On the Maximal Ideals in the Banach Space of +Quasicontinuous Functions

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Abstract - In this paper some interesting properties of +Quasicontinuous functions are presented. The maximal ideals in the Banach space of bounded real valued +Quasicontinuous functions defined on [0,1] are investigated.

Keywords : +quasicontinuity, maximal ideal, space of maximal ideals, weak* topology, compact hausdorff space, bounded linear functional, cliquish function.

On the Maximal Ideals in the Banach Space of \(^{+}\)Quasicontinuous Functions

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Keywords: \(^{+}\)Quasicontinuity, maximal ideal, space of maximal ideals, weak* topology, compact hausdorff space, bounded linear functional, cliquish function.

I. Introduction

In this paper, it is shown that the set of all bounded real \(^{+}\)Quasicontinuous functions defined on [0,1] forms a commutative Banach algebra with identity under the supremum norm. The maximal ideals in this Banach algebra are identified to be of the form \(M_x = \{ f / f(x) = 0 \} \) or \(M^+_x = \{ f / f(x+) = 0 \} \) for \(x \in [0,1] \).

In what follows \(\mathbb{R}, \mathbb{I}\) and \(J\) stand for the real line, the unit closed interval [0,1] and any closed and bounded interval \([a,b]\) respectively.

a) Preliminaries

1.1 Definition: Let \(f : J \to \mathbb{R}\). We define \(f(a-) = f(a)\) and \(f(b+) = f(b)\). We say that \(f(p+)\) exists at \(p \in [a,b]\) and we write \(f(p+) = L\), where \(L \in \mathbb{R}\) if for every \(\epsilon > 0\) there exists a \(\delta > 0\) such that \(|f(x) - L| < \epsilon \ \forall \ x \in (p, p+\delta) \subset J\)

Similarly for \(p \in (a,b]\) we write \(f(p-) = l \in \mathbb{R}\) if for every \(\epsilon > 0\) there exists a \(\delta > 0\) such that \(|f(x) - l| < \epsilon \ \forall \ x \in (p-\delta, p) \subset J\)

1.2 Definition: A function \(f : J \to \mathbb{R}\) is said to be \(^{+}\)Quasicontinuous on \(J\) if

(i) \(f(p+)\) exists at every \(p \in [a,b]\)
(ii) \( f(b-) = f(b) \)

1.3 Definition: A function \( f : J \to \mathbb{R} \) is said to be cliquish at a point \( p \in J \) if for every \( \varepsilon > 0 \) and every neighborhood \( U \) of \( p \) in \( J \) there exists a non-empty open set \( W \subset U \) such that \( |f(x) - f(y)| < \varepsilon \ \forall \ x, y \in W \). We say that \( f \) is cliquish on \( J \) if it is cliquish at every point of \( J \).

1.4 Definition: A mapping \( T \) from a linear space \( \mathcal{V} \) into a linear space \( \mathcal{W} \) is said to be linear if \( T(cx + dy) = cT(x) + dT(y) \) for all \( x \) and \( y \) in \( \mathcal{V} \) and constants \( c \) and \( d \).

1.5 Definition: Let \( \mathcal{V} \) and \( \mathcal{W} \) be normed linear spaces. A linear map \( T : \mathcal{V} \to \mathcal{W} \) is said to be bounded if there exists a real number \( K \geq 0 \) such that \( \|T(x)\| \leq K\|x\| \ \forall \ x \in \mathcal{V} \).

1.6 Definition: A linear functional on a vector space \( \mathcal{V} \) over a field \( \mathcal{K} \) is a linear mapping from \( \mathcal{V} \) to \( \mathcal{K} \).

b) Properties of \(^+\)Quasicontinuous functions

2.1 Proposition: Let \( c \in \mathbb{R} \). If \( f : J \to \mathbb{R} \) and \( g : J \to \mathbb{R} \) are

\(^+\)Quasicontinuous on \( J \) then \( f + g, cf, fg, f \lor g \) and \( f \land g \) are

\(^+\)Quasicontinuous on \( J \), where \( (f \lor g)(x) = \max\{f(x), g(x)\} \) and \( (f \land g)(x) = \min\{f(x), g(x)\} \).

Proof: Let \( p \in [a, b] \). (i) Let \( \varepsilon > 0 \) be given. Then there exist \( \delta_1 > 0 \) and \( \delta_2 > 0 \) such that \( |f(x) - f(p+)| < \frac{\varepsilon}{2} \ \forall \ x \in (p, p + \delta_1) \subset J \) and

\[ |g(x) - g(p+)| < \frac{\varepsilon}{2} \ \forall \ x \in (p, p + \delta_2) \subset J . \]

Put \( \delta = \min\{\delta_1, \delta_2\} \).

Then \( x \in (p, p + \delta) \Rightarrow |(f + g)(x) - (f(p+) + g(p+))| \leq |f(x) - f(p+)| + |g(x) - g(p+)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \).

Thus for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[ |(f + g)(x) - (f(p+) + g(p+))| < \varepsilon \ \forall \ x \in (p, p + \delta) . \]
Hence \((f + g)(p+)\) exists and \((f + g)(p+) = f(p+) + g(p+)\). Since \(f\) and \(g\) are continuous at \(b\), \(f + g\) is continuous at \(b\).

