NS, TS and WC are of oscillatory behavior. Amplitude ratio for TS remains greater than the values of amplitude ratio for PSR, NS and WC in range $\theta_0 \geq 3$, whereas values of amplitude ratio for WC remains less than the values of amplitude ratio for PSR, NS, and TS in range $\theta_0 \geq 5$. The values of amplitude ratio for PSR, NS and WC oscillate in the whole range whereas for WC it decrease in range $1 \leq \theta_0 \leq 70$ and then for $\theta_0 \geq 71$ it starts increasing.

Fig. 2(b) shows that the values of amplitude ratio $|Z_2|$ for PFR, TS and WC decreases with increase in angle of incidence, whereas the values of amplitude ratio $|Z_2|$ for NS initially oscillates and then decrease with angle of incidence. The values of amplitude ratios for WC remains greater than the values obtained for PFR, NS and TS in whole range. The values of amplitude ratio for NS remain less than the values of amplitude ratio for PFR, TS and WC in whole range.

Fig. 2(c) depicts the variation in amplitude ratio $|Z_3|$ due to incidence of PS wave. From the fig we notice that amplitude ratio for SVSR oscillates in the region $1 \leq \theta_0 \leq 50$. Then for $\theta_0 \geq 51$ it decrease. The values of amplitude ratio for TS oscillate in the region $1 \leq \theta_0 \leq 30$ then for $31 \leq \theta_0 \leq 80$ it decrease, for $\theta_0 \geq 81$it is of oscillatory behavior. The values of amplitude ratio for NS decreases for $1 \leq \theta_0 \leq 5$, for $6 \leq \theta_0 \leq 45$ it increase and for $\theta_0 \geq 46$ it decreases with angle of incidence. The values of amplitude ratio for WC remain less than the values obtain for SVSR, NS and TS in whole range and keeps increasing with increase in angle of incidence.

Fig. 2(d) depicts the variation in amplitude ratio $|Z_4|$ due to the incidence of PS wave. From the figure, we notice that amplitude ratio for SVFR initially oscillates, then decreases in range $\theta_0 \geq 4$. The values of amplitude ratio for NS, TS and WC decrease in whole range. For the $\theta_0 \geq 4$ values of amplitude ratio for WC remains greater than the values obtain for SVFR, NS and WC. The values of amplitude ratio for TS remain less than the values of amplitude ratio for SVFR, NS and WC in whole range.

From Fig. 2(e) we notice that the values of amplitude ratio $|Z_5|$ for PST initially oscillate then decreases in range $\theta_0 \geq 4$. The values of amplitude ratio for NS oscillates in the region $1 \leq \theta_0 \leq 3$, then decreases in the range $\theta_0 \geq 4$, whereas for TS it initially oscillates, then decrease in the range $4 \leq \theta_0 \leq 80$ and remains greater than the values obtain for PST, NS and WC in whole range. The values of amplitude ratio for WC remain less than the values of amplitude ratio for PST, TS in whole range.

From Fig. 2(f) we notice that the values of amplitude ratio $|Z_6|$ for PFT, NS, TS and WC decreases with increase in angle of incidence. The values of amplitude ratio for WC remain greater than the values of amplitude ratio for PFT, NS and TS. The values of amplitude ratios for NS remain less than the values of amplitude ratios for PFT, TS and WC in whole range.

From Fig. 2(g), we notice that values of amplitude ratio $|Z_7|$ for SVST and WC decreases with increase in angle of incidence. In range $1 \leq \theta_0 \leq 35$ the values of amplitude ratio for SVST remains greater than the values of amplitude ratio for NS, TS and WC. The values of amplitude ratio for WC remain less than the values of amplitude ratio for SVST, NS and TS.

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue II Version I February 2012

© 2012 Global Journals Inc. (US)

From Fig. 2(h), we notice that values of amplitude ratio $|Z_8|$ for SVFT, NS, TS and WC decreases with angle of incidence. The values of amplitude ratio for WC remain greater than the values of amplitude ratio for SVFT, NS and TS, whereas for SVFT remains less than the values obtain for NS, TS and WC.

VIII. Incidence of SVS-Wave

Fig. 3(a)-3(h) depicts the variation in values of amplitude ratios , $|Z_s|$, $s = 1,2,3,4,5,6,7,8$ when SVS wave is incident.

Notes

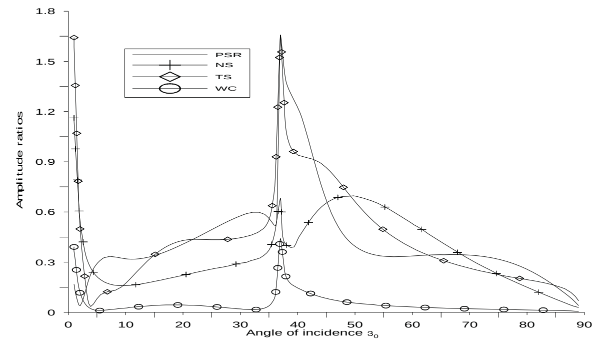


Fig. 3(a) Variation in amplitude ratios of Reflected PS wave when SVS wave is incident

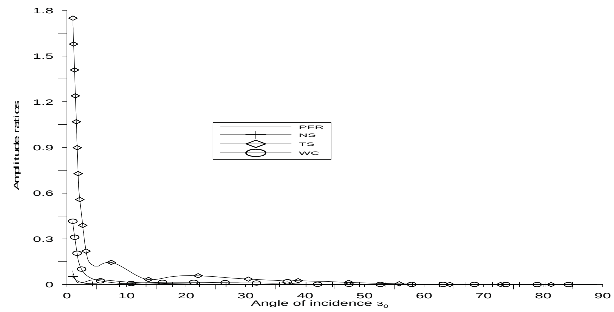


Fig. 3(b) Variation in amplitude ratios of Reflected PF wave when SVS wave is incident

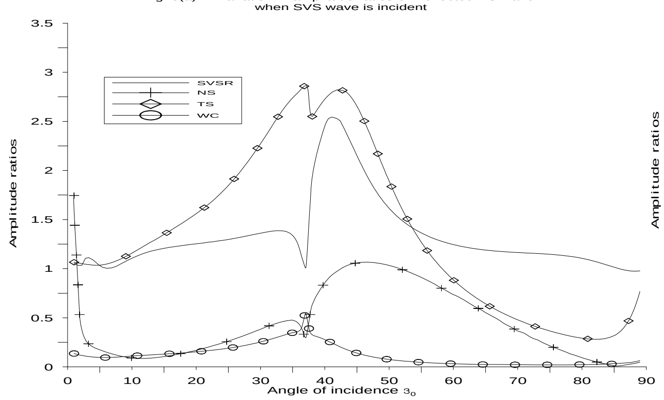


Fig. 3(c)Variation in amplitude ratios of Reflected SVS wave when SVS wave is incident

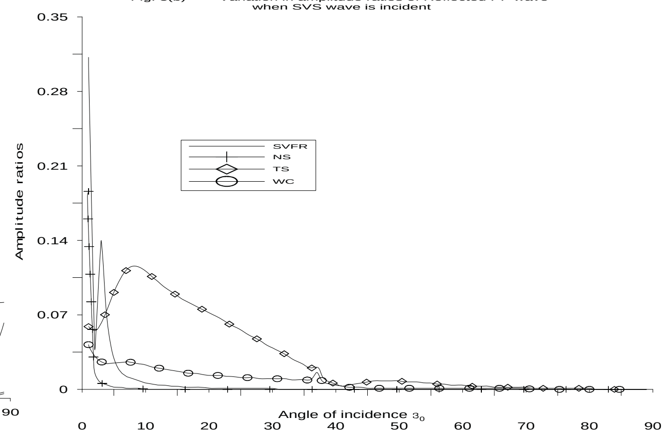


Fig. 3(d) Variation in amplitude ratios of Reflected SVF wave when SVS wave is incident

