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Characterization of Partial Lattices on Countable Boolean Lattice

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Characterization of Partial Lattices on Countable Boolean Lattice

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Abstract - In this paper new concepts countable join property, countable meet property, P_σ–lattice and P_δ –lattice are introduced. We established that P_σ–lattice and P_δ –lattice are measureable partial lattices and characterized partial lattices of a lattice through countable join and meet properties. We also established some interesting result on the injective property of the lattice measurable functions defined over countable Boolean lattices.

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I. Introduction

The origin of a lattice concept can be traced back to Boole's analysis of thought and Dedekind's study of divisibility, Schroder and Pierce contributed substantially to this area. Though some of the work in this direction was done around 1930, much momentum was gained in 1967 with the contributions of Birkhoff's [2]. In 1963, Gabor szasz [9] introduced the generalization of the lattice measure concepts. To study σ-additive set functions on a lattice of sets, Gena A. DE Both [3] introduced σ - lattice in 1973. The concept of partial lattices was introduced by George Gratzer [5] in 1978. In 2000, Pao - Sheng Hus [8] characterized outer measures associated with lattice measure. The Hann decomposition theorem of a signed lattice measure by Jun Tanaka [10] defined a signed lattice measure on a lattice σ - algebras and the concept of sigma algebras are extensively studied by [4]. D.V.S.R. Anil Kumar etal [1] introduce the concept of measurable Borel lattices, σ - lattice and δ –lattice to characterize a class of Measurable Borel Lattices. This paper is organized as follows. Section 2 presents the preliminaries definitions and results. In Section 3 we proved that

P_σ – lattice and P_δ –lattice are measurable partial lattices and all partial lattices of a lattice satisfy both countable join and meet properties. Some interesting result on the injective property of the lattice measurable functions defined over countable Boolean lattices are established in Section 4.

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2. Birkhoff. G, Lattice Theory 3rd ed., AMS Colloquim Publications, Providence, RI, 1967.

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II. Preliminaries

Consider a lattice $(L,\wedge,\vee)$ with the operations meet $\wedge$ and join $\vee$ and usual ordering $\leq$, where L is a collection of subset of a non empty set X. Now this lattice $(L,\wedge,\vee)$ is denoted by L and satisfy the commutative law, the associative law and the absorption law. A lattice L is called distributive if the distributive law is satisfied. The zero and one elements of the lattice L are denoted by 0 and 1 respectively. A distributive lattice L is called a Boolean lattice if for any element x in L, there exists a unique complement x^c such that $x \vee x^c = 1$ and $x \wedge x^c = 0$. An operator $C: L \to L$, where L is a lattice is called a lattice complement in L if the law of complementation, the law of contra positive and the law of double negation are satisfied. The following are very important examples of Boolean lattice.

Example2.1. Let $(\{0,1\},\leq)$ be the set consisting of the two elements 0,1 equipped with the usual order relation $0 \leq 1$.This poset is a Boolean lattice with respect to the operations presented in the tables below (at the left the lattice operations and at the right the complementation):

a	b	$a\wedge b$	$a\vee b$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

x	x^c
0	1
1	0

This is usually known as the two valued or two elements Boolean lattice, denoted by $B=(\{0,1\},\vee,\wedge,{}^c,0,1)$.

Example2.2. The power set P(X) of a universe X a Boolean lattice if we choose the set theoretic complement $A^c = X\setminus A:=\{x\in X: x\in X \text{ and } x\notin A\}$ as the complement of a given set A in the universe X. Such a Boolean lattice is $P = (P(X), \vee, \wedge, {}^c, \phi, X)$.

Example2.3. $E = (2^X ,\vee, \wedge,{}^c\ 0, 1)$ is the collection 2^X of all two valued functional on the universe X is a Boolean lattice if we choose the functional $\chi^c = 1\text{-}\chi$ as the complement of a given functional χ.

