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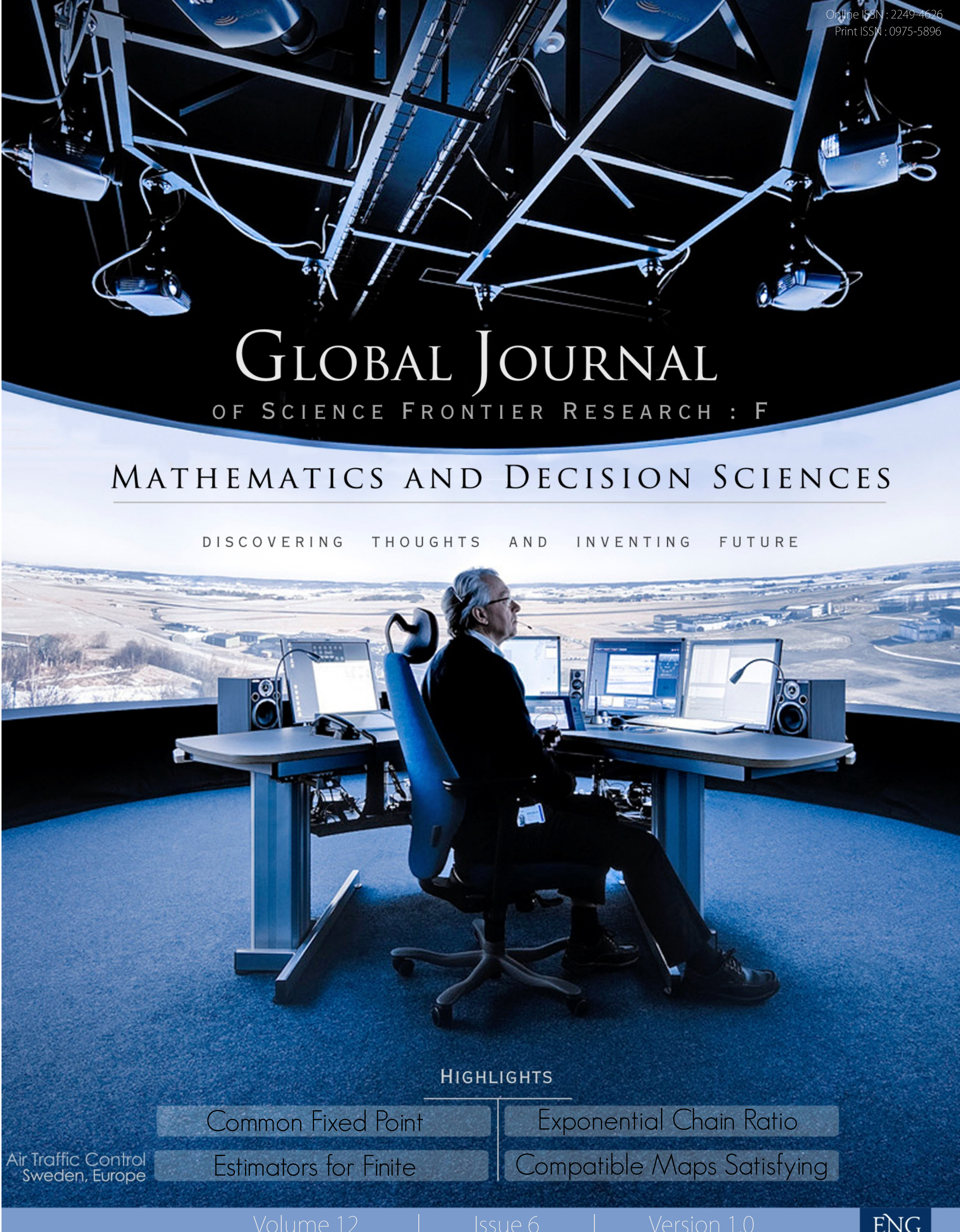

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- Estimators for Finite
- Exponential Chain Ratio
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Keywords : Occasionally weakly compatible maps, implicit relation, common fixed point theorems, Strict contractive condition, fuzzy metric space.

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Common Fixed Point Theorems for Single and Set – Valued Maps in Non – Archimedean Fuzzy Metric Spaces

T. K. Samanta[α] & Sumit Mohinta[σ]

Abstract - The intent of this paper is to establish a common fixed point theorem for two pairs of occasionally weakly compatible single and set-valued maps satisfying a strict contractive condition in a non - Archimedean fuzzy metric space.
Keywords : Occasionally weakly compatible maps, implicit relation, common fixed point theorems, Strict contractive condition, fuzzy metric space.

I. Introduction

Ever since the concept of fuzzy sets was coined by Zadeh [9] in 1965 to describe the situation in which data are imprecise or vague or uncertain. Consequently, the last three decades remained productive for various authors [1, 11, 13] etc. and they have extensively developed the theory of fuzzy sets due to a wide range of application in the field of population dynamics , chaos control , computer programming , medicine , etc.

Kramosil and Michalek [10] introduced the concept of fuzzy metric spaces (briefly, FM-spaces) in 1975, which opened a new avenue for further development of analysis in such spaces. Later on it is modified and a few concepts of mathematical analysis have been developed in fuzzy metric space by George and Veeramani [1, 2]. In fact, the concepts of fixed point theorem have been developed in fuzzy metric space in the paper [12]. In recent years several fixed point theorems for single and set valued maps are proved and have numerous applications and by now, there exists a considerable rich literature in this domain.

Various authors [7, 8, 3] have discussed and studied extensively various results on coincidence, existence and uniqueness of fixed and common fixed points by using the concept of weak commutativity, compatibility, non - compatibility and weak compatibility for single and set valued maps satisfying certain contractive conditions in different spaces and that have been applied to diverse problems.

The intent of this paper is to establish a common fixed point theorem for two pairs of casionally weakly compatible single and set - valued maps satisfying a strict contractive condition in a non - Archimedean fuzzy metric space.

II. Preliminaries

We quote some definitions and statements of a few theorems which will be needed in the sequel.

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Definition 2.1 [5] A binary operation $*$: $[0,1] \times [0,1] \longrightarrow [0,1]$ is continuous t – norm if $*$ satisfies the following conditions :

i.) is commutative and associative;
ii.) is continuous;
iii.) $a * 1 = a \quad \forall \ a \in [0,1]$;
iv.) $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0,1]$.

Result 2.2 [6]

i.) For any $r_1, r_2 \in (0,1)$ with $r_1 > r_2$, there exist $r_3 \in (0,1)$ such that $r_1 * r_2 > r_2$,
ii.) For any $r_5 \in (0,1)$, there exist $r_6 \in (0,1)$ such that $r_6 * r_6 \geq r_5$.

Definition 2.3 [1] The 3-tuple $(X, \mu, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t - norm and μ is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions :

i.) $\mu(x, y, t) > 0$;
ii.) $\mu(x, y, t) = 1$ if and only if $x = y$;
iii.) $\mu(x, y, t) = \mu(y, x, t)$;
iv.) $\mu(x, y, s) * \mu(y, z, t) \leq \mu(x, z, s+t)$;
v.) $\mu(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous ;

for all $x, y, z \in X$ and $t, s > 0$.

Note that $\mu(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.

Example 2.4 Let $X = [0, \infty)$, $a * b = ab$ for every $a, b \in [0,1]$ and d be the usual metric defined on X. Define $\mu(x, y, t) = e^{-\frac{d(x,y)}{t}}$ for all x, y in X. Then clearly $(X, \mu, *)$ is a fuzzy metric space.

Example 2.5 Let (X, d) be a metric space, and let $a * b = ab$ or $a * b = \min\{a, b\}$ for all $a, b \in [0,1]$. Let $\mu(x, y, t) = \dfrac{t}{t + d(x,y)}$ or all $x, y \in X$ and $t > 0$. Then $(X, \mu, *)$ is a fuzzy metric space and this fuzzy metric μ induced by d is called the standard fuzzy metric [1].

Note 2.6 George and Veeramani [1] proved that every fuzzy metric space is a metrizable topological space. In this paper, also they have proved, if (X, d) is a metric space then the topology generated by d coincides with the topology generated by the fuzzy metric μ of example (2.5). As a result, we can say that an ordinary metric space is a special case of fuzzy metric space.

Note 2.7 Consider the following condition :

$$(iv') \ \ \mu(x, y, s) * \mu(y, z, t) \leq \mu(x, z, \max\{s, t\});$$

If the condition (iv) in the definition (2.3) is replaced by the condition (iv'), the fuzzy

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metric space $(X, \mu, *)$ is said to be a non - Archimedean fuzzy metric space .

Remark 2.8 In fuzzy metric space X , for all $x, y \in X$, $\mu(x, y, \cdot)$ is non- decreasing with respect to the variable t. It is easy to see that every non - Archimedean fuzzy metric space is also a fuzzy metric space.

In fact, in a non - Archimedean fuzzy metric space,

$$\mu(x, y, t) \geq \mu(x, z, t) * \mu(z, y, t) \text{ for all } x, y, z \in X,\ t > 0.$$

Throughout the paper X will represent the non - Archimedean fuzzy metric space $(X, \mu, *)$ and $CB(X)$, the set of all non-empty closed and bounded sub-set of X. We recall these usual notations : for $x \in X$, $A \subseteq X$ and for every $t > 0$,

$$\mu(x, A, t) = \max\{\mu(x, y, t) : y \in A\}$$

and let H be the associated Hausdorff fuzzy metric on $CB(X)$, for every A, B in $CB(X)$

$$H(A, B, t) = \min\left\{\min_{x \in A} \mu(x, B, t),\ \min_{x \in B} \mu(A, y, t)\right\}$$

Definition2.9 A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if

i.) given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for $n = 1, 2, \cdots$, and $\{a_n\}$ converges to a.

ii.) given $\in > 0$ there exists a positive integer N such that $A_n \subseteq A_\in$ for $n > N$ where $A_\in$ is the union of all open spheres with centers in A and radius $\in$.

Definition 2.10 A point $x \in X$ is called a coincidence point (resp. fixed point) of

$$A : X \to X,\ B : X \to CB(X) \text{ if } Ax \in Bx \text{ (resp. } x = Ax \in Bx \text{).}$$

Definition 2.11 Maps $A : X \to X$, $B : X \to CB(X)$ are said to be compatible if $ABx \in CB(X)$ for all $x \in X$ and

$$\lim_{n \to \infty} H\left(ABx_n, BAx_n, t\right) = 1$$

whenever $\{x_n\}$ is a sequence in X such that $Bx_n \to M \in CB(X)$ and $Ax_n \to x \in M$.

Definition 2.12 Maps $A : X \to X$ and $B : X \to CB(X)$ are said to be weakly compatible if they commute at coincidence points. ie., if $ABx = BAx$ whenever $Ax \in Bx$.

Definition 2.13 Maps $A : X \to X$ and $B : X \to CB(X)$ are said to be occasionally weakly compatible (**owc**) if there exists some point $x \in X$ such that $Ax \in Bx$ and $ABx \subseteq BAx$.

Example 2.14 Let $X = [1, \infty[$ with the usual metric. Define $f : X \to X$ and $F : X \to CB(X)$ by, for all $x \in X$, $fx = x + 1$, $Fx = [1, x + 1]$. We see that $fx = x + 1 \in Fx$ and $fFx = [2, x+2] \subset Ffx = [1, x + 2]$.

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Hence, f and F are occasionally weakly compatible but not weakly compatible.

Definition 2.15 Let $F : X \to 2^X$ be a set - valued map on X. $x \in X$ is a fixed point of F if $x \in Fx$, and is a strict fixed point of F if $Fx = \{x\}$.

Property 2.16 Let A and $B \in CB(X)$, then for any $a \in A$, we have $\mu(a, B, t) \geq H(A, B, t)$.

Proof Obvious.

III. A Strict Fixed Point Theorem

Theorem 3.1 Let $f, g : X \to X$ be mappings and $F, G : X \to CB(X)$ be set - valued mappings such that the pairs $\{f, F\}$ and $\{g, G\}$ are **owc**. Let $\varphi : R^6 \to R$ be a real valued map satisfying the following conditions

$(\varphi_1) : \varphi$ is increasing in variables t_2, t_5 and t_6;

$(\varphi_2) : \varphi\ (u(t), u(t), 1, 1, u(t), u(t)) > 1$ for all $u(t) \in [0, 1)$.

If , for all x and $y \in X$ for which

$$(*)\ \varphi(H(Fx, Gy, t), \mu(fx, gy, t), \mu(fx, Fx, t), \mu(gy, Gy, t), \mu(fx, Gy, t)$$
$$\mu(gy, Fx, t)) < 1$$

then f, g, F and G have a unique fixed point which is a strict fixed point for F and G.

Proof

i.) We begin to show existence of a common fixed point . Since the pairs $\{f, F\}$ and $\{g, G\}$ are **owc** then , there exist u, v in X such that $fu \in Fu$, $gv \in Gv$, $fFu \subseteq Ffu$ and $gGv \subseteq Ggv$. Also , using the triangle inequality and Property (2.16) , we obtain

$$\mu(fu, gv, t) \geq H(Fu, Gv, t) \tag{1}$$

and

$$\mu\left(f^2u, gv, t\right) \geq H(Ffu, Gv, t) \tag{2}$$

First we show that $gv = fu$. The condition $(*)$ implies that

$$\varphi(H(Fu, Gv, t), \mu(fu, gv, t), \mu(fu, Fu, t), \mu(gv, Gv, t), \mu(fu, Gv, t)$$
$$\mu(gv, Fu, t)) < 1$$

$$\Rightarrow \varphi(H(Fu, Gv, t), \mu(fu, gv, t), 1, 1, \mu(fu, Gv, t), \mu(gv, Fu, t)) < 1$$

By (φ_1) we have

$$\varphi(H(Fu, Gv, t), H(Fu, Gv, t), 1, 1, H(Fu, Gv, t), H(Fu, Gv, t)) < 1$$

which from (φ_2) gives $H(Fu, Gv, t) = 1$. So , $Fu = Gv$ and by (1) $fu = gv$.
Again by (2), we have

$$\mu\left(f^2 u, fu, t\right) \geq H(Ffu, Gv, t).$$

Next , we claim that $f^2 u = fu$. The condition $(*)$ implies that

$$\varphi\Big(H(Ffu, Gv, t), \mu\left(f^2 u, gv, t\right), \mu\left(f^2 u, Ffu, t\right), \mu(gv, Gv, t), \mu\left(f^2 u, Gv, t\right)$$
$$\mu(gv, Ffu, t)\Big) < 1$$

$\Rightarrow \varphi\left(H(Ffu, Gv, t), \mu\left(f^2 u, fu, t\right), 1, 1, \mu\left(f^2 u, Gv, t\right), \mu(fu, Ffu, t)\right) < 1$B

y (φ_1) we have

$\Rightarrow \varphi(H(Ffu, Gv, t), H(Ffu, Gv, t), 1, 1, H(Ffu, Gv, t), H(Ffu, Gv, t)) < 1$
Which, from (φ_2) gives $H(Ffu, Gv, t) = 1$.

By (2) we obtain $f^2 u = fu$. Since $\{f, F\}$ and $\{g, G\}$ have the same role , we have $gv = g^2 v$. Therefore ,

$$ffu = fu = gv = ggv = gfu$$

and $fu = f^2 u \in fFu \subset Ffu$

So $fu \in Ffu$ and $fu = gfu \in Gfu$. Then fu is common fixed point of f, g, F and G .

ii.) Now , we show uniqueness of the common fixed point . Put $fu = w$ and let w' be another common fixed point of the four maps, then we have

$$\mu(w, w', t) = \mu(fw, gw', t) \geq H(Fw, Gw', t)$$

by $(*)$ we get

$$\varphi(H(Fw, Gw', t), \mu(fw, gw', t), \mu(fw, Fw, t), \mu(gw', Gw', t), \mu(fw, Gw', t)$$
$$\mu(gw', Fw, t)) < 1$$

$\Rightarrow \varphi(H(Fw, Gw', t), \mu(fw, gw', t), 1, 1, \mu(fw, Gw', t), \mu(gw', Fw, t)) < 1$By
(φ_1) we get

$\varphi(H(Fw, Gw', t), H(Fw, Gw', t), 1, 1, H(Fw, Gw', t), H(Fw, Gw', t)) < 1$So ,
by (φ_2) , $H(Fw, Gw', t) = 1$ and from (3) , we have

$$\mu(fw, gw', t) = \mu(w, w', t) = 1 \Rightarrow w = w' .$$

iii.) Let $w \in F\,f\,u$. Using the triangle inequality and Property (2.16) , we have

$$\mu(f\,u, w, t) \geq \mu(f\,u\,,\,F\,f\,u\,,t) * H(F\,f\,u, Gv, t) * \mu(w, Gv, t)$$

Since $f\,u \in F\,f\,u$ and $H(F\,f\,u, Gv, t) = 1$,

$$\mu(w\,,\,f\,u\,,\,t) \geq \mu(w, Gv, t) \geq H(Ffu, Gv, t) = 1$$

$$\text{So, } w = f\,u \text{ and } Ffu = \{f\,u\} = \{g\,v\} = G\,g\,v\ .$$

This completes the proof .

IV. A Type Gregus Fixed Point Theorem

Theorem 4.1 Let $f, g : X \to X$ be mappings and $F, G : X \to CB(X)$ be set - valued mappings such that that the pairs $\{f, F\}$ and $\{g, G\}$ are **owc** . Let $\psi : \square \to \square$ be a non-decreasing map such that for every $0 \leq l < 1$, $\psi(l) > l$ and satisfies the following condition :

$$(*)\quad H^p(Fx, Gy, t) \geq \psi\left[a\,\mu^p(f\,x, g\,y, t) + (1-a)\,\mu^{\frac{p}{2}}(g\,y, F\,x, t)\,\mu^{\frac{p}{2}}(f\,x, G\,y, t)\right]$$

for all x and $y \in X$, where $0 \leq a < 1$ and $p \geq 1$.

Then f , g , F and G have a unique fixed point which is a strict fixed point for F and G .

Proof

Since the pairs $\{f, F\}$ and $\{g, G\}$ are **owc** , as in proof of theorem (3.1) , there exist $u, v \in X$ such that $f\,u \in F\,u$, $g\,v \in G\,v$, $f\,Fu \subseteq F\,f\,u$ and $g\,Gv \subseteq G\,g\,v$ and (1) , (2) holds.

i.) As in proof of theorem $(3{\cdot}1)$, we begin to show existence of a common fixed point . We have ,

$$H^p(Fu, Gv, t) \geq \psi\left[a\,\mu^p(f\,u, g\,v, t) + (1-a)\,\mu^{\frac{p}{2}}(g\,v, F\,u, t)\,\mu^{\frac{p}{2}}(f\,u, Gv, t)\right]$$

and by (1) and Property$(2{\cdot}16)$,

$$H^p(Fu, Gv, t) \geq \psi\left[a\,H^p(Fu, Gv, t) + (1-a)H^p(Gv, Fu, t)\right]$$

$$= \psi\left(H^p(Fu, Gv, t)\right)$$

So , if $0 \leq H(Fu, Gv, t) < 1$, $\psi(l) > l$ for $0 \leq l < 1$, we obtain

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$$H^{p}(Fu,Gv,t) \geq \psi\left[H^{p}(Fu,Gv,t)\right] > H^{p}(Fu,Gv,t)$$

which is a contradiction , thus we have $H(Fu,Gv,t)=1$ and hence $fu = gv$.

Again , if $0 \leq H(F\,fu,Gv,t) < 1$ then by (2) and $(*)$ we have

$$H^{p}(Ffu,Gv,t) \geq \psi\left[a\,\mu^{p}\left(f^{2}u\,,\,g\ \ yt\right)+(1-a)\,\mu^{\frac{p}{2}}(gv,Ffu,t)\,\mu^{\frac{p}{2}}(f^{2}u,Gv,t)\right]$$

$$\geq \psi\left[aH^{p}(Ffu,Gv,t)+(1-a)H^{p}(Ffu,Gv,t)\right]$$

$$= \psi\left(H^{p}(Ffu,Gv,t)\right)$$

If $0 \leq H(Ffu,Gv,t) < 1$, we obtain

$$H^{p}(Ffu,Gv,t) \geq \psi\left[H^{p}(Ffu,Gv,t)\right] > H^{p}(Ffu,Gv,t)$$

which is a contradiction , thus we have $H(Ffu,Gv,t)=1 \Rightarrow Ffu = Gv \Rightarrow f^{2}u = fu$.

Similarly , we can prove that $g^{2}v = gv$.

Let $fu = w$ then $fw = w = gw$, $w \in Fw$ and $w \in Gw$, this completes the proof of the existence .

ii.) For the uniqueness , let w' be a second common fixed point of f,g,F and G . Then

$$\mu(w,w',t) = \mu(fw,gw',t) \geq H(Fw,Gw',t)$$

and by assumption $(*)$, we obtain

$$H^{p}(Fw,Gw',t) \geq \psi\left[a\,\mu^{p}(fw,gw',t)+(1-a)\mu^{\frac{p}{2}}(gw',Fw,t)\,\mu^{\frac{p}{2}}(fw,Gw',t)\right]$$

$$\geq \psi\left(H^{p}(Fw\,,\,Gw',t)\right)$$

$$> H^{p}(Fw,Gw',t) \quad \text{if } 0 \leq H(Fw,Gw',t) < 1$$

which is a contradiction. So , $Fw = Gw'$. Since w and w' are common fixed point of f,g,F and G, we have

$$\mu(fw\,,\,gw',t) \geq \mu(fw,Fw,\,t) * H(Fw,Gw',t) * \mu(gw',Gw',t)$$

$$\geq H(Fw,Gw',\,t)$$

So , $w = fw = gw' = w'$ and there exists a unique common fixed point of f,g,F and G .

iii.) The proof that the fixed point of F and G is a strict fixed point is identical of that of theorem (3.1).

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b) Theorem

Let $f, g : X \to X$ and $F, G : X \to CB(X)$ be single and set - valued maps respectively such that the pairs $\{f, F\}$ and $\{g, G\}$ are **owc** and satisfy inequality

$$(*)\ H^{p}(Fx, Gy, t)$$

$$\geq a(\mu(fx, gy, t))\Big[\min\Big\{\mu(fx, gy, t)\,\mu^{p-1}(fx, Fx, t),\ \mu(fx, gy, t)\,\mu^{p-1}(gy, Gy, t),$$
$$\mu(fx, Fx, t)\,\mu^{p-1}(gy, Gy, t),\ \mu^{p-1}(fx, Gy, t)\,\mu(gy, Fx, t)\Big\}\Big]$$

for all $x, y \in X$, where $p \geq 2$ and $a : [0,1] \to [0, \infty)$ is decreasing and satisfies the condition

$$a(t) > 1 \text{ forall } 0 \leq t < 1 \text{ and } a(t) = 1 \text{ iff } t = 1.$$

Then f, g, F and G have a unique fixed point which is a strict fixed point for F and G.

Proof

Since the pairs $\{f, F\}$ and $\{g, G\}$ are **owc**, there exist two elements u and v in X such that $fu \in Fu$, $gv \in Gv$, $fFu \subseteq Ffu$, $gGv \subseteq Ggv$.