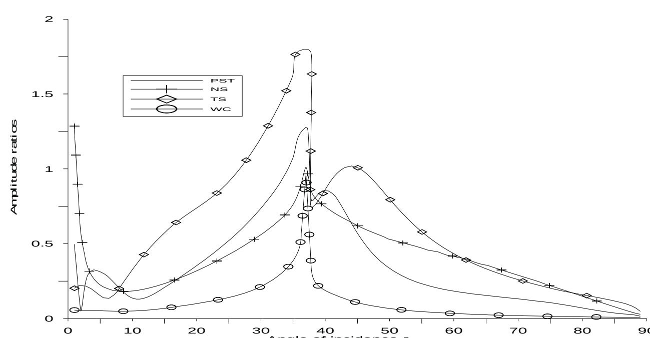


Fig. 3(e) Variation in amplitude ratios of Transmitted PS wave when SVS wave is incident

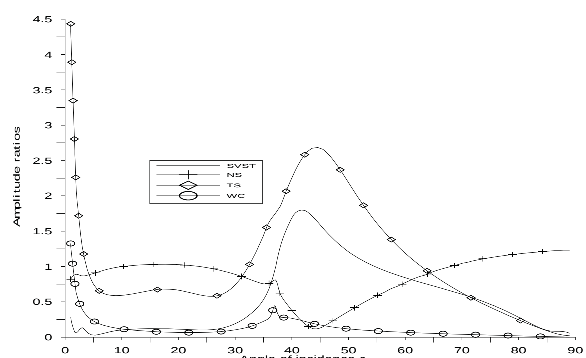


Fig. 3(g) Variation in amplitude ratios of Transmitted SVS wave when SVS wave is incident

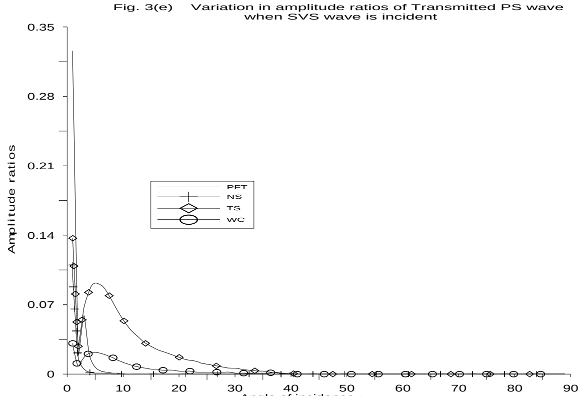


Fig. 3(f) Variation in amplitude ratios of Transmitted PF wave when SVS wave is incident

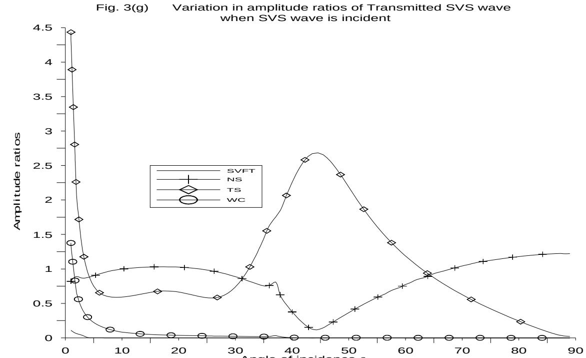


Fig. 3(h) Variation in amplitude ratios of Transmitted SVF wave when SVS wave is incident

© 2012 Global Journals Inc. (US)

The values of amplitude ratios $|Z_1|$ for PSR and TS are of oscillatory behavior. They attains peak value at θ_0= 37 and then for θ_0≥38 it decrease, whereas for WC it attains peak value at θ_0=36 and then starts decreasing. For $\theta_0 \geq 5$the values of amplitude ratio for WC remains less than the values for PSR, NS and TS.

From Fig. 3(b) we notice that values of amplitude ratio$|Z_2|$for PFR, NS, TS and WC decreases in whole range. The values of amplitude ratio$|Z_2|$for TS remain greater than the values of amplitude ratio$|Z_2|$for PFR, NS and WC.

From Fig. 3(c) we notice, that values of amplitude ratio$|Z_3|$for SVSR, NS, TS and WC is of oscillatory behavior. For $\theta_0 \geq 36$ the values of amplitude ratio for WC remains less than the values of amplitude ratio for SVSR, NS, and TS. For $5 \leq \theta_0 \leq 50$ the values of amplitude ratio for TS remains greater than the values of amplitude ratio for SVSR, NS, WC.

Fig. 3(d), depicts that values of amplitude ratio$|Z_4|$for SVFR oscillates in the region $1\leq \theta_0 \leq 5$ then decrease with angle of incidence. The values of amplitude ratio for NS and WC decrease with angle of incidence. For $\theta_0 \geq 3$the values of amplitude ratio for NS remains less than the values obtain for SVFR, TS and WC. The values of amplitude ratio for TS initially increase then decrease with angle of incidence. For $\theta_0 \geq 5$the values of amplitude ratio for TS remains greater than the values of amplitude ratio for SVFR, NS and WC.

From Fig. 3(e) we notice, that values of amplitude ratios$|Z_5|$for PST, NS, TS and WC are of oscillatory behavior. For $10 \leq \theta_0 \leq 35$ the values of amplitude ratio for NS remains greater than the values of amplitude ratio for PST, NS and WC. The values of amplitude ratio for WC remains less than the values obtain for PST, NS, and TS in whole range. The values of amplitude ratio for NS and WC attains maximum value at $\theta_0 = 35$.

From Fig. 3(f) we notice that values of amplitude ratio$|Z_6|$for PFT initially oscillates in region $1\leq \theta_0 \leq 4$, then starts decreasing . The values of amplitude ratio for NS decrease with angle of incidence. The values of amplitude ratio for WC and TS initially oscillates then decrease with angle of incidence. The values of amplitude ratio for TS remains greater than the values of amplitude ratio obtain for PFT, NS and WC for $\theta_0 \geq 4$.

From Fig. 3(g), we notice that values of amplitude ratio$|Z_7|$for SVST, NS, TS and WC are of oscillatory behavior. In range $5 \leq \theta_0 \leq 30$ the values of amplitude ratio for NS remains greater than the values of amplitude ratio for SVST, TS and WC. For $31\leq \theta_0 \leq 64$ the values of amplitude ratio for TS remain greater than the values of amplitude ratio obtain for SVST, NS and WC.

From Fig. 3(h), we notice that values of amplitude ratio$|Z_8|$for NS and TS are of oscillatory behavior, whereas for SVFT and WC it decrease with angle of incidence. The values of amplitude ratio for SVFT remain less than the values of amplitude ratio for NS, TS and WC in whole range.

IX. Conclusion

When PS wave is incident the values of amplitude ratio for$|Z_2|,|Z_4|,|Z_6|,|Z_8|$decrease with angle of incidence, whereas for $|Z_1|$, $|Z_3|$, $|Z_5|$, $|Z_7|$ are of oscillatory behavior. When SVS wave is incident the values of amplitude ratio for $|Z_1|,|Z_3|,|Z_4|,|Z_5|,|Z_6|,|Z_7|,|Z_8|$are of oscillatory behavior, whereas $|Z_2|$ decrease with angle of incidence.

References Références Referencias

1. Von Terzaghi, K., Die Berechnug der Durchlassigkeit des Tones aus dem verlauf der hydromechanischen Spannungserscheinungen, Sitzungsber. Akad. Wiss. (Wien), Math- Naturwiss. Kl., Abt. Iia, 132, 125-138, (1923).
2. Von Terzaghi, K., Erdbaumechanik. Auf Bodenphysikalischer Grundlage, 399 p.p., Leipzig-wien: Franz Deuticke, (1925).

