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A New Efficient Approach to Analytical Solutions of Parabolic Equations

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1. E. Hesameddini, H. Latifizadeh, Reconstruction of Variational Iteration Algorithms using the Laplace Transform, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009)1377-1382.

A New Efficient Approach to Analytical Solutions of Parabolic Equations

Esmail Hesameddini [α] & Habibolla Latifizadeh [σ]

Abstract - In this work the new algorithm entitled Reconstruction of Variational Iteration Method (RVIM) is used as an efficient technique in finding the approximate solutions of the linear and nonlinear equations using only few terms of its own iteration. The algorithm can help overcome the difficulty arising in calculating nonlinear intricate terms. Reconstruction of Variational Iteration Method (RVIM) is independent of any small parameters. Besides, it provides us with a simple way to ensure the convergence of series solution, so that we can always get accurate enough approximations. All of these verify the great potential and validity of the RVIM technique in comparison with those of Variational Iteration Method (VIM) for strongly nonlinear problems in science and engineering.

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I. Introduction

It is very difficult to solve nonlinear problems, either numerically or theoretically, and even more difficult to establish a real model for nonlinear problems. Much assumption has to be made artificially or unnecessarily to make the practical engineering problems solvable, leading to loss of most important information. we propose a new kind of analytical method for nonlinear problems called the reconstruction of variational iteration method, in which is pointing out of similarities and differences with VIM and Homotopy Perturbation Method (HPM), not requiring small parameter in an equation as the perturbation techniques conduct and don't use the Lagrange multiplier. It has been shown the capability of this method to solve effectively, easily, and accurately, a large class of nonlinear problems with approximations converging rapidly to accurate solutions.

The method that has been used gives rapidly convergent successive approximations. As stated before, we aim to obtain analytical solutions to problems. We also aim to confirm that the reconstruction of variational iteration method is powerful, efficient, and promising in handling scientific and engineering problems. The RVIM technique is independent of any small parameters in general. In addition, it provides us with a simple way to ensure the convergence of series solution; therefore we can always get accurate enough approximations. The RVIM technique has been successfully applied to many nonlinear problems in science and engineering. All of these facts verifying the great potential and validity of the RVIM technique for strongly nonlinear problems in science and engineering.

II. Description of the New Method

To clarify the basic ideas of our proposed method in [1], we consider the following differential equation the same as VIM based on Lagrange multiplier [2-5]:

$$Lu(x_1,\cdots,x_k)+Nu(x_1,\cdots,x_k)=f(x_1,\cdots,x_k), \tag{1}$$

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Subject to:

$$Lu(x_1,\cdots,x_k)=\sum_{i=0}^{k}L_{x_i}u(x_i), \tag{2}$$

where L is a linear operator, N is a nonlinear operator and $f(x_1,\cdots,x_k)$ is an inhomogeneous term. we can rewrite equation (1) down a correction functional as follows:

$$L_{x_j}u(x_j)=\underbrace{f(x_1,\cdots,x_{kn})-Nu(x_1,\cdots,x_k)-\sum_{\substack{i=0\\ i\neq j}}^{k}L_{x_i}u(x_i)}_{h((x_1,\cdots,x_k),u(x_1,\cdots,x_k))}. \tag{3}$$

Notes

Therefore:

$$L_{x_j}u(x_j)=h\big((x_1,\cdots,x_k),u(x_1,\cdots,x_k)\big), \tag{4}$$

with artificial initial conditions being zero regarding the independent variable x_j .