First we prove that $fu = gv$. By property (2.16) and the triangle inequality we have

$$\mu(fu, g\ yt) \geq H(Fu, Gv, t), \mu(fu, Gv, t) \geq H(Fu, Gv, t) \text{ and}$$
$$\mu(Fu, gv, t) \geq H(Fu, Gv, t).$$

Suppose that $H(Fu, Gv, t) < 1$. Then by inequality $(*)$ we get

$$H^{p}(Fu, Gv, t)$$

$$\geq a(\mu(fu, gv, t))\Big[\min\Big\{\mu(fu, gv, t)\,\mu^{p-1}(fu, Fu, t),\ \mu(fu, gv, t)\,\mu^{p-1}(gv, Gv, t),$$
$$\mu(fu, Fu, t)\,\mu^{p-1}(gv, Gv, t),\ \mu^{p-1}(fu, Gv, t)\,\mu(gv, Fu, t)\Big\}\Big]$$

$$= a(\mu(fu, gv, t))\Big[\min\Big\{\mu(fu, gv, t), \mu(fu, gv, t), 1, \mu^{p-1}(fu, Gv, t)\,\mu(gv, Fu, t)\Big\}\Big]$$
$$\geq a(H(Fu, Gv, t))\Big[H(Fu, Gv, t), 1, H^{p}(Fu, Gv, t)\Big] > H^{p}(Fu, Gv, t)$$

which is a contradiction . Hence $H(Fu, Gv, t) = 1$ which implies that $fu = gv$.

Again by property (2.16) and the triangle inequality we have

$$\mu\left(f^{2}u, fu, t\right) = \mu\left(f^{2}u, gv, t\right) \geq H(Ffu, Gv, t)$$

We prove that $f^{2}u = fu$. Suppose $H(Ffu, Gv, t) < 1$ and by $(*)$, property (2.16) we obtain

$$H^{p}(Ffu, Gv, t)$$

$$\geq a\Big(\mu\big(f^{2}u, gv, t\big)\Big)\Big[\min\Big\{\mu\big(f^{2}u, gv, t\big)\mu^{p-1}\big(f^{2}u, Ffv, t\big), \mu\big(f^{2}u, gv, t\big)\mu^{p-1}(gv, Gv, t),$$

$$\mu\big(f^{2}u, Ffu, t\big)\mu^{p-1}(gv, Gv, t), \mu^{p-1}\big(f^{2}u, Gv, t\big)\mu(gv, Ffu, t)\Big\}\Big]$$

$$a\Big(\mu\big(f^{2}u, gv, t\big)\Big)\Big[\min\Big\{\mu\big(f^{2}u, gv, t\big), \mu\big(f^{2}u, gv, t\big), 1, \mu^{p-1}\big(f^{2}u, Gv, t\big)\mu(gv, Ffu, t)\Big\}\Big]$$

$$= \geq a\big(H(Ffu, Gv, t)\big)\Big[\min\Big\{H(Ffu, Gv, t), H^{p}(Ffu, Gv, t)\Big\}\Big] > H^{p}(Ffu, Gv, t)$$

which is a contradiction . Hence $H^{p}(Ffu, Gv, t) = 1$ which implies that

$$f^{2}u = gv = fu \ .$$

Similarly , we can prove that $g^{2}v = gv$. Putting $fu = gv = z$, we have $fz = gz = z$, $z \in Fz$ and $z \in Gz$. Therefore z is a common fixed point of maps f, g, F and G . Now , suppose that f, g, F and G have another common fixed point $z' \neq z$. Then , by property (2.16) and the triangle inequality we have $\mu(z, z', t) = \mu(fz, gz', t) \geq H \;\; Fz, Gz', t)$.

Assume that $H(Fz, Gz', t) < 1$. Then the use of inequality $(*)$ gives

$$H^{p}(Fz, Gz', t)$$

$$\geq a\big(\mu(fz, gz', t)\big)\Big[\min\Big\{\mu(fz, gz', t)\mu^{p-1}(fz, Fz, t), \mu(fz, gz', t)\mu^{p-1}(gz', Gz', t),$$

$$\mu(fz, Fz, t)\mu^{p-1}(g\,'z, Gz', t), \mu^{p-1}(fz, Gz', t)\mu(g\,'z, Fz, t)\Big\}\Big]$$

$$= a\big(\mu(fz, gz', t)\big)\Big[\min\Big\{\mu(fz, gz', t), \mu\big(f^{2}z, gz', t\big), 1, \mu^{p-1}\big(f^{2}z, Gz', t\big)\mu(gz', Ffz, t)$$

$$\geq a\big(H(Fz, Gz', t)\big)\Big[\min\Big\{H(Fz, Gz', t), H^{p}(Fz, Gz', t)\Big\}\Big] > H^{p}(Fz, Gz', t)$$

which is a contradiction . Hence $H(Fz, Gz', t) = 1$ which implies that $z' = z$.

iv.) The proof that the fixed point of F and G is a strict fixed point is identical of that of theorem(3.1).

V. Another Type Fixed Point Theorem

Theorem. 5.1

Let $f, g : X \to X$ be mappings and $F, G : X \to CB(X)$ be set - valued maps and ϕ be non -decreasing function of $[0,1]$ into itself such that $\phi(t) = 1$ iff $t = 1$ and for all $t \in [0,1)$, ϕ satisfies the following inequality

$$(*) \quad \phi\big(H(Fx, Gy, t)\big) \geq a\big(\mu(fx, gy, t)\big)\,\phi\big(\mu(fx, gy, t)\big)$$

$$+ b\big(\mu(fx, gy, t)\big)\min\Big\{\phi\big(\mu(fx, Gy, t)\big), \phi\big(\mu(gy, Fx, t)\big)\Big\}$$

for all x and y in X, where $a, b : [0,1] \to [0,1]$ are satisfying the conditions

$$a(t) + b(t) > 1 \text{ for all } t > 0$$

and

$$a(t) + b(t) = 1 \text{ iff. } t = 1.$$

If the pairs $\{f, F\}$ and $\{g, G\}$ are **owc** then f, g, F and G have a unique common fixed point in X which is a strict fixed point for F and G.

Proof.

Since $\{f, F\}$ and $\{g, G\}$ are **owc**, as in proof of theorem(3.1), there exist u and v in X such that

$$fu \in Fu,\ gv \in Gv,\ fFu \subseteq Ffu,\ gGv \subseteq Ggv \text{ and } (1),\ (2) \text{ holds.}$$

i.) First we prove that $fu = gv$. Suppose $H(Fu, Gv, t) < 1$. By $(*)$ and Property (2.16), we have

$$\phi(H(Fu, Gv, t)) \geq a(\mu(fu, gv, t))\, \phi(\mu(fu, gv, t))$$

$$+ b(\mu(fu, gv, t)) \min\{\phi(\mu(fu, Gv, t)), \phi(\mu(gv, Fu, t))\}$$

$$\geq [a(\mu(fu, gv, t)) + b(\mu(fu, gv, t))]\, \phi(H(Fu, Gv, t))$$

$$> \phi(H(Fu, Gv, t))$$

which is a contradiction. Hence $H(Fu, Gv, t) = 1$ and thus $fu = gv$.

Now we prove that $f^2u = fu$. Suppose $H(Ffu, Gv, t) < 1$. By $(*)$ and Property (2.16), we have

$$\phi(H(Ffu, Gv, t)) \geq a\left(\mu\left(f^2u, gv, t\right)\right) \phi\left(\mu\left(f^2u, gv, t\right)\right)$$

$$+ b\left(\mu\left(f^2u, gv, t\right)\right) \min\left\{\phi\left(\mu\left(f^2u, Gv, t\right)\right), \phi(\mu(gv, Ffu, t))\right\}$$

$$\geq \left[a\left(\mu\left(f^2u, fu, t\right)\right) + b\left(\mu\left(f^2u, fu, t\right)\right)\right] \phi(H(Ffu, Gv, t))$$

$$> \phi(H(Ffu, Gv, t))$$

which is a contradiction. Hence $H(Ffu, Gv, t) = 1$ and this implies that $f^2u = fu$.

Similarly, we can prove that $g^2v = gv$. So, if $w = fu = gv$ then $fw = w = gw$, $w \in Fw$ and $w \in Gw$. Existence of a common fixed point is proved.

ii.) Assume that there exists a second common fixed point w' of f, g, F and G. We see that

$$\mu(w, w', t) = \mu(fw, gw', t) \geq H(Fw, Gw', t)$$

If $H(Fw, Gw', t) < 1$, by inequality $(*)$, we obtain

$$\phi(H(Fw, Gw', t)) \geq a(\mu(fw, g'wt))\,\phi(\mu(fw, g'wt))$$

$$\geq [a(\mu(w, w', t)) + b(\mu(w, w', t))]\,\phi(H(Fw, Gw', t))$$

$$> \phi(H(Fw, Gw', t))$$

this contradiction implies that $H(Fw, Gw', t) = 1$ and hence $w = w'$.

a) This part of the proof is analogous of that of theorem (3.1).

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Exponential Chain Ratio and Product type Estimators for Finite Population Mean Under Double Sampling Scheme

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Abstract - In this paper an exponential chain ratio and product type estimators in double sampling have been developed for estimating finite population mean of the study variable when the information on another additional auxiliary character is available along with the main auxiliary character. The bias and mean square error of the proposed estimators have been obtained in two different cases. Theoretical and empirical studies have been done to demonstrate the efficiency of the proposed strategy with respect to the strategies which utilizes the information on one and two auxiliary characteristics.

Keywords: *Auxiliary information, Exponential chain ratio and product estimators in double sampling, Study variate, Mean square error.*

EXPONENTIAL CHAIN RATIO AND PRODUCT TYPE ESTIMATORS FOR FINITE POPULATION MEAN UNDER DOUBLE SAMPLING SCHEME

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B. K. Singh[α] & Sanjib Choudhury[σ]

Abstract - In this paper an exponential chain ratio and product type estimators in double sampling have been developed for estimating finite population mean of the study variable when the information on another additional auxiliary character is available along with the main auxiliary character. The bias and mean square error of the proposed estimators have been obtained in two different cases. Theoretical and empirical studies have been done to demonstrate the efficiency of the proposed strategy with respect to the strategies which utilizes the information on one and two auxiliary characteristics.

Keywords : Auxiliary information, Exponential chain ratio and product estimators in double sampling, Study variate, Mean square error.

I. Introduction

Information on variables correlated with the main variable under study is popularly known as auxiliary information which may be fruitfully utilized either at planning stage or at design stage or at the information stage to arrive at an improved estimator compared to those, not utilizing auxiliary information. Use of auxiliary information for forming ratio and regression method of estimation were introduced during the 1930's with a comprehensive theory provided by Cochran (1942).

When information on any auxiliary variable x highly correlated with y is readily available on all units of the population, it is well known that ratio and regression estimators provide more efficient estimators of population mean of y, having advance information on population mean $\bar{X}$ of x. However, in many situations of practical importance, the population mean $\bar{X}$ is not known before the start of a survey. In such a situation, the usual thing to do is to estimate it by the sample mean $\bar{x}_1$ based on a preliminary sample of size n_1 of which n is a sub sample $(n<n_1)$. At the most, we use only knowledge of the population mean of another auxiliary character, which is comparatively less correlated to the main characters. That is, if the population mean $\bar{Z}$ of another auxiliary variate Z, closely related to X but compared to X remotely related to Y is known, it is advisable to estimate $\bar{X}$ by $\bar{X}=\bar{x}_1\bar{Z}/\bar{z}_1$, which would provide better estimate of $\bar{X}$ than $\bar{x}_1$.

Chand (1975) and Sukhatme and Chand (1977) proposed a technique of chaining the available information on auxiliary characteristics with the main characteristics. Kiregyera (1980, 1984) also proposed some chain type ratio and regression estimators based on two auxiliary variates. Further contribution are due to Srivastava and Jhajj

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(1980, 1995), Isaki (1983), Singh and Kataria (1990), Prasad and Singh (1990, 1992), Ahmed *et al.* (2000), Singh and Singh (2001), Al-Jararha and Ahmed (2002), Ahmed *et al.* (2003), Pradhan (2005), Singh *et al.* (2006), Singh *et al.* (2009) and many others.

Let us consider a finite population $U=\{U_1,U_2,...U_N\}$ of size N units and the value of the variables on the i[th] unit U_i, $i=1,2,...,N$, be (y_i,x_i). Let $\bar{Y}=\sum_{i=1}^{N}\frac{y_i}{N}$ and $\bar{X}=\sum_{i=1}^{N}\frac{x_i}{N}$ be the population means of the study variable y and the auxiliary variable x, respectively. For estimating the population mean $\bar{Y}$ of y, a simple random sample of size n is drawn without replacement from the population U. Then the classical ratio and product estimators are defined by

$$\hat{\bar{Y}}_R=\bar{y}\frac{\bar{X}}{\bar{x}},\text{ if }\bar{x}\neq 0\quad\text{and}\quad\hat{\bar{Y}}_P=\bar{y}\frac{\bar{x}}{\bar{X}}$$

where $\bar{y}$ and $\bar{x}$ are the sample means of y and x respectively based on a sample of size n out of the population of size N units and $\bar{X}$ is the known population mean of x. With known population mean $\bar{X}$, Bahl and Tuteja (1991) suggested the exponential ratio-type estimator as

$$\hat{\bar{Y}}_{Re}=\bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$$

and the exponential product-type estimator as

$$\hat{\bar{Y}}_{Pe}=\bar{y}\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)$$

for the population mean $\bar{Y}$.

If the population mean $\bar{X}$ of the auxiliary variable x is not known before start of the survey, a first-phase sample of size n_1 is drawn from the population, on which only the auxiliary variable x is observed. Then a second phase sample of size n is drawn, on which both study variable y and auxiliary variable x are observed. Let $\bar{x}_1=\sum_{i=1}^{n_1}\frac{x_i}{n_1}$ denotes the sample mean of size n_1 based on the first phase sample and $\bar{y}=\sum_{i=1}^{n}\frac{y_i}{n}$ and $\bar{x}=\sum_{i=1}^{n}\frac{x_i}{n}$ denote the sample means of variables y and x respectively, obtained from the second phase sample of size n. Then the double sampling ratio and product estimators of population mean $\bar{Y}$ are given by

$$\hat{\bar{Y}}_R^d=\bar{y}\frac{\bar{x}_1}{\bar{x}}\quad\text{and}\quad\hat{\bar{Y}}_P^d=\bar{y}\frac{\bar{x}}{\bar{x}_1}.$$

Singh and Vishwakarma (2007) suggested the exponential ratio and product type estimators for $\bar{Y}$ in double sampling respectively, as

$$\hat{\bar{Y}}_{Re}^d=\bar{y}\exp\left(\frac{\bar{x}_1-\bar{x}}{\bar{x}_1+\bar{x}}\right)\quad\text{and}\quad\hat{\bar{Y}}_{Pe}^d=\bar{y}\exp\left(\frac{\bar{x}-\bar{x}_1}{\bar{x}+\bar{x}_1}\right).$$

Notes

Notes

In the present paper, we have proposed an exponential chain ratio and product type estimators in double sampling for estimating finite population mean $\bar{Y}$ using two auxiliary characters.

II. The Proposed Estimator

Let us consider a finite population $U=\{U_1,U_2,...U_N\}$ of size N units. A first phase large sample of size n_1 units is drawn from population U following simple random sampling without replacement (SRSWOR) scheme, while in the second phase; a subsample of size n $(n_1>n)$ is drawn by SRSWOR from either n_1 units or directly from the population U. We assume that $\rho_{yx}>\rho_{yz}>0$.

The exponential chain ratio estimator in double sampling is defined as

$$\hat{\bar{Y}}_{Re}^{dc}=\bar{y}\exp\left(\frac{\bar{x}_1\dfrac{\bar{Z}}{\bar{z}_1}-\bar{x}}{\bar{x}_1\dfrac{\bar{Z}}{\bar{z}_1}+\bar{x}}\right) \tag{1}$$

and exponential chain product estimator in double sampling as

$$\hat{\bar{Y}}_{Pe}^{dc}=\bar{y}\exp\left(\frac{\bar{x}-\bar{x}_1\dfrac{\bar{Z}}{\bar{z}_1}}{\bar{x}+\bar{x}_1\dfrac{\bar{Z}}{\bar{z}_1}}\right) \tag{2}$$

The properties of proposed estimators are obtained for the following two cases.

Case I : When the second phase sample of size n is a subsample of the first phase of size n_1.

Case II : When the second phase sample of size n is drawn independently of the first phase sample of size n_1, Bose (1943).

III. Case I

Bias and Mean Square Error of $\hat{\bar{Y}}_{Re}^{dc}$ and $\hat{\bar{Y}}_{Pe}^{dc}$

To obtain the bias (B) and mean square error (M) of estimators $\hat{\bar{Y}}_{Re}^{dc}$ and $\hat{\bar{Y}}_{Pe}^{dc}$, we write $e_0=(\bar{y}-\bar{Y})/\bar{Y}$, $e_1=(\bar{x}-\bar{X})/\bar{X}$, $e_1'=(\bar{x}_1-\bar{X})/\bar{X}$ and $e_2=(\bar{z}_1-\bar{Z})/\bar{Z}$ such that

$$\left.\begin{aligned}
&E(e_0)=E(e_1)=E(e_1')=E(e_2)=0,\quad E(e_0^2)=\frac{1-f}{n}C_y^2,\quad E(e_1^2)=\frac{1-f}{n}C_x^2,\\
&E(e_1'^2)=\frac{1-f_1}{n_1}C_x^2,\quad E(e_2^2)=\frac{1-f_1}{n_1}C_z^2,\quad E(e_0e_1)=\frac{1-f}{n}C_{yx}C_x^2,\\
&E(e_0e_1')=\frac{1-f_1}{n_1}C_{yx}C_x^2,\quad E(e_0e_2)=\frac{1-f_1}{n_1}C_{yz}C_z^2,\quad E(e_1e_1')=\frac{1-f_1}{n_1}C_x^2,\\
&E(e_1e_2)=\frac{1-f_1}{n_1}C_{xz}C_z^2,\quad E(e_1'e_2)=\frac{1-f_1}{n_1}C_{xz}C_z^2
\end{aligned}\right\} \tag{3}$$

where $f=\dfrac{n}{N}$, $f_1=\dfrac{n_1}{N}$;

$C_y=\dfrac{S_y}{\bar{Y}}$, $C_x=\dfrac{S_x}{\bar{X}}$ and $C_z=\dfrac{S_z}{\bar{Z}}$ are the coefficients of variation of the study variate y, auxiliary variates x and z respectively.

$\rho_{yx}=\dfrac{S_{yx}}{S_xS_y}$, $\rho_{yz}=\dfrac{S_{yz}}{S_yS_z}$ and $\rho_{zx}=\dfrac{S_{zx}}{S_zS_x}$ are the correlation coefficients between y and x, y and z and x and z respectively.

$S_y^2=\dfrac{1}{N-1}\sum_{i=1}^{N}\left(y_i-\bar{Y}\right)^2$, $S_x^2=\dfrac{1}{N-1}\sum_{i=1}^{N}\left(x_i-\bar{X}\right)^2$ and $S_z^2=\dfrac{1}{N-1}\sum_{i=1}^{N}\left(z_i-\bar{Z}\right)^2$ are the population variances of study variate y, auxiliary variates x and z respectively.

$S_{yx}=\dfrac{1}{N-1}\sum_{i=1}^{N}\left(y_i-\bar{Y}\right)\left(x_i-\bar{X}\right)$, $S_{yz}=\dfrac{1}{N-1}\sum_{i=1}^{N}\left(y_i-\bar{Y}\right)\left(z_i-\bar{Z}\right)$ and $S_{xz}=\dfrac{1}{N-1}\sum_{i=1}^{N}\left(x_i-\bar{X}\right)\left(z_i-\bar{Z}\right)$ are the co-variances between y and x, y and z; and x and z respectively; and

$C_{yx}=\dfrac{\rho_{yx}C_y}{C_x}$, $C_{yz}=\dfrac{\rho_{yz}C_y}{C_z}$ and $C_{xz}=\dfrac{\rho_{xz}C_x}{C_z}$.

Expanding the right hand side of equations (1) and (2), multiplying out and neglecting terms of e's having power greater than two, we have

$$\hat{\bar{Y}}_{Re}^{dc}-\bar{Y}\cong\bar{Y}\left[e_0+\frac{1}{2}\left(e_1'-e_1-e_2-e_1e_2\right)-\frac{1}{8}e_1'^2+\frac{3}{8}\left(e_1^2+e_2^2+2e_1e_2\right)-\frac{1}{4}\left(e_1e_1'+e_2e_1'\right)\right.$$
$$\left.+\frac{1}{2}\left(e_0e_1'-e_0e_1-e_0e_2\right)\right] \tag{4}$$

$$\hat{\bar{Y}}_{Pe}^{dc}-\bar{Y}\cong\bar{Y}\left[e_0+\frac{1}{2}\left(e_1-e_1'+e_2\right)+\frac{1}{2}\left(e_1e_2+e_0e_1-e_0e_1'+e_0e_2\right)-\frac{1}{4}\left(e_1^2+e_2^2-e_1'^2+2e_1e_2\right)\right.$$
$$\left.+\frac{1}{8}\left(e_1^2+e_2^2+e_1'^2+2e_1e_2-2e_1'e_2-2e_1e_1'\right)\right] \tag{5}$$

Therefore, the bias of the estimators $\hat{\bar{Y}}_{Re}^{dc}$ and $\hat{\bar{Y}}_{Pe}^{dc}$ can be obtained by using the results of (3) in equations (4) and (5) are respectively as

$$B\left(\hat{\bar{Y}}_{Re}^{dc}\right)_I=\bar{Y}\left[\frac{3}{8}\left(\frac{1-f^*}{n}C_x^2+\frac{1-f_1}{n_1}C_z^2\right)+\frac{1}{2}\left(\frac{1-f^*}{n}C_{yx}C_x^2-\frac{1-f_1}{n_1}C_{yz}C_z^2\right)\right]$$

and

$$B\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_I=\bar{Y}\left[-\frac{1}{8}\left(\frac{1-f^*}{n}C_x^2+\frac{1-f_1}{n_1}C_z^2\right)+\frac{1}{2}\left(\frac{1-f^*}{n}C_{yx}C_x^2+\frac{1-f_1}{n_1}C_{yz}C_z^2\right)\right]$$

where $f^*=n/n_1$.