Notes

© 2012 Global Journals Inc. (US)

3. Biot M.A., General Theory of Three dimensional consolidation J. Appl. Phys., 12, 155 - 161, (1941).
4. Biot M.A., Theory of Propagation of elastic waves in a fluid-saturated Porous solid-I Low frequency range, J. Acoust. Soc. Am. 28, 168-178, (1956).
5. Biot M.A., Theory of Propagation of elastic waves in a fluid-saturated Porous solid-II Higher frequency range. J. Acoust. Soc. Am. 28, 179-191, (1956).
6. Biot M.A., Mechanics of deformation and acoustic Propagation in Porous media. J. Appl. Phys. 33, 1482-1498, (1962).
7. Biot M.A., Generalized theory of acoustic propagation in porous dissipation media, J. Acoust. Soc. Of America 34, 1254-1264, (1962).
8. Fillunger,P., Der Auftrieb in Talsperren Osterr Wochenschrift fur den oftemtl. Bauelienst. I. Teil 532-552, III Teil 567-570, (1933).
9. Bowen R.M., Incompressible Porous Media Models by use of the Theory of Mixtures, Int. J. Engg. Sci., 18, 1129-1148, (1980).
10. de Boer R., and Ehlers W., The development of the concept of Effective Stress, Acta Mechanica, 83, 77-92, (1990).
11. de Boer R., and Ehlers W., Uplift, Friction and Capillarity-Three Fundamental Effects for Liquid-saturated porous solida, Int. J. Solid Structures, 26, 43-57, (1990).
12. Ehlers W., Compressible, incompressible and hybrid two-phase models in porous theories, ASME; AMD, 158, 25-38, (1993).
13. Eringen A.C., A Continuum theory of Swelling porous elastic soils, Int. J. Engng. Sci., 32, 1337-1349, (1994).
14. Bofill F. and Quintanilla R., Anti-plane shear deformations of swelling porous elastic soils, Int. Journal of Engng. Sci. 41,801-816, (2003).
15. Gales C., Spatial decay estimates for solutions describing harmonic vibrations in the theory of swelling porous elastic soils, Acta Mechanica, 161, 151-163, (2003).
16. Gales C., Waves and vibrations in the theory of swelling porous elastic soils, European Journal of Mechanics A solida 23, 345-357, (2004).
17. Kleintelter N., Park M., Cushman J., Mixture theory and unsaturated flow in swelling soils, Transp porous Med. 68, 69-89, (2007).
18. Kumar R., Singh M., Propagation of generalized thermoelastic plane waves at an imperfect interface, Int. J of Applied Mechanics and Engineering, 12(3), 713-732, 2007.
19. Kumar R., Singh M., Reflection and transmission of thermoelastic plane waves at an imperfect boundary, Bull. Cal. Math. Soc., 101(4), 419-436, 2009.
20. Kumar R., Sharma N., Ram P., Response of Imperfections at the boundary surface, IEJEMTA, 3(1), 90-109, 2008.
21. Lavrentyev A.I., Rokhlin S.I., Ultrasonic spectroscopy of imperfect contact interfaces between a layer and two solids, J. Acoust, Soc. Am. 103(2), 657-664, 1998.

Notes

© 2012 Global Journals Inc. (US)

Notes

This page is intentionally left blank

© 2012 Global Journals Inc. (US)

Global Journal of Science Frontier Research
Mathematics & Decision Sciences
Volume 12 Issue 2 Version 1.0 February 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Certain Derivation on Lorentzian α - Sasakian Manifolds

By S.Yadav & D.L.Suthar

Alwar Institute of Engineering &Technology, Rajasthan India

Abstract - We classify Lorentzian α - Sasakian manifolds, which satisfy the derivation and $Z(\zeta, X)\cdot Z = 0$, $Z(\zeta, X)\cdot R = 0$, $R(\zeta, X)\cdot Z = 0$, $Z(\zeta, X)\cdot S = 0$, and $Z(\zeta, X)\cdot C = 0$.

Keywords and phrases : Lorentzian α - Sasakian manifold, Concircular curvature tensor and Weyl conformal curvature.

GJSFR-F Classification: Mathematical Subject Classification (2000) : 10 54 , 25 53 , 20 53 , 10

CERTAIN DERIVATION ON LORENTZIAN -SASAKIANMANIFOLDS

Strictly as per the compliance and regulations of :

© 2012. S.Yadav & D.L.Suthar.This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Ref.

[5] J.C.Marrero, The local structure of trans-Sasakian manifolds, Ann.Mat.Pura Appl.162, 4, 1992, pp.77- 86.
[6] J.C.Marrero and D.Chinea, On trans-Sasakian manifolds, In: Proceeding of the XIV-th Spanish-Portuguese Conference on Mathematics, Volumes I-III, In Spanish (Puerto de la Cruz, 1989), 655-659, Univ.La Laguna, a Laguna, 1990.

Certain Derivation on Lorentzian α-Sasakian Manifolds

S.Yadav[α] & D.L.Suthar[σ]

Abstract - We classify Lorentzian α - Sasakian manifolds, which satisfy the derivation $Z(\zeta,X)\cdot Z=0$, $Z(\zeta,X)\cdot R=0$, $R(\zeta,X)\cdot Z=0$, $Z(\zeta,X)\cdot S=0$, and $Z(\zeta,X)\cdot C=0$.

Keywords and phrases : Lorentzian α - Sasakian manifold, Concircular curvature tensor and Weyl conformal curvature.

I. Introduction

In [11], S.Tanno classified connected almost contact metric manifolds whose automorphism group possesses the maximum dimension. For such a manifold, the sectional curvature of a plain sections containing ζ is a constant, say c .He showed that they can be divided into three classes:

(1.1) homogeneous normal contact Riemannian manifolds with $c<0$,
(1.2) global Riemannian products of a line or a circle with a Kaehlar manifold of constant holomorphic sectional curvature if $c=0$ and
(1.3) A warped product space $\mathfrak{R}\times_f C$ if $c>0$.

It is well known that the manifolds of class (1.1) are characterized by admitting a Sasakian structure. Kenmotsu [8] characterized the differential geometric properties of the manifolds of class (1.3); the structure so obtained is now known as Kenmotsu structure. In general these structures are not Sasakian [8].The Gray-Hervella classication of almost Hermitian manifolds [2], there appears a class W_4 , of Hermitian manifolds which are closely related to locally conformal Kaehlar manifolds [10]. An almost contact metric structure on the manifold M is called a trans-Sasakian structure [7] if the product manifold $M\times\mathfrak{R}$ belongs to the class$W4$. The class $C_6\oplus C_5$ (see [5], [6]) coincides with the class of trans-Sasakian structure of type (α, β) .We note that trans-Sasakian structure of type $(0,0)$,$(0, \beta)$ and $(\alpha,0)$ are cosymplectic [4], β -Kenmotsu [8] and α -Sasakian [8] respectively.

In 2005, Ahmet Yildiz [1] studied Lorentzian α –Sasakian manifolds and proved that conformally flat and quasi conformally flat Lorentzian α -Ssaskian manifolds are locally isometric with a sphere.

A Riemannian manifold M are locally symmetric if its curvature tensor R satisfies $\nabla R=0$, where Levi-Civita connection of the Riemannian metric. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold M is said to be semi-symmetric if its curvature tensor R satisfies

$$R(X,Y)\cdot R=0, \qquad X,Y\in TM,$$

where $R(X,Y)$ acts on R as a derivation.

Locally symmetric and semi-symmetric P-Sasakian manifolds are studied in [14] .After curvature tensor, the Weyl conformal curvature tensor C and the concircular curvature tensor Z are the next important curvature tensor .In this paper, we study several derivation conditions on Lorentzian α – Sasakian manifolds. The

Author α σ : Department of Applied Science, Faculty of Mathematics, Alwar Institute of Engineering &Technology, M.I.A.Alwar-301030, Rajasthan India. E-mails : prof_sky16@yahoo.com,dd_suthar@yahoo.co.in

© 2012 Global Journals Inc. (US)

paper is organized as follows. In section2, we give a brief account of Lorentzian α –Sasakian manifolds, the Wey conformal curvature tensor and the concircular curvature tensor. In section 3, we find the necessary and sufficient condition for Lorentzian α–Sasakian manifolds satisfying the condition $Z(\zeta, X)\cdot Z=0$, $Z(\zeta, X)\cdot R=0$, $R(\zeta, X)\cdot Z=0$, $Z(\zeta, X)\cdot S=0$, and $Z(\zeta, X)\cdot C=0$.

