Global Journal of Science Frontier Research
Physics and Space Science
Volume 13 Issue 8 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

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GJSFR-A Classification : FOR Code: 240000

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Klein-Gordon Equation for a Particle in Brane Model

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I. Introduction

The Klein-Gordon equation describing motion of a scalar particle is known in quantum field theory that does not account the changes of space metrics and changes of particles behavior connected with it [1]. These changes can be accounted by Einstein's equation. Wheeler - deWitt equation occupies the place of Einstein's equation in quantum theory [2]. The approach of Wheeler - de Witt is applied to brane theory of Universe [3], [4] in papers [5, 6].

In present paper, we will derive starting from the symmetry properties of the brane [7, 8, 9] the equation of motion for a particle in the framework of brane model with the account of its radius variation in universal space. This equation has a form of Klein-Gordon equation in curved space [10] accounting the field of brane fluctuations and describes particle temporal behavior with Einstein's or time dependent Wheeler - de Witt equation [11].

II. Energy Conservation Law

Let's consider our space as four dimensional hyper-surface that is the insertion in the space of higher dimension (Fig. 1). Then interval for a moving particle in normal Gauss coordinates can be written as

$$ds = \sqrt{g_{ij}dx^i dx^j - c^2 dt^2}, \tag{2.1}$$

where g_{ij} is metric tensor dx^i, dx^j are differentials of coordinates (i, j = 0,1,2,3) on brane, dt is differential of universal time that is proportional to extra dimensional coordinate. Greek symbols will denote all indexes (a = 0,1,2,3,4). Then action can be written as

$$S = mc\int_0^T ds = \int_0^T L dt, \tag{2.2}$$

where m is mass of particle, c is speed of light,

$$L = \sqrt{g_{ij}(m\dot{x}^i)(m\dot{x}^j) - m^2c^2}, \tag{2.3}$$

is Lagrangian, T is current value of universal time in multidimensional space (proportional to the brane radius). Let's introduce the symmetry of configuration space as single parametric transformation group $f(q,\varepsilon)$:

$$t \to t+\varepsilon, x_i \to x_i(t+\varepsilon), \tag{2.4}$$

$$f(q_i,\varepsilon) = x_i(t+\varepsilon), f(q_i,0) = x_i(t) \tag{2.5}$$

conserving Lagrangian (2.3). According to Netter's theorem, first integral

$$I = \frac{\partial L}{\partial \dot{x}_i}h^i, \tag{2.6}$$

where

$$h^i = \frac{\partial f^i}{\partial \varepsilon_{|\varepsilon=0}}, \tag{2.7}$$

can be put in correspondence to each symmetry. Then

$$I = \frac{1}{L}g_{ij}m^2\dot{x}^j\{\frac{\partial x_i(t+\varepsilon)}{\partial(t+\varepsilon)}\frac{\partial(t+\varepsilon)}{\partial\varepsilon}\}_{|\varepsilon=0}, \tag{2.8}$$

or

$$g_{ij}m\dot{x}^i m\dot{x}^j = const, \tag{2.9}$$

If the particle moves uniformly and rectilinearly on the background of Lorenz's metrics than we can choose reference system where $\dot{x}^{(1)} = \dot{x}^{(2)} = \dot{x}^{(3)} = 0$ and $\dot{x}^{(0)} = c$ when brane is expanding with velocity c. Then Eq. (2.9) yields

$$g_{ij}m\dot{x}^i m\dot{x}^j = m^2c^2. \tag{2.10}$$

The same equation can be derived in the framework of quantum mechanical treatment. The wave function of particle in quasi-classical approximation is

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$$\Psi = ae^{\frac{iS}{\hbar}}, \tag{2.11}$$

where a is slowly varying amplitude, S is action expressed by formulas (2.2), (2.3). Let's differentiate both sides of expression (2.11) by T neglecting the dependence of amplitude on time

$$\frac{d\Psi}{dt} = a\frac{i}{\hbar}e^{\frac{iS}{\hbar}}\frac{dS}{dT} = i\frac{c}{\hbar}\sqrt{g_{ij}p^i p^j - m^2c^2\Psi}, \tag{2.12}$$