Notes

Notes

From equations (4) and (5), we have

$$\hat{\bar{Y}}_{Re}^{dc}-\bar{Y}\cong\bar{Y}\left\{e_0+\frac{1}{2}\left(e_1'-e_1-e_2\right)\right\} \tag{6}$$

and

$$\hat{\bar{Y}}_{Pe}^{dc}-\bar{Y}\cong\bar{Y}\left\{e_0+\frac{1}{2}\left(e_1-e_1'+e_2\right)\right\} \tag{7}$$

Squaring both sides of equations (6) and (7), taking expectations and using the results of (3), we get the MSE of $\hat{\bar{Y}}_{Re}^{dc}$ and $\hat{\bar{Y}}_{Pe}^{dc}$ to the first degree of approximation as

$$M\left(\hat{\bar{Y}}_{Re}^{dc}\right)_I=\bar{Y}^2\left[\frac{1-f}{n}C_y^2+\frac{1}{4}\left(\frac{1-f^*}{n}C_x^2+\frac{1-f_1}{n_1}C_z^2\right)-\frac{1-f^*}{n}C_{yx}C_x^2-\frac{1-f_1}{n_1}C_{yz}C_z^2\right] \tag{8}$$

$$M\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_I=\bar{Y}^2\left[\frac{1-f}{n}C_y^2+\frac{1}{4}\left(\frac{1-f^*}{n}C_x^2+\frac{1-f_1}{n_1}C_z^2\right)+\frac{1-f^*}{n}C_{yx}C_x^2+\frac{1-f_1}{n_1}C_{yz}C_z^2\right] \tag{9}$$

and the MSE of the usual unbiased estimator $\bar{y}$ under the SRSWOR scheme is

$$M\left(\bar{y}\right)=\bar{Y}^2\frac{1-f}{n}C_y^2. \tag{10}$$

IV. Efficiency Comparisons

a) Efficiency comparisons of exponential chain ratio estimator in double sampling

(i) *with chain ratio estimator in double sampling (Chand, 1975)*

The MSE of chain ratio estimator in double sampling is

$$M\left(\hat{\bar{Y}}_{R}^{dc}\right)_I=\bar{Y}^2\left[\frac{1-f}{n}C_y^2+\frac{1-f^*}{n}C_x^2\left(1-2C_{yx}\right)+\frac{1-f_1}{n_1}C_z^2\left(1-2C_{yz}\right)\right] \tag{11}$$

From equations (8) and (11), we have

$$M\left(\hat{\bar{Y}}_{R}^{dc}\right)_I-M\left(\hat{\bar{Y}}_{Re}^{dc}\right)_I=\bar{Y}^2\left[\frac{1-f^*}{n}C_x^2\left(\frac{3}{4}-C_{yx}\right)+\frac{1-f_1}{n_1}C_z^2\left(\frac{3}{4}-C_{yz}\right)\right]$$

$$>0,\ \text{if}\ \frac{3}{4}-C_{yx}>0\ \text{and}\ \frac{3}{4}-C_{yz}>0$$

Thus, the proposed estimator $\hat{\bar{Y}}_{Re}^{dc}$ is more efficient than $\hat{\bar{Y}}_{R}^{dc}$ if $\frac{3}{4}-C_{yx}>0$ and $\frac{3}{4}-C_{yz}>0$. (12)

(ii) *with chain product estimator in double sampling*

The MSE of chain product estimator in double sampling is

$$M\left(\hat{\bar{Y}}_{P}^{dc}\right)_I=\bar{Y}^2\left[\frac{1-f}{n}C_y^2+\frac{1-f^*}{n}C_x^2\left(1+2C_{yx}\right)+\frac{1-f_1}{n_1}C_z^2\left(1+2C_{yz}\right)\right] \tag{13}$$

From equations (8) and (13), we have

$$M\left(\hat{\bar{Y}}_P^{dc}\right)_I - M\left(\hat{\bar{Y}}_{Re}^{dc}\right)_I = 3\bar{Y}^2\left[\frac{1-f^*}{n}C_x^2\left(\frac{1}{4}+C_{yx}\right)+\frac{1-f_1}{n_1}C_z^2\left(\frac{1}{4}+C_{yz}\right)\right]$$

$$>0\text{, if } \frac{1}{4}+C_{yx}>0 \text{ and } \frac{1}{4}+C_{yz}>0.$$

Thus, the proposed estimator $\hat{\bar{Y}}_{Re}^{dc}$ is better than $\hat{\bar{Y}}_P^{dc}$ if $\frac{1}{4}+C_{yx}>0$ and $\frac{1}{4}+C_{yz}>0$. (14)

Notes

(iii) *with sample mean per unit estimator* $\bar{y}$

From equations (8) and (10), we have

$$M(\bar{y}) - M\left(\hat{\bar{Y}}_{Re}^{dc}\right)_I = \bar{Y}^2\left[\frac{1-f^*}{n}C_x^2\left(-\frac{1}{4}+C_{yx}\right)+\frac{1-f_1}{n_1}C_z^2\left(-\frac{1}{4}+C_{yz}\right)\right]$$

$$>0\text{, if } C_{yx}-\frac{1}{4}>0 \text{ and } C_{yz}-\frac{1}{4}>0.$$

Thus the proposed estimator $\hat{\bar{Y}}_{Re}^{dc}$ has smaller MSE than sample mean per unit estimator $\bar{y}$ if $C_{yx}-\frac{1}{4}>0$ and $C_{yz}-\frac{1}{4}>0$. (15)

b) Efficiency comparisons of exponential chain product estimator in double sampling

(i) *with chain ratio estimator in double sampling (Chand, 1975)*

From equations (9) and (11), we have

$$M\left(\hat{\bar{Y}}_R^{dc}\right)_I - M\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_I = 3\bar{Y}^2\left[\frac{1-f^*}{n}C_x^2\left(\frac{1}{4}-C_{yx}\right)+\frac{1-f_1}{n_1}C_z^2\left(\frac{1}{4}-C_{yz}\right)\right]$$

Thus the proposed estimator $\hat{\bar{Y}}_{Pe}^{dc}$ will dominate over the estimator $\hat{\bar{Y}}_R^{dc}$ if $\frac{1}{4}-C_{yx}>0$ and $\frac{1}{4}-C_{yz}>0$. (16)

(ii) *with chain product estimator in double sampling*

From equations (9) and (13), we have

$$M\left(\hat{\bar{Y}}_P^{dc}\right)_I - M\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_I = \bar{Y}^2\left[\frac{1-f^*}{n}C_x^2\left(\frac{3}{4}+C_{yx}\right)+\frac{1-f_1}{n_1}C_z^2\left(\frac{3}{4}+C_{yz}\right)\right]$$

Thus, the proposed estimator $\hat{\bar{Y}}_{Pe}^{dc}$ is better than $\hat{\bar{Y}}_P^{dc}$ if $\frac{3}{4}+C_{yx}>0$ and $\frac{3}{4}+C_{yz}>0$. (17)

(iii) *with sample mean per unit estimator* $\bar{y}$

From equations (9) and (10), we have

$$M(\bar{y}) - M\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_I = -\bar{Y}^2\left[\frac{1-f^*}{n}C_x^2\left(\frac{1}{4}+C_{yx}\right)+\frac{1-f_1}{n_1}C_z^2\left(\frac{1}{4}+C_{yz}\right)\right]$$

Thus the proposed estimator $\hat{\bar{Y}}_{Pe}^{dc}$ has smaller MSE than that of the sample mean per unit estimator $\bar{y}$ if $\frac{1}{4}+C_{yx}<0$ and $\frac{1}{4}+C_{yz}<0$. (18)

V. Case II

Bias and Mean Square Error of $\hat{\hat{Y}}_{Re}^{dc}$ and $\hat{\hat{Y}}_{Pe}^{dc}$

In case II, we have

Notes

$$\left.\begin{aligned} &E(e_0)=E(e_1)=E(e_1')=E(e_2)=0,\quad E(e_0^2)=\frac{1-f}{n}C_y^2,\\ &E(e_1^2)=\frac{1-f}{n}C_x^2,\quad E(e_1'^2)=\frac{1-f_1}{n_1}C_x^2,\quad E(e_2^2)=\frac{1-f_1}{n_1}C_z^2,\\ &E(e_0e_1)=\frac{1-f}{n}C_{yx}C_x^2,\quad E(e_1'e_2)=\frac{1-f_1}{n_1}C_{xz}C_z^2,\\ &E(e_0e_1')=E(e_0e_2)=E(e_1e_1')=E(e_1e_2)=0\quad . \end{aligned}\right\} \tag{19}$$

Taking expectation in equations (4) and (5) and using the results of equation (19), we get the bias of the estimators $\hat{\hat{Y}}_{Re}^{dc}$ and $\hat{\hat{Y}}_{Pe}^{dc}$ to the first degree of approximation as

$$B\left(\hat{\hat{Y}}_{Re}^{dc}\right)_{II}=\bar{Y}\left[-\frac{1}{8}\frac{1-f_1}{n_1}C_x^2+\frac{3}{8}\left(\frac{1-f}{n}C_x^2+\frac{1-f_1}{n_1}C_z^2\right)-\frac{1}{4}\frac{1-f_1}{n_1}C_{xz}C_z^2-\frac{1}{2}\frac{1-f}{n}C_{yx}C_x^2\right]$$

and

$$B\left(\hat{\hat{Y}}_{Pe}^{dc}\right)_{II}=\bar{Y}\left[\frac{1}{2}\frac{1-f}{n}C_{yx}C_x^2-\frac{1}{8}\left(\frac{1-f}{n}C_x^2+\frac{1-f_1}{n_1}C_z^2\right)+\frac{3}{8}\frac{1-f_1}{n_1}C_x^2-\frac{1}{4}\frac{1-f_1}{n_1}C_{xz}C_z^2\right]$$

Squaring both sides of equations (6) and (7), taking expectations and using the results of (19), we get the MSE of $\hat{\hat{Y}}_{Re}^{dc}$ and $\hat{\hat{Y}}_{Pe}^{dc}$ to the first degree of approximation as

$$M\left(\hat{\hat{Y}}_{Re}^{dc}\right)_{II}=\bar{Y}^2\left[\frac{1-f}{n}C_y^2+\frac{1}{4}\left(f^{**}C_x^2+\frac{1-f_1}{n_1}C_z^2\right)-\frac{1-f}{n}C_{yx}C_x^2-\frac{1}{2}\frac{1-f_1}{n_1}C_{xz}C_z^2\right] \tag{20}$$

and

$$M\left(\hat{\hat{Y}}_{Pe}^{dc}\right)_{II}=\bar{Y}^2\left[\frac{1-f}{n}C_y^2+\frac{1}{4}\left(f^{**}C_x^2+\frac{1-f_1}{n_1}C_z^2\right)+\frac{1-f}{n}C_{yx}C_x^2-\frac{1}{2}\frac{1-f_1}{n_1}C_{xz}C_z^2\right] \tag{21}$$

where $f^{**}=\frac{1-f}{n}+\frac{1-f_1}{n_1}$.

VI. Efficiency Comparisons

a) Efficiency comparisons of exponential chain ratio estimator in double sampling

(i) *with chain ratio estimator in double sampling (Chand, 1975)*

The MSE of chain ratio estimator in double sampling is

$$M\left(\hat{\hat{Y}}_{R}^{dc}\right)_{II}=\bar{Y}^2\left[\frac{1-f}{n}C_y^2+\frac{1-f}{n}C_x^2\left(1-2C_{yx}\right)+\frac{1-f_1}{n_1}C_x^2+\frac{1-f_1}{n_1}C_z^2\left(1-2C_{xz}\right)\right] \tag{22}$$

From equations (20) and (22), we have

$$M\left(\hat{\bar{Y}}_{R}^{dc}\right)_{II}-M\left(\hat{\bar{Y}}_{Re}^{dc}\right)_{II}=\bar{Y}^{2}\left[\frac{1-f}{n}C_{x}^{2}\left(\frac{3}{4}-C_{yx}\right)+\frac{3}{4}\frac{1-f_{1}}{n_{1}}C_{x}^{2}+\frac{3}{4}\frac{1-f_{1}}{n_{1}}C_{z}^{2}\left(1-C_{xz}\right)\right]$$

Therefore, the proposed estimator $\hat{\bar{Y}}_{Re}^{dc}$ is better than $\hat{\bar{Y}}_{R}^{dc}$ if

$\frac{3}{4}-C_{yx}>0$ and $1-C_{xz}>0$. (23)

(ii) *with chain product estimator in double sampling*

The MSE of chain product estimator in double sampling is

$$M\left\{\bar{y}_{P}^{(dc)}\right\}_{II}=\bar{Y}^{2}\left[\frac{1-f}{n}C_{y}^{2}+\frac{1-f}{n}C_{x}^{2}\left(1+2C_{yx}\right)+\frac{1-f_{1}}{n_{1}}C_{x}^{2}+\frac{1-f_{1}}{n_{1}}C_{z}^{2}\left(1-2C_{xz}\right)\right] \quad (24)$$

From equations (20) and (24), we have

$$M\left(\hat{\bar{Y}}_{P}^{dc}\right)_{II}-M\left(\hat{\bar{Y}}_{Re}^{dc}\right)_{II}=\bar{Y}^{2}\left[\frac{3}{4}f^{**}C_{x}^{2}+3\frac{1-f}{n}C_{yx}C_{x}^{2}+\frac{3}{4}\frac{1-f_{1}}{n_{1}}C_{z}^{2}\left(1-C_{xz}\right)\right]$$

which is positive if $C_{yx}>0$ and $1-C_{xz}>0$.

i.e.,the estimator $\hat{\bar{Y}}_{Re}^{dc}$ is more efficient than $\bar{y}_{P}^{(dc)}$ if $C_{yx}>0$ and $1-C_{xz}>0$. (25)

(iii) *with sample mean per unit estimator* $\bar{y}$

From equations (10) and (20), we have

$$M\left(\bar{y}\right)-M\left(\hat{\bar{Y}}_{Re}^{dc}\right)_{II}=\bar{Y}^{2}\left[-\frac{1}{4}\left(f^{**}C_{x}^{2}+\frac{1-f_{1}}{n_{1}}C_{z}^{2}\right)+\frac{1-f}{n}C_{yx}C_{x}^{2}+\frac{1}{2}\frac{1-f_{1}}{n_{1}}C_{xz}C_{z}^{2}\right]$$

Therefore, the estimator $\hat{\bar{Y}}_{Re}^{dc}$ is better than $\bar{y}$ if

$$\frac{1-f}{n}C_{yx}C_{x}^{2}+\frac{1}{2}\frac{1-f_{1}}{n_{1}}C_{xz}C_{z}^{2}-\frac{1}{4}\left(f^{**}C_{x}^{2}+\frac{1-f_{1}}{n_{1}}C_{z}^{2}\right)>0. \quad (26)$$

b) Efficiency comparisons of exponential chain product estimator in double sampling

(i) *with chain ratio estimator in double sampling (Chand, 1975)*

From equations (21) and (22), we have

$$M\left(\hat{\bar{Y}}_{R}^{dc}\right)_{II}-M\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_{II}=3\bar{Y}^{2}\left[\frac{1-f}{n}C_{x}^{2}\left(\frac{1}{4}-C_{yx}\right)+\frac{1}{4}\frac{1-f_{1}}{n_{1}}C_{x}^{2}+\frac{1}{4}\frac{1-f_{1}}{n_{1}}C_{z}^{2}\left(1-2C_{xz}\right)\right]$$

Therefore, the estimator $\hat{\bar{Y}}_{Pe}^{dc}$ is more efficient than the estimator $\hat{\bar{Y}}_{R}^{dc}$ if

$\frac{1}{4}-C_{yx}>0$ and $\frac{1}{2}-C_{xz}>0$. (27)

(ii) *with chain Product estimator in double sampling*

From equations (21) and (24), it is found that the estimator $\hat{\bar{Y}}_{Pe}^{dc}$ will dominate over the estimator $\hat{\bar{Y}}_{P}^{dc}$ if

$$M\left(\hat{\bar{Y}}_{P}^{dc}\right)_{II}-M\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_{II}=\bar{Y}^{2}\left[\frac{1-f}{n}C_x^2\left(\frac{3}{4}+C_{yx}\right)+\frac{3}{4}\frac{1-f_1}{n_1}C_x^2+\frac{3}{4}\frac{1-f_1}{n_1}C_z^2\left(1-2C_{xz}\right)\right]$$

$$>0 \text{ i.e., if } \frac{3}{4}+C_{yx}>0 \text{ and } \frac{1}{2}-C_{xz}>0. \tag{28}$$

(iii) *with sample mean per unit estimator* $\bar{y}$

From equations (10) and (21), we have

$$M\left(\bar{y}\right)-M\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_{II}=\bar{Y}^{2}\left[-\frac{1}{4}\left(f^{**}C_x^2+\frac{1-f_1}{n_1}C_z^2\right)-\frac{1-f}{n}C_{yx}C_x^2+\frac{1}{2}\frac{1-f_1}{n_1}C_{xz}C_z^2\right]$$

which is positive if

$$\frac{1}{2}\frac{1-f_1}{n_1}C_{xz}C_z^2-\frac{1}{4}\left(f^{**}C_x^2+\frac{1-f_1}{n_1}C_z^2\right)-\frac{1-f}{n}C_{yx}C_x^2>0 \tag{29}$$

Therefore, the estimator $\hat{\bar{Y}}_{Pe}^{dc}$ is more efficient than $\bar{y}$ if the condition (29) is satisfied.

VII. Empirical Study

To examine the merits of the proposed estimators, we have considered four natural population data sets. The sources of populations, nature of the variates y, x and z; and the values of the various parameters are given as.

Population I - Source : Cochran (1977)

Y : Number of 'placebo' children

X : Number of paralytic polio cases in the placebo group

Z : Number of paralytic polio cases in the 'not inoculated' group

$N=34,\ n=10,\ n_1=15,\ \bar{Y}=4.92,\ \bar{X}=2.59,\ \bar{Z}=2.91,\ \rho_{yx}=0.7326,\ \rho_{yz}=0.6430,$

$\rho_{xz}=0.6837,\ C_y^2=1.0248,\ C_x^2=1.5175,\ C_z^2=1.1492.$

Population II - Source: Sukhatme and Chand (1977)

Y : Apple trees of bearing age in 1964

X : Bushels of apples harvested in 1964

Z : Bushels of apples harvested in 1959

$N=200,\ n=20,\ n_1=30,\ \bar{Y}=0.103182\times10^4,\ \bar{X}=0.293458\times10^4,\ \bar{Z}=0.365149\times10^4,$

$\rho_{yx}=0.93,\ \rho_{yz}=0.77,\ \rho_{xz}=0.84,\ C_y^2=2.55280,\ C_x^2=4.02504,\ C_z^2=2.09379.$

Population III - Source: Srivastava *et al.* (1989, Page 3922)

Y : The measurement of weight of children

X : Mid arm circumference of children

Z : Skull circumference of children.

$N=82,\ n=25,\ n_1=43,\ \bar{Y}=5.60\ kg,\ \bar{X}=11.90\ cm,\ \bar{Z}=39.80\ cm,\ \rho_{yx}=0.09,$

$\rho_{yz}=0.12,\ \rho_{xz}=0.86,\ C_y^2=0.0107,\ C_x^2=0.0052,\ C_z^2=0.0008.$

Notes

Population IV - Source: Srivastava *et al.* (1989, Page 3922)

Y : The measurement of weight of children

X : Mid arm circumference of children

Z : Skull circumference of children.

$$N=55,\quad n=18,\quad n_1=30,\quad \bar{Y}=17.08\quad kg,\quad \bar{X}=16.92\quad cm,\quad \bar{Z}=50.44\quad cm,\quad \rho_{yx}=0.54,$$

$$\rho_{yz}=0.51,\quad \rho_{xz}=-0.08,\quad C_y^2=0.0161,\quad C_x^2=0.0049,\quad C_z^2=0.0007.$$

To establish the theoretical conditions for efficiencies of proposed estimators obtained in Section 4 and Section 6, empirically, we have conducted empirical studies and these are shown in Table 1 and Table 2.

Table 1 : Empirical study of theoretical conditions explained in Section-4

Population	$M\left(\hat{\bar{Y}}_R^{dc}\right)_I > M\left(\hat{\bar{Y}}_{Re}^{dc}\right)_I$ Condition (12)	$M\left(\hat{\bar{Y}}_P^{dc}\right)_I > M\left(\hat{\bar{Y}}_{Re}^{dc}\right)_I$ Condition (14)	$M(\bar{y}) > M\left(\hat{\bar{Y}}_{Re}^{dc}\right)_I$ Condition (15)	$M\left(\hat{\bar{Y}}_R^{dc}\right)_I > M\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_I$ Condition (16)	$M\left(\hat{\bar{Y}}_P^{dc}\right)_I > M\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_I$ Condition (17)	$M(\bar{y}) > M\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_I$ Condition (18)
I	0.15>0, 0.14>0	0.85>0, 0.86>0	0.35>0, 0.36>0	*, *	1.35>0, 1.36>0	*, *
II	0.01>0, *	0.99>0, 1.10>0	0.49>0, 0.60>0	* , *	1.49>0, 1.60>0	*, *
III	0.62>0, 0.31>0	0.38>0, 0.69>0	*, 0.19>0	0.12>0, *	0.88>0, 1.19>0	*, *
IV	*, *	1.23>0, 2.70>0	0.73>0, 2.20>0	*, *	1.73>0, 3.20>0	*, *

* Does not satisfy theoretical conditions empirically

Table 2 : Empirical study of theoretical conditions explained in Section-6

Population	$M\left(\hat{\bar{Y}}_R^{dc}\right)_{II} > M\left(\hat{\bar{Y}}_{Re}^{dc}\right)_{II}$ Condition (23)	$M\left(\hat{\bar{Y}}_P^{dc}\right)_{II} > M\left(\hat{\bar{Y}}_{Re}^{dc}\right)_{II}$ Condition (25)	$M(\bar{y}) > M\left(\hat{\bar{Y}}_{Re}^{dc}\right)_{II}$ Condition (26)	$M\left(\hat{\bar{Y}}_R^{dc}\right)_{II} > M\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_{II}$ Condition (27)	$M\left(\hat{\bar{Y}}_P^{dc}\right)_{II} > M\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_{II}$ Condition (28)	$M(\bar{y}) > M\left(\hat{\bar{Y}}_{Pe}^{dc}\right)_{II}$ Condition (29)
I	0.15>0, 0.21>0	0.21>0, 0.60>0	0.72>0	*, *	1.35, *	*
II	0.01>0, *	*, 0.74>0	85249>0	*, *	1.49>0 *	*
III	0.62>0, *	*, 0.13>0	0.000002>0	0.12>0, *	0.88>0, *	*
IV	*, 1.21 >0	1.21>0, 0.98>0	0.03>0	*, 0.71>0	1.73>0, 0.71>0	*

* Does not satisfy theoretical conditions empirically

Notes

To observe the relative performance of different estimators of $\overline{Y}$, we have computed the percentage relative efficiencies of the proposed estimators $\hat{\bar{Y}}_{Re}^{dc}$ and $\hat{\bar{Y}}_{Pe}^{dc}$, chain ratio estimator $\hat{\bar{Y}}_{R}^{dc}$, product estimator $\hat{\bar{Y}}_{P}^{dc}$ in double sampling and sample mean per unit estimator $\bar{y}$ with respect to usual unbiased estimator $\bar{y}$ in *Case I* and *Case II* and the findings are presented in Table 3.