Example2.4. Let $(D, \vee, \wedge, {}^c,1,70)$ is a Boolean lattice where $D=\{1,2,5,7,10,14,35,70\}$ is the set of all divisors of 70, $x \wedge y$ = Greatest Common Devisor of x and y, $x \vee y$ = Least Common Multiple of x and y and $x^c = \frac{70}{x}$.

Definition2.1. A Boolean lattice L is called a countable Boolean lattice if L is closed under countable join and is denoted by σ (L).

Example2.5. {empty setϕ, X}, Power set of X, Let $X = \Re$, L = {measurable subsets of $\Re$} with usual ordering ($\leq$) are all countable Boolean lattice.

Definitition2.2. The entire set X together with countable Boolean lattice is called lattice measurable space and is denoted by the ordered pair $(X, \sigma (L))$.

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Example2.6. $X = \Re$, where $\Re$ is extended real number system and L = {All Lebesgue measurable sub sets of $\Re$ },($\Re$,σ (L)) is a lattice measurable space.

Definitition2.3. If μ: σ (L) → R ∪ {∞} satisfies the following properties (i) μ(ϕ) = μ (0) = 0 (ii) for all h, g ∈ σ (L), such that μ (h), μ (g) ≥ 0; h ≤ g ⇒ μ (h) ≤ μ (g) (iii) for all h, g ∈ σ (L): μ (h ∨ g) + μ (h ∧ g) = μ (h) + μ (g) (iv) If $h_n \in \sigma(L)$, n ∈ N such that $h_1 \leq h_2 \leq \ldots \leq h_n \leq \ldots$, then $\mu(\vee_{n=1}^{\infty} h_n) = \lim \mu(h_n)$ then μ is called a lattice measure on the countable Boolean lattice σ (L).

The following is definition given in [5]

Definition2.4. Let σ (L) be a countable Boolean lattice, H ⊆ σ (L), and restrict ∧ and ∨ to H as follows. For a, b, c ∈ H, if a ∧ b = c (dually, a ∨ b = c), then we say that in H, a ∧ b(dually a ∨ b) is defined and it equals c, if, for a, b ∈ H, a ∧ b ∉ H(dually a ∨ b ∉ H), then we say that a ∧ b(dually a ∨ b) is not defined in H. Thus (H, ∧,∨) is a set with two binary partial operations. (H, ∧,∨) is called a partial lattice, a relative sublattice of σ (L).

Observation2.1. Every subset of a countable Boolean lattice determines a partial lattice. Every sublattice of σ (L) is a partial lattice and the converse need not be true.

Definition2.5.[7] A set A is said to be measurable partial lattice, if A is in σ (L).

Example2.7. ($\Re$, σ (L)) be lattice measurable space. Then the interval (a, ∞) is a measurable partial lattice under usual ordering.

Example2.8. [0, 1) < $\Re$ is a measurable partial lattice under usual ordering.

Definition2.9. A P_σ - lattice is a poset for which sup exist for any countable collection of its partial lattices.

Example2.9. $R = \vee_{n=1}^{\infty} (-n, n)$ is a p_σ -lattice.

Definition2.10. A P_δ - lattice is a poset for which inf exist for any countable collection of its partial lattices.

Example2.11. (i) $\wedge_{n=1}^{\infty} (-n, n) = (-1, 1)$ and (ii) $\wedge_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ are P_δ – lattices.

Definition2.11.Countable join property (CJP): If $\{E_k\}$ is monotonic increasing sequence of partial lattices of a lattice L and $E = \vee_{k=1}^{\infty} E_k$. Then $\mu(E) = \underset{n\to\infty}{Lt}\, \mu(E_n)$.

Definition2.12.Countable meet property (CMP): If $\{E_k\}$ is a monotonic decreasing sequence of partial lattices of a lattice L and $E = \wedge_{k=1}^{\infty} E_k$. Then $\mu(E) = \underset{n\to\infty}{Lt}\, m(E_n)$.