If evolution of particle in brane does not depend on brane radius then $\frac{d\Psi}{dT} = \frac{dS}{dT} = 0$ and

$$g_{ij}p^i p^j = m^2c^2. \tag{2.13}$$

III. Klein-Gordon Equation

Expression (2.13) can be rewritten in the following form:

$$p_i p^j = m^2c^2. \tag{3.1}$$

Let's consider functional variation [1] of relation (3.1) in the vicinity of x. Complete variation of momentum vector can be written as the sum of functional variation δp of vector p at the comparison of $p\prime$ of with $p\prime\prime$ in the vicinity of p at the parallel transfer of momentum vector in universal space and ordinary variation $d\,p$. Then, it can be written that

$$\triangle p = p'(x') - p(x'') = p'(x') - \tilde{p}(x') + \tilde{p}(x') - p(x'') = \delta p + dp, \tag{3.2}$$

Where

$$\delta p = p'(x') - \tilde{p}(x') \tag{3.3}$$

And

$$dp = \tilde{p}(x') - p(x''), \tag{3.4}$$

$\tilde{p}(x')$ is momentum vector at its parallel transfer in the universal space from point $x'' = x - \delta x$ to point $x' = x + \delta x$. If trajectory of particle is geodetic one then according [1].

$$dp_i = \frac{\partial p_i}{\partial x^k}dx^k = 0, \tag{3.5}$$

$$\delta p_i = \tilde{p}_k \Gamma^k_{i\alpha}\delta x^\alpha, \tag{3.6}$$

where $\delta x^\alpha = \frac{1}{2}(x'^\alpha - x''^\alpha)$. Rome indexes numerate here coordinates of usual four-coordinate space and Greek indexes numerate coordinates of universal five-coordinate space. It was assumed at formulation of (3.6) that $\Gamma^4_{i\alpha} = 0$.

Then, it can be written, omitting stroked index of momentum vector,

$$p(x') = p(x) + \frac{1}{2}\delta p. \tag{3.7}$$

At the transform $x \to x'$, relation (3.1) is transforming accounting (3.7) to the following form:

$$p_i p^i + \frac{1}{2}(p_i\delta p^i + \delta p_i p^i) + \frac{1}{4}\delta p_i \delta p^i = m^2c^2. \tag{3.8}$$

Let's pass in relation (3.8) to operators acting in Hilbert space of wave functions $\psi(x)$. We represent for this sake the components of vector p as

$$p_i = -i\hbar\frac{\partial}{\partial x^i}, \tag{3.9}$$

and rewrite relation (3.6) as

$$\delta p_i = -i\hbar\left\{\Gamma^k_{i\alpha}\delta x^\alpha\right\}_{;k}, \tag{3.10}$$

assuming that $\tilde{p}_k$ is a covariant derivative because of brane curvature.

Let's consider the first term in the left side of equation (3.8). For this purpose, we represent it in the form

$$p_i p^i = p_i g^{ij} p_j. \tag{3.11}$$

Using expression (3.9), we get

$$p_i p^i = -\hbar^2\left(\frac{\partial g^{ij}}{\partial x^i}\frac{\partial}{\partial x^j} + g^{ij}\frac{\partial^2}{\partial x^i \partial x^j}\right) \tag{3.12}$$

Let's use well known relation

$$\frac{\partial g^{ij}}{\partial x^k} = -\Gamma^i_{mk}g^{mj} - \Gamma^j_{mk}g^{im}. \tag{3.13}$$

Then

$$p_i p^i = -\hbar^2\left(g^{ij}\frac{\partial^2}{\partial x^i \partial x^j} - g^{mj}\Gamma^i_{mi}\frac{\partial}{\partial x^j} - g^{im}\Gamma^j_{mi}\frac{\partial}{\partial x^i}\right). \tag{3.14}$$

Changing indexes of summation, we get

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$$p_i p^i = -\hbar^2 g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ik}\frac{\partial}{\partial x^j} - \Gamma^k_{ij}\frac{\partial}{\partial x^k}\right). \quad (3.15)$$

Let's consider second term in the left side of equation (3.2), rewriting it in the form

$$p_i \delta p^i = p_i g^{ij}\delta p_j. \quad (3.16)$$

Using formula (3.13), we get

$$p_i \delta p^i = g^{ij}(p_i \delta p_j) + i\hbar\left(g^{ij}\Gamma^m_{im} + g^{im}\Gamma^j_{im}\right)\delta p_j + g^{ij}\delta p_j p_i. \quad (3.17)$$