II. Lorentzian α -Sasakian Manifolds

An n- dimension differentiable manifold M is called Lorentzian α –Sasakian manifold if it admits a $(1,1)$ tensor field φ ,a contravarient vector field ζ , a covariant vector field η and a Lorentzian metric g which satisfy (see [1])

$$\eta(\zeta)=-1, \tag{2.1}$$

$$\varphi^2=I+\eta\otimes\zeta, \tag{2.2}$$

$$g(\varphi X,\varphi Y)=g(X,Y)+\eta(X)\eta(Y), \tag{2.3}$$

$$g(X,\zeta)=\eta(X), \tag{2.4}$$

$$\varphi\zeta=0, \qquad \eta(\varphi X)=0, \tag{2.5}$$

for all $X, Y \in TM$.
Also Lorentzian α -Sasakian manifold is satisfying (see [1])

$$(a)\ \nabla_X\zeta=-\alpha\varphi X, \qquad (b)\ (\nabla_X\eta)(Y)=-\alpha g(\varphi X,Y), \tag{2.6}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g Further on Lorentzian α – Sasakian manifold M the following relations holds ([1]).

$$\eta(R(X,Y)Z)=\alpha^2\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}, \tag{2.7}$$

$$g(R(\zeta,X)Y,\zeta)=-\alpha^2\{g(X,Y)-\eta(X)\eta(Y)\}, \tag{2.8}$$

$$(R(\zeta,X)Y=\alpha^2\{g(X,Y)\zeta-\eta(Y)X\}, \tag{2.9}$$

$$(R(X,Y)\zeta=\alpha^2\{\eta(Y)X-\eta(X)Y\}, \tag{2.10}$$

$$R(\zeta,Y)\zeta=\alpha^2\{\eta(Y)Y+Y\}, \tag{2.11}$$

$$(\nabla_X\varphi)(Y)=\alpha^2\{g(X,Y)\zeta-\eta(Y)X\}, \tag{2.12}$$

$$S(X,\zeta)=(n-1)\alpha^2\eta(X), \tag{2.13}$$

An almost para contact Riemannian manifold M is said to be η -Einstein if the Ricci operator Q satisfies

$$Q=aId+b\eta\otimes\zeta,$$

where a and b are smooth functions on the manifold. In particular if $b=0$, then M is an Einstein manifold. Let (M, g) be an n – dimensional Riemannian manifold .Then the concircular curvature tensor and the Wey conformal curvature tensor are defined by 9.

$$Z(X,Y)U=R(X,Y)U-\frac{\tau}{n(n-1)}\left[g(Y,U)X-g(X,U)Y\right], \tag{2.14}$$

$$C(X,Y)U=R(X,Y)U-\frac{1}{(n-2)}\left[S(Y,U)X-S(X,U)Y+g(Y,U)QX-g(X,U)QY\right] + \frac{\tau}{(n-1)(n-2)}\left[g(Y,U)X-g(X,U)Y\right], \tag{2.15}$$

Ref.

[1] Ahmit Yildiz and Cengizhan Murathan, On Lorentzianα–Sasakian manifolds,Kyungpook Math.J.,45,2005,pp.95-103.

[9] K.Yano and M.Kon, Structure on manifolds, Series in Pure Math, 3, Words Sci., 1984

© 2012 Global Journals Inc. (US)

for all $X, Y, U \in TM$, ,respectively, where R is the curvature tensor, S is the Ricci tensor and τ is the scalar curvature tensor of M .

III. Main Results

In this section, we obtain necessary and sufficient condition for Lorentzian α –Sasakian manifolds satisfying the derivations conditions $Z(\zeta, X) . Z = 0$, $Z(\zeta, X) . R = 0$, $R(\zeta, X) . Z = 0$, $Z(\zeta, X) . S = 0$, and $Z(\zeta, X) . C = 0$.

Notes

Theorem 3.1. *An n – dimensional Lorentzian α–Sasakian manifold* (M^n, g) *satisfies*

$$Z(\zeta, X) \cdot Z = 0$$

if and only if either the scalar curvature of (M^n, g) *is* $\tau = \alpha^2 n (1\text{-}n)$ *or* (M^n, g) *is locally isometric to the Hyperbolic space* $H^n(-\alpha^2)$.

Proof. In a Lorentz an α – Sasakian manifold (M^n, g) , we have

$$Z(X,Y)\zeta = \left[\left(\alpha^2 - \frac{\tau}{n(n-1)}\right)\right](\eta(Y)X - \eta(X)Y), \quad (3.1)$$

$$Z(\zeta, X)Y = \left[\left(\alpha^2 - \frac{\tau}{n(n-1)}\right)\right](g(X,Y)\zeta - \eta(Y)X). \quad (3.2)$$

The condition $Z(\zeta, X) . Z = 0$ implies that

$$[Z(\zeta,U), Z(X,Y)]\zeta - Z(Z(\zeta,U)X,Y)\zeta - Z(X, Z(\zeta,U)Y)\zeta = 0,$$

This in view of (3.1) and (3.2) gives

$$\left(\alpha^2 + \frac{\tau}{n(n-1)}\right)\left[Z(X,Y)U - \left(\alpha^2 - \frac{\tau}{n(n-1)}\right)\right]\{(g(Y,U)X - g(X,U)Y)\} = 0.$$

Therefore either the scalar curvature $\tau = \alpha^2 n (1-n)$ or

$$Z(X,Y)U = \left(\alpha^2 - \frac{\tau}{n(n-1)}\right)((g(Y,U)X - g(X,U)Y)) = 0,$$

This in view of (2.14) gives

$$R(X,Y)U = -\alpha^2(g(X,U)Y - g(Y,U)X).$$

The above equation implies that is of constant curvature $-\alpha^2$ and consequently it is locally isometric to the Hyperbolic space $H^n(-\alpha 2)$. Conversely, if has scalar curvature $\tau = \alpha^2 n (1-n)$ .Then from (3.2), it follows that $Z(\zeta, X) = 0$ .Similarly in the second case, since is of constant curvature $\tau = \alpha^2 n (1=n)$ therefore we again get $Z(\zeta, X) = 0$ .In view of the fact $Z(\zeta, X).R$ denotes acting on R as a derivation, we state the following result as the theorem

Theorem3.2. *An n –dimensional Lorentzian α –Sasakian manifold* (M^n, g) *satisfies*

$$Z(\zeta, X) \cdot R = 0$$

if and only if either (M^n, g) *is locally isometric to the Hyperbolic space* $H^n(-\alpha^2)$.*or the scalar curvature of* (M^n, g) *is* $\tau = \alpha^2 n (1=n)$.

Proposition3.3. In an n –dimensional Riemannian manifold, we have $R \cdot Z = R \cdot R$

Proof. We suppose that $X, Y, U, V, W \in TM$.Therefore

© 2012 Global Journals Inc. (US)

$$(R(X,Y)\cdot Z(U,V,W)) = R(X,Y)Z(U,V)W - Z(R(X,Y)U,V - Z(U,R(X,Y)V)W - Z(U,V)R(X,Y)W.$$

which in view of (3.1) and symmetric properties of R ,we get

$$(R(X,Y)\cdot Z(U,V,W)) = R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W.$$

$$= (R(X,Y)\cdot R)(U,V,W).$$

This proves the proposition3.3
Now, in view of theorem2.1 12 and the proposition3.3 we have the following result as the theorem:

Theorem3.4. An $n-$ dimensional Lorentzian $\alpha-$ Sasakian manifold (M^n, g) satisfies

$$R(\zeta, X)\cdot Z = 0$$

if and only if either (M^n,g) is locally isometric to the Hyperbolic space Hn$(-\alpha^2)$.

Next we prove the following result

Theorem3.5 An $n-$ dimensional Lorentzian $\alpha-$ Sasakian manifold (M^n,g) satisfies

$$Z(\zeta, X)\cdot S = 0$$

if and only if either (M^n,g) has the curvature $\tau = \alpha^2$ n (1–n) or M^n is an Einstein manifold.