By taking Laplace transform to both sides of the equation (4) in the usual way and using the artificial initial conditions, the result is as follows:

$$P(s).U(x_1,\cdots,x_{i-1},s,x_{i+1},x_k)=H\big((x_1,\cdots,x_{i-1},s,x_{i+1},x_k),u\big), \tag{5}$$

where P(s) is a polynomial with the degree of the highest derivative in the equation (5), (the same as the highest order of the linear operator L_{x_j}). The following relations are possible;

$$\mathcal{L}[h]=H, \tag{6-a}$$

$$B(s)=\frac{1}{P(s)}, \tag{6-b}$$

$$\mathcal{L}[b(x_i)]=B(s), \tag{6-c}$$

where in the equation (6-a) the function $H\big((x_1,\cdots,x_{i-1},s,x_{i+1},x_k),u\big)$ and

, $h\big((x_1,\cdots,x_{i-1},x_i,x_{i+1},x_k),u\big)$ have been abbreviated as H,h, respectively.

Hence, the equation (5) can be rewritten as:

$$U(x_1,\cdots,x_{i-1},s,x_{i+1},x_k)=H\big((x_1,\cdots,x_{i-1},s,x_{i+1},x_k),u\big).B(s). \tag{7}$$

Now, by applying the inverse Laplace transform to both sides of the equation (7) and using the (6-a) - (6-c), we obtain:

$$u(x_1,\cdots,x_{i-1},x_i,x_{i+1},x_k)=\int_0^{x_i}h\big((x_1,\cdots,x_{i-1},\tau,x_{i+1},x_k),u\big).b(x_i-\tau)d\tau. \tag{8}$$

Now, we must impose the actual initial conditions to obtain solution of the equation (1). Thus, we have the following iteration formulation:

$$u_{n+1}(x_1,\cdots,x_{i-1},x_i,x_{i+1},x_k)=u_0(x_1,\cdots,x_{i-1},x_i,x_{i+1},x_k)+ \tag{9}$$

$$+\int_0^{x_i}\{h\big((x_1,\cdots,x_{i-1},\tau,x_{i+1},x_k),u_n\big).b(x_i-\tau)\}d\tau,$$

Notes

where u_0 is an initial solution with or without unknown parameters. Assuming u_0 is the solution of, Lu, with initial/boundary conditions of the main problem. In case of no unknown parameters, u_0 should satisfy initial/ boundary conditions. When some unknown parameters are involved in u_0, the unknown parameters can be identified by initial/boundary conditions after few iterations, this technology is very effective in dealing with boundary problems. It is worth mentioning that, in fact, the Lagrange multiplier in the He's variational iteration method is $\lambda(\tau) = b(x_i - \tau)$ as shown in [1]. The initial values are usually used for selecting the zeroth approximation u_0. When u_0 is determined, then several approximations u_n $n > 0$, follow immediately. Consequently, the exact solution may be obtained by using

$$u(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, x_k) = lim_{n\to} \; u_n(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, x_k). \tag{10}$$

In what follows, we will apply the RVIM method to homogeneous/non-homogeneous parabolic partial differential equations to illustrate the strength of the method and to establish exact solutions for these problems.

In an algorithmic form, the new presented method can be expressed and implemented the solutions as follows:

Algorithm. Let k be the iteration index, set an appropriate value for the tolerance (Tol.) for numerical purposes.

Step 0: Choose an appropriate u_0 so that $L_{x_j}(u_0) = 0$,

Step 1: Set $k = 0$.

Step 2: Use the calculated values of u_n to compute u_{k+1} from Eq. (9).

Step 3: Define $u_n \coloneqq u_{k+1}$.

Step 4: If $\mathrm{Max}|u_k - u_{k-1}| < Tol$ stop, otherwise continue.

Step 5: Define $u_{k+1} \coloneqq u_k$.

Step 6: Set $n = n + 1$, and return to step **2**.

