Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 12 Issue 7 Version 1.0 June 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

New Representations in Terms of q-product Identities for Ramanujan's Results III

By M.P. Chaudhary

International Scientific Research and Welfare Organization, New Delhi, India

Abstract - In this paper author has established four q-product identities with the help of Jacobi's triple product identity using elementary method. These identities are new and not available in the literature of special functions.

Keywords : Generating functions, triple product identities.

AMS Subject Classifications: Primary 05A17, 05A15; Secondary 11P83.

NEW REPRESENTATIONS IN TERMS OF Q-PRODUCT IDENTITIES FOR RAMANUJANS RESULTS III

Strictly as per the compliance and regulations of :

OPEN ASSOCIATION OF RESEARCH SOCIETY, USA
RESEARCH | DIVERSITY | ETHICS

New Representations in Terms of q-product Identities for Ramanujan's Results III

M.P. Chaudhary

Abstract - In this paper author has established four q-product identities with the help of Jacobi's triple product identity using elementary method. These identities are new and not available in the literature of special functions.

Keywords : Generating functions, triple product identities.

I. Introduction

For $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) \tag{1.1}$$

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{(n-1)}) \tag{1.2}$$

$$(a_1,a_2,a_3,...,a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty...(a_k;q)_\infty \tag{1.3}$$

Ramanujan [2, p.1(1.2)]has defined general theta function, as

$$f(a,b) = \sum_{-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \quad ; \quad |ab| < 1, \tag{1.4}$$

Jacobi's triple product identity [3,p.35] is given, as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{1.5}$$

Special cases of Jacobi's triple products identity are given, as

$$\phi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty \tag{1.6}$$

$$(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.7}$$

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.8}$$

Equation (1.8) is known as Euler's pentagonal number theorem. Euler's another well known identity is as

$$(q;q^2)^{-1}_\infty = (-q;q)_\infty \tag{1.9}$$

Author : International Scientific Research and Welfare Organization, New Delhi, India. E-mail: mpchaudhary 2000@yahoo.com

Ref.

2. B.C. Berndt; What is a q-series?, preprint.
3. B.C. Berndt; Ramanujan's notebook Part III, Springer-Verlag, New York, 1991.

Throughout this paper we use the following representations

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty=(q^a,q^b,q^c\cdots q^t;q^n)_\infty \tag{1.10}$$

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty=(q^a,q^b,q^c\cdots q^t;q^n)_\infty \tag{1.11}$$

$$(-q^a;q^n)_\infty(-q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty=(-q^a,-q^b,q^c\cdots q^t;q^n)_\infty \tag{1.12}$$

Now we can have following q-products identities, as

$$(q^2;q^2)_\infty=\prod_{n=0}^{\infty}(1-q^{2n+2})$$

$$=\prod_{n=0}^{\infty}(1-q^{2(4n)+2})\times\prod_{n=0}^{\infty}(1-q^{2(4n+1)+2})\times$$

$$\times\prod_{n=0}^{\infty}(1-q^{2(4n+2)+2})\times\prod_{n=0}^{\infty}(1-q^{2(4n+3)+2})$$

$$=\prod_{n=0}^{\infty}(1-q^{8n+2})\times\prod_{n=0}^{\infty}(1-q^{8n+4})\times\prod_{n=0}^{\infty}(1-q^{8n+6})\times\prod_{n=0}^{\infty}(1-q^{8n+8})$$

or,

$$(q^2;q^2)_\infty=(q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty$$

$$=(q^2,q^4,q^6,q^8;q^8)_\infty \tag{1.13}$$

$$(q^4;q^4)_\infty=\prod_{n=0}^{\infty}(1-q^{4n+4})$$

$$=\prod_{n=0}^{\infty}(1-q^{4(3n)+4})\times\prod_{n=0}^{\infty}(1-q^{4(3n+1)+4})\times\prod_{n=0}^{\infty}(1-q^{4(3n+2)+4})$$

$$=\prod_{n=0}^{\infty}(1-q^{12n+4})\times\prod_{n=0}^{\infty}(1-q^{12n+8})\times\prod_{n=0}^{\infty}(1-q^{12n+12})$$