Result2.1.[1]. If E is measurable lattice so is E^c.

III. P_σ-Lattice and P_δ -Lattice

Theorem3.1. Every P_σ - lattice is lattice measurable.

Proof. Let $E_1, E_2, \ldots\ldots$ are pair wise disjoint measurable partial lattices and $E = \vee_{k=1}^{\infty} E_k$,

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5. Gratzer. G, General Lattice Theory, Academic Press Inc., 1978.

Evidently,

$$\mu\,(\mathop{\vee}_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty}\mu(E_k) \qquad (1)$$

and

$$\mu\,(\mathop{\vee}_{k=1}^{\infty} E_k) \geq \mu(\mathop{\vee}_{k=1}^{n} E_k) \qquad (2)$$

From definition 2.3. We have $\mu\,(E_1 \vee E_2) = \mu\,(E_1) + \mu\,(E_2)$. By the principle of mathematical induction on number of pair wise disjoint measurable partial lattices, n, we have $\mu\,(\mathop{\vee}_{k=1}^{n} E_k) = \sum_{k=1}^{n}\mu(E_k)$. As $n \to \infty$, from (2) it follows that

$$\mu\,(\mathop{\vee}_{k=1}^{\infty} E_k) \geq \sum_{k=1}^{\infty}\mu(E_k) \qquad (3)$$

From (1) and (3), we have $\mu\,(\mathop{\vee}_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty}\mu(E_k)$. Now $E = \mathop{\vee}_{k=1}^{\infty} E_k = E_1 \vee (E_2 \wedge E_1^c) \vee \ldots \vee (E_k \wedge (\mathop{\vee}_{k=1}^{n-1} E_k^c) \vee \ldots$. Since $E_1, E_2 \wedge E_1^c, \ldots$ are disjoint measurable partial lattices, we have, $\mathop{\vee}_{k=1}^{\infty} E_k$ is a measurable partial lattice. Hence every P_σ - lattice is a lattice measurable.

Theorem3.2. Every P_σ - lattice satisfies CJP.

Proof. Suppose that $\{E_k\}$ is monotonic increasing sequence of partial lattices of a σ (L) and $E = \mathop{\vee}_{k=1}^{\infty} E_k$. Write $E = E_1 \vee (E_2 \wedge E_1^c) \vee \ldots \vee (E_k \wedge (\mathop{\vee}_{k=1}^{n-1} E_k^c) \vee \ldots$

So we have $E = E_1 \vee (\mathop{\vee}_{k=1}^{\infty}(E_{k+1} \wedge E_k^c)$ (a disjoint joint). By Theorem 3.1.

Now, $\mu\,(E) = \mu\,(E_1) + \sum_{k=1}^{\infty}\mu(E_{k+1} - E_k) = \mu\,(E_1) + \mathop{Lt}_{n\to\infty} \sum_{k=1}^{n} [\mu(E_{k+1}) - \mu(E_k)] = \mu\,(E_1) + \mathop{Lt}_{n\to\infty} [\mu(E_2) - \mu(E_1) + \ldots\ldots + \mu(E_n) - \mu(E_{n-1})] = \mu\,(E_1) + \mathop{Lt}_{n\to\infty} [-\mu(E_1) + \mu(E_n)] = \mu\,(E_1) - \mu\,(E_1) + \mathop{Lt}_{n\to\infty} \mu(E_n) = \mathop{Lt}_{n\to\infty} \mu(E_n)$.

Theorem3.3. Every P_δ -lattice is lattice measurable.

Proof. Let $E_1, E_2, \ldots$ are measurable partial lattices.

By theorem 3.1. $E = \mathop{\vee}_{k=1}^{\infty} E_k$ is a measurable partial lattice. Let $G = \mathop{\wedge}_{k=1}^{\infty} E_k$.