III. Application of the Proposed Method for the Parabolic Equations

Example1. Consider the following one dimensional, variable coefficient of the fourth-order parabolic partial differential equations [6, 7, and 8]

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4} = 0, \qquad \frac{1}{2} < x < 1, t > 0. \tag{11}$$

Subject to the initial conditions:

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 1 + \frac{x^5}{120}, \tag{12}$$

and the boundary conditions:

$$u\left(\tfrac{1}{2}, t\right) = \left(1 + \tfrac{(0.5)^5}{120}\right)\sin(t), \qquad u(1,t) = \left(\tfrac{121}{120}\right)\sin(t), \tag{13}$$

$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2}, t\right) = \frac{1}{6}\left(\frac{1}{2}\right)^3 \sin(t), \qquad \frac{\partial^2 u}{\partial x^2}(1,t) = \frac{1}{6}\sin(t).$$

By selecting auxiliary linear operator, the Eq. (11) is rewritten as:

$$L_t u(x,t) = \frac{\partial^2 u}{\partial t^2} = \overbrace{-\left(\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4}\right)}^{h(x,\xi,u)}. \tag{14}$$

Now by applying the Laplace transformation with respect to independent variable t on both sides of Eq. (11), using the artificial initial condition [1], solution of the governing equation (11) has aforementioned in Eq. (8),

$$u(x,t) = \int_0^t (t-\xi)\, h(x,\xi,u)\, d\xi. \tag{15}$$

Therefore, using the Eqs. (8), (9) one can obtain the following RVIM's iteration formula in the t-direction

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t (t-\xi)\, h(x,\xi,u_n)\, d\xi. \tag{16}$$

where, $h(x,\xi,u_n)$ is indicated as:

$$h(x,\xi,u_n) = -\left(\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u_n}{\partial x^4}(x,\xi)\right). \tag{17}$$

Now let's, start with an arbitrary initial approximation $u_0(x,t) = \left(1 + \frac{x^5}{120}\right)t$ that satisfies the initial condition. Using the RVIM iteration formula (16), we can have the following successive approximation:

$$u_1(x,t) = \left(1 + \frac{x^5}{120}\right)\left(t - \frac{t^3}{3!}\right), \tag{18}$$

$$u_2(x,t) = \left(1 + \frac{x^5}{120}\right)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!}\right), \tag{19}$$

$$u_3(x,t) = \left(1 + \frac{x^5}{120}\right)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right). \tag{20}$$

$$\vdots$$

Whereas the RVIM method admits the use of

$$u = lim_{n\to}\ u_n.$$

Which gives the exact solution:

$$u(x,t) = \left(1 + \frac{x^5}{120}\right)\sin(t). \tag{21}$$

Obtained upon using the Taylor expansion of $\sin(t)$.

Example 2. Consider the following parabolic equation [6, 7, and 8]

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{x}{\sin(x)} - 1\right)\frac{\partial^4 u}{\partial x^4} = 0, \qquad 0 < x < 1, t > 0, \tag{22}$$

subject to the initial conditions:

$$u(x,0) = x - \sin(x), \quad \frac{\partial u}{\partial t}(x,0) = -(x - \sin(x)), \tag{23}$$

Ref.

1. E. Hesameddini, H. Latifizadeh, Reconstruction of Variational Iteration Algorithms using the Laplace Transform, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009)1377-1382.

and the boundary conditions:

$$u(0,t)=0, \qquad u(1,t)=e^{-t}(1-\sin(1)), \tag{24}$$

$$\frac{\partial^2 u}{\partial x^2}(0,t)=0, \qquad \frac{\partial^2 u}{\partial x^2}(1,t)=e^{-t}\sin(1).$$

As in the previous example, to implement the RVIM technique, first of all we need to choose the auxiliary linear operator as:

$$L_t u(x,t)=\frac{\partial^2 u}{\partial t^2}=\overbrace{-\left(\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u}{\partial x^4}\right)}^{h(x,\xi,u)}.$$

Accordingly, after taking Laplace transform using the artificial initial condition as in [1], on both side of the Eq. (22), the following RVIM iteration formula in the *t*-direction can be obtained as:

$$u_{n+1}(x,t)=u_0(x,t)-\int_0^t (t-\xi)\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 u_n}{\partial x^4}(x,\xi)d\xi. \tag{25}$$