Notes

Table 3 : Percentage relative efficiencies of different estimators with respect to $\bar{y}$

Estimator	$\bar{y}$	$\hat{\bar{Y}}_{R}^{dc}$	$\hat{\bar{Y}}_{P}^{dc}$	$\hat{\bar{Y}}_{Re}^{dc}$	$\hat{\bar{Y}}_{Pe}^{dc}$
Case I					
Population I	100.00	136.91	25.96	184.36	47.55
Population II	100.00	279.93	26.02	247.82	46.58
Population III	100.00	81.92	70.22	97.11	88.38
Population IV	100.00	131.91	61.01	120.57	78.75
Case II					
Population I	100.00	87.63	21.24	141.68	42.15
Population II	100.00	182.67	19.16	220.59	37.90
Population III	100.00	68.82	58.68	91.06	82.82
Population IV	100.00	116.68	48.81	122.79	70.87

VIII. Results and Discussion

We have analyzed the exponential chain ratio and product type estimators in double sampling and obtained its bias and MSE equations in two different cases. The MSEs of the proposed estimators have been compared with the MSEs of classical estimators (ratio, product and sample mean per unit estimator) on a theoretical basis, and conditions have been obtained under which the proposed estimators have smaller MSE than the classical estimators.

Section 4 and section 6 provides the theoretical conditions under which the proposed estimators $\hat{\bar{Y}}_{Re}^{dc}$ and $\hat{\bar{Y}}_{Pe}^{dc}$ are more efficient than other estimators. Table 1 and Table 2 establish these theoretical conditions empirically. It shows that almost all theoretical conditions obtained in section 4 and section 6 are satisfied with respect to the population data sets.

From Table 3, it clearly indicates that the proposed estimators $\hat{\bar{Y}}_{Re}^{dc}$ and $\hat{\bar{Y}}_{Pe}^{dc}$ are more efficient than the estimators $\hat{\bar{Y}}_{R}^{dc}$, $\hat{\bar{Y}}_{P}^{dc}$ and $\bar{y}$ in both the *Cases* *I* and *II*, except for the data sets of population II and IV in *Case I*, where $\hat{\bar{Y}}_{R}^{dc}$ is slightly better than $\hat{\bar{Y}}_{Re}^{dc}$.

Thus, the uses of the proposed estimators are preferable over other estimators.

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Notes

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Common Fixed Point Theorem for Weakly Compatible Maps Satisfying E.A Property in Intuitionistic Fuzzy Metric Spaces using Implicit Relation

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Abstract - In this paper, we use the notion of E.A. property in intuitionistic fuzzy metric space to prove a common fixed point theorem which generalizes Theorem-2 of Turkoglu, Alaca,Cho and Yildiz [12].

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Subject classification: 2001 AMS: 47H10, 54H25

COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPS SATISFYING E.A PROPERTY IN INTUITIONISTIC FUZZY METRIC SPACES USING IMPLICIT RELATION

Strictly as per the compliance and regulations of :

Common Fixed Point Theorem for Weakly Compatible Maps Satisfying E.A Property in Intuitionistic Fuzzy Metric Spaces using Implicit Relation

Sanjay Kumar [α], S. S. Bhatia [σ] & Saurabh Manro [σ]

Abstract - In this paper, we use the notion of E.A. property in intuitionistic fuzzy metric space to prove a common fixed point theorem which generalizes Theorem-2 of Turkoglu, Alaca,Cho and Yildiz [12].

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I. Introduction

In 1986, Jungck[6] introduced the notion of compatible maps for a pair of self mappings. Several papers have come up involving compatible maps in proved the existence of common fixed points in the classical and fuzzy metric spaces. Aamri and El. Moutawakil[1] generalized the concept of non compatibility by defining the notion of property E.A. and proved common fixed point theorems under strict contractive conditions. Atanassove [3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets and later there has been much progress in the study of intuitionistic fuzzy sets by many authors [4,5]. In 2004, Park [9] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani [5]. Fixed point theory has important applications in diverse disciplines of mathematics, statistics, engineering, and economics in dealing with problems arising in: Approximation theory, potential theory, game theory, mathematical economics, etc. Several authors [5,8] proved some fixed point theorems for various generalizations of contraction mappings in probabilistic and fuzzy metric space. Turkoglu [12] gave a generalization of Jungck's common fixed point theorem [6] to intuitionistic fuzzy metric spaces. In this paper, we use the notion of E.A property in intuitionistic fuzzy metric space to prove a common fixed point theorem for a pair of self mappings in intuitionistic fuzzy metric space. Our result generalizes Theorem-2 of Turkoglu, Alaca,Cho and Yildiz [12].

II. Preliminaries

The concepts of triangular norms (t–norm) and triangular conorms (t-conorm) are known as the axiomatic skelton that we use are characterization fuzzy intersections and union respectively. These concepts were originally introduced by Menger [8] in study of statistical metric spaces.

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Definition 2.1. A binary operation $*$: $[0, 1]\times[0, 1] \to [0, 1]$ is continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$ for all $a \in [0, 1]$;

(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.2. A binary operation $\diamond$: $[0, 1] \times [0, 1] \to [0, 1]$ is continuous t-conorm if $\diamond$ satisfies the following conditions:

(i) $\diamond$ is commutative and associative;

(ii) $\diamond$ is continuous;

(iii) $a \diamond 0 = a$ for all $a \in [0, 1]$;

(iv) $a \diamond b \le c \diamond d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Alaca et al. [2] using the idea of Intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous tconorms as a generalization of fuzzy metric space due to Kramosil and Michalek [7] as :

Definition 2.3 [2]: A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous tconorm and M, N are fuzzy sets on $X2\times[0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \le 1$ for all $x, y \in X$ and $t > 0$;

(ii) $M(x, y, 0) = 0$ for all $x, y \in X$;

(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(v) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;

(vi) for all $x, y \in X$, $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;

(vii) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$;

(viii) $N(x, y, 0) = 1$ for all $x, y \in X$;

(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(xi) $N(x, y, t) \diamond N(y, z, s) \ge N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;

(xii) for all $x, y \in X$, $N(x, y, .) : [0, \infty)\to[0, 1]$ is right continuous;

(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$ for all x, y in X:

Then (M, N) is called an intuitionistic fuzzy metric space on X. The functions M (x, y,t) and N(x, y, t) denote the degree of nearness and the degree of non - nearness between x and y w.r.t. t respectively.

Ref.

2. C. Alaca, D. Turkoglu, and C. Yildiz, Fixed points in Intuitionistic fuzzy metric spaces, Chaos, Solitons & Fractals 29, 1073-1078 (2006),.
7. I.Kramosil and J. Michalek, Fuzzy metric and Statistical metric spaces, Kybernetica 11, 326-334 (1975).

Remark 2.1 [2]: Every fuzzy metric space (X, M, *) is an intuitionistic fuzzy metric space of the form (X, M, 1-M, *, ◊) such that t-norm * and t-conorm ◊ are associated as $x \diamond y = 1-((1-x) * (1-y))$ for all x, y ∈ X.

Remark 2.2 [2]: In intuitionistic fuzzy metric space (X, M, N, *, ◊), M(x, y, *) is nondecreasing and N(x, y, ◊) is non-increasing for all x, y ∈ X.

Notes

Proof : Suppose that M(x, y, *) is non-increasing, therefore for $t \le s$, we have

$$M(x, y, t) \ge M(x, y, s).$$

For all x, y, z ∈ X, we have $M(x, z, t + s) \ge M(x, y, t) * M(y, z, s)$.
In particular for z = y, we have $M(x, y, t + s) \ge M(x, y, t) * M(y, y, s)$.
$M(x, y, t + s) \ge M(x, y, t) * 1 = M(x, y, t)$, a contradiction, hence M(x, y, *) is non-decreasing.

Again, suppose N(x, y, ◊) is non-decreasing, therefore for $t \le s$,
we have $N(x, y, s) \ge N(x, y, t)$.

For all x, y, z ∈ X, we have $N(x, z, t + s) \le N(x, y, t) \diamond N(y, z, s)$.

In particular for z = y, we have $N(x, y, t + s) \le N(x, y, t) \diamond N(y, y, s)$

$N(x, y, t + s) \le N(x, y, t) \diamond 0 = N(x, y, t)$, a contradiction, hence N(x, y, ◊) is nonincreasing.
Hence the result.
Alaca, Turkoglu and Yildiz [2] introduced the following notions:

Definition 2.4 [2] : Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space. Then
(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all t > 0 and p > 0,

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1 \text{ and } \lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0.$$

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point x ∈ X if, for all t > 0,

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \text{ and } \lim_{n\to\infty} N(x_n, x, t) = 0.$$

Definition 2.5 [2] : An intuitionistic fuzzy metric space (X, M, N, *, ◊) is said to be complete if and only if every Cauchy sequence in X is convergent.

Example 2.1 [2] : Let X = {1/n : n ∈ N}∪{0} and let * be the continuous t-norm and ◊ be the continuous t-conorm defined by $a * b = ab$ and $a \diamond b = \min\{1, a + b\}$ respectively, for all $a, b \in [0,1]$. For each $t \in (0, \infty)$ and $x, y \in X$, define (M, N) by

$$M(x, y, t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0, \\ 0 & t = 0 \end{cases} \quad \text{and } N(x, y, t) = \begin{cases} \dfrac{|x-y|}{t+|x-y|}, & t > 0, \\ 1 & t = 0 \end{cases}$$

Clearly, (X, M, N, *, ◊) is complete intuitionistic fuzzy metric space.

Definition 2.6 [2] : A pair of self mappings (f, g) of a intuitionistic fuzzy metric space (X, M, N, *, ◊) is said to be commuting if

$$M(fgx, gfx, t) = 1 \text{ and } N(fgx, gfx, t) = 0 \text{ for all } x \in X.$$

Definition 2.7 [1] : A pair of self mappings (f, g) of a intuitionistic fuzzy metric space (X, M, N, *, ◊) is said to satisfy the E.A property if there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(fx_n, gx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(fx_n, gx_n, t) = 0.$$

Example 2.2 [1] : Let X =[0, ∞). Consider (X, M, N, *, ◊) be an intuitionistic fuzzy metric space as in Example 2.1. Define f, g : X →X by $fx = \frac{x}{5}$ and $gx = \frac{2x}{5}$ for all x ∈ X. Then for sequence $\{x_n\} = \left\{\frac{1}{n}\right\}$,

$$\lim_{n\to\infty} M(Sx_n, Tx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(Sx_n, Tx_n, t) = 0.$$

Then f and g satisfies E.A property.

Definition 2.8 [12] : A pair of self mappings (f, g) of a intuitionistic fuzzy metric space (X, M, N, *, ◊) is said to be weakly compatible if they commute at coincidence points i.e. if fu = gu for some u in X, then fgu = gfu.

It is easy to see that two compatible maps are weakly compatible.

Lemma 1[2] : Let (X, M, N, *, ◊) be intuitionistic fuzzy metric space and for all x, y in X, t > 0 and if for a number k ∈ (0, 1),

$$M(x, y, kt) \geq M(x, y, t) \text{ and } N(x, y, kt) \leq N(x, y, t)$$

Then x = y.

III. Main Results

Turkoglu et al.[12] proved the following Theorem:

Theorem 3.1 : Let (X, M, N, *, ◊) be a complete intuitionistic fuzzy metric space. Let f and g be self mappings of X satisfying the following conditions:

(a) g(X) ⊆ f(X)

(b) there exist 0 < k < 1 such that M(gx, gy, kt) ≥ M(fx , fy, t) and N(gx, gy, kt) ≤ N(fx , fy, t),

(c) f is continuous

Then f and g have a unique common fixed point provided f and g commute.

Now, we prove a common fixed point theorem using E.A property in intuitionistic fuzzy metric space, which is a generalization of Theorem-3.1 in the following way.

I. to relax the continuity requirement of maps completely,

II. E.A property buys containment of ranges.

Theorem 3.2 : Let (X, M, N, *, ◊) be a intuitionistic fuzzy metric space with continuous t-norm and continuous t-conorm defined by a * a ≥ a and (1 – a) ◊ (1 – a) ≤ (1 – a) where a, b in [0, 1]. Let S and T be two weakly compatible self mappings of X satisfying the following conditions:

(i) T and S satisfy the E.A property,

(ii) for each x, y ∈ X, t >0, there exist 0 < k < 1 such that

$$M(Tx , Ty , kt) \geq M(Sx , Sy, t) \text{ and}$$
$$N(Tx , Ty , kt) \leq N(Sx , Sy, t),$$

Ref.

12. D. Turkoglu, C. Alaca, Y. J. Cho, and C. Yildiz, Common fixed point theorems in Intuitionistic fuzzy metric spaces, J. Appl. Math. & Computing 22, 411-424 (2006).

Notes

(iii) S(X) or T(X) is complete subspace of X.

Then S and T have a unique common fixed point.

Proof : In view of (i), there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}Tx_n = \lim_{n\to\infty}Sx_n, = x_0$ for some x_0 in X. Suppose that S(X) is complete subspace of X, therefore, every convergent sequence of points of S(X) has a limit point in S(X) implies $\lim_{n\to\infty} Sx_n = Sa = u = \lim_{n\to\infty} Tx_n$, for some $a \in X$, which implies that $u = S\,a \in S\,(X)$.

Now, we prove that Ta = S a.

From (ii) take $x = x_n$, $y = a$, we get

$M(Tx_n, Ta, kt) \geq M(Sx_n, Sa, t)$ and $N(Tx_n, Ta, kt) \leq N(Sx_n, Sa, t)$.

Taking limit $n\to\infty$ on both sides,

We get, $M(Sa, Ta, kt) \geq M(Sa, Sa, t)$ and $N(Sa, Ta, kt) \leq N(Sa, Sa, t)$

This implies by using definition of IFS, Sa = Ta.

Therefore, u = S a = T a.

This shows that 'a' is coincident point of T and S.

As T and S are weakly compatible, therefore, TS(a) = ST(a) = SS(a) = TT(a).

Now, we show that Ta is the common fixed point of T and S.

From (ii) take x = a, y = Ta,

$$M(Ta, TTa, kt) \geq M(Sa, STa, t) \text{ and } N(Ta, TTa, kt) \leq N(Sa, STa, t),$$

$$M(Ta, TTa, kt) \geq M(Ta, TTa, t) \text{ and } N(Ta, TTa, kt) \leq N(Ta, TTa, t),$$

This implies by Lemma 1, TTa = Ta = STa.

This proves that Ta is the common fixed point of T and S.

Now, we prove the uniqueness of common fixed point of T and S. If possible, let x_0 and y_0 be two common fixed points of S and T. Then by condition (ii),

$$M(x_0, y_0, kt) = M(Tx_0, Ty_0, kt) \geq M(Sx_0, Sy_0, t) = M(x_0, y_0, t)$$

$$N(x_0, y_0, kt) = N(Tx_0, Ty_0, kt) \leq N(Sx_0, Sy_0, t) = N(x_0, y_0, t),$$

Then by Lemma 1, we have $x_0 = y_0$.

Therefore, the mappings S and T have a unique common fixed point.

This completes the proof.

Example 3.1: Let $X = \left\{\frac{1}{n} : n = 1, 2, 3, \ldots\right\} \cup \{0\}$ with the usual metric d defined by d (x, y) = $|x-y|$ for all $x, y \in X$ and $t>0$, define

$$M(x, y, t) = \begin{cases} \dfrac{t}{(t+|x-y|)}, & t > 0 \\ 0, & t = 0 \end{cases}$$

and

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$$N(x, y, t) = \begin{cases} \dfrac{|x-y|}{(kt+|x-y|)}, & t > 0, k > 0 \\ 1, & t = 0 \end{cases}$$

Clearly, (X, M, N, *, ◊) is intuitionistic fuzzy metric space, where * and ◊ are defined by a * b = ab and a ◊ b = min –1, a + b″ respectively. Define $T(x) = \frac{x}{12}$, $S(x) = \frac{x}{4}$ for all x ∈ X. Clearly, S and T are weakly compatible mappings on X,

(1) S and T satisfy the E.A property for the sequence $\{x_n\} = \left\{\frac{1}{n}\right\}$,

(2) Also for $k = \frac{1}{3}$, the condition (ii) of above theorem is satisfied ,

(3) S(X) is complete subspace of X and

Thus all the conditions of theorem 3.2 are satisfied and so S and T have the common fixed point x = 0.

Theorem 3.3 : Let (X, M, N, *, ◊) be a intuitionistic fuzzy metric space with continuous t-norm and continuous t-conorm defined by a * a ≥ a and (1 – a) ◊ (1 – a) ≤ (1 – a), where a, b in [0, 1]. Let f and g be two weakly compatible self mappings of X satisfying the following conditions:

(i) f and g satisfy the E.A property ,

(ii) for each x, y ∈ X, t >0, there exist 0 < k < 1 such that

$$M(fx,fy,kt) \geq \phi\ (M(gx,gy,t), M(fx,gx,t), M(fy,gy,t), M(fx,gy,t), M(fy,gx,t)) \text{ and}$$

$$N(fx,fy,kt) \leq \psi\ (N(gx,gy,t), N(fx,gx,t), N(fy,gy,t), N(fx,gy,t), N(fy,gx,t))$$

where ϕ, ψ is a mapping from [0,1] to [0,1], which is upper semi-continuous, nondecreasing in each coordinate variable and such that

$\phi\ (1,1,t,1,\ t) \geq t$, $\phi\ (t,1,1,t,\ t) \geq t$ and y $(1,1,t,1,\ t) \leq t$, $\psi\ (t,1,1,t,\ t) \leq t$ where t in [0,1]

(iii) the range of g is a closed subspace of X.

Then f and g have a unique common fixed point.

Proof. In view of (i), there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n, = p$ for some p in X. As g(X) is a closed subspace of X, there is u in X such that p = gu .

Therefore, $\lim_{n\to\infty} fx_n = p = gu = \lim_{n\to\infty} gx_n$.

Now, we prove that fu = gu.

From (ii) take $x = x_n$, y = u,

$M(fx_n, fu, kt) \geq \phi\ (M(gx_n, gu, t), M(fx_n, gx_n, t), M(fu, gu, t), M(fx_n, gu, t), M(fu, gx_n, t))$

$N(fx_n, fu, kt) \leq \psi\ (N(gx_n, gu, t), N(fx_n, gx_n, t), N(fu, gu, t), N(fx_n, gu, t), N(fu, gx_n, t))$

As n→∞, we get

$$M(gu,fu,kt) \geq \phi\ (M(gu,gu,t),M(gu,gu,t),M(fu,gu,t),M(gu,gu,t),M(fu,gu,t))$$

$$= \phi\ (1,1,M(gu,fu,t),1,M(gu,fu,t)) \geq M(gu,fu,t). \text{ (By using (ii))}$$

Notes

$$N(gu,fu,kt)) \leq \psi\ (N(gu,gu,t),N(gu,gu,t),N(fu,gu,t),N(gu,gu,t),N(fu,gu,t))$$

$$= \psi\ (1,1,M(gu,fu,t),1,M(gu,fu,t)) \leq N(gu,fu,t).\ \text{(By using (ii))}$$

By using lemma 1, we deduce that fu = gu. Denote fu by z.

Therefore, fu = gu = z.

This shows that 'u' is coincident point of f and g.

Notes

From weak compatibility of the mappings f and g it follows that fg(u) = gf(u)

This implies, fz = gz.

Now, we show that z is the common fixed point of f and g.

From (ii) take x = z, y = u,

$$M(fz,z,t) = M(fz,fu,t)$$

$$\geq \phi\ (M(gz,gu,t),M(fz,gz,t),M(fu,gu,t),M(fz,gu,t),M(fu,gz,t)),$$

that is,

$$M(fz,z,t) \geq \phi\ (M(fz,z,t),1,1,M(fz,z,t),M(z,fz,t)) \geq M(z,fz,t).$$

And

$$N(fz,z,t) = N(fz,fu,t)$$

$$\leq \psi\ (N(gz,gu,t),N(fz,gz,t),N(fu,gu,t),N(fz,gu,t),N(fu,gz,t)),$$

that is,

$$N(fz,z,t) = N(fz,fu,t)$$

$$\leq \psi\ (\ N(fz,z,t),\ 1\ ,\ 1\ ,\ N(fz,z,t),N(z,fz,t))\ \Theta\ N(fz,z,t),$$

By using lemma 1, we deduce that, fz = z = gz and thus we obtain that z is a common fixed point of f and g.

Now, we prove the uniqueness of common fixed point of f and g. If possible, let 'a' and 'b' be two common fixed points of f and g. Then by condition (ii) take x = a, y = b we get,

$$M(fa,fb,kt) \geq \phi\ (M(ga,gb,t),M(fa,ga,t),M(fb,gb,t),M(fa,gb,t),M(fb,ga,t))$$

$$M(a,b,kt) \geq \phi\ (M(a,b,t),M(a,a,t),M(b,b,t),M(a,b,t),M(b,a,t))$$

$$M(a,b,kt) \geq \phi\ (M(a,b,t),\ 1\ ,1\ ,M(a,b,t),M(a,b,t)) \geq M(a,b,t)$$

and

$$N(fa,fb,kt) \leq \psi\ (N(ga,gb,t),N(fa,ga,t),N(fy,gb,t),N(fa,gb,t),N(fb,ga,t))$$

$$N(a,b,kt) \leq \psi\ (N(a,b,t),N(a,a,t),N(b,b,t),N(a,b,t),N(b,a,t))$$

$$N(a,b,kt) \leq \psi\ (N(a,b,t)\ 1,\ 1,N(a,b,t),\ N(a,\ b\ ,t)) \leq N(a,b,t)$$

Then by Lemma 1, we have a = b.

Therefore, the mappings f and g have a unique common fixed point.

This completes the proof.

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Notes

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Hypergeometric Form of Certain Indefinite Integrals

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Abstract - In this paper we have given Hypergeometric form of some indefinite integrals involving Hypergeometric function. The results represent here are assume to be new.

Keywords :Pochhammer Symbol; Gaussian Hypergeometric Function; Kampé de Fériet double Hypergeometric Function and Srivastava's Triple Hypergeometric Function.

GJSFR-F Classification 2010 MSC NO : 33C05, 33C45, 33C15, 33D50, 33D60

HYPERGEOMETRIC FORM OF CERTAIN INDEFINITE INTEGRALS

Strictly as per the compliance and regulations of :

Notes

Hypergeometric Form of Certain Indefinite Integrals

Salahuddin

Abstract - In this paper we have given Hypergeometric form of some indefinite integrals involving Hypergeometric function. The results represent here are assume to be new.
Keywords and Phrases : Pochhammer Symbol; Gaussian Hypergeometric Function; Kampé de Fériet double Hypergeometric Function and Srivastava's Triple Hypergeometric Function.

I. Introduction and Preliminaries

The Pochhammer's symbol or Appell's symbol or shifted factorial or rising factorial or generalized factorial function is defined by

$$(b,\ k)=(b)_k=\frac{\Gamma(b+k)}{\Gamma(b)}=\begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } \ k=1,2,3,\cdots \\ 1 \qquad ; & \text{if } \ k=0 \\ k! \qquad ; & \text{if } \ b=1,\ k=1,2,3,\cdots \end{cases}$$

where b is neither zero nor negative integer and the notation Γ stands for Gamma function.