Then $G^c = (\mathop{\wedge}_{k=1}^{\infty} E_k)^c = \mathop{\vee}_{k=1}^{\infty} E_k^c$. Given that each E_k is a measurable partial lattice.

Hence by Result 2.1., each E_k^c is a measurable partial lattice. Which implies $\mathop{\vee}_{k=1}^{\infty} E_k^c$ is a measurable partial lattice (Every P_σ - lattice is a measurable partial lattice). This leads to G^c is measurable partial lattice. Hence G is measurable partial lattice (By Result 2.1.).

Notes

Theorem 3.4. Every P_δ -lattice satisfies CMP**.**

Proof. Suppose that $\{ E_k \}$ is a monotonic decreasing sequence of partial lattices of σ (L) and $E = \bigwedge_{k=1}^{\infty} E_k$. Let $E = \bigwedge_{k=1}^{\infty} E_k$. Evidently $E_1 = E \vee (E_1 \wedge E_2^c) \vee (E_2 \wedge E_3^c) \vee \ldots$

Then $\mu(E_1) = \mu(E) + \sum_{k=1}^{\infty} \mu(E_k) - \mu(E_{k+1}) = \mu(E) + \underset{n\to\infty}{\mathrm{Lt}} \sum_{k=1}^{n} \mu(E_k) - \mu(E_{k+1})$

$= \mu(E) + \underset{n\to\infty}{\mathrm{Lt}} [\mu(E_1) - \mu(E_2) + \ldots\ldots\ldots + \mu(E_n) - \mu(E_{n+1})] = \mu(E) + \underset{n\to\infty}{\mathrm{Lt}} [\mu(E_1) - \mu(E_{n+1})]$

$= \mu(E) + \mu(E_1) - \underset{n\to\infty}{\mathrm{Lt}} \mu(E_{n+1})$. Which implies $\mu(E) = \underset{n\to\infty}{\mathrm{Lt}} \mu(E_n)$.

IV. The Injective and Projective Properties of Lattice Measurable Functions

Definition4.1. An extended real value function f defined on a lattice measurable E is said to be lattice measurable function if the set $\{x \in E / f(x) > \alpha\}$ is lattice measurable for all real numbers α.

Example4.1. Constant functions, Continuous functions and Characteristic functions are lattice measurable functions.

Result4.1. If f and g are lattice measurable functions then $f \vee g$ and $f \wedge g$ are also lattice measurable functions.

Proof. For any real number α we have $\{x \in L / (f \vee g)(x) > \alpha \} = \{ x \in L / f(x) > \alpha \} \vee \{ x \in L / g(x) > \alpha \}$ and $\{x \in L / (f \wedge g)(x) > \alpha \} = \{ x \in L / f(x) > \alpha \} \wedge \{ x \in L / g(x) > \alpha \}$. Since $\{ x \in L / f(x) > \alpha \}$ and $\{ x \in L / g(x) > \alpha \}$ are lattice measurable sets implies the sets of RHS are lattice measurable implies $f \vee g$ and $f \wedge g$ are lattice measurable functions.

The following interesting property can easily be verified from the works of [6] by considering lattice measurable functions f and g defined over countable Boolean lattice.

Property4.1. A Countable Boolean lattice A is a Retrace of Countable Boolean lattice B if there exist homomorphism g: $A \to B$ and f: $B \to A$ such that fg is the identity on A. Here g and f are necessarily a monomorphism (injection) and epimorphism(projective) respectively. That is A Countable Boolean lattice is Retrace injective if it is a Retrace of every Countable Boolean lattice that contains it

V. Conclusion

New concepts like countable join property, countable meet property, P_σ–lattice and P_δ–lattice are introduced. Characterized partial lattices of a lattice through countable join and meet properties and proved that P_σ–lattice and P_δ–lattice are measureable partial lattices. Interesting result on the injective property of the lattice measurable functions defined over Countable Boolean lattices are established.

VI. Acknowledgements

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