Proof. The condition $Z(\zeta, X) . S = 0$ implies that

$$S(Z(\zeta,X)Y,\zeta) + S(Y,Z(\zeta,X)\zeta) = 0,$$

This in view of (2.13) and (3.2) gives

$$\left(\alpha^2 - \frac{\tau}{n(n-1)}\right)\left[S(X,Y) + \alpha^2(n-1)g(X,Y)\right]$$

Therefore either the scalar curvature of (M^n,g) is $\tau = \alpha^2 n\,(1-n)$ which is of constant or $S = \alpha^2(1-n)g(X,Y)$ which implies that (M^n,g) is an Einstein manifold with $\tau = \alpha^2 n\,(1-n)$.

which proves that theorem3.5.

Theorem3.6 *.An $n-$dimensional conformally flat Lorentzian $\alpha-$Sasakian manifold (M^n,g) is locally isometric to the hyperbolic space $H^n(-\alpha^2)$.*

Proof. In this section we suppose that $Z(X,Y)\ .U=0$ .Then from (2.14) we get

$$R(X,Y)U = \frac{\tau}{n(n-1)}\left[g(Y,U)X - g(X,U)Y\right], \tag{3.3}$$

From (3.3), we have

$$\tilde{R}(X,Y,U,W) = \frac{\tau}{n(n-1)}\left[g(Y,U)g(X,W) - g(X,U)g(Y,W)\right], \tag{3.4}$$

where $\tilde{R}(X,Y,U,W) = g(R(X,Y,U)W)$.

Putting $X=W=\zeta$ in (3.4) and by use of (2.4) and (2.8), we obtain

$$\left(\alpha^2 - \frac{\tau}{n(n-1)}\right)\left[g(Y,U) + \eta(Y)\eta(U)\right] = 0,$$

Ref.

[12] Sunil Kumar Yadav, Praduman K.Dwivedi and Dayalal Suthar, On $(LCS)_{2n+1}$ – Manifolds Satisfying Certain Conditions on the Concircular Curvature Tensor, ThiJournal of Mathematics, 9,2011,pp. 597-603, Thailand.

© 2012 Global Journals Inc. (US)

This shows that either $\tau = \alpha^2 n(n-1)$ or $g(Y,U) = -\eta(Y)\eta(U)$ .But if $g(Y,U) = -\eta(Y)\eta(U)$ .Then from (2.3) we get $g\varphi(Y,\varphi U) = 0$, which is not possible. Therefore, $\tau = \alpha^2 n(n-1)$ .Now putting $\tau = \alpha^2 n(n-1)$ in (3.3), we find

$$R(X,Y)U = \alpha^2[g(Y,U)X - g(X,U)Y]$$

This proves the theorem3.6

$$Z(\zeta,X)\cdot C = 0$$

Notes

Theorem6. *An $n-$ dimensional Lorentzian $\alpha-$Sasakian manifold (M^n,g) satisfies*

if and only if either (M^n,g) has the scalar curvature $\tau = \alpha^2 n(n-1)$ or (M^n,g) is an η-Einstein manifold.

Proof. The condition $Z(\zeta, X).C = 0$ implies that

$$[Z(\zeta,U),C(X,Y)]W - C(Z(\zeta,U)X,Y)W - C(X,Z(\zeta,U)Y)W = 0,$$

This in view of (3.1) gives

$$\left(\alpha^2 - \frac{\tau}{n(n-1)}\right)\begin{bmatrix} C(X,Y,W,U)\zeta - \eta(C(X,Y)W)U - g(U,X)C(\zeta,Y)W \\ +\eta(X)C(U,Y,W) - g(U,Y)C(X,\zeta,W) + \eta(Y)C(X,U,W) \end{bmatrix} = 0,$$

So either scalar curvature of (M^n,g) is $\tau = \alpha^2 n(n-1)$ or the equation

$$\begin{bmatrix} C(X,Y,W,U)\zeta - \eta(C(X,Y)W)U - g(U,X)C(\zeta,Y)W \\ +\eta(X)C(U,Y,W) - g(U,Y)C(X,\zeta,W) + \eta(Y)C(X,U,W) \end{bmatrix} = 0,$$

holds on M .Taking inner product of above last equations with ζ , we get

$$\begin{bmatrix} -C(X,Y,W,U)\zeta - \eta(C(X,Y)W)\eta(U) - g(U,X)\eta(C(\zeta,Y)W) \\ +\eta(X)\eta(C(U,Y,W)) - g(U,Y)\eta(C(X,\zeta,W)) + \eta(Y)\eta(C(X,U,W)) \end{bmatrix} = 0,$$

Hence by using (2.7)(2.13)and(2.15) in above equations we get

$$S(X,U) = \left(\alpha^2 + \frac{\tau}{(n-1)(n-2)}\right)g(X,U) + \left(\alpha^2 + \frac{\tau}{(n-1)(n-2)} + \alpha^2(n-1)\right)\eta(X)\eta(U),$$

which implies that (M^n,g) is an η -Einstein manifold

This proves the theorem 6.

References Références Referencias

[1] Ahmit Yildiz and Cengizhan Murathan, On Lorentzianα–Sasakian manifolds,Kyungpook Math.J.,45,2005,pp.95-103.

[2] A.Gray and L.M.Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann.Mat.Pura Appl.123, 4, 1980, pp.35-58.

[3] Cihan Ozgur and M.M.Tripathi, On P–Sasakian manifold satisfying certain condition on the concircula curvature tensor, Turk.J.Math.30, 2006, pp.1-9.

[4] D.E.Blair, Contact Manifold in Riemannian Geometry, Lecture Notes in Mathematics,Springe Verlag,Berlin,509,1976,146.s

[5] J.C.Marrero, The local structure of trans-Sasakian manifolds, Ann.Mat.Pura Appl.162, 4, 1992, pp.77-86.

[6] J.C.Marrero and D.Chinea, On trans-Sasakian manifolds, In: Proceeding of the XIV-th Spanish-Portuguese Conference on Mathematics, Volumes I-III, In Spanish (Puerto de la Cruz, 1989), 655-659, Univ.La Laguna, a Laguna, 1990.

[7] J.A.Oubina, New classes of contact metric structures, Publ.Math.Debrecen, 32, No.3-4, 1985, pp.187-193.

© 2012 Global Journals Inc. (US)

[8] K.Kenmotsu, A class of almost contact Riemannian manifolds, Tohoku Math., 24, 1972, pp. 93-103.
[9] K.Yano and M.Kon, Structure on manifolds, Series in Pure Math, 3, Words Sci., 1984
[10] S.Dragomir and L.Ornea, Locally conformal Kaehler geometry, In: Progress in Mathematics, 155, Birkhauser Boston, Inc.M.A, 1998.
[11] S.Tanno, The automorphisam groups of almost contact Riemannian manifolds, Tohoku Math.J.21, 1969, pp.21-38.
[12] Sunil Kumar Yadav, Praduman K.Dwivedi and Dayalal Suthar, On $(LCS)_{2n+1}$ – Manifolds Satisfying Certain Conditions on the Concircular Curvature Tensor, ThiJournal of Mathematics, 9,2011,pp. 597-603, Thailand.
[13] S.Yadav D.L.Suthar and A.K Srivastava, Some Results on $M(f_1, f_2, f_3)_{2n+1}$–Manifolds.Int. J. of Pure &Applied Mathematics, 70, 3, 2011, pp.415-423, Sofia, Bulgaria.
[14] T.Adati and Miyazawa, T., On P–Sasakian manifold satisfying certain conditions Tensor (N.S.), 33, 1929, pp.173-178.