Generalized Gaussian Hypergeometric Function

Generalized ordinary hypergeometric function of one variable is defined by

$${}_AF_B\left[\begin{array}{ccc} a_1,a_2,\cdots,a_A & ; & \\ & & z \\ b_1,b_2,\cdots,b_B & ; & \end{array}\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_A)_k z^k}{(b_1)_k(b_2)_k\cdots(b_B)_k k!}$$

or

$${}_AF_B\left[\begin{array}{ccc} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right]\equiv {}_AF_B\left[\begin{array}{ccc} (a_j)_{j=1}^{A} & ; & \\ & & z \\ (b_j)_{j=1}^{B} & ; & \end{array}\right]=\sum_{k=0}^{\infty}\frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1.1}$$

where denominator parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.

Kampé de Fériet's General Double Hypergeometric Function

In 1921, Appell's four double hypergeometric functions F_1, F_2, F_3, F_4 and their confluent forms $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$ were unified and generalized by Kampé de Fériet.

We recall the definition of general double hypergeometric function of Kampé de Fériet in slightly modified notation of H.M.Srivastava and R.Panda:

$$\mathrm{F}^{A:B;D}_{E:G;H}\left[\begin{array}{ccc} (a_A):(b_B);(d_D) & ; & \\ & & x,\ y \\ (e_E):(g_G);(h_H) & ; & \end{array}\right]=\sum_{m,n=0}^{\infty}\frac{((a_A))_{m+n}\,((b_B))_m\,((d_D))_n\ x^m\,y^n}{((e_E))_{m+n}\,((g_G))_m\,((h_H))_n\ m!\,n!} \tag{1.2}$$

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where for convergence

(i) $A+B<E+G+1,\ A+D<E+H+1\quad ;|x|<\infty,\quad |y|<\infty,$ or

(ii) $A+B=E+G+1,\ A+D=E+H+1,$ and

$$\begin{cases} |x|^{\frac{1}{(A-E)}} + |y|^{\frac{1}{(A-E)}} < 1 & ,\text{if } E<A \\ \max\ \{|x|,|y|\}<1 & ,\text{if } E\geq A \end{cases}$$

Srivastava's General Triple Hypergeometric Function

In 1967, H. M. Srivastava defined a general triple hypergeometric function $F^{(3)}$ in the following form

$$F^{(3)}\left[\begin{array}{ll}(a_A) :: (b_B);\ (d_D);\ (e_E) : (g_G);\ (h_H);\ (l_L); & \\ & x,\ y,\ z \\ (m_M) :: (n_N);\ (p_P);\ (q_Q) : (r_R);\ (s_S);\ (t_T); & \end{array}\right]$$

$$=\sum_{i,j,k=0}^{\infty}\frac{((a_A))_{i+j+k}\ ((b_B))_{i+j}\ ((d_D))_{j+k}\ ((e_E))_{k+i}\ ((g_G))_i\ ((h_H))_j\ ((l_L))_k\ x^i\ y^j\ z^k}{((m_M))_{i+j+k}\ ((n_N))_{i+j}\ ((p_P))_{j+k}\ ((q_Q))_{k+i}\ ((r_R))_i\ ((s_S))_j\ ((t_T))_k\ i!\ j!\ k!} \tag{1.3}$$

Wright's Generalized Hypergeometric Function

$${}_p\Psi_q\left[\begin{array}{ll}(\alpha_1,A_1),\cdots,(\alpha_p,A_p) & ; \\ & x \\ (\lambda_1,B_1),\cdots,(\lambda_q,B_q) & ;\end{array}\right]=\sum_{m=0}^{\infty}\frac{\Gamma(\alpha_1+mA_1)\Gamma(\alpha_2+mA_2)\cdots\Gamma(\alpha_p+mA_p)x^m}{\Gamma(\lambda_1+mB_1)\Gamma(\lambda_2+mB_2)\cdots\Gamma(\lambda_q+mA_q)m!} \tag{1.4}$$

$${}_p\Psi_q^*\left[\begin{array}{ll}(\alpha_1,A_1),\cdots,(\alpha_p,A_p) & ; \\ & x \\ (\lambda_1,B_1),\cdots,(\lambda_q,B_q) & ;\end{array}\right]=\sum_{m=0}^{\infty}\frac{(\alpha_1)_{mA_1}(\alpha_2)_{mA_2}\cdots(\alpha_p)_{mA_p}x^m}{(\lambda_1)_{mB_1}(\lambda_2)_{mB_2}\cdots(\lambda_q)_{mB_q}m!} \tag{1.5}$$

II. Main Integrals

$$\int\frac{\mathrm{d}x}{\sqrt{(1-x\sinh^4 x)}}=$$

$$=\cosh x\sinh^{4m-1}x\ (-\sinh^2 x)^{\frac{1-4m}{2}}\ F_{0;1}^{1;2}\left[\begin{array}{ll}\frac{1}{2}\ ;\frac{1}{2},\ \frac{1-4m}{2} & ; \\ & x,\ \cosh^2 x \\ -;\ \frac{-3}{2} & ;\end{array}\right]+Constant \tag{2.1}$$

$$\int\frac{\mathrm{d}x}{\sqrt{(1-x\cosh^4 x)}}=$$

$$=\frac{\sqrt{-\sinh^2 x}\ \ \operatorname{cosech} x\cosh^{4m+1}x}{4m+1}\ F_{0;1}^{1;2}\left[\begin{array}{ll}\frac{1}{2}\ ;\frac{1}{2},\ \frac{4m+1}{2} & ; \\ & x,\ \cosh^2 x \\ -;\ \frac{4m+3}{2} & ;\end{array}\right]+Constant \tag{2.2}$$

$$\int\frac{\mathrm{d}x}{\sqrt{(1-x\tanh^4 x)}}=\frac{\tanh^{4m+1}x}{(4m+1)}\ F_{0;1}^{1;2}\left[\begin{array}{ll}\frac{1}{2}\ ;1,\ \frac{4m+1}{2} & ; \\ & x,\ \tanh^2 x \\ -;\ \frac{4m+3}{2} & ;\end{array}\right]+Constant \tag{2.3}$$

Notes

$$\int \frac{dx}{\sqrt{(1 - x\coth^4 x)}} = \frac{\coth^{4m+1} x}{(4m+1)} F^{1;2}_{0;1}\left[\begin{array}{lll} \frac{1}{2} & ;1, \frac{4m+1}{2} & ; \\ & & x, \coth^2 x \\ —; & \frac{4m+3}{2} & ; \end{array}\right] + Constant \qquad (2.4)$$

$$\int \frac{dx}{\sqrt{(1 - x\ \mathrm{sech}^4\ x)}} =$$

$$= \sinh x \cosh^2(x)^{\frac{4m+1}{2}} \mathrm{sech}^{4m+1} x\ F^{1;2}_{0;1}\left[\begin{array}{lll} \frac{1}{2} & ;\frac{1}{2}, \frac{1+4m}{2} & ; \\ & & x, -\sinh^2 x \\ —; & \frac{3}{2} & ; \end{array}\right] + Constant \qquad (2.5)$$

$$\int \frac{dx}{\sqrt{(1 - x\ \mathrm{cosech}^4\ x)}} =$$

$$= \cosh x(-\sinh^2(x))^{\frac{4m+1}{2}} \mathrm{cosech}^{4m+1} x\ F^{1;2}_{0;1}\left[\begin{array}{lll} \frac{1}{2} & ;\frac{1}{2}, \frac{1+4m}{2} & ; \\ & & x, \cosh^2 x \\ —; & \frac{3}{2} & ; \end{array}\right] + Constant \qquad (2.6)$$

III. Derivation of Integrals

Derivation of integral (2.1)

$$\int \frac{dx}{\sqrt{(1 - x\sinh^4 x)}} = \int (1 - x\sinh^4 x)^{-\frac{1}{2}} dx$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} \sinh^{4m} x\ dx = \sum_{\mathrm{m}=0}^{\infty} \frac{(\frac{1}{2})_{\mathrm{m}} (\mathrm{x})^{\mathrm{m}}}{\mathrm{m}!} \int \sinh^{4\mathrm{m}} \mathrm{x}\ \mathrm{dx}$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} (\cosh x)\ \sinh^{4m-1} x(-\sinh^2 x)^{\frac{1-4m}{2}} {}_2F_1\left[\begin{array}{ll} \frac{1}{2}, \frac{1-4m}{2} & ; \\ \frac{-3}{2} & ; \end{array} \cosh^2 x\right] + Constant$$

$$= \cosh x \sinh^{4m-1} x\ (-\sinh^2 x)^{\frac{1-4m}{2}}\ F^{1;2}_{0;1}\left[\begin{array}{lll} \frac{1}{2} & ;\frac{1}{2}, \frac{1-4m}{2} & ; \\ & & x, \cosh^2 x \\ —; & \frac{-3}{2} & ; \end{array}\right] + Constant \qquad (3.1)$$

Derivation of integral (2.2)

$$\int \frac{dx}{\sqrt{(1 - x\cosh^4 x)}} = \int (1 - x\cosh^4 x)^{-\frac{1}{2}} dx$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} \cosh^{4m} x\ dx = \sum_{\mathrm{m}=0}^{\infty} \frac{(\frac{1}{2})_{\mathrm{m}} (\mathrm{x})^{\mathrm{m}}}{\mathrm{m}!} \int \cosh^{4\mathrm{m}} \mathrm{x}\ \mathrm{dx}$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} \frac{\sqrt{-\sinh^2 x} \cosh^{4m+1} x\ \mathrm{cosech}\ x}{(4m+1)} {}_2F_1\left[\begin{array}{ll} \frac{1}{2}, \frac{4m+1}{2} & ; \\ \frac{4m+3}{2} & ; \end{array} \cosh^2 x\right] + Constant$$

$$= \frac{\sqrt{-\sinh^2 x}\ \ \mathrm{cosech}\ x \cosh^{4m+1} x}{4m+1} F^{1;2}_{0;1}\left[\begin{array}{lll} \frac{1}{2} & ;\frac{1}{2}, \frac{4m+1}{2} & ; \\ & & x, \cosh^2 x \\ —; & \frac{4m+3}{2} & ; \end{array}\right] + Constant \qquad (3.2)$$

Derivation of integral (2.3)

$$\int \frac{\mathrm{d}x}{\sqrt{(1 - x\tanh^4 x)}} = \int (1 - x\tanh^4 x)^{-\frac{1}{2}}\mathrm{d}x$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} \tanh^{4m} x \ \mathrm{d}x = \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} \int \tanh^{4m} x \ \mathrm{d}x$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} \frac{\tanh^{4m+1} x}{(4m+1)} {}_2F_1 \left[\begin{array}{ccc} 1, \frac{4m+1}{2} & ; & \\ \frac{4m+3}{2} & ; & \tanh^2 x \end{array} \right] + Constant$$

$$= \frac{\tanh^{4m+1} x}{(4m+1)} F^{1;2}_{0;1} \left[\begin{array}{ccc} \frac{1}{2} \ ;1, \frac{4m+1}{2} & ; & \\ & & x,\ \tanh^2 x \\ -;\ \frac{4m+3}{2} & ; & \end{array} \right] + Constant \tag{3.3}$$

Derivation of integral (2.4)

$$\int \frac{\mathrm{d}x}{\sqrt{(1 - x\coth^4 x)}} = \int (1 - x\coth^4 x)^{-\frac{1}{2}}\mathrm{d}x$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} \coth^{4m} x \ \mathrm{d}x = \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} \int \coth^{4m} x \ \mathrm{d}x$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} \frac{\coth^{4m+1} x}{(4m+1)} {}_2F_1 \left[\begin{array}{ccc} 1, \frac{4m+1}{2} & ; & \\ \frac{4m+3}{2} & ; & \coth^2 x \end{array} \right] + Constant$$

$$= \frac{\coth^{4m+1} x}{(4m+1)} F^{1;2}_{0;1} \left[\begin{array}{ccc} \frac{1}{2} \ ;1, \frac{4m+1}{2} & ; & \\ & & x,\ \coth^2 x \\ -;\ \frac{4m+3}{2} & ; & \end{array} \right] + Constant \tag{3.4}$$

Derivation of integral (2.5)

$$\int \frac{\mathrm{d}x}{\sqrt{(1 - x\ \mathrm{sech}^4\ x)}} = \int (1 - x\ \mathrm{sech}^4\ x)^{-\frac{1}{2}}\mathrm{d}x$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} \mathrm{sech}^{4m} x \ \mathrm{d}x = \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} \int \mathrm{sech}^{4m} x \ \mathrm{d}x$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} \sinh x \ \cosh^2(x)^{\frac{4m+1}{2}} \mathrm{sech}^{4m+1}\ x \ {}_2F_1 \left[\begin{array}{ccc} \frac{1}{2}, \frac{4m+1}{2} & ; & \\ \frac{3}{2} & ; & -\sinh^2 x \end{array} \right] + Constant$$

$$= \sinh x \cosh^2(x)^{\frac{4m+1}{2}} \mathrm{sech}^{4m+1} x \ F^{1;2}_{0;1} \left[\begin{array}{ccc} \frac{1}{2} \ ; \frac{1}{2}, \frac{1+4m}{2} & ; & \\ & & x,\ -\sinh^2 x \\ -;\ \frac{3}{2} & ; & \end{array} \right] + Constant \tag{3.5}$$

Derivation of integral (2.6)

$$\int \frac{\mathrm{d}x}{\sqrt{(1 - x\ \mathrm{cosech}^4\ x)}} = \int (1 - x\ \mathrm{cosech}^4\ x)^{-\frac{1}{2}}\mathrm{d}x$$

$$\int \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (x)^m}{m!} \mathrm{cosech}^{4m} x \ \mathrm{d}x = \sum_{\mathrm{m}=0}^{\infty} \frac{(\frac{1}{2})_{\mathrm{m}} (\mathrm{x})^{\mathrm{m}}}{\mathrm{m}!} \int \mathrm{cosech}^{4\mathrm{m}} \mathrm{x} \ \mathrm{dx}$$

Notes

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m(-x)^m}{m!} \cosh x \, (-\sinh^2(x))^{\frac{4m+1}{2}} \mathrm{cosech}^{4m+1} x \; {}_2F_1 \left[\begin{array}{cc} \frac{1}{2}, \frac{4m+1}{2} & ; \\ \frac{3}{2} & ; \end{array} \cosh^2 x \right] + Constant$$

$$= \cosh x(-\sinh^2(x))^{\frac{4m+1}{2}} \mathrm{cosech}^{4m+1} x \; F^{1;2}_{0;1} \left[\begin{array}{ccc} \frac{1}{2} \;; \frac{1}{2}, \frac{1+4m}{2} & ; & \\ & & x, \cosh^2 x \\ -; \frac{3}{2} & ; & \end{array} \right] + Constant \quad (3.6)$$

Notes

IV. Conclusion

In our work we have established hypergeometric form of some indefinite integrals . However, the formulae presented herein may be further developed to extend the work .Thus we can only hope that the development presented in this work will stimulate further interest and research in this important area of classical special functions. Just as the mathematical properties of the Gauss hypergeometric function are already of immense and significant utility in mathematical sciences and numerous other areas of pure and applied mathematics, the elucidation and discovery of the formula of hypergeometric functions considered herein should certainly eventually prove useful to further developments in the broad areas alluded to above.

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Applications of Laplace Homotopy Analysis Method for Solving Fractional Heat- And Wave-like Equations

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Abstract - In this paper, we apply Laplace homotopy analysis method for solving various fractional heat- and wave-like equations. This method is combined form of homotopy analysis method and Laplace transform. The proposed algorithm presents a procedure of construct the base function and gives a high order deformation equation in simple form. The purpose of using the Laplace transform is to overcome the deficiency that is mainly caused by unsatisfied conditions in the other analytical techniques. Numerical examples demonstrate the capability of LHAM for fractional partial differential equations.

Keywords: *Laplace homotopy analysis method, homotopy analysis method, fractional heat- and wave-like equations.*

GJSFR-F Classification: *MSC 2010: 80A20*

APPLICATIONS OF LAPLACE HOMOTOPY ANALYSIS METHOD FOR SOLVING FRACTIONAL HEAT-AND WAVE-LIKE EQUATIONS

Strictly as per the compliance and regulations of :

Applications of Laplace Homotopy Analysis Method for Solving Fractional Heat- And Wave-like Equations

V.G.Gupta[α] & Pramod Kumar[σ]

Abstract - In this paper, we apply Laplace homotopy analysis method for solving various fractional heat- and wave-like equations. This method is combined form of homotopy analysis method and Laplace transform. The proposed algorithm presents a procedure of construct the base function and gives a high order deformation equation in simple form. The purpose of using the Laplace transform is to overcome the deficiency that is mainly caused by unsatisfied conditions in the other analytical techniques. Numerical examples demonstrate the capability of LHAM for fractional partial differential equations.

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I. Introduction

In recent years, fractional calculus has been given considerable popularity, due mainly to its various applications in fluid mechanics, viscoelasticity, biology, electrical network, optics and signal processing, electrochemistry and so on. A review of some applications in continuum and statistical mechanics is given by Mainardi [2].One of the most recent works on the subject of fractional calculus i.e. the theory of derivatives and integrals of fractional order, is the book of Podlubny [7], which deals principally with fractional differential equations and today there are many works on fractional calculus [11,12,20,32].The importance of obtaining the exact and approximate solutions of fractional linear or nonlinear differential equations is still significant problem that needs new methods to discover the exact and approximate solutions. But these nonlinear differential equations are difficult to get their exact solutions so numerical methods have been used to handle these equations, some numerical methods have been developed, such as Laplace transform method [7,12], differential transform method [1,4,10], Adomian decomposition method [5,6,25,26,28], variational iteration method [3,15,34], homotopy perturbation method [9,17,18,27,35], homotopy perturbation transform method [29,31]. Another analytical approach that can be applied to solve linear or nonlinear equations is homotopy analysis method [21-24]. A systematic and clear exposition on HAM is given in [24]. This method has been successfully applied to solve many types of nonlinear problems, such as nonlinear Riccati differential equation with fractional order [8], nonlinear Vakhnenko equation [33], the Gluert-jet problem [30], fractional KdV-Bergers-Kurumoto equation [13], and so on.

The objective of present paper is to apply the modified homotopy analysis method namely Laplace homotopy analysis method [16], to provide symbolic approximate

Ref.

1. A. Al-rabtah, V.S. Ertürk. Momani, S. Solutions of a fractional oscillator by using differential transform method, Computers and Mathematics with Applications 59 (2010), no.3, 1356-1362.

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solutions for heat-like and wave-like fractional differential equations with variable coefficients. The Laplace homotopy analysis method is a combination of HAM and Laplace transform. This method is characterized by choosing the identity auxiliary linear operator. The proposed method work efficiently and the result so far are very encouraging and reliable. We would like to emphasize that the LHAM may be considered as an important and significant refinement of the previously developed techniques and can be viewed as an alternative to the recently developed methods such as Adomian decomposition method , variational iteration method , homotopy perturbation method , homotopy perturbation transform method. In this paper we have considered the effectiveness of LHAM for solving various heat-like and wave-like fractional differential equation variable coefficients. The organization of this paper is as follows: Basic definitions of fractional calculus is given in next section, the LHAM is presented in section 3. In section 4, heat-like and wave-like equations are solved to illustrate the applicability of considered method.

II. Basic Definitions

For the concept of fractional derivatives, we will adopt Caputo's definition which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variable and their integral order which is the case in most physical processes. Some basic definitions and properties of fractional calculus theory which we have used in this paper are given in this section.

Definition 2.1 : A real function $f(x)$, $x > 0$ is said to be in the space C_μ, $\mu \in R$, if there exist a real number $p\ (>\mu)$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C\ [0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu, m \in N \cup \{0\}$.

Definition 2.2 : The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-t)^{\alpha-1} f(t)\,dt\ ,\quad \alpha > 0,\ x > 0 \qquad \ldots(1)$$

$$J^0 f(x) = f(x) \qquad \ldots(2)$$

Properties of the operator J^α can be found in [14, 19], we mention only the following:

(i) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$

(ii) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$

(iii) $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$.

Definition 2.3 : The fractional derivative of f(x) in the Caputo sense is defined as [14]

$$D_*^\alpha f(x) = J^{m-\alpha} D_*^m f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt \qquad \ldots(3)$$

Ref.

19. R. Gorenflow, F. Mainardi. Fractional Calculus: integral and differential equations of fractional order, in: A Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York (1997), 223-276.

For $m-1<\alpha\le m,\ m\in N,\ x>0,\ f\in C_{-1}^{m}$.

Also, we need here three basic properties

(i) $D_*^{\alpha} J^{\alpha} f(x)=f(x)$

(ii) $J^{\alpha} D_*^{\alpha} f(x)=f(x)-\sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!},\ \ x>0$

(iii) $D_*^{\alpha} x^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}x^{\gamma-\alpha}$; $x>0, \gamma>0.$

For $m-1<\alpha\le m, m\in N, \mu\ge -1$ and $f\in C_{\mu}^{m}$.

Lemma 2.1. If $m-1<\alpha\le m, m\in N,$ then the Laplace transform of the fractional derivative $D_*^{\alpha} f(t)$ is $L(D_*^{\alpha} f(t))=s^{\alpha}\bar{f}(s)-\sum_{k=0}^{m-1} f^{(k)}(0^+)s^{\alpha-k-1},\ t>0$ …(4)

Where $\bar{f}(s)$ is the Laplace transform of f (t).

III. Laplace Homotopy Analysis Method

The homotopy analysis method which provides an analytical approximate solution has been applied to various linear or nonlinear problems by many workers. In this section, we apply the modified homotopy analysis method [16].This modification is based on the Laplace transform of the fractional differential equations. To illustrate the basic idea of this method, let us consider the following fractional differential equation

$$D_t^{\alpha}u(t)=f(u,u_x,u_{xx})\ ,1<\alpha\le 2\ ,t\ge 0 \quad …(5)$$

Subject to the initial conditions

$$u(x,0)=g_1(x)\quad ,\quad u_t(x,0)=g_2(x) \quad …(6)$$

Where f a linear or nonlinear function and D_t^{α} is a fractional differential operator.

The operator form of nonlinear fPDE (5) can be written as follows:

$$D_t^{\alpha}u(x,t)=A(u,u_x,u_{xx})+B(u,u_x,u_{xx})+C(x,t)\quad ,1<\alpha\le 2\ ,t>0 \quad …(7)$$

Where A and B are linear and nonlinear operators respectively which might include other fractional derivatives of order less than ∝ and C is the known analytic function.

Now Taking the Laplace transform of both sides of eq. (5) and using (6), we have

$$L\left(D_t^{\alpha}u(x,t)\right)=L\left(A(u,u_x,u_{xx})+B(u,u_x,u_{xx})+C(x,t)\right) \quad …(8)$$

$$s^{\alpha}\bar{u}(x,s)-s^{\alpha-1}g_1(x)-s^{\alpha-2}g_2(x)=L\left(A(u,u_x,u_{xx})+B(u,u_x,u_{xx})+C(x,t)\right)$$

$$\bar{u}(x,s)=\frac{g_1(x)}{s}+\frac{g_2(x)}{s^2}+\frac{1}{s^{\alpha}}L\left(A(u,u_x,u_{xx})+B(u,u_x,u_{xx})+C(x,t)\right) \quad …(9)$$

Where $L\left(u(x,t)\right)=\bar{u}(x,s)$

The so-called zero-order deformation equation of the Laplace equation (9) has the form

Ref.