Let's write in its direct form the covariant derivative in the expression (3.10):

$$\delta p_j = -i\hbar\left(\Gamma^k_{jk} + \frac{\partial \Gamma^k_{j\alpha}}{\partial x^k}\delta x^\alpha - \Gamma^k_{lk}\Gamma^k_{l\alpha}\delta x^\alpha + \Gamma^k_{lk}\Gamma^l_{j\alpha}\delta x^\alpha\right). \quad (3.18)$$

We get from formula (3.18)

$$\delta p_j = -i\hbar\left(\frac{1}{2}\Gamma^k_{jk} + R_{j\alpha}\delta x^\alpha + \frac{\partial \Gamma^k_{jk}}{\partial x^\alpha}\delta x^\alpha\right), \quad (3.19)$$

It can be shown that $\delta p_l = 0$ (see Appendix). Then

$$\delta p_j = -i\hbar R_{i\alpha}\delta x^\alpha. \quad (3.20)$$

Substituting expression (3.20) into formula (3.17), we get

$$p_i \delta p^i = -\hbar^2 g^{ij}\left(\frac{\partial R_{j\alpha}}{\partial x^i}\delta x^\alpha - \Gamma^l_{ij}R_{l\alpha}\delta x^\alpha + R_{i\alpha}\delta x^\alpha \frac{\partial}{\partial x^j} - \frac{1}{2}R_{ij}\right) \quad (3.21)$$

And

$$\delta p_i p^i = -\hbar^2 g^{ij} R_{i\alpha}\delta x^\alpha \frac{\partial}{\partial x^j} \quad (3.22)$$

Where $\delta_{p^l} = 0$ was assumed after taking the derivatives. From now up to the end of paper, we will denote by α only the extra dimensional coordinate.

Obviously,

$$\delta p_i \delta p^i = -\hbar^2 g^{ij} R_{i\alpha} R_{j\alpha'}\delta x^\alpha \delta x^{\alpha'} \quad (3.23)$$

Using equations (3.8, 3.15, 3.21, 3.22, 3.23), we get

$$\hbar^2 g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial}{\partial x^k} + \frac{1}{2}\delta x^\alpha\left(\frac{\partial R_{j\alpha}}{\partial x^i} - \Gamma^l_{ij}R_{l\alpha} + 2R_{i\alpha}\frac{\partial}{\partial x^l}\right) + \right.$$
$$\left. + \frac{1}{4}R_{i\alpha}R_{j\alpha'}\delta x^\alpha \delta x^{\alpha'} - \frac{1}{4}R_{ij}\right)\psi + m^2c^2\psi = 0 \quad (3.24)$$

where ψ is a wave function. Equation (3.24) can be rewritten as

$$g^{ij}\left(D_i + \frac{1}{2}\delta x^\alpha R_{i\alpha}\right)\left(D_j + \frac{1}{2}\delta x^\alpha R_{j\alpha}\right)\psi = \left\{\frac{1}{4}R - \left(\frac{mc}{\hbar}\right)^2\right\}\psi, \quad (3.25)$$

where D_i,D_j are covariant derivatives and R is scalar curvature.

IV. Approximate Solutions

Assuming that the metrics of space-time is almost Galileo's one, we can rewrite equation (3.25) in single dimensional approximation for brane as

$$\{(g_0^{11} + h^{11})\frac{\partial^2}{\partial x^2} + 2h^{10}\frac{\partial^2}{\partial x \partial t} + (g_0^{11} + h^{11})\gamma\frac{\partial}{\partial x} + a\}\psi = -(g_0^{00} + h^{00})\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \quad (4.1)$$

where

$$\gamma = R_{1\alpha}\delta x^\alpha, \quad (4.2)$$

$$a = \frac{1}{2}(g_0^{11} + h^{11})\{\left(\frac{\partial}{\partial x}R_{1\alpha}\right)\delta x^\alpha + \frac{1}{2}R_{1\alpha}R_{1\alpha}\delta x^\alpha \delta x^\alpha - \frac{1}{2}R_{11}\} + \left(\frac{mc}{\hbar}\right)^2, \quad (4.3)$$

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Taking $g_0^{00}=-1, g_0^{11}=1, h^{11}=h, h^{10}=h^{00}=0$, we get

$$(1-h)\frac{\partial^2\psi}{\partial x^2}+(1-h)\gamma\frac{\partial\psi}{\partial x}-a\psi=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \tag{4.4}$$