Notes

© 2012 Global Journals Inc. (US)

Global Journals Inc. (US) Guidelines Handbook 2012

www.GlobalJournals.org

FELLOW OF ASSOCIATION OF RESEARCH SOCIETY IN SCIENCE (FARSS)

- 'FARSS' title will be awarded to the person after approval of Editor-in-Chief and Editorial Board. The title 'FARSS" can be added to name in the following manner. eg. Dr. John E. Hall, Ph.D., FARSS or William Walldroff Ph. D., M.S., FARSS
- Being FARSS is a respectful honor. It authenticates your research activities. After becoming FARSS, you can use 'FARSS' title as you use your degree in suffix of your name. This will definitely will enhance and add up your name. You can use it on your Career Counseling Materials/CV/Resume/Visiting Card/Name Plate etc.
- 60% Discount will be provided to FARSS members for publishing research papers in Global Journals Inc., if our Editorial Board and Peer Reviewers accept the paper. For the life time, if you are author/co-author of any paper bill sent to you will automatically be discounted one by 60%
- FARSS will be given a renowned, secure, free professional email address with 100 GB of space eg.johnhall@globaljournals.org. You will be facilitated with Webmail, SpamAssassin, Email Forwarders, Auto-Responders, Email Delivery Route tracing, etc.
- FARSS member is eligible to become paid peer reviewer at Global Journals Inc. to earn up to 15% of realized author charges taken from author of respective paper. After reviewing 5 or more papers you can request to transfer the amount to your bank account or to your PayPal account.
- Eg. If we had taken 420 USD from author, we can send 63 USD to your account.
- FARSS member can apply for free approval, grading and certification of some of their Educational and Institutional Degrees from Global Journals Inc. (US) and Open Association of Research,Society U.S.A.
- After you are FARSS. You can send us scanned copy of all of your documents. We will verify, grade and certify them within a month. It will be based on your academic records, quality of research papers published by you, and 50 more criteria. This is beneficial for your job interviews as recruiting organization need not just rely on you for authenticity and your unknown qualities, you would have authentic ranks of all of your documents. Our scale is unique worldwide.
- FARSS member can proceed to get benefits of free research podcasting in Global Research Radio with their research documents, slides and online movies.
- After your publication anywhere in the world, you can upload you research paper with your recorded voice or you can use our professional RJs to record your paper their voice. We can also stream your conference videos and display your slides online.
- FARSS will be eligible for free application of Standardization of their Researches by Open Scientific Standards. Standardization is next step and level after publishing in a journal. A team of research and professional will work with you to take your research to its next level, which is worldwide open standardization.

- FARSS is eligible to earn from their researches: While publishing his paper with Global Journals Inc. (US), FARSS can decide whether he/she would like to publish his/her research in closed manner. When readers will buy that individual research paper for reading, 80% of its earning by Global Journals Inc. (US) will be transferred to FARSS member's bank account after certain threshold balance. There is no time limit for collection. FARSS member can decide its price and we can help in decision.

MEMBER OF ASSOCIATION OF RESEARCH SOCIETY IN SCIENCE (MARSS)

- 'MARSS' title will be awarded to the person after approval of Editor-in-Chief and Editorial Board. The title 'MARSS" can be added to name in the following manner. eg. Dr. John E. Hall, Ph.D., MARSS or William Walldroff Ph. D., M.S., MARSS
- Being MARSS is a respectful honor. It authenticates your research activities. After becoming MARSS, you can use 'MARSS' title as you use your degree in suffix of your name. This will definitely will enhance and add up your name. You can use it on your Career Counseling Materials/CV/Resume/Visiting Card/Name Plate etc.
- 40% Discount will be provided to MARSS members for publishing research papers in Global Journals Inc., if our Editorial Board and Peer Reviewers accept the paper. For the life time, if you are author/co-author of any paper bill sent to you will automatically be discounted one by 60%
- MARSS will be given a renowned, secure, free professional email address with 30 GB of space eg.johnhall@globaljournals.org. You will be facilitated with Webmail, SpamAssassin, Email Forwarders, Auto-Responders, Email Delivery Route tracing, etc.
- MARSS member is eligible to become paid peer reviewer at Global Journals Inc. to earn up to 10% of realized author charges taken from author of respective paper. After reviewing 5 or more papers you can request to transfer the amount to your bank account or to your PayPal account.
- MARSS member can apply for free approval, grading and certification of some of their Educational and Institutional Degrees from Global Journals Inc. (US) and Open Association of Research,Society U.S.A.
- MARSS is eligible to earn from their researches: While publishing his paper with Global Journals Inc. (US), MARSS can decide whether he/she would like to publish his/her research in closed manner. When readers will buy that individual research paper for reading, 40% of its earning by Global Journals Inc. (US) will be transferred to MARSS member's bank account after certain threshold balance. There is no time limit for collection. MARSS member can decide its price and we can help in decision.

ANNUAL MEMBER

- Annual Member will be authorized to receive e-Journal GJSFR for one year (subscription for one year).
- The member will be allotted free 1 GB Web-space along with subDomain to contribute and participate in our activities.
- A professional email address will be allotted free 500 MB email space.

PAPER PUBLICATION

- The members can publish paper once. The paper will be sent to two-peer reviewer. The paper will be published after the acceptance of peer reviewers and Editorial Board.

Process of submission of Research Paper

The Area or field of specialization may or may not be of any category as mentioned in 'Scope of Journal' menu of the GlobalJournals.org website. There are 37 Research Journal categorized with Six parental Journals GJCST, GJMR, GJRE, GJMBR, GJSFR, GJHSS. For Authors should prefer the mentioned categories. There are three widely used systems UDC, DDC and LCC. The details are available as 'Knowledge Abstract' at Home page. The major advantage of this coding is that, the research work will be exposed to and shared with all over the world as we are being abstracted and indexed worldwide.

The paper should be in proper format. The format can be downloaded from first page of 'Author Guideline' Menu. The Author is expected to follow the general rules as mentioned in this menu. The paper should be written in MS-Word Format (*.DOC,*.DOCX).

The Author can submit the paper either online or offline. The authors should prefer online submission.Online Submission: There are three ways to submit your paper:

(A) (I) First, register yourself using top right corner of Home page then Login. If you are already registered, then login using your username and password.

(II) Choose corresponding Journal.

(III) Click 'Submit Manuscript'. Fill required information and Upload the paper.

(B) If you are using Internet Explorer, then Direct Submission through Homepage is also available.

(C) If these two are not conveninet , and then email the paper directly to dean@globaljournals.org.

Offline Submission: Author can send the typed form of paper by Post. However, online submission should be preferred.

© Copyright by Global Journals Inc.(US)| Guidelines Handbook

MANUSCRIPT STYLE INSTRUCTION **<u>(Must be strictly followed)</u>**

Page Size: 8.27" X 11'"

- Left Margin: 0.65
- Right Margin: 0.65
- Top Margin: 0.75
- Bottom Margin: 0.75
- Font type of all text should be Swis 721 Lt BT.
- Paper Title should be of Font Size 24 with one Column section.
- Author Name in Font Size of 11 with one column as of Title.
- Abstract Font size of 9 Bold, “Abstract” word in Italic Bold.
- Main Text: Font size 10 with justified two columns section
- Two Column with Equal Column with of 3.38 and Gaping of .2
- First Character must be three lines Drop capped.
- Paragraph before Spacing of 1 pt and After of 0 pt.
- Line Spacing of 1 pt
- Large Images must be in One Column
- Numbering of First Main Headings (Heading 1) must be in Roman Letters, Capital Letter, and Font Size of 10.
- Numbering of Second Main Headings (Heading 2) must be in Alphabets, Italic, and Font Size of 10.

You can use your own standard format also.
Author Guidelines:

1. General,

2. Ethical Guidelines,

3. Submission of Manuscripts,

4. Manuscript’s Category,

5. Structure and Format of Manuscript,

6. After Acceptance.

1. GENERAL

Before submitting your research paper, one is advised to go through the details as mentioned in following heads. It will be beneficial, while peer reviewer justify your paper for publication.

Scope

The Global Journals Inc. (US) welcome the submission of original paper, review paper, survey article relevant to the all the streams of Philosophy and knowledge. The Global Journals Inc. (US) is parental platform for Global Journal of Computer Science and Technology, Researches in Engineering, Medical Research, Science Frontier Research, Human Social Science, Management, and Business organization. The choice of specific field can be done otherwise as following in Abstracting and Indexing Page on this Website. As the all Global

Journals Inc. (US) are being abstracted and indexed (in process) by most of the reputed organizations. Topics of only narrow interest will not be accepted unless they have wider potential or consequences.