16. M. Zurigat. Solving fractional oscillators using Laplace homotopy analysis method. Annals of the University of Craiova, Mathematics and Computer Science Series, 38(4) (2011), 1-11.

$$(1-q)\left[\bar{\phi}(x,s;q)-\bar{u}_0(x,s)\right]=q\,h\left[\begin{array}{l}\bar{\phi}(x,s;q)-\frac{g_1(x)}{s}-\frac{g_2(x)}{s^2}-\\ \frac{1}{s^\alpha}L\left(A(\phi(x,t),\phi_x(x,t),\phi_{xx}(x,t))+B(\phi(x,t),\phi_x(x,t),\phi_{xx}(x,t))+C(x,t)\right)\end{array}\right] \quad (10)$$

Where $q\in[0,1]$ is an embedding parameter, when q=0 and q=1, we have $\bar{\phi}(x,s;0)=\bar{u}_0(x,s)$ and $\bar{\phi}(x,s;1)=\bar{u}(x,s)$ respectively. Thus, as q increasing from 0 to 1, $\bar{\phi}(x,s;q)$ varies from $\bar{u}_0(x,s)$ *to* $\bar{u}(x,s)$. Expanding $\bar{\phi}(x,s;q)$ in Taylor series with respect to q, one has

$$\bar{\phi}(x,s,q)=\bar{u}_0(x,s)+\sum_{m=1}^{\infty}\bar{u}_m(x,s)\,q^m \quad \ldots(11)$$

Where

$$\overline{u_m}(x,s)=\frac{1}{m!}\frac{\partial^m}{\partial x^m}\bar{\phi}(x,s,q)\ \Big|_{q=0} \quad \ldots(12)$$

Define the vectors

$$\vec{\bar{u}}_0(x,s)=\left\{\bar{u}_0(x,s),\bar{u}_1(x,s),\bar{u}_2(x,s),\ldots\bar{u}_m(x,s)\right\} \quad \ldots(13)$$

Differentiating Equation (10) m times with respect to the embedding parameter q, and then
Setting q=0, h= -1 and finally dividing them by m!, we have the so-called m^th^-order deformation

$$\bar{u}_m(x,s)=\chi_m\bar{u}_{m-1}(x,s)-\Re_m(\vec{\bar{u}}_{m-1}(x,s) \quad \ldots(14)$$

Where

$$\Re_m(\vec{\bar{u}}_{(m-1)}(x,s))=\bar{u}_{(m-1)}(x,s)-\frac{1}{s^\alpha}\left(\frac{1}{m-1!}\frac{\partial^{m-1}}{\partial q^{m-1}}[L(A(\phi,\phi_x(x,t,q),\phi_{xx}(x,t,q))+B(\phi,\phi_x(x,t,q),\phi_{xx}(x,t,q)))\right)_{q=0}$$

$$-\left(\frac{g_1(x)}{s}+\frac{g_2(x)}{s^2}+\frac{1}{s^\alpha}L(C(x,t))\right)(1-\chi_m) \quad \ldots(15)$$

And $$\chi_m=\begin{cases}0 & ,\ m\le 1\\ 1 & ,\ m>1\end{cases} \quad \ldots(16)$$

Applying the inverse Laplace transform of both sides of (14), then we have a power series solution $u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)$ *o f*(5)

IV. Numerical Results

In this section we shall illustrate the Laplace homotopy analysis method (LHAM) to fractional heat- and wave-like equations.

Example 4.1 : Consider the one dimensional fractional Heat-like equation with variable coefficient.

$$D_t^\alpha u(x,t)=\frac{1}{2}x^2u_{xx}(x,t) \qquad ;0<x<1\,,0<\alpha\le 1\,,t>0 \quad \ldots(17)$$

Notes

Notes

Subject to the initial conditions $u(x,0)=x^2$ ...(18)

Taking the Laplace transform of both sides of eq. (17) and using (18) , we have

$$\bar{u}(x,s)-\frac{1}{s}x^2-\frac{1}{2}\frac{1}{s^\alpha}L\left(x^2u_{xx}\right)=0 \quad \ldots(19)$$

Now in view of eq. (14) and (15), we have

$$\bar{u}_m(x,s)=\chi_m u_{m-1}(x,s)-\left[\bar{u}_{m-1}(x,s)-\frac{1}{2s^\alpha}L\left(x^2u_{(m-1)xx}(x,t)\right)-\frac{1}{s}x^2(1-\chi_m)\right] \quad \ldots(20)$$

$$\bar{u}_0(x,s)=\frac{x^2}{s}$$

$$\bar{u}_1(x,s)=\frac{x^2}{s^{\alpha+1}}$$

$$\bar{u}_2(x,s)=\frac{x^2}{s^{2\alpha+1}}$$

Now

$$\bar{u}(x,s)=\bar{u}_0(x,s)+\bar{u}_1(x,s)+\bar{u}_2(x,s)+\ldots \quad \ldots(21)$$

$$\bar{u}(x,s)=x^2\left(\frac{1}{s}+\frac{1}{s^{\alpha+1}}+\frac{1}{s^{2\alpha+1}}+\ldots\right) \quad \ldots(22)$$

Taking the inverse Laplace transform of both sides of (22), we have

$$u(x,t)=x^2\left(1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots\right) \quad \ldots(23)$$

For the special case $\alpha=1$

$$u(x,t)=x^2e^t$$

Example 4.2 *:* Consider the two dimensional fractional Heat-like equation:

$$D_t^\alpha u(x,t)=u_{xx}(x,t)+u_{yy}(x,t) \quad ;0<x,y<2\pi\ ,0<\alpha\le 1\ ,t>0 \quad \ldots(24)$$

Subject to the initial conditions

$$u(x,y,0)=\sin x\sin y \quad \ldots(25)$$

Taking the Laplace transform of both sides of eq. (24) and using (25), we have

$$\bar{u}(x,s)-\frac{1}{s}\sin x\sin y-\frac{1}{s^\alpha}L\left(u_{xx}+u_{yy}\right)=0 \quad \ldots(26)$$

In view of equation (14) and (15), we have

$$\bar{u}_m(x,y,s)=\chi_m\bar{u}_{m-1}(x,y,s)-\left[\bar{u}_{m-1}(x,y,s)-\frac{1}{s^\alpha}L\left(u_{(m-1)xx}+u_{(m-1)yy}\right)-\frac{1}{s}\sin x\sin y(1-\chi_m)\right] \ldots(27)$$

$$\bar{u}_0(x,y,s)=\frac{1}{s}\sin x\sin y$$

$$\bar{u}_1(x,y,s)=-2\sin x\sin y\frac{1}{s^{\alpha+1}}$$

$$\bar{u}_2(x,y,s) = 4\sin x \sin y \frac{1}{s^{2\alpha+1}}$$

$$\bar{u}(x,s) = \bar{u}_0(x,s) + \bar{u}_1(x,s) + \bar{u}_2(x,s) + \ldots \quad \ldots(28)$$

Taking the inverse Laplace transform of both side of eq. (28), we have

$$u(x,t) = \sin x \sin y \left(1 - 2\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots\right) \quad \ldots(29)$$

Notes

If we take ∝=1

$$u(x,t) = e^{-2t} \sin x \sin y$$

Example 4.3 *:* Consider the three dimensional inhomogeneous fractional Heat-like equation

$$D_t^{\alpha} u(x,y,z,t) = x^4 y^4 z^4 + \frac{1}{36}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right) \quad \ldots(30)$$

$$0 < x,y,z \le 1 \ ,0 < \alpha \le 1 \,, t > 0$$

Subject to the initial conditions

$$u(x,y,z,0) = 0 \quad \ldots(31)$$

Taking the Laplace transform of both sides of eq. (30) and using (31), we have

$$\bar{u}(x,y,z,s) = \frac{1}{s^{\alpha}} L\left(x^4 y^4 z^4 + \frac{1}{36}(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right) \quad \ldots(32)$$

Now in view of eq. (14) and (15), we have

$$\bar{u}_m(x,y,z,s) = \chi_m u_{m-1}(x,y,z,s) - \left[\begin{array}{l} u_{m-1}(x,y,z,s) - \frac{1}{s^{\alpha}} L\left(\frac{1}{36}(x^2 u_{(m-1)xx} + y^2 u_{(m-1)yy} + z^2 u_{(m-1)zz})\right) \\ -\frac{1}{s^{\alpha}} x^4 y^4 z^4 (1-\chi_m) \end{array}\right] \quad \ldots(33)$$

$$\bar{u}_0(x,y,z,s) = 0$$

$$\bar{u}_1(x,y,z,s) = -\frac{1}{s^{\alpha+1}} x^4 y^4 z^4$$

$$\bar{u}_2(x,y,z,s) = \frac{1}{s^{2\alpha+1}} x^4 y^4 z^4$$

$$\bar{u}_3(x,y,z,s) = \frac{1}{s^{3\alpha+1}} x^4 y^4 z^4$$

Now

$$\bar{u}(x,y,z,s) = \bar{u}_0(x,y,z,s) + \bar{u}_1(x,y,z,s) + \bar{u}_2(x,y,z,s) + \ldots \quad \ldots(34)$$

Now taking the inverse Laplace transform of both sides of eq. (34), we have

$$u(x,y,z,t) = x^4 y^4 z^4 \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right) \quad \ldots(35)$$

If we take ∝=1

$$u(x,y,z,t) = x^4 y^4 z^4 (e^t - 1)$$

Notes

Example 4.4 : Consider the one dimensional wave-like equation:

$$D_t^{\alpha} u(x,t) = \frac{1}{2} x^2 u_{xx}(x,t) \quad , 1 < \alpha \leq 2 \ , t > 0 \qquad \ldots(36)$$

Subject to the initial conditions

$$u(x,0) = x \ \ , u_t(x,0) = x^2 \qquad \ldots(37)$$

Taking the Laplace transform of both sides of eq. (36) and using (37), we have

$$\bar{u}(x,s) = \frac{x}{s} + \frac{x^2}{s^2} + \frac{1}{2s^{\alpha}} L\left(x^2 u_{xx}\right) \qquad \ldots(38)$$

Now in view of equation (14) and (15), we have

$$\bar{u}_m(x,s) = \chi_m \bar{u}_{m-1}(x,s) - \left[\bar{u}_{m-1}(x,s) - \frac{1}{2s^{\alpha}} L\left(x^2 \bar{u}_{(m-1)}(x,t)\right) - \left(\frac{x}{s} + \frac{x^2}{s^2}\right)(1-\chi_m) \right] \qquad \ldots(39)$$

$$\bar{u}_0(x,s) = \frac{x}{s} + \frac{x^2}{s^2}$$

$$\bar{u}_1(x,s) = \frac{x^2}{s^{\alpha+2}}$$

$$\bar{u}_2(x,s) = \frac{x^2}{s^{2\alpha+2}}$$

Now

$$\bar{u}(x,s) = \bar{u}_0(x,s) + \bar{u}_1(x,s) + \bar{u}_2(x,s) + \ldots \qquad \ldots(40)$$

Now taking the inverse Laplace transform of both sides of eq. (40), we have

$$u(x,t) = x + x^2\left(t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots\right) \qquad \ldots(41)$$

If we take $\alpha=1$

$$u(x,t) = x + x^2(e^t - 1)$$

Example 4.5 : Consider the two dimensional fractional Wave-like equation:

$$D_t^{\alpha} u(x,y,t) = \frac{1}{12}\left(x^2 u_{xx}(x,y,t) + y^2 u_{yy}(x,y,t)\right) , \ 0 < x, y < 1, 1 < \alpha \leq 2 \ , t > 0 \qquad \ldots(42)$$

Subject to the initial conditions

$$u(x,y,0) = x^4 \ \ , u_t(x,y,0) = y^4 \qquad \ldots(43)$$

Taking the inverse Laplace transform of both sides of eq. (42) and using (43), we have

$$\bar{u}(x,y,s) = \left(\frac{1}{s} x^4 + \frac{1}{s^2} y^4\right) + \frac{1}{12s^{\alpha}} L\left(x^2 u_{xx} + y^2 u_{yy}\right) \qquad \ldots(44)$$

Now in view of eq. (14) and (15), we have

$$\bar{u}_m(x,y,s) = \chi_m \bar{u}_{m-1}(x,y,s) - \left[\bar{u}_{m-1}(x,y,s) - \frac{1}{12s^{\alpha}} L\left(x^2 \bar{u}_{(m-1)xx} + y^2 \bar{u}_{(m-1)yy}\right) - \left(\frac{x^4}{s} + \frac{y^4}{s^2}\right)(1-\chi_m) \right]$$

$$\ldots(45)$$

$$\bar{u}_0(x,y,s)=\frac{1}{s}x^4+\frac{1}{s^2}y^4$$

$$\bar{u}_1(x,y,s)=\frac{1}{s^{\alpha+1}}x^4+\frac{1}{s^{\alpha+2}}y^4$$

$$\bar{u}_2(x,y,s)=\frac{1}{s^{2\alpha+1}}x^4+\frac{1}{s^{2\alpha+2}}y^4$$

$$\bar{u}(x,y,s)=\bar{u}_0(x,y,s)+\bar{u}_1(x,y,s)+\bar{u}_2(x,y,s)+.\ . \qquad \ldots(46)$$

Taking the inverse Laplace transform of both sides of eq. (46), we have

$$u(x,y,t)=x^4\left(1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots\right)+y^4\left(t+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right) \quad \ldots(47)$$

If we take ∝=1

$$u(x,y,t)=x^4e^t+y^4(e^t-1)$$

Example 4.6 : Consider the three dimensional fractional Wave-like equation of the form:

$$D_t^\alpha u(x,y,z,t)=(x^2+y^2+z^2)+\frac{1}{2}(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}) \qquad \ldots(48)$$

$$0<x,y,z<1 \quad ,\ 1<\alpha\le 2\ ,\ t>0$$

Subject to the initial conditions

$$u(x,y,z,0)=0 \quad ,\ u_t(x,y,z,0)=(x^2+y^2-z^2) \qquad \ldots(49)$$

Taking the Laplace transform of both sides of the eq. (48) and using (49), we have

$$\bar{u}(x,s)=\frac{1}{s^2}(x^2+y^2-z^2)+\frac{1}{s^\alpha}L\left((x^4+y^4+z^4)+\frac{1}{2}(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right) \qquad \ldots(50)$$

Now in view of eq. (14) and (15), we have

$$\bar{u}_m(x,y,z,s)=\chi_m\bar{u}_{m-1}-\left[\begin{array}{l}\bar{u}_{m-1}-\dfrac{1}{2s^\alpha}L\left(x^2u_{(m-1)xx}+y^2u_{(m-1)yy}+z^2u_{(m-1)zz}\right)-\\ \left(\dfrac{1}{s^2}(x^2+y^2-z^2)+\dfrac{1}{s^{\alpha+1}}(x^2+y^2+z^2)\right)(1-\chi_m)\end{array}\right] \qquad \ldots(51)$$

$$\bar{u}_0(x,y,z,s)=\frac{1}{s^2}(x^2+y^2-z^2)+\frac{1}{s^{\alpha+1}}(x^2+y^2+z^2)$$

$$\bar{u}_1(x,y,z,s)=\frac{1}{s^{\alpha+2}}(x^2+y^2-z^2)+\frac{1}{s^{2\alpha+1}}(x\ \ +y\ \ +z\ \ )$$

$$\bar{u}_2(x,y,z,s)=\frac{1}{s^{2\alpha+2}}(x^2+y^2-z^2)+\frac{1}{s^{3\alpha+1}}(x\ \ +y\ \ +z\ \ )$$

$$\bar{u}(x,y,z,s)=\bar{u}_0(x,y,z,s)+\bar{u}_1(x,y,z,s)+\bar{u}_2(x,y,z,s)+.\ . \qquad \ldots(52)$$

Now taking the inverse Laplace transform of both sides of eq. (52), we have

$$u(x,y,z,t)=(x^2+y^2)\left(t+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+1)}+\ldots\right)$$

Notes

$$+z^2(-t+\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{t^{2\alpha+1}}{\Gamma(2\alpha+1)}+...) \quad ...(53)$$

If we take ∝=2

$$u(x,y,z,t)=(x^2+y^2)e^t+z^2e^{-t}-(x^2+y^2+z^2)$$

V. Conclusion

Notes

The main concern of this article is to apply the Laplace homotopy analysis method to construct an analytical solution for heat- and wave-like partial differential equations of fractional order with variable coefficients. The method was used in a direct way without using linearization, perturbation, or restrictive assumptions. Also its small size of computation in comparison with the computational size required in other numerical methods and its rapid convergence shows that the LHAM is reliable and introduces a significant improvement in solving partial differential equations over existing methods. Finally, the appearance of fractional differential equations as models in some field of applied mathematics makes it necessary to investigate the method (analytical or numerical) for such equations and we hope the LHAM can be applied for some other engineering system with less computational work.

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Deformation Due to Various Sources in Saturated Porous Media with Incompressible Fluid

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Abstract - The present investigation deals with the deformation of various sources in fluid saturated porous medium with incompressible fluid. The normal mode analysis is used to obtain the components of displacement, stress and pore pressure. The variations of normal stress, tangential stress and pore pressure with the distance x has been shown graphically. A particular case of interest has also been deduced from the present investigation.

Keywords: Deformation, porous medium, Normal mode analysis, pore pressure, source, incompressible fluid.

GJSFR-F Classification: MSC 2010 : 74J20

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Deformation Due to Various Sources in Saturated Porous Media with Incompressible Fluid

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Abstract - The present investigation deals with the deformation of various sources in fluid saturated porous medium with incompressible fluid. The normal mode analysis is used to obtain the components of displacement, stress and pore pressure. The variations of normal stress, tangential stress and pore pressure with the distance x has been shown graphically. A particular case of interest has also been deduced from the present investigation.

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I. Introduction

Wave propagation in saturated porous media and the dynamic response of such media are of great interest in geophysics, acoustic, soil and rock mechanics and many earthquake engineering problems.

Biot [1] derived the basic equations of poroelastisity on the basis of energy principles. Privost[17] rederived these equations by use of mixture theory. Zienkiewicz ,Chang [18] and Zienkiewicz, Shiomi [19] derived the basic equations of poroelasticity by the use of principal of continuum mechanics . Gatmiri and Kamalian [4] adopted the later approach because it is more flexible and is based on a set of parameters with a clear physical interpretation to discuss different type of problem. Gatmiri and Nguyen [5] investigated two dimensional problem for saturated porous media with incompressible fluid.

Gatmiri and Jabbari [7,8] discuss time domain Green'’s functions for unsaturated soil for two dimensional and three dimensional solution. Gatmiri ,Maghoul and Duhamel[6] also discuss the two dimensional transient thermo-hydro-mechanical fundamental solution of multiphase porous media in frequency and time domains. Gatmiri and Eslami [9] discuss the scattering of harmonic waves by a circular cavity in a porous medium by using complex function theory approach.

Normal mode analysis approach has been successfully applied by different authors e.g. Ezzat, Othman and Karamang[3],Othman,Ezzat,Zaki and Karamang[12], Othman and Oman[14], Othman and Singh[15], Othman,Farouk and Hamied[11], Othman and Lotfy[13], Othman,Lotfy and Farouk[16], Ezzat,Zakaria and Karamang[2].Resently Kumar,Miglani and Kumar[10] investigated the different problems by using normal mode analysis in fluid saturated porous medium.

In the present paper, we obtain the components of stress and pore pressure for homogeneous isotropic porous saturated medium with incompressible fluid due to various sources. The resulting quantities are shown graphically to depict the effect of porosity.

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II. Governing Equations

Following Gatmiri and Nguyen [5],the basic equations are

Equation of motion :

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i + \rho_f \ddot{w}_i \tag{1}$$

Constitutive relation :

$$\sigma_{ij} = \lambda u_{i,i}\, \delta_{ij} + \mu\, (u_{i,j} + u_{j,i}) - \alpha p \tag{2}$$

Flow conservation for the fluid phase :

$$-\dot{w}_{i,i} + \gamma = \alpha\, \dot{u}_{i,i} + \frac{\dot{p}}{M} \tag{3}$$

Generalized Darcy's law :

$$p_{,i} = -\frac{1}{\kappa}\dot{w}_i - \rho_f \ddot{u}_i - m\ddot{w}_i \tag{4}$$

where u_i is the displacement of the solid skeleton, p denote the fluid pressure, w_i represents the average displacement of the fluid relative to the solid. The elastic constants λ and μ are drained Lame's constant. ρ_f is the fluid density, ρ_s is the solid density, $\rho = 1-n\ \rho_s + n\rho_f$ is the density of solid-fluid mixture and $m = \frac{\rho_f}{n}$ is the mass parameter where n is the porosity, κ is the permeability coefficient. α and M are material parameters which describes the relative compressibility of the constituents. f_i and γ denotes the body force and the rate of fluid injection in to the media.

Equations (1) and (4) with the aid of (2) and (3) in the absence of body force and the rate of fluid injection in to the media, reduce to

$$\mu u_{i,} + (\lambda+\mu)\, u_{i,ij} - \rho_1\, \ddot{u}_i - \alpha^* p_{,i} = 0\,, \tag{5}$$

$$\tau p_{,ii} - \frac{1}{M}\frac{\partial p}{\partial t} - \alpha^* \dot{u}_{i,i} = 0\,, \tag{6}$$

where

$$\rho_1 = \rho - {\rho_f}^2 \tau \frac{\partial}{\partial t}, \quad \alpha^* = \alpha - \rho_f \tau \frac{\partial}{\partial t}, \tau = \left[\frac{1}{\kappa} + m\frac{\partial}{\partial t}\right]^{-1}.$$

Formulation of the problem

We consider a homogeneous, isotropic conducting porous elastic layer of thickness 2H initially undisturbed. The origin of the coordinate system (x_1, x_2, x_3) is taken at the middle surface of the plate and x_3 - axis normal to it along the thickness. The surface $x_3 = \pm H$ is subjected to different sources. The $x_1 - x_2$ plane is chosen to coincide with the middle surface and x_3- axis is normal to it along the thickness.

For two dimensional problem, we take

$$u = (u_1, 0, u_3) \tag{7}$$

We define the non-dimensional quantities

$$x_1' = \frac{\omega}{c_1}x_1, x_3' = \frac{\omega}{c_1}x_3\ , u_1' = \frac{\omega}{c_1}u_1\ , u_3' = \frac{\omega}{c_1}u_3\ , p' = \frac{p}{\lambda}\ , c_1^2 = \frac{\lambda+2\mu}{\rho},\ t' = \omega t \tag{8}$$

where ω is the constant having the dimensions of frequency.