By the RVIM's recurrent formula in the Eq. (25), the terms of the sequence $\{u_n\}$ are constructed as follows, after choosing its initial approximate solution as $u_0(x,t)=(x-\sin(x))(1-t)$.

$$u_1(x,t)=(x-\sin(x))\left(1-t+\frac{t^2}{2!}\right), \tag{26}$$

$$u_2(x,t)=(x-\sin(x))\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}\right), \tag{27}$$

$$u_3(x,t)=(x-\sin(x))\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}\right). \tag{28}$$

The next terms of $\{u_n\}$ can be determined in a similar way and the n-th approximation of u can be constructed as:

$$u_n=(x-\sin(x))\sum_{i=0}^{n}\frac{t^i}{i!}. \tag{29}$$

And since

$$\lim_{n\to} u_n=(x-\sin(x))e^{-t}. \tag{30}$$

Therefore, approximate solution that is obtained by RVIM procedure converges to the exact solution.

Example3. Now, we solve the following one dimensional non-homogeneous fourth-order equation [6, 7, and 8]**:**

$$\frac{\partial^2 u}{\partial t^2}+(x+1)\frac{\partial^4 u}{\partial x^4}=\left(x^4+x^3-\frac{6}{7!}x^7\right)\cos(t), \qquad 0<x<1, t>0. \tag{31}$$

Subject to the initial conditions:

$$u(x,0)=\frac{6}{7!}x^7, \quad \frac{\partial u}{\partial t}(x,0)=0, \tag{32}$$

and the boundary conditions:

$$u(0,t)=0, \qquad u(1,t)=\frac{6}{7!}\cos(t), \tag{33}$$

$$\frac{\partial^2 u}{\partial x^2}(0,t)=0, \qquad \frac{\partial^2 u}{\partial x^2}(1,t)=\frac{1}{20}\cos(t).$$

Ref.

1. E. Hesameddini, H. Latifizadeh, Reconstruction of Variational Iteration Algorithms using the Laplace Transform, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009)1377-1382.

Applying RVIM to this equation with the given initial and boundary conditions, according to (3) and (4), we can have:

$$L_t u(x,t) = \frac{\partial^2 u}{\partial t^2} = \overbrace{\left(x^4 + x^3 - \frac{6}{7!}x^7\right)\cos(t) - (x+1)\frac{\partial^4 u}{\partial x^4}}^{h(x,\xi,u)}. \tag{34}$$

Now, by applying the Laplace Transform with respect to independent variable t on both sides of Eq. (31) using the artificial initial condition [1], therefore, using the Eqs. (8) and (9), one obtain the following RVIM's iteration formula in the t-direction:

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t (t-\xi)\, h(x,\xi,u_n)d\xi. \tag{35}$$

Where, $h(x,\xi,u_n)$ is indicated as:

$$h(x,\xi,u_n) = \left(x^4 + x^3 - \frac{6}{7!}x^7\right)\cos(t) - (x+1)\frac{\partial^4 u_n}{\partial x^4}(x,\xi). \tag{36}$$

With the aid of initial approximation $u_0(x,t) = \frac{6}{7!}x^7$ and using the RVIM iteration, we can obtain directly the rest of the other components as follows:

$$u_1(x,t) = x^3 + x^4 - \frac{t^2}{2!}(x^4 + x^3) - x^3\cos(t) - x^4\cos(t) + \frac{1}{7!}x^7\cos(t), \tag{37}$$

$$u_2(x,t) = 24 + 24x - 12t^2(x+1) - 24\cos(t) + \frac{1}{7!}x^7\cos(t) + t^4(1+x), \tag{38}$$

$$u_3(x,t) = \frac{1}{7!}x^7\cos(t). \tag{39}$$

It can be easily shown that $u_n = \frac{1}{7!}x^7\cos(t)$, $n = 4,5,\ldots$; thus, the solution of Eq. (31) is obtained which reads:

$$\lim_{n\to}\ u_n = \frac{1}{7!}x^7\cos(t). \tag{40}$$

That is the exact solution.