2. ETHICAL GUIDELINES

Authors should follow the ethical guidelines as mentioned below for publication of research paper and research activities.

Papers are accepted on strict understanding that the material in whole or in part has not been, nor is being, considered for publication elsewhere. If the paper once accepted by Global Journals Inc. (US) and Editorial Board, will become the copyright of the Global Journals Inc. (US).

Authorship: The authors and coauthors should have active contribution to conception design, analysis and interpretation of findings. They should critically review the contents and drafting of the paper. All should approve the final version of the paper before submission

The Global Journals Inc. (US) follows the definition of authorship set up by the Global Academy of Research and Development. According to the Global Academy of R&D authorship, criteria must be based on:

1) Substantial contributions to conception and acquisition of data, analysis and interpretation of the findings.

2) Drafting the paper and revising it critically regarding important academic content.

3) Final approval of the version of the paper to be published.

All authors should have been credited according to their appropriate contribution in research activity and preparing paper. Contributors who do not match the criteria as authors may be mentioned under Acknowledgement.

Acknowledgements: Contributors to the research other than authors credited should be mentioned under acknowledgement. The specifications of the source of funding for the research if appropriate can be included. Suppliers of resources may be mentioned along with address.

Appeal of Decision: The Editorial Board's decision on publication of the paper is final and cannot be appealed elsewhere.

Permissions: It is the author's responsibility to have prior permission if all or parts of earlier published illustrations are used in this paper.

Please mention proper reference and appropriate acknowledgements wherever expected.

If all or parts of previously published illustrations are used, permission must be taken from the copyright holder concerned. It is the author's responsibility to take these in writing.

Approval for reproduction/modification of any information (including figures and tables) published elsewhere must be obtained by the authors/copyright holders before submission of the manuscript. Contributors (Authors) are responsible for any copyright fee involved.

3. SUBMISSION OF MANUSCRIPTS

Manuscripts should be uploaded via this online submission page. The online submission is most efficient method for submission of papers, as it enables rapid distribution of manuscripts and consequently speeds up the review procedure. It also enables authors to know the status of their own manuscripts by emailing us. Complete instructions for submitting a paper is available below.

Manuscript submission is a systematic procedure and little preparation is required beyond having all parts of your manuscript in a given format and a computer with an Internet connection and a Web browser. Full help and instructions are provided on-screen. As an author, you will be prompted for login and manuscript details as Field of Paper and then to upload your manuscript file(s) according to the instructions.

To avoid postal delays, all transaction is preferred by e-mail. A finished manuscript submission is confirmed by e-mail immediately and your paper enters the editorial process with no postal delays. When a conclusion is made about the publication of your paper by our Editorial Board, revisions can be submitted online with the same procedure, with an occasion to view and respond to all comments.

Complete support for both authors and co-author is provided.

4. MANUSCRIPT'S CATEGORY

Based on potential and nature, the manuscript can be categorized under the following heads:

Original research paper: Such papers are reports of high-level significant original research work.

Review papers: These are concise, significant but helpful and decisive topics for young researchers.

Research articles: These are handled with small investigation and applications

Research letters: The letters are small and concise comments on previously published matters.

5.STRUCTURE AND FORMAT OF MANUSCRIPT

The recommended size of original research paper is less than seven thousand words, review papers fewer than seven thousands words also.Preparation of research paper or how to write research paper, are major hurdle, while writing manuscript. The research articles and research letters should be fewer than three thousand words, the structure original research paper; sometime review paper should be as follows:

Papers: These are reports of significant research (typically less than 7000 words equivalent, including tables, figures, references), and comprise:

(a)Title should be relevant and commensurate with the theme of the paper.

(b) A brief Summary, "Abstract" (less than 150 words) containing the major results and conclusions.

(c) Up to ten keywords, that precisely identifies the paper's subject, purpose, and focus.

(d) An Introduction, giving necessary background excluding subheadings; objectives must be clearly declared.

(e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition; sources of information must be given and numerical methods must be specified by reference, unless non-standard.

(f) Results should be presented concisely, by well-designed tables and/or figures; the same data may not be used in both; suitable statistical data should be given. All data must be obtained with attention to numerical detail in the planning stage. As reproduced design has been recognized to be important to experiments for a considerable time, the Editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned un-refereed;

(g) Discussion should cover the implications and consequences, not just recapitulating the results; conclusions should be summarizing.

(h) Brief Acknowledgements.

(i) References in the proper form.

Authors should very cautiously consider the preparation of papers to ensure that they communicate efficiently. Papers are much more likely to be accepted, if they are cautiously designed and laid out, contain few or no errors, are summarizing, and be conventional to the approach and instructions. They will in addition, be published with much less delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and to make suggestions to improve briefness.

It is vital, that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

Format

Language: The language of publication is UK English. Authors, for whom English is a second language, must have their manuscript efficiently edited by an English-speaking person before submission to make sure that, the English is of high excellence. It is preferable, that manuscripts should be professionally edited.

Standard Usage, Abbreviations, and Units: Spelling and hyphenation should be conventional to The Concise Oxford English Dictionary. Statistics and measurements should at all times be given in figures, e.g. 16 min, except for when the number begins a sentence. When the number does not refer to a unit of measurement it should be spelt in full unless, it is 160 or greater.

Abbreviations supposed to be used carefully. The abbreviated name or expression is supposed to be cited in full at first usage, followed by the conventional abbreviation in parentheses.

Metric SI units are supposed to generally be used excluding where they conflict with current practice or are confusing. For illustration, 1.4 l rather than 1.4 × 10-3 m3, or 4 mm somewhat than 4 × 10-3 m. Chemical formula and solutions must identify the form used, e.g. anhydrous or hydrated, and the concentration must be in clearly defined units. Common species names should be followed by underlines at the first mention. For following use the generic name should be constricted to a single letter, if it is clear.

Structure

All manuscripts submitted to Global Journals Inc. (US), ought to include:

Title: The title page must carry an instructive title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) wherever the work was carried out. The full postal address in addition with the e-mail address of related author must be given. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining and indexing.

Abstract, used in Original Papers and Reviews:

Optimizing Abstract for Search Engines

Many researchers searching for information online will use search engines such as Google, Yahoo or similar. By optimizing your paper for search engines, you will amplify the chance of someone finding it. This in turn will make it more likely to be viewed and/or cited in a further work. Global Journals Inc. (US) have compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Key Words

A major linchpin in research work for the writing research paper is the keyword search, which one will employ to find both library and Internet resources.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy and planning a list of possible keywords and phrases to try.

Search engines for most searches, use Boolean searching, which is somewhat different from Internet searches. The Boolean search uses "operators," words (and, or, not, and near) that enable you to expand or narrow your affords. Tips for research paper while preparing research paper are very helpful guideline of research paper.

Choice of key words is first tool of tips to write research paper. Research paper writing is an art.A few tips for deciding as strategically as possible about keyword search:

- One should start brainstorming lists of possible keywords before even begin searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in research paper?" Then consider synonyms for the important words.
- It may take the discovery of only one relevant paper to let steer in the right keyword direction because in most databases, the keywords under which a research paper is abstracted are listed with the paper.
- One should avoid outdated words.

Keywords are the key that opens a door to research work sources. Keyword searching is an art in which researcher's skills are bound to improve with experience and time.

Numerical Methods: Numerical methods used should be clear and, where appropriate, supported by references.

Acknowledgements: Please make these as concise as possible.

References

References follow the Harvard scheme of referencing. References in the text should cite the authors' names followed by the time of their publication, unless there are three or more authors when simply the first author's name is quoted followed by et al. unpublished work has to only be cited where necessary, and only in the text. Copies of references in press in other journals have to be supplied with submitted typescripts. It is necessary that all citations and references be carefully checked before submission, as mistakes or omissions will cause delays.

References to information on the World Wide Web can be given, but only if the information is available without charge to readers on an official site. Wikipedia and Similar websites are not allowed where anyone can change the information. Authors will be asked to make available electronic copies of the cited information for inclusion on the Global Journals Inc. (US) homepage at the judgment of the Editorial Board.