Looking for the solution in the form of plane wave

$$\psi=A^{i(kx+\omega t)}, \tag{4.5}$$

we obtain the following dispersion equation:

$$(1-h)(k^2-i\gamma k)+a-\frac{\omega^2}{c^2}=0. \tag{4.6}$$

It has a solution

$$k=i\frac{\gamma}{2}+\frac{\omega}{c}\sqrt{\frac{1}{1-h}}\sqrt{1-\frac{c^2}{\omega^2}a-\frac{c^2\gamma^2}{4\omega^2}(1-h)}. \tag{4.7}$$

Assuming that h, a and γ are small, we approximately get

$$k=\frac{\omega}{c}+\frac{1}{2c}\left(\omega h-\frac{ac^2}{2\omega}-\frac{c^2\gamma^2}{8\omega}+ic\gamma\right). \tag{4.8}$$

So, we have for the frequency shift

$$\Delta\omega=\omega_0-\omega=\frac{1}{2}\omega h-\frac{1}{4}\frac{c^2}{\omega}\left(a+\frac{\gamma}{4}\right), \tag{4.9}$$

Using expressions (4.27,4.28), we get

$$\Delta\omega=\frac{1}{2}\omega h+\frac{1}{4}\frac{c^2}{\omega}\left(\frac{1}{2}\delta R_{1\alpha}+\frac{1}{4}R_{11}-\left(\frac{mc}{\hbar}\right)^2\right), \tag{4.10}$$

where

$$\delta R_{1\alpha}=\left(\frac{\partial}{\partial x}R_{1\alpha}\right)\delta x^{\alpha}. \tag{4.11}$$

For a photon at $m=R=0$ we have

$$\Delta\omega=\frac{1}{2}\omega h+\frac{1}{8}\frac{c^2}{\omega}\delta R_{1\alpha}. \tag{4.12}$$

The first term in (4.37) is usual gravitational shift while the other two terms are connected with variation of external brane curvature.

Also, we get from the formula (4.36) the expression for the group speed of a particle

$$\frac{\partial\omega}{\partial k}=\frac{c}{1-\frac{c^2}{4\omega^2}\left\{\frac{1}{2}\delta R_{1\alpha}+\frac{1}{4}R_{11}-\left(\frac{mc}{\hbar}\right)^2\right\}} \tag{4.38}$$

For a photon, we have

$$\frac{\partial\omega}{\partial k}=\frac{c}{1-\frac{c^2}{8\omega^2}\delta R_{1\alpha}} \tag{4.39}$$

We see that the group speed of light is less than c when the variation of external brane curvature is negative but is more than c when the external brane curvature is positive. So, formula (4.39) shows that the change of light speed depends on the external brane curvature variation.

V. Conclusion

Thus, we have derived Klein-Gordon equation for a particle on brane using variation principle. It can be verified that the Dirac decomposition of obtained Klein-Gordon equation yields Dirac-Fock-Ivanenko equation [12] at zero external curvature that can be solved with Einstein's equation [13]. Indeed, squaring Dirac-Fock-Ivanenko equation gives wave equation [14, 15] coinciding in its main part with that obtained in the present paper but we have obtained additional brane curvature depending terms. Solution of this equation for a photon on the background of almost Galileo's metrics yields the changes in frequency and speed of particle's wave packet depending on external curvature that can be verified experimentally.

VI. Acknowledgments

We are grateful to V.V. Bochkarev and V.Kurbanova for the helpful discussions.

VII. Appendix

If the wave function were vector ψ_n the first order covariant derivative on ψ_n will be

$$\{\psi^n\}_{;i}=\frac{\partial\psi^n}{\partial x}+\Gamma^n_{ik}\psi^k, \tag{A.1}$$

Let's denote the "geometrical" part of partial wave function derivative as

$$(\psi^n)'=\Gamma^n_{ik}\psi^k, \tag{A. 2}$$

Then we can write using orthogonal character of wave functions

$$(\psi^n)'=\psi^n\sum_k\psi_k(\psi^k)'=\psi^n\sum_k\psi_k\sum_l\Gamma^k_{il}\psi^l=\sum_k\Gamma^k_{ik}\psi^n \tag{A. 3}$$

And

$$\Gamma^n_{ik}\psi^k=\Gamma^k_{ik}\psi^n \tag{A. 4}$$

But the wave function is a scalar and $(\psi^n)'=\Gamma^n_{ik}\psi^k=\Gamma^k_{ik}\psi^n=0$. Hence, $\Gamma^k_{ik}=0$.

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