or,

$$(q^4;q^4)_\infty=(q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty$$

$$=(q^4,q^8,q^{12};q^{12})_\infty \tag{1.14}$$

$$(q^4;q^{12})_\infty=\prod_{n=0}^{\infty}(1-q^{12n+4})$$

$$=\prod_{n=0}^{\infty}(1-q^{12(5n)+4})\times\prod_{n=0}^{\infty}(1-q^{12(5n+1)+4})\times$$

$$\times\prod_{n=0}^{\infty}(1-q^{12(5n+2)+4})\times\prod_{n=0}^{\infty}(1-q^{12(5n+3)+4})\times\prod_{n=0}^{\infty}(1-q^{12(5n+4)+4})$$

$$=\prod_{n=0}^{\infty}(1-q^{60n+4})\times\prod_{n=0}^{\infty}(1-q^{60n+16})\times\prod_{n=0}^{\infty}(1-q^{60n+28})\times$$

$$\times\prod_{n=0}^{\infty}(1-q^{60n+40})\times\prod_{n=0}^{\infty}(1-q^{60n+52})$$

or,

Notes

$$(q^4;q^{12})_\infty = (q^4;q^{60})_\infty(q^{16};q^{60})_\infty(q^{28};q^{60})_\infty(q^{40};q^{60})_\infty(q^{52};q^{60})_\infty$$
$$= (q^4,q^{16},q^{28},q^{40},q^{52};q^{60})_\infty \quad (1.15)$$

Similarly we can compute following as

$$(q^5;q^5)_\infty = (q^5;q^{15})_\infty(q^{10};q^{15})_\infty(q^{15};q^{15})_\infty$$
$$= (q^5,q^{10},q^{15};q^{15})_\infty \quad (1.16)$$
$$(q^6;q^6)_\infty = (q^6;q^{24})_\infty(q^{12};q^{24})_\infty(q^{18};q^{24})_\infty(q^{24};q^{24})_\infty$$
$$= (q^6,q^{12},q^{18},q^{24};q^{24})_\infty \quad (1.17)$$
$$(q^6;q^{12})_\infty = (q^6;q^{60})_\infty(q^{18};q^{60})_\infty(q^{30};q^{60})_\infty(q^{42};q^{60})_\infty(q^{54};q^{60})_\infty$$
$$= (q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty \quad (1.18)$$

The outline of this paper is as follows. In sections 2, some recent results obtained by the author [1], and also some well known results are recorded in [6;7], those are useful to the rest of the paper. In section 3, we state and prove four q-product identities, which are new and not recorded in the literature of special functions.

II. Preliminaries

In [1], following identities are being established

$$(q;q^2)_\infty = (q,q^3,q^5;q^6)_\infty \quad (2.1)$$

$$\left[\frac{(-q;q^2)^8_\infty - (q;q^2)^8_\infty}{q}\right]^{\frac{1}{4}} = \frac{2}{\left[(q^2;q^4)_\infty\right]^2} \quad (2.2)$$

$$\frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty} = (q,-q;q^2)_\infty \quad (2.3)$$

$$(q^2;q^2)_\infty = (q^2;q^4)_\infty(q^4;q^4)_\infty \quad (2.4)$$

In Ramanujan's notebook [7, p. 209], following entry is recorded as

$$f(-q)f(-q^2) = \phi(-q)\boldsymbol{\psi}(q) \quad (2.5)$$

In Ramanujan's notebook [6, p. 245], following entry is recorded as

$$\frac{\boldsymbol{\psi}(q)\,\boldsymbol{\psi}(-q)}{\boldsymbol{\psi}(q^2)\,\boldsymbol{\psi}(-q^2)} = \frac{\boldsymbol{\psi}(-q^2)}{\boldsymbol{\psi}(q^4)} \quad (2.6)$$

In Ramanujan's notebook [6, p. 254], following entry is recorded as

$$[3\phi(-q^9) - \phi(-q)]^3 = 8\,\frac{\boldsymbol{\psi}^3(q)}{\boldsymbol{\psi}(q^3)}\phi(-q^3) \quad (2.7)$$

In Ramanujan's notebook [6, p. 222], following entry is recorded as

$$\frac{\frac{\phi^5(q)}{\phi(q^5)} + 4\frac{{}^5(q)}{(q^5)}}{\phi(q)\phi^3(q^5) + 4q^2\boldsymbol{\psi}(q)\boldsymbol{\psi}^3(q^5)} = 5\,\frac{\phi^2(q)}{\phi^2(q^5)} \quad (2.8)$$

Ref.