The Editorial Board and Global Journals Inc. (US) recommend that, citation of online-published papers and other material should be done via a DOI (digital object identifier). If an author cites anything, which does not have a DOI, they run the risk of the cited material not being noticeable.

The Editorial Board and Global Journals Inc. (US) recommend the use of a tool such as Reference Manager for reference management and formatting.

Tables, Figures and Figure Legends

Tables: Tables should be few in number, cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g. Table 4, a self-explanatory caption and be on a separate sheet. Vertical lines should not be used.

Figures: Figures are supposed to be submitted as separate files. Always take in a citation in the text for each figure using Arabic numbers, e.g. Fig. 4. Artwork must be submitted online in electronic form by e-mailing them.

Preparation of Electronic Figures for Publication

Even though low quality images are sufficient for review purposes, print publication requires high quality images to prevent the final product being blurred or fuzzy. Submit (or e-mail) EPS (line art) or TIFF (halftone/photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Do not use pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings) in relation to the imitation size. Please give the data for figures in black and white or submit a Color Work Agreement Form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution (at final image size) ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs) : >350 dpi; figures containing both halftone and line images: >650 dpi.

Color Charges: It is the rule of the Global Journals Inc. (US) for authors to pay the full cost for the reproduction of their color artwork. Hence, please note that, if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a color work agreement form before your paper can be published.

Figure Legends: Self-explanatory legends of all figures should be incorporated separately under the heading 'Legends to Figures'. In the full-text online edition of the journal, figure legends may possibly be truncated in abbreviated links to the full screen version. Therefore, the first 100 characters of any legend should notify the reader, about the key aspects of the figure.

6. AFTER ACCEPTANCE

Upon approval of a paper for publication, the manuscript will be forwarded to the dean, who is responsible for the publication of the Global Journals Inc. (US).

6.1 Proof Corrections

The corresponding author will receive an e-mail alert containing a link to a website or will be attached. A working e-mail address must therefore be provided for the related author.

Acrobat Reader will be required in order to read this file. This software can be downloaded

(Free of charge) from the following website:

www.adobe.com/products/acrobat/readstep2.html. This will facilitate the file to be opened, read on screen, and printed out in order for any corrections to be added. Further instructions will be sent with the proof.

Proofs must be returned to the dean at dean@globaljournals.org within three days of receipt.

As changes to proofs are costly, we inquire that you only correct typesetting errors. All illustrations are retained by the publisher. Please note that the authors are responsible for all statements made in their work, including changes made by the copy editor.

6.2 Early View of Global Journals Inc. (US) (Publication Prior to Print)

The Global Journals Inc. (US) are enclosed by our publishing's Early View service. Early View articles are complete full-text articles sent in advance of their publication. Early View articles are absolute and final. They have been completely reviewed, revised and edited for publication, and the authors' final corrections have been incorporated. Because they are in final form, no changes can be made after sending them. The nature of Early View articles means that they do not yet have volume, issue or page numbers, so Early View articles cannot be cited in the conventional way.

6.3 Author Services

Online production tracking is available for your article through Author Services. Author Services enables authors to track their article - once it has been accepted - through the production process to publication online and in print. Authors can check the status of their articles online and choose to receive automated e-mails at key stages of production. The authors will receive an e-mail with a unique link that enables them to register and have their article automatically added to the system. Please ensure that a complete e-mail address is provided when submitting the manuscript.

6.4 Author Material Archive Policy

Please note that if not specifically requested, publisher will dispose off hardcopy & electronic information submitted, after the two months of publication. If you require the return of any information submitted, please inform the Editorial Board or dean as soon as possible.

6.5 Offprint and Extra Copies

A PDF offprint of the online-published article will be provided free of charge to the related author, and may be distributed according to the Publisher's terms and conditions. Additional paper offprint may be ordered by emailing us at: editor@globaljournals.org .

the search? Will I be able to find all information in this field area? If the answer of these types of questions will be "Yes" then you can choose that topic. In most of the cases, you may have to conduct the surveys and have to visit several places because this field is related to Computer Science and Information Technology. Also, you may have to do a lot of work to find all rise and falls regarding the various data of that subject. Sometimes, detailed information plays a vital role, instead of short information.

2. Evaluators are human: First thing to remember that evaluators are also human being. They are not only meant for rejecting a paper. They are here to evaluate your paper. So, present your Best.

3. Think Like Evaluators: If you are in a confusion or getting demotivated that your paper will be accepted by evaluators or not, then think and try to evaluate your paper like an Evaluator. Try to understand that what an evaluator wants in your research paper and automatically you will have your answer.

4. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

5. Ask your Guides: If you are having any difficulty in your research, then do not hesitate to share your difficulty to your guide (if you have any). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work then ask the supervisor to help you with the alternative. He might also provide you the list of essential readings.

6. Use of computer is recommended: As you are doing research in the field of Computer Science, then this point is quite obvious.

7. Use right software: Always use good quality software packages. If you are not capable to judge good software then you can lose quality of your paper unknowingly. There are various software programs available to help you, which you can get through Internet.

8. Use the Internet for help: An excellent start for your paper can be by using the Google. It is an excellent search engine, where you can have your doubts resolved. You may also read some answers for the frequent question how to write my research paper or find model research paper. From the internet library you can download books. If you have all required books make important reading selecting and analyzing the specified information. Then put together research paper sketch out.

9. Use and get big pictures: Always use encyclopedias, Wikipedia to get pictures so that you can go into the depth.

10. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right! It is a good habit, which helps to not to lose your continuity. You should always use bookmarks while searching on Internet also, which will make your search easier.

11. Revise what you wrote: When you write anything, always read it, summarize it and then finalize it.

12. Make all efforts: Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

13. Have backups: When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

14. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several and unnecessary diagrams will degrade the quality of your paper by creating "hotchpotch." So always, try to make and include those diagrams, which are made by your own to improve readability and understandability of your paper.

15. Use of direct quotes: When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

16. Use proper verb tense: Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

17. Never use online paper: If you are getting any paper on Internet, then never use it as your research paper because it might be possible that evaluator has already seen it or maybe it is outdated version.

18. **Pick a good study spot:** To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

19. Know what you know: Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

20. Use good quality grammar: Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

21. Arrangement of information: Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

23. Multitasking in research is not good: Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

24. Never copy others' work: Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

25. Take proper rest and food: No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

26. Go for seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

27. Refresh your mind after intervals: Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

29. **Think technically:** Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

30. Think and then print: When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

31. Adding unnecessary information: Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

32. Never oversimplify everything: To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

33. Report concluded results: Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

34. After conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page

- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.

Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to

© Copyright by Global Journals Inc.(US)| Guidelines Handbook

shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.
- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.

- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

Administration Rules Listed Before
Submitting Your Research Paper to Global Journals Inc. (US)

Please carefully note down following rules and regulation before submitting your Research Paper to Global Journals Inc. (US):

Segment Draft and Final Research Paper: You have to strictly follow the template of research paper. If it is not done your paper may get rejected.

- The **major constraint** is that you must independently make all content, tables, graphs, and facts that are offered in the paper. You must write each part of the paper wholly on your own. The Peer-reviewers need to identify your own perceptive of the concepts in your own terms. NEVER extract straight from any foundation, and never rephrase someone else's analysis.
- Do not give permission to anyone else to "PROOFREAD" your manuscript.
- Methods to avoid Plagiarism is applied by us on every paper, if found guilty, you will be blacklisted by all of our collaborated research groups, your institution will be informed for this and strict legal actions will be taken immediately.)
- To guard yourself and others from possible illegal use please do not permit anyone right to use to your paper and files.

Criterion for Grading a Research Paper (Compilation) by Global Journals Inc. (US)

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals Inc. (US).

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

INDEX

M

N

O

P

Q

R

S

T

save our planet

Global Journal of Science Frontier Research

Visit us on the Web at www.GlobalJournals.org | www.JournalofScience.org
or email us at helpdesk@globaljournals.org

ISSN 9755896

© 2012 by Global Journals