Example4. Consider the fourth-order parabolic equation in two space variables [9]

$$\frac{\partial^2 u}{\partial t^2} + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4} + 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4} = 0 \qquad \frac{1}{2} < x < 1, t > 0. \tag{41}$$

Subject to the initial conditions:

$$u(x,y,0) = 0, \quad \frac{\partial u}{\partial t}(x,y,0) = 2 + \frac{1}{6!}x^6 + \frac{y^6}{6!}, \tag{42}$$

and the boundary conditions:

$$u\left(\frac{1}{2}, y, t\right) = \left(2 + \frac{(0.5)^6}{6!} + \frac{y^6}{6!}\right)\sin(t), \qquad u(1,y,t) = \left(2 + \frac{1}{6!} + \frac{y^6}{6!}\right)\sin(t), \tag{43}$$

$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2}, y, t\right) = \left(\frac{(0.5)^4}{4!}\right)\sin(t), \qquad \frac{\partial^2 u}{\partial x^2}(1,y,t) = \frac{1}{4!}\sin(t),$$

$$\frac{\partial^2 u}{\partial x^2}\left(x, \frac{1}{2}, t\right) = \left(\frac{(0.5)^4}{4!}\right)\sin(t), \qquad \frac{\partial^2 u}{\partial x^2}(x,1,t) = \frac{1}{4!}\sin(t).$$

Ref.

1. E. Hesameddini, H. Latifizadeh, Reconstruction of Variational Iteration Algorithms using the Laplace Transform, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009)1377-1382.

Notes

To implement the RVIM method on this differential equation with the given initial and boundary conditions, according to (3) and (4), the auxiliary linear operator is selected as:

$$L_t u(x,t) = \frac{\partial^2 u}{\partial t^2} = \overbrace{-\left(2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4} + 2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4}\right)}^{h(x,\xi,u)}. \quad (44)$$

Therefore, as in the previous examples the RVIM iterative formula can be expressed as:

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t (t-\xi)\, h(x,\xi,u_n) d\xi. \quad (45)$$

So that $h(x,\xi,u_n)$, is indicated as:

$$h(x,\xi,u_n) = -\left(2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4}(x,\xi) + 2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4}(x,\xi)\right). \quad (46)$$

We start with an initial approximation $u_0(x,t) = t\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)$; by the iteration formula (45), one can obtain the first few components as follows:

$$u_1(x,t) = \left(t - \frac{t^3}{3!}\right)\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right), \quad (47)$$

$$u_2(x,t) = \left(t - \frac{t^3}{3!} + \frac{t^5}{5!}\right)\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right), \quad (48)$$

$$u_3(x,t) = \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right)\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right). \quad (49)$$

The rest of the other components are obtained using the iteration formula (45) and then the solution of $u(x,t)$ in closed form is given by:

$$u(x,y,t) = \lim_{n\to} u_n(x,y,t) = \left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\sin(t). \quad (50)$$

That follows immediately upon using the Taylor expansion for $\sin(t)$.

IV. Discussion

There are two main goals pursued in this work. The first was employing the powerful Reconstruction of the Variational Iteration to investigate fourth-order parabolic differential equations and the second one was showing the power of this method and its significant features. The two goals are achieved.

It is obvious that the method gives rapidly convergent successive approximations without any restrictive assumptions or transformation that may change the physical behavior of the problem. Reconstruction of the variational iteration gives several successive approximations through using the RVIM's iteration relation. Moreover, the RVIM reduces the size of calculations and the method is direct and straightforward. The RVIM uses the initial values for selecting the zeroth approximation, and boundary conditions, when given for bounded domains, and can be used for justification only.

For nonlinear equations that arise frequently in expressing nonlinear phenomena, RVIM facilitates the computations and gives the solution rapidly. As for concrete problems where an exact solution does not exist, a few approximations can be used for numerical purposes So by these advantages the RVIM method seems to be reliable and promising.

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Notes