6. S. Ramanujan; Notebooks (Volume I), Tata Institute of Fundamental Research, Bombay, 1957.
7. S. Ramanujan; Notebooks (Volume II), Tata Institute of Fundamental Research, Bombay, 1957.

III. Main Results

In this section, we established following new results with the help of $\psi(.)$ and $\phi(.)$ functions or in more general language we can say that by using the properties of Jacobi's triple product identity, as $\psi(.)$ and $\phi(.)$ functions are its special cases. These results are not recorded in the literature of special functions

$$(q;q)_\infty = (q,q;q^2)_\infty \tag{3.1}$$

$$(q^4;q^4)_\infty = (-q^2,q^2;q^4)_\infty (q^8;q^8)_\infty \tag{3.2}$$

$$(q;q^2)^3_\infty + 2(q^3;q^6)_\infty = \frac{3(q;q^2)_\infty (q^9;q^9)_\infty (q^9;q^{18})_\infty}{(q^2;q^2)_\infty} \tag{3.3}$$

$$\frac{5(q^2;q^2)^2_\infty}{(q^5;q^{10})^3_\infty} = \frac{(-q;q^2)^5_\infty (q^2;q^2)^5_\infty (q^{20};q^{20})_\infty + 4(-q^5;q^{10})_\infty (q^{10};q^{10})_\infty (q^4;q^5)^5_\infty}{(-q;q^2)_\infty (q^2;q^2)_\infty (-q^5;q^{10})^3_\infty (q^{10};q^{10})^3_\infty + 4q^2 (q^4;q^4)_\infty (q^{20};q^{20})^3_\infty} \tag{3.4}$$

Proof of (3.1): By substituting, $q = q^2$ in (1.8), we have also by substituting, $q = -q$ in (1.6), we get

$$f(-q^2) = (q^2;q^2)_\infty$$

$$\phi(-q) = (q;q^2)^2_\infty (q^2;q^2)_\infty$$

by substituting the values of $f(-q^2), \phi(-q)$, employing (1.7) and (1.8) in (2.5), we get

$$(q;q)_\infty (q^2;q^2)_\infty = (q;q^2)^2_\infty (q^2;q^2)_\infty \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

after simplification, we get (3.1).

Proof of (3.2): By substituting, $q = -q, q = -q^2, q = q^2$ and $q = q^4$ respectively in (1.7), we get

$$\psi(-q) = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \ ; \ \psi(-q^2) = \frac{(q^4;q^4)_\infty}{(-q^2;q^4)_\infty}$$

and

$$\psi(q^2) = \frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty} \ ; \ \psi(q^4) = \frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty}$$

by substituting the values of $\psi(-q), \psi(-q^2), \psi(q^2), \psi(q^4)$ and employing (1.7) in (2.6), and further using (2.3), we get

$$\frac{(q^2;q^2)_\infty (q^8;q^8)_\infty}{(q^4;q^8)_\infty} = \frac{(q^4;q^4)_\infty (q^4;q^4)_\infty}{(-q^2;q^4)_\infty (-q^2;q^4)_\infty (q^2;q^4)_\infty}$$

further using (2.4) in above expression, and after simplification, we get (3.2).