Hence \(f + g\) is \(Q^{+}\)Quasicontinuous on \(J\).

(ii) If \(c = 0\) then \(cf = O\), where \(O : J \to \mathbb{R}\) is defined by \(O(x) = 0\).

Then \(cf\) is \(Q^{+}\)Quasicontinuous on \(J\). Now suppose that \(c \neq 0\).

Let \(\varepsilon > 0\) be given. Then there exists a \(\delta > 0\) such that

\[
|f(x) - f(p+)| < \frac{\varepsilon}{|c|} \quad \forall x \in (p, p+\delta) \subset J
\]

\[
\Rightarrow \quad |(cf)(x) - (cf)(p+)| < \varepsilon \quad \forall x \in (p, p+\delta)
\]

Hence \((cf)(p+)\) exists and \((cf)(p+) = c f(p+)\). Since \(f\) is continuous at \(b\), \(cf\) is continuous at \(b\). Hence \(cf\) is \(Q^{+}\)Quasicontinuous on \(J\).

(iii) Since \(f\) and \(g\) are \(Q^{+}\)Quasicontinuous at \(p\), for every \(\varepsilon > 0\) there exists a \(\delta > 0\) such that

\[
|f(x) - f(p+)| < \varepsilon \quad \text{and} \quad |g(x) - g(p+)| < \varepsilon \quad \forall x \in (p, p+\delta) \subset J
\]

\[
\Rightarrow \quad |(fg)(x) - f(p+)g(p+)| = |f(x)g(x) - f(x)g(p+) + f(x)g(p+) - f(p+)g(p+)|
\]

\[
\leq \left|f(x)\right| |g(x) - g(p+)| + |g(p+)| \left|f(x) - f(p+)\right|
\]

\[
< \left|f(x)\right| \varepsilon + |g(p+)| \varepsilon \quad \forall x \in (p, p+\delta)
\]

\[
= \left|f(x) - f(p+) + f(p+)|\varepsilon + |g(p+)|\varepsilon
\]

\[
< \varepsilon \left(\varepsilon + |f(p+)| + |g(p+)|\right) \quad \forall x \in (p, p+\delta).
\]

Hence \((fg)(p+)\) exists and \((fg)(p+) = f(p+)g(p+)\). Since \(f\) and \(g\) are continuous at \(b\), \(fg\) is continuous at \(b\).

Hence \(fg\) is \(Q^{+}\)Quasicontinuous on \(J\).

It is easy to verify that \(f \vee g\) and \(f \wedge g\) are \(Q^{+}\)Quasicontinuous on \(J\) and we have the following.

\((f \vee g)(p+) = \max \{f(p+), g(p+)\}\) and \((f \wedge g)(p+) = \min \{f(p+), g(p+)\}\).

2.2 **Proposition:** Let \(f_n : J \to \mathbb{R}\), \(n = 1, 2, 3, \ldots\), be \(Q^{+}\)Quasicontinuous on \(J\) and \(f_n \to f\) uniformly on \(J\). Then \(f\) is \(Q^{+}\)Quasicontinuous on \(J\).
Proof: Let \( p \in [a, b) \). Let \( \varepsilon > 0 \) be given. Then there exists an integer \( N \) such that \( n \geq N \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \forall \ x \in J \).

Since \( f_N \) is \( ^+ \) Quasicontinuous at \( p \), there exists a \( \delta > 0 \) such that
\[
|x - p| < \delta \quad \Rightarrow \quad |f(x) - f_N(p)| < \varepsilon \quad \forall \ x \in (p, p + \delta) \subset J.
\]

Thus for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that
\[
|f(x) - f_N(p)| < \varepsilon \quad \forall \ x \in (p, p + \delta) \subset J.
\]
Hence \( f(p+) \) exists for every \( p \in [a, b) \).

Since each \( f_n \) is continuous at \( b \) and \( f_n \to f \) uniformly on \( J \), \( f \) is continuous at \( b \). Hence \( f \) is \( ^+ \) Quasicontinuous on \( J \).

2.3 Remark: It is not necessary that a \( ^+ \) Quasicontinuous function defined on a compact domain is bounded. It can be seen from the following example.

2.4 Example: Define \( f : [-1,1] \to \mathbb{R} \) by \( f(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq 1 \\ x & \text{if} \quad -1 \leq x < 0 \end{cases} \)

This function \( f \) is \( ^+ \) Quasicontinuous on \( [-1,1] \) but it is not bounded.

2.5 Remark: We denote the set of all bounded real valued \( ^+ \) Quasicontinuous functions defined on \( I \) by the symbol \( \mathcal{CE}^+(I) \). By the propositions 2.1 and 2.2 it follows that \( \mathcal{CE}^+(I) \) forms a commutative Banach algebra with identity under the supremum norm, where the identity \( e : I \to \mathbb{R} \) is defined by \( e(x) = 1 \quad \forall \ x \in I \).

2.6 Proposition: Let \( f : J