The displacement components are related by the potential functions φ and Ψ as

$$u_1 = \frac{\partial\varphi}{\partial x_1} - \frac{\partial\Psi}{\partial x_3}\ ,\ u_3 = \frac{\partial\varphi}{\partial x_3} + \frac{\partial\Psi}{\partial x_1} \tag{9}$$

Making use of equations (8) and (9), the equations (5) and (6) with aid of (7) after suppressing the prime for convenience, reduce to

$$(1+a_1)\nabla^2\varphi - a_2 p - a_3\frac{\partial^2\varphi}{\partial t^2} = 0 \tag{10}$$

Ref.

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$$a_1\nabla^2\Psi - a_3\frac{\partial^2\Psi}{\partial t^2} = 0 \tag{11}$$

$$b_1\nabla^2 \mathrm{p} - \mathrm{b}_2\frac{\partial \mathrm{p}}{\partial \mathrm{t}} - \frac{\partial}{\partial t}\left[\nabla^2\varphi\right] = 0 \tag{12}$$

Where $a_1 = \frac{\mu}{\lambda+\mu}$, $a_2 = \frac{\alpha^*\lambda}{\lambda+\mu}$, $a_3 = \frac{c_1^2\rho_1}{\lambda+\mu}$, $b_1 = \frac{\tau\omega\lambda}{\alpha^* c_1^2}$ and $\mathrm{b}_2 = \frac{\lambda}{\alpha^* M}$.

We assume the solution of equations (10) - (12) of the form

$$(\varphi, \Psi, \mathrm{p}) = [\mathrm{f(z)}, \mathrm{g(z)}, \mathrm{h(z)}]\mathrm{e}^{\mathrm{i}\xi\,(\mathrm{x}_1 - \mathrm{ct})} \tag{13}$$

where

$c = \frac{\omega}{\xi}$, where ξ is the wave number.

Making use of (13) in equations (10) - (12), eliminating h(z) from the resulting equations, we obtain

$$\left(\frac{d^2}{dz^2} - m_n^2\right)f(z) = 0 \tag{14}$$

where

$m_n^2 = \frac{-A_1 \pm\sqrt{A_1^2+4B_1}}{2}$, (n = 1,2) are the roots of equation (14) and

$$m_3^2 = A_2 - \xi^2 \tag{15}$$

and

$$\nabla = \frac{d}{dz}\ , A_1 = A - 2\xi^2\ , B_1 = \xi^4 - A\xi^2 + B\ ,$$

$$A_2 = \frac{a_3}{a_1}\xi^2c^2\ , A = \left[\frac{a_3}{1+a_1}\xi^2c^2 + \frac{b_2}{b_1}i\xi c + \frac{a_2}{b_1(1+a_1)}i\xi c\right] \text{ and } B = \frac{b_2a_3}{b_1(1+a_1)}i\omega^3c^3 .$$

The appropriate potential φ, Ψ, p can be written as

$$\varphi = [C_1 cos m_1z + C_2 sin\, m_1z + D_1cos m_2z + D_2 sin\, m_2z]e^{i\xi(\mathrm{x}_1-\mathrm{ct})} \tag{16}$$

$$\Psi = [E_1cos m_3z + E_2sin\, m_3z]e^{i\xi(\mathrm{x}_1-\mathrm{ct})} \tag{17}$$

$$p = [r_1C_1cos m_1z + r_1C_2 sin\, m_1z + r_2D_1cos m_2z + r_2D_2 sin\, m_2z]e^{i\xi(\mathrm{x}_1-\mathrm{ct})} \tag{18}$$

where

$$r_i = \frac{(1+a_1)}{a_2}\left[m_i^2 - \xi^2\right] + \frac{a_3}{a_2}\xi^2\mathrm{c}^2 \qquad (\mathrm{i}=1,2)\,. \tag{19}$$

With the help of equations (16) and (17), we obtain the displacement components u_1 and u_3 as

$$\begin{aligned} u_1 = [i\xi(C_1cos m_1z + C_2 sin\, m_1z + D_1cos m_2z + D_2 sin\, m_2z) + m_3(E_1cos m_3z \\ - E_2 sin\, m_3z)]e^{i\xi(\mathrm{x}_1-\mathrm{ct})} \end{aligned} \tag{20}$$

$$\begin{aligned} u_3 = [(-C_1m_1 sin\, m_1z + C_2m_1 cos\, m_1z - D_1m_2 sin\, m_2z + D_2m_2 cos\, m_2z) \\ + i\xi(E_1Dos m_3z + E_2 sin\, m_3z)]\, e^{i\xi(\mathrm{x}_1-\mathrm{ct})} \end{aligned} \tag{21}$$

III. Boundary Conditions

The boundary conditions at $x_3 = \pm H$ are

$$\sigma_{33} = -F_1e^{i\xi(\mathrm{x}_1-\mathrm{ct})}, \sigma_{31} = -F_2e^{i\xi(\mathrm{x}_1-\mathrm{ct})}, p = F_3e^{i\xi(\mathrm{x}_1-\mathrm{ct})} \tag{22}$$

Notes

where F_1 ,F_2 are the magnitudes of the forces and F_3 is the constant pressure applied on the boundary and

$$\sigma_{33} = R_1\frac{\partial u_1}{\partial x_1} + R_2\frac{\partial u_3}{\partial x_3} - \alpha p\,,$$
$$\sigma_{31} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \qquad (23)$$

where

$$R_1 = \frac{\lambda}{\mu},\ R_2 = \frac{\lambda+2\mu}{\mu}.$$

Notes

Case 1 : For normal force $F_2 = F_3 = 0$,

Case 2 : For tangential force $F_1 = F_3 = 0$,

Case 3 : For pressure $F_1 = F_2 = 0$

Derivation of the secular equations

Substituting the value of u_1, u_3 and p from (20), (21) and (18) in the boundary condition (22) and with help of (23) after some simplifications, we obtain

$$\sigma_{33} = \Big[R_3\Big\{\Big(\frac{F_1a_{11}-F_3a_{22}-F_3a_{33}}{\Delta_{10}}\Big)\cos m_1z + \Big(\frac{F_2a_{44}}{\Delta_{20}}\Big)\sin m_1z\Big\} + R_4\Big\{\Big(\frac{F_3a_{55}-F_1a_{66}-F_3a_{77}}{\Delta_{10}}\Big)\cos m_2z - \Big(\frac{F_2a_{88}}{\Delta_{20}}\Big)\sin m_2z\Big\} - d_3\Big\{-\Big(\frac{F_2a_{99}}{\Delta_{20}}\Big)\sin m_3z + \cos m_3z\Big(\frac{-F_3b_{11}-F_3b_{22}+F_1b_{33}+F_1b_{44}}{\Delta_{10}}\Big)\Big\}\Big]e^{i\xi(x_1-ct)} \qquad (24)$$

$$\sigma_{31} = \Big[2i\xi\Big\{m_1\Big(\frac{F_1a_{11}-F_3a_{22}-F_3a_{33}}{\Delta_{10}}\Big)\sin m_1z + m_1\Big(\frac{-F_2a_{44}}{\Delta_{20}}\Big)\cos m_1z - m_2\Big(\frac{-F_3a_{55}+F_1a_{66}+F_3a_{77}}{\Delta_{10}}\Big)\sin m_2z + m_2\Big(\frac{F_2a_{88}}{\Delta_{20}}\Big)\cos m_2z\Big\} + d_6\Big\{\Big(\frac{F_2a_{99}}{\Delta_{20}}\Big)\cos m_3z + \sin m_3z\Big(\frac{-F_3b_{11}-F_3b_{22}+F_1b_{33}+F_1b_{44}}{\Delta_{10}}\Big)\Big\}\Big]e^{i\xi(x_1-ct)} \qquad (25)$$

For normal force: F_1=1, F_2=F_3 = 0

$$\sigma_{33} = \Big[R_3\Big\{\Big(\frac{F_1a_{11}}{\Delta_{10}}\Big)\cos m_1z\Big\} + R_4\Big\{\Big(\frac{-F_1a_{66}}{\Delta_{10}}\Big)\cos m_2z\Big\} - d_3\Big\{\cos m_3z\Big(\frac{F_1b_{33}+F_1b_{44}}{\Delta_{10}}\Big)\Big\}\Big]e^{i\xi(x_1-ct)} \qquad (26)$$

$$\sigma_{31} = \Big[2i\xi\Big\{m_1\Big(\frac{F_1a_{11}}{\Delta_{10}}\Big)\sin m_1z - m_2\Big(\frac{F_1a_{66}}{\Delta_{10}}\Big)\sin m_2z\Big\} + d_6\Big\{\sin m_3z\Big(\frac{F_1b_{33}+F_1b_{44}}{\Delta_{10}}\Big)\Big\}\Big]e^{i\xi(x_1-ct)} \qquad (27)$$

For Tangential Force: F_2=1 , F_1=F_3=0

$$\sigma_{33} = \Big[R_3\Big\{\Big(\frac{F_2a_{44}}{\Delta_{20}}\Big)\sin m_1z\Big\} + R_4\Big\{-\Big(\frac{F_2a_{88}}{\Delta_{20}}\Big)\sin m_2z\Big\} - d_3\Big\{-\Big(\frac{F_2a_{99}}{\Delta_{20}}\Big)\sin m_3z\Big\}\Big]e^{i\xi(x_1-ct)} \qquad (28)$$

$$\sigma_{31} = \Big[2i\xi\Big\{m_1\Big(\frac{-F_2a_{44}}{\Delta_{20}}\Big)\cos m_1z + m_2\Big(\frac{F_2a_{88}}{\Delta_{20}}\Big)\cos m_2z\Big\} + d_6\Big\{\Big(\frac{F_2a_{99}}{\Delta_{20}}\Big)\cos m_3z\Big\}\Big]e^{i\xi(x_1-ct)} \qquad (29)$$

For Pressure : F_1=F_2=0, F_3=1

$$\sigma_{33} = \Big[R_3\Big\{\Big(\frac{-F_3a_{22}-F_3a_{33}}{\Delta_{10}}\Big)\cos m_1z\Big\} + R_4\Big\{\Big(\frac{F_3a_{55}-F_3a_{77}}{\Delta_{10}}\Big)\cos m_2z\Big\} - d_3\Big\{\cos m_3z\Big(\frac{-F_3b_{11}-F_3b_{22}}{\Delta_{10}}\Big)\Big\}\Big]e^{i\xi(x_1-ct)} \qquad (30)$$

$$\sigma_{31} = \Big[2i\xi\Big\{m_1\Big(\frac{-F_3a_{22}-F_3a_{33}}{\Delta_{10}}\Big)\sin m_1z - m_2\Big(\frac{-F_3a_{55}+F_3a_{77}}{\Delta_{10}}\Big)\sin m_2z\Big\} + d_6\Big\{\sin m_3z\Big(\frac{-F_3b_{11}-F_3b_{22}}{\Delta_{10}}\Big)\Big\}\Big]e^{i\xi(x_1-ct)} \qquad (31)$$

Where

$$R_3 = R_1\xi^2 + R_2m_1^2 + \alpha r_1,\ R_4 = R_1\xi^2 + R_2m_2^2 + \alpha r_2,$$

Notes

$a_{11} = r_2 d_6 cos\, m_2 H sin\, m_3 H$,$a_{22} = d_2 d_6 cos\, m_2 H sin\, m_3 H$,$a_{33} = d_3 d_5 sin\, m_2 H cos\, m_3 H$,$a_{44} = r_2 d_3 sin\, m_2 H cos\, m_3 H$

$a_{55} = d_1 d_6 cos\, m_1 H sin\, m_3 H$,$a_{66} = r_1 d_6 cos\, m_1 H sin\, m_3 H$,$a_{77} = d_3 d_4 sin\, m_1 H cos\, m_3 H$,$a_{88} = r_1 d_3 sin\, m_1 H sin\, m_3 H$,

$a_{99} = (r_1 d_2 - r_2 d_1) sin\, m_1 H sin\, m_2 H$, $b_{11} = d_1 d_5 cos\, m_1 H sin\, m_2 H$, $b_{22} = d_2 d_4 sin\, m_1 H cos\, m_2 H$, $b_{33} = r_2 d_4 sin\, m_1 H cos\, m_2 H$, $b_{44} = r_1 d_5 cos\, m_1 H sin\, m_2 H$.

$\Delta_{10} = (-r_2 d_1 d_6 + r_1 d_2 d_6) cosm_1 H cosm_2 H sinm_3 H + r_2 d_3 d_4 sin\, m_1 H cosm_2 H\, cos\, m_3 H + r_1 d_3 d_5 cosm_1 H cos\, m_3 H\, sin\, m_2 H$,

$\Delta_{20} = (r_2 d_1 d_6 - r_1 d_2 d_6) sin\, m_1 H sin\, m_2 H cosm_3 H + r_2 d_3 d_4 cosm_1 H sinm_2 H sinm_3 H - r_1 d_3 d_5 sin\, m_1 H sinm_3 H cosm_2 H$,

$d_1 = R_1 \xi^2 - R_2 m_1^2$, $d_2 = R_1 \xi^2 - R_2 m_2^2$, $d_3 = (R_1 - R_2)\, i\xi\, m_3$, $d_4 = 2i\xi m_1$, $d_5 = 2i\xi m_2$, $d_6 = m_3^2 - \xi^2$.

IV. Special Case

In the absence of incompressible fluid, the boundary conditions reduce to

$$\sigma_{33} = -F_1 e^{i\xi(x_1 - ct)} , \sigma_{31} = -F_2 e^{i\xi(x_1 - ct)} \tag{32}$$

and we obtain the constituting expressions for stress components for elastic layer as

$$\sigma_{33} = \left[R_5 \left\{-\frac{F_1 d_6 sinm_3 H}{\Delta_{50}} cosm_4 z - \frac{F_2 d_3 sin\, m_3 H}{\Delta_{60}} sin\, m_4 z\right\} - d_3 \left\{-\frac{F_2 d_1 sinm_4 H}{\Delta_{60}} sinm_3 z + \frac{F_1 d_4 sinm_4 H}{\Delta_{50}} cos\, m_3 z\right\}\right] e^{i\xi(x_1 - ct)} \tag{33}$$

$$\sigma_{31} = \left[2i\xi\left\{m_4 \frac{F_2 d_3 sin\, m_3 H}{\Delta_{60}} cos\, m_4 z\right\} + d_6 \left\{\frac{F_2 d_1 sinm_4 H}{\Delta_{60}} cosm_3 z + \frac{F_1 d_4 sinm_4 H}{\Delta_{50}} sin\, m_3 z\right\}\right] e^{i\xi(x_1 - ct)} \tag{34}$$

For normal force: F_1=1 and F_2=0

$$\sigma_{33} = \left[R_5 \left\{-\frac{F_1 d_6 sinm_3 H}{\Delta_{50}} cosm_4 z\right\} - d_3 \left\{\frac{F_1 d_4 sinm_4 H}{\Delta_{50}} cos\, m_3 z\right\}\right] e^{i\xi(x_1 - ct)} \tag{35}$$

$$\sigma_{31} = d_6 \left[\frac{F_1 d_4 sinm_4 H}{\Delta_{50}} sin\, m_3 z\right] e^{i\xi(x_1 - ct)} \tag{36}$$

For Tangential Force: F_1=0 and F_2=1

$$\sigma_{33} = \left[R_5 \left\{-\frac{F_2 d_3 sin\, m_3 H}{\Delta_{60}} sin\, m_4 z\right\} - d_3 \left\{-\frac{F_2 d_1 sinm_4 H}{\Delta_{60}} sinm_3 z\right\}\right] e^{i\xi(x_1 - ct)} \tag{37}$$

$$\sigma_{31} = \left[2i\xi\left\{m_4 \frac{F_2 d_3 sin\, m_3 H}{\Delta_{60}} cos\, m_4 z\right\} + d_6 \left\{\frac{F_2 d_1 sinm_4 H}{\Delta_{60}} cosm_3 z\right\}\right] e^{i\xi(x_1 - ct)} \tag{38}$$

Where

$R_5 = R_1 \xi^2 + R_2 m_4^2$

$\Delta_{50} = d_1 d_6 cosm_4 H sinm_3 H + d_3 d_4 sin\, m_4 H\, cos\, m_3 H$,

$\Delta_{60} = -\ d_1 d_6 cos\, m_3 H sin\, m_4 H - d_3 d_4 sin\, m_3 H\, cos\, m_4 H$

V. Numerical Results and Discussion

With the view of illustrating the theoretical results and for numerical discussion we take a model for which the value of the various physical parameters are taken from Gatmiri and Ngyun[2007]:

$$\lambda = 12.5\text{MPa} ,\mu = 8.33MPa ,K_s = 10^5MPa , ,K_f = 0.22 \times 10^4MPa ,\rho_s = 2600Kg/m^3$$

$$\rho_f = 1000Kg/m^3 ,k = 0.001m/s ,\alpha = 1, n = 0.3$$

The values of normal stress σ_{33} , tangential stress σ_{31} and pore pressure p for homogeneous isotropic porous saturated medium with incompressible fluid and elastic medium are obtained for t=1 and z=1 in the range $0 \leq x \leq 10$.

The solid line represent the value of σ_{33} in fluid saturated porous medium with incompressible fluid for normal force(NFSPM), long dash line represent the value of σ_{31} in fluid saturated porous medium with incompressible fluid for tangential force (TFSPM) and small dash line represent the value of p in fluid saturated porous medium with incompressible fluid for pressure (PFSPM) where as solid line with central symbol(NFEM) and small dash line with central symbol(TFEM) represent the value of σ_{33} and σ_{31} in elastic medium for normal and tangential force respectively.

Fig.1 shows the variation of normal stress component σ_{33} w.r.t distance x in fluid saturated porous medium with incompressible fluid and elastic medium. The value of σ_{33} in fluid saturated porous medium with incompressible fluid, in case of normal force, first increase and then starts decrease and in case of tangential force, it remains linear with small decrease and in case of normal pressure source, it first increase and then start decreasing. The value of σ_{33} in elastic medium first increase and then starts decreasing in case of normal force where as in case of tangential force there is sharp increase and then starts decreasing.

Fig.2 shows the variations of tangential stress component σ_{31} w.r.t distance x in fluid saturated porous medium with incompressible fluid and elastic medium. The value of σ_{31} in fluid saturated porous medium with incompressible fluid, in case of normal force, first starts with small increase and then starts decreasing. In case of tangential force, it shows small decrease where as there is a sharp decrease in case of normal pressure. The values of σ_{31} in elastic medium, show small increase in case of normal force and there is a sharp decrease and then starts increasing and ends with small decrease in case of tangential force.

Fig.3 shows the variation of pore pressure w.r.t distance x in fluid saturated porous medium with incompressible fluid. The values of p start with small decrease and increase in case of normal force and become linear in case of tangential force. There is sharp increase in case of pressure force.

VI. Conclusion

It is observed that the behaviour ofσ_{33} in case of normal force and tangential force is same although the value due pore pressure is more. Appreciable porosity effect is observed on normal stress component. The behaviour of σ_{31} in case of normal force and tangential force is opposite. In case of normal pressure the value of normal force is initially less as compared with tangential force.

Notes

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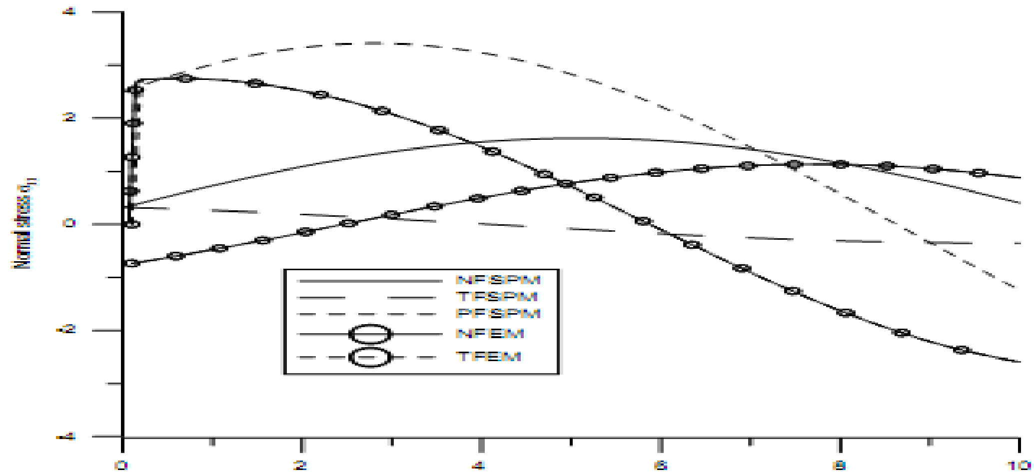

Fig.1 : Variation of normal stress component $\boldsymbol{\sigma}_{33}$ w.r.t horigentol distance x.

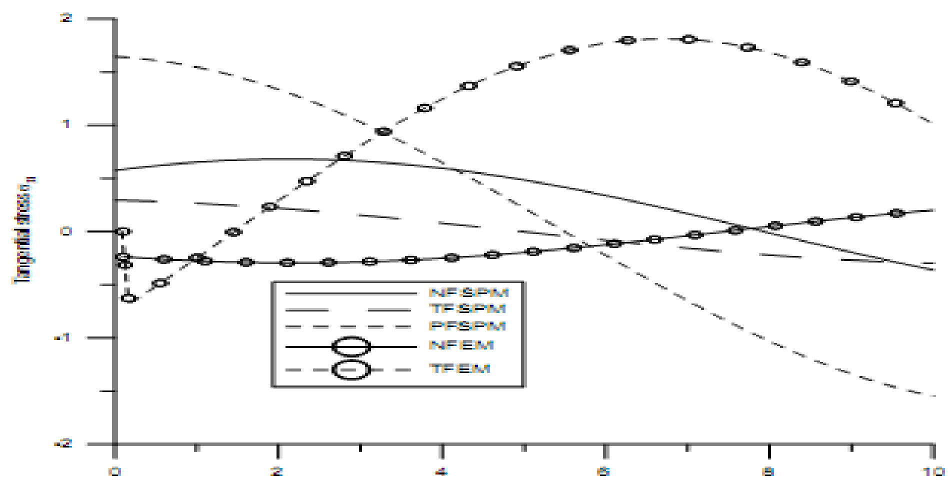

Fig.2 : Variation of tangential stress component $\boldsymbol{\sigma}_{31}$ w.r.t. horizontal distance x.

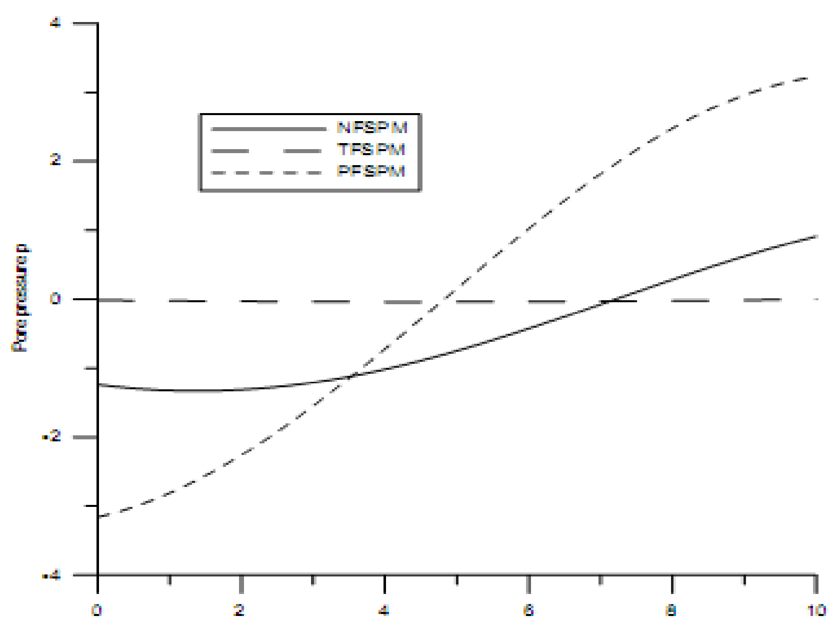

Fig.3 : Variation of pore pressure p w.r.t horizontal distance x.