Proof of (3.3): By rearranging (2.7) can be written as

$$\frac{[3\phi(-q^9) - \phi(-q)]^3}{[2\psi(q)]^3} = \frac{\phi(-q^3)}{\psi(q^3)} \tag{3.3.1}$$

by substituting $q = -q^3$ in (1.6) and $q = q^3$ in (1.7) respectively, we get

$$\phi(-q^3) = (q^3;q^6)^2_\infty (q^6;q^6)_\infty \ ; \psi(q^3) = \frac{(q^6;q^6)_\infty}{(q^3;q^6)_\infty}$$

Notes

Global Journal of Science Frontier Research (F) Volume XII Issue VII Version I June 2012

now substituting the values of $\phi(-q^3)$ and $\boldsymbol{\psi}(q^3)$ in (3.3.1), and after simplification, we get

$$\frac{[3\phi(-q^9)-\phi(-q)]}{[2\boldsymbol{\psi}(q)]}=(q^3;q^6)_\infty \tag{3.3.2}$$

by substituting $q=-q^9$ and $q=-q$ respectively in (1.6), we get

$$\phi(-q^9)=(q^9;q^{18})^2_\infty(q^{18};q^{18})_\infty\ ;\ \phi(-q)=(q;q^2)^2_\infty(q^2;q^2)_\infty$$

Notes

now substituting the values of $\phi(-q^9)$, $\phi(-q)$ and employing (1.7) in (3.3.2), after little algebra, we get

$$3(q^9;q^{18})^2_\infty(q^{18};q^{18})_\infty=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}[2(q^3;q^6)_\infty+(q;q^2)^3_\infty] \tag{3.3.3}$$

by substituting $q=q^{\frac{9}{2}}$ in (2.4), we get

$$(q^9;q^9)_\infty=(q^9;q^{18})_\infty(q^{18};q^{18})_\infty$$

by using the values of $(q^9;q^9)_\infty$ in (3.3.3), and after simplification, we get (3.3).

Proof of (3.4): By substituting $q=q^5$ in (1.6) and (1.7) respectively, we get

$$\phi(q^5)=(-q^5;q^{10})^2_\infty(q^{10};q^{10})_\infty\ ;\ \boldsymbol{\psi}(q^5)=\frac{(q^{10};q^{10})_\infty}{(q^5;q^{10})_\infty}$$

with values of $\phi(q^5),\boldsymbol{\psi}(q^5)$, and further employing (1.6) and (1.7), we can compute following identities

$$\frac{\phi^5(q)}{\phi(q^5)}+4\frac{\boldsymbol{\psi}^5(q)}{\boldsymbol{\psi}(q^5)}=\frac{(q^2;q^2)^5_\infty}{(q^{10};q^{10})_\infty}\left[\frac{(-q;q^2)^{10}_\infty}{(-q^5;q^{10})^2_\infty}+4\frac{(q^5;q^{10})_\infty}{(q;q^2)^5_\infty}\right] \tag{3.4.1}$$

$$\phi(q)\phi^3(q^5)+4q^2\boldsymbol{\psi}(q)\boldsymbol{\psi}^3(q^5)=(q^2;q^2)_\infty(q^{10};q^{10})^3_\infty\times$$
$$\times\left[(-q;q^2)^2_\infty(-q^5;q^{10})^6_\infty+\frac{4q^2}{(q;q^2)_\infty(q^5;q^{10})^3_\infty}\right] \tag{3.4.2}$$

$$5\frac{\phi^2(q)}{\phi^2(q^5)}=5\frac{(-q;q^2)^4_\infty}{(-q^5;q^{10})^4_\infty} \tag{3.4.3}$$

by substituting the values from (3.4.1), (3.4.2) and (3.4.3) in (2.8), further using the property (2.3), after simplification, we get (3.4).

References Références Referencias

1. M.P. Chaudhary; Development on q-product identities, preprint.
2. B.C. Berndt; What is a q-series?, preprint.
3. B.C. Berndt; Ramanujan's notebook Part III, Springer-Verlag, New York, 1991.
4. B.C. Berndt; Ramanujan's notebook Part V, Springer-Verlag, New York, 1998.
5. G.E. Andrews, R. Askey and R. Roy; Special Functions, Cambridge University Press, Cambridge, 1999.
6. S. Ramanujan; Notebooks (Volume I), Tata Institute of Fundamental Research, Bombay, 1957.
7. S. Ramanujan; Notebooks (Volume II), Tata Institute of Fundamental Research, Bombay, 1957.