Notes

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New Representations in Terms of q-product Identities for Ramanujan's Results II

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Abstract - In this paper author has established seven q-product identities, which are not available in the literature of special functions.

Keywords: *Generating functions, triple product identities.*

AMS Subject Classifications : *Primary 05A17, 05A15; Secondary 11P83*

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M.P. Chaudhary

Abstract - In this paper author has established seven q-product identities, which are not available in the literature of special functions.

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I. Introduction

For $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) \tag{1.1}$$

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{(n-1)}) \tag{1.2}$$

$$(a_1,a_2,a_3,...,a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty...(a_k;q)_\infty \tag{1.3}$$

Ramanujan [2, p.1(1.2)]has defined general theta function, as

$$f(a,b) = \sum_{-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \quad ; \quad |ab| < 1, \tag{1.4}$$

Jacobi's triple product identity [3,p.35] is given, as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{1.5}$$

Special cases of Jacobi's triple products identity are given, as

$$\phi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty \tag{1.6}$$

$$(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.7}$$

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.8}$$

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Equation (1.8) is known as Euler's pentagonal number theorem. Euler's another well known identity is as

$$(q;q^2)_\infty^{-1} = (-q;q)_\infty \tag{1.9}$$

Throughout this paper we use the following representations

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty = (q^a,q^b,q^c\cdots q^t;q^n)_\infty \tag{1.10}$$

$$(q^a;q^n)_\infty(q^a;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty = (q^a,q^a,q^c\cdots q^t;q^n)_\infty \tag{1.11}$$

Now we can have following q-products identities, as

$$(q^2;q^2)_\infty = \prod_{n=0}^{\infty}(1-q^{2n+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{2(4n)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+1)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+2)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+3)+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{8n+2}) \times \prod_{n=0}^{\infty}(1-q^{8n+4}) \times \prod_{n=0}^{\infty}(1-q^{8n+6}) \times \prod_{n=0}^{\infty}(1-q^{8n+8})$$

or,

$$(q^2;q^2)_\infty = (q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty = (q^2,q^4,q^6,q^8;q^8)_\infty \tag{1.12}$$

$$(q^4;q^4)_\infty = \prod_{n=0}^{\infty}(1-q^{4n+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{4(3n)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+1)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+2)+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{12n+4}) \times \prod_{n=0}^{\infty}(1-q^{12n+8}) \times \prod_{n=0}^{\infty}(1-q^{12n+12})$$

or,

$$(q^4;q^4)_\infty = (q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty = (q^4,q^8,q^{12};q^{12})_\infty \tag{1.13}$$

$$(q^4;q^{12})_\infty = \prod_{n=0}^{\infty}(1-q^{12n+4}) = \prod_{n=0}^{\infty}(1-q^{12(5n)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+1)+4}) \times$$

$$\times \prod_{n=0}^{\infty}(1-q^{12(5n+2)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+3)+4}) \times \prod_{n=0}^{\infty}(1-q^{12(5n+4)+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{60n+4}) \times \prod_{n=0}^{\infty}(1-q^{60n+16}) \times \prod_{n=0}^{\infty}(1-q^{60n+28}) \times \prod_{n=0}^{\infty}(1-q^{60n+40}) \times \prod_{n=0}^{\infty}(1-q^{60n+52})$$

or,

$$(q^4;q^{12})_\infty = (q^4;q^{60})_\infty(q^{16};q^{60})_\infty(q^{28};q^{60})_\infty(q^{40};q^{60})_\infty(q^{52};q^{60})_\infty$$
$$= (q^4,q^{16},q^{28},q^{40},q^{52};q^{60})_\infty \tag{1.14}$$

Similarly we can compute following as

$$(q^6;q^6)_\infty = (q^6;q^{24})_\infty(q^{12};q^{24})_\infty(q^{18};q^{24})_\infty(q^{24};q^{24})_\infty = (q^6,q^{12},q^{18},q^{24};q^{24})_\infty \tag{1.15}$$

Notes

$$(q^6;q^{12})_\infty = (q^6;q^{60})_\infty(q^{18};q^{60})_\infty(q^{30};q^{60})_\infty(q^{42};q^{60})_\infty(q^{54};q^{60})_\infty$$
$$= (q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty \quad (1.16)$$

$$(q^8;q^8)_\infty = (q^8;q^{48})_\infty(q^{16};q^{48})_\infty(q^{24};q^{48})_\infty(q^{32};q^{48})_\infty(q^{40};q^{48})_\infty(q^{48};q^{48})_\infty$$
$$= (q^8,q^{16},q^{24},q^{32},q^{40},q^{48};q^{48})_\infty \quad (1.17)$$

$$(q^8;q^{12})_\infty = (q^8;q^{60})_\infty(q^{20};q^{60})_\infty(q^{32};q^{60})_\infty(q^{44};q^{60})_\infty(q^{56};q^{60})_\infty$$
$$= (q^8,q^{20},q^{32},q^{44},q^{56};q^{60})_\infty \quad (1.18)$$

$$(q^8;q^{16})_\infty = (q^8;q^{48})_\infty(q^{24};q^{48})_\infty(q^{40};q^{48})_\infty = (q^8,q^{24},q^{40};q^{48})_\infty \quad (1.19)$$

$$(q^{10};q^{20})_\infty = (q^{10};q^{60})_\infty(q^{30};q^{60})_\infty(q^{50};q^{60})_\infty = (q^{10},q^{30},q^{50};q^{60})_\infty \quad (1.20)$$

$$(q^{12};q^{12})_\infty = (q^{12};q^{60})_\infty(q^{24};q^{60})_\infty(q^{36};q^{60})_\infty(q^{48};q^{60})_\infty(q^{60};q^{60})_\infty$$
$$= (q^{12},q^{24},q^{36},q^{48},q^{60};q^{60})_\infty \quad (1.21)$$

$$(q^{16};q^{16})_\infty = (q^{16};q^{48})_\infty(q^{32};q^{48})_\infty(q^{48};q^{48})_\infty = (q^{16},q^{32},q^{48};q^{48})_\infty \quad (1.22)$$

$$(q^{20};q^{20})_\infty = (q^{20};q^{60})_\infty(q^{40};q^{60})_\infty(q^{60};q^{60})_\infty = (q^{20},q^{40},q^{60};q^{60})_\infty \quad (1.23)$$

The outline of this paper is as follows. In sections 2, we have recorded some recent results obtained by the author and also some well known results, those are useful to the rest of the paper. In section 3, we state and prove seven new q-product identities, which are not available in the literature of special functions.

II. Preliminaries

In [1], following identities are being established

$$(q^2,q^4,q^6;q^8)_\infty[(-q;q^2)^2_\infty + (q;q^2)^2_\infty] = 2(-q^4;q^8)^2_\infty \quad (2.1)$$

$$(q^2,q^4,q^6,q^8;q^8)_\infty[(-q;q^2)^2_\infty - (q;q^2)^2_\infty] = 4q\frac{(q^{16},q^{32},q^{48};q^{48})_\infty}{(q^8,q^{24},q^{40};q^{48})_\infty} \quad (2.2)$$

$$\frac{(-q;q^2)^2_\infty + (q;q^2)^2_\infty}{(-q;q^2)^2_\infty - (q;q^2)^2_\infty} = \frac{(-q^4;q^8)^2_\infty(q^8,q^8,q^{24},q^{24},q^{40},q^{40};q^{48})_\infty}{2q} \quad (2.3)$$

$$(-q;q^2)^2_\infty(q;q^2)^2_\infty(q^2;q^2)^2_\infty = (q^2,q^2,q^4;q^4)_\infty \quad (2.4)$$

$$\frac{(-q;q^2)_\infty(-q^3;q^6)_\infty - (q;q^2)_\infty(q^3;q^6)_\infty}{(-q;q^2)_\infty \times (-q^3;q^6)_\infty \times (q;q^2)_\infty \times (q^3;q^6)_\infty} = \frac{2q(-q^2;q^4)^2_\infty(q^4,q^8,q^{16},q^{20},q^{24};q^{24})_\infty}{(q^2,q^4,q^6,q^8;q^8)_\infty(q^6,q^{12},q^{18};q^{24})_\infty} \quad (2.5)$$

$$\frac{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty - (q^3;q^6)_\infty(q^5;q^{10})_\infty}{(-q^3;q^6)_\infty \times (-q^5;q^{10})_\infty \times (q^3;q^6)_\infty \times (q^5;q^{10})_\infty} = \frac{(q^4,q^8,q^{12};q^{12})_\infty}{(q^6,q^{12},q^{18},q^{24};q^{24})_\infty}\times$$
$$\times\frac{2q^3}{(q^2,q^6,q^{10};q^{12})_\infty(q^{10},q^{20},q^{30},q^{30},q^{40},q^{50};q^{60})_\infty} \quad (2.6)$$

$$\frac{[(q;q^2)_\infty(q^{15};q^{30})_\infty] + [(-q;q^2)_\infty(-q^{15};q^{30})_\infty]}{[(q;q^2)_\infty(q^{15};q^{30})_\infty][(-q;q^2)_\infty(-q^{15};q^{30})_\infty]} = \frac{(q^{12},q^{20},q^{24},q^{36},q^{40},q^{48},q^{60},q^{60};q^{60})_\infty}{(q^{10},q^{30},q^{30},q^{50},q^{60};q^{60})_\infty}\times$$

Notes

1. M.P. Chaudhary; on q-product identities, preprint.

$$\times \frac{2}{(q^2, q^4, q^6, q^8, q^8; q^8)_\infty (q^6, q^{18}, q^{30}, q^{42}, q^{54}; q^{60})_\infty} \tag{2.7}$$

In Ramanujan's notebooks [6, p.245], the following entries are recorded as

$$\phi(q) \quad (q^2) = \quad {}^2(q) \tag{2.8}$$

$$\phi^2(q) - \phi^2(-q) = 8q\psi^2(q^4) \tag{2.9}$$

$$\phi^2(q) + \phi^2(-q) = 2\phi^2(q^2) \tag{2.10}$$

$$\phi^4(q) - \phi^4(-q) = 16q\psi^4(q^2) \tag{2.11}$$

$$ {}^2(q) + \quad {}^2(-q) = 2 \quad (q^2)\phi(q^4) \tag{2.12}$$

III. Main Results

In this paper, we have established following new results, which are not recorded in the literature of special functions

$$(-q; q^2)^2_\infty (q; q^2)^2_\infty = (-q, -q, q, q; q^2)_\infty = \frac{(q^2, q^2, q^4, q^6, q^6, q^8; q^8)_\infty}{(q^4, q^8, q^{12}; q^{12})_\infty} \tag{3.1}$$

$$(-q; q^2)^4_\infty - (q; q^2)^4_\infty = \frac{8q}{(q^2, q^2, q^4, q^4, q^4, q^4, q^6, q^6; q^8)_\infty} \tag{3.2}$$

$$(-q; q^2)^4_\infty + (q; q^2)^4_\infty = \frac{2(-q^2, -q^2, -q^2, -q^2; q^4)_\infty}{(q^2, q^2; q^4)_\infty} \tag{3.3}$$

$$\frac{(-q; q^2)^4_\infty - (q; q^2)^4_\infty}{(-q; q^2)^4_\infty + (q; q^2)^4_\infty} = \frac{4q(q^2, q^2; q^4)_\infty}{(q^2, q^2, q^4, q^4, q^4, q^4, q^6, q^6; q^8)_\infty (-q^2, -q^2, -q^2, -q^2; q^4)_\infty} \tag{3.4}$$

$$(q^2; q^4)^8_\infty [(-q; q^2)^8_\infty - (q; q^2)^8_\infty] = 1 \tag{3.5}$$

$$(q^2, q^4, q^6; q^8)_\infty [(-q; q^2)^2_\infty + (q; q^2)^2_\infty] = 2\left[\frac{(q; q^2)_\infty (-q; q^2)_\infty (-q^4; q^8)_\infty}{(q^2; q^4)_\infty}\right]^2 \tag{3.6}$$

$$\frac{1}{(q^2; q^4)^6_\infty} = \frac{16q(-q^2, -q^2, -q^2, -q^2; q^4)_\infty}{(q^2, q^2, q^4, q^4, q^4, q^4, q^6, q^6; q^8)_\infty} \tag{3.7}$$

Proof of (3.1): Employing (1.6) and (1.7) in (2.8), we have

$$(-q; q^2)^2_\infty (q^2; q^2)_\infty \frac{(q^4; q^4)_\infty}{(q^2; q^4)_\infty} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \times \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}$$

$$(-q; q^2)^2_\infty (q; q^2)_\infty (q; q^2)_\infty (q^4; q^4)_\infty = (q^2; q^2)_\infty (q^2; q^4)_\infty$$

employing equations (1.15) and (1.16), and after little algebra, we get

$$(-q; q^2)^2_\infty (q; q^2)^2_\infty = (-q, -q, q, q; q^2)_\infty = \frac{(q^2, q^2, q^4, q^6, q^6, q^8; q^8)_\infty}{(q^4, q^8, q^{12}; q^{12})_\infty}$$

Ref.

6. S. Ramanujan, Notebooks (Volume I), Tata Institute of Fundamental Research, Bombay, 1957.

which established (3.1).

Proof of (3.2): Employing (1.6) and (1.7) in (2.9), we have

$$(-q;q^2)_\infty^2(q^2;q^2)_\infty(-q;q^2)_\infty^2(q^2;q^2)_\infty - (q;q^2)_\infty^2(q^2;q^2)_\infty(q;q^2)_\infty^2(q^2;q^2)_\infty$$

$$= 8q\frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty} \times \frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty}$$

$$(q^2;q^2)_\infty(q^2;q^2)_\infty[(-q;q^2)_\infty^2(-q;q^2)_\infty^2 - (q;q^2)_\infty^2(q;q^2)_\infty^2] = 8q\frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty}\frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty}$$

employing equation (1.15), and after little algebra, we get

$$(-q;q^2)_\infty^4 - (q;q^2)_\infty^4 = \frac{8q}{(q^2,q^2,q^4,q^4,q^4,q^4,q^6,q^6;q^8)_\infty}$$

which established (3.2).

Proof of (3.3): Employing (1.6) in (2.10), we have

$$(-q;q^2)_\infty^2(q^2;q^2)_\infty(-q;q^2)_\infty^2(q^2;q^2)_\infty + (q;q^2)_\infty^2(q^2;q^2)_\infty(q;q^2)_\infty^2(q^2;q^2)_\infty$$

$$= 2(-q^2;q^4)_\infty^2(q^4;q^4)_\infty(-q^2;q^4)_\infty^2(q^4;q^4)_\infty$$

$$(q^2;q^2)_\infty(q^2;q^2)_\infty[(-q;q^2)_\infty^4 + (q;q^2)_\infty^4] = 2(-q^2;q^4)_\infty^2(q^4;q^4)_\infty(-q^2;q^4)_\infty^2(q^4;q^4)_\infty$$

$$(q^2;q^4)_\infty(q^4;q^4)_\infty(q^2;q^4)_\infty(q^4;q^4)_\infty[(-q;q^2)_\infty^4 + (q;q^2)_\infty^4]$$

$$= 2(-q^2;q^4)_\infty^2(q^4;q^4)_\infty(-q^2;q^4)_\infty^2(q^4;q^4)_\infty$$

after simplification by using little algebra, we get

$$(-q;q^2)_\infty^4 + (q;q^2)_\infty^4 = \frac{2(-q^2,-q^2,-q^2,-q^2;q^4)_\infty}{(q^2,q^2;q^4)_\infty}$$

which established (3.3).

Proof of (3.4): Dividing (3.2) by (3.3), and after little simplification, we get

$$\frac{(-q;q^2)_\infty^4 - (q;q^2)_\infty^4}{(-q;q^2)_\infty^4 + (q;q^2)_\infty^4} = \frac{8q}{(q^2,q^2,q^4,q^4,q^4,q^4,q^6,q^6;q^8)_\infty} \times \frac{(q^2,q^2;q^4)_\infty}{2(-q^2,-q^2,-q^2,-q^2;q^4)_\infty}$$

$$\frac{(-q;q^2)_\infty^4 - (q;q^2)_\infty^4}{(-q;q^2)_\infty^4 + (q;q^2)_\infty^4} = \frac{4q(q^2,q^2;q^4)_\infty}{(q^2,q^2,q^4,q^4,q^4,q^4,q^6,q^6;q^8)_\infty(-q^2,-q^2,-q^2,-q^2;q^4)_\infty}$$

which established (3.4).

Proof of (3.5): Employing (1.6) and (1.7) in (2.11), we have

$$(-q;q^2)_\infty^2(q^2;q^2)_\infty(-q;q^2)_\infty^2(q^2;q^2)_\infty(-q;q^2)_\infty^2(q^2;q^2)_\infty(-q;q^2)_\infty^2(q^2;q^2)_\infty -$$

Notes

$$-(q;q^2)^2_\infty(q^2;q^2)_\infty(q;q^2)^2_\infty(q^2;q^2)_\infty(q;q^2)^2_\infty(q^2;q^2)_\infty(q;q^2)^2_\infty(q^2;q^2)_\infty$$

$$= 16q\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty}\times\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\times\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\times\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

$$(q^2;q^2)^4_\infty[(-q;q^2)^8_\infty-(q;q^2)^8_\infty] = 16q\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty}\times\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\times\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\times\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}$$

using $(q^2;q^2)_\infty=(q^2;q^4)_\infty(q^4;q^4)_\infty$, and after simplification, we get

$$(q^2;q^4)^8_\infty[(-q;q^2)^8_\infty-(q;q^2)^8_\infty]=1$$

which established (3.5).

Proof of (3.6): Employing (1.6) and (1.7) in (2.12), we have

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\times\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}+\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\times\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}=2\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty}\times(-q^4;q^8)^2_\infty(q^8;q^8)_\infty$$

$$\frac{(q^2;q^2)^2_\infty}{(q;q^2)^2_\infty(-q;q^2)^2_\infty}[(-q;q^2)^2_\infty+(q;q^2)^2_\infty]=2\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty}\times(-q^4;q^8)^2_\infty(q^8;q^8)_\infty$$

$$\frac{(q^2;q^2)_\infty\times(q^2;q^2)_\infty}{(q;q^2)^2_\infty(-q;q^2)^2_\infty}[(-q;q^2)^2_\infty+(q;q^2)^2_\infty]=2\frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty}\times(-q^4;q^8)^2_\infty(q^8;q^8)_\infty$$

using $(q^2;q^2)_\infty=(q^2;q^4)_\infty(q^4;q^4)_\infty$ and $(q^2;q^2)_\infty=(q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty$, after simplification, we get

$$(q^2,q^4,q^6;q^8)_\infty[(-q;q^2)^2_\infty+(q;q^2)^2_\infty]=2\left[\frac{(q;q^2)_\infty(-q;q^2)_\infty(-q^4;q^8)_\infty}{(q^2;q^4)_\infty}\right]^2$$

which established (3.6).

Proof of (3.7): On multiplying (3.2) and (3.3), we have

$$[(-q;q^2)^4_\infty-(q;q^2)^4_\infty]\times[(-q;q^2)^4_\infty+(q;q^2)^4_\infty]$$

$$=\frac{8q}{(q^2,q^2,q^4,q^4,q^4,q^4,q^6,q^6;q^8)_\infty}\times\frac{2(-q^2,-q^2,-q^2,-q^2;q^4)_\infty}{(q^2,q^2;q^4)_\infty}$$

$$[(-q;q^2)^8_\infty-(q;q^2)^8_\infty]=\frac{16q(-q^2,-q^2,-q^2,-q^2;q^4)_\infty}{(q^2,q^2,q^4,q^4,q^4,q^4,q^6,q^6;q^8)_\infty}\times\frac{1}{(q^2;q^4)_\infty(q^2;q^4)_\infty}$$

using (3.5) in the left hand side, we get

$$\frac{1}{(q^2;q^4)^8_\infty}=\frac{16q(-q^2,-q^2,-q^2,-q^2;q^4)_\infty}{(q^2,q^2,q^4,q^4,q^4,q^4,q^6,q^6;q^8)_\infty}\times\frac{1}{(q^2;q^4)_\infty(q^2;q^4)_\infty}$$

after simplification, we get

$$\frac{1}{(q^2;q^4)^6_\infty}=\frac{16q(-q^2,-q^2,-q^2,-q^2;q^4)_\infty}{(q^2,q^2,q^4,q^4,q^4,q^4,q^6,q^6;q^8)_\infty}$$

which established (3.7).

Notes

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Notes

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- Line Spacing of 1 pt
- Large Images must be in One Column
- Numbering of First Main Headings (Heading 1) must be in Roman Letters, Capital Letter, and Font Size of 10.
- Numbering of Second Main Headings (Heading 2) must be in Alphabets, Italic, and Font Size of 10.

You can use your own standard format also.
Author Guidelines:

1. General,

2. Ethical Guidelines,

3. Submission of Manuscripts,

4. Manuscript’s Category,

5. Structure and Format of Manuscript,

6. After Acceptance.

1. GENERAL

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Scope

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Papers: These are reports of significant research (typically less than 7000 words equivalent, including tables, figures, references), and comprise:

(a)Title should be relevant and commensurate with the theme of the paper.

(b) A brief Summary, "Abstract" (less than 150 words) containing the major results and conclusions.

(c) Up to ten keywords, that precisely identifies the paper's subject, purpose, and focus.

(d) An Introduction, giving necessary background excluding subheadings; objectives must be clearly declared.

(e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition; sources of information must be given and numerical methods must be specified by reference, unless non-standard.

(f) Results should be presented concisely, by well-designed tables and/or figures; the same data may not be used in both; suitable statistical data should be given. All data must be obtained with attention to numerical detail in the planning stage. As reproduced design has been recognized to be important to experiments for a considerable time, the Editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned un-refereed;

(g) Discussion should cover the implications and consequences, not just recapitulating the results; conclusions should be summarizing.

(h) Brief Acknowledgements.

(i) References in the proper form.

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Acknowledgements: Please make these as concise as possible.

References

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22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

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28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

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Key points to remember:

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A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

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· Use paragraphs to split each significant point (excluding for the abstract)

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· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

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- Significant conclusions or questions that track from the research(es)

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- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

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- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
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- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

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- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.

- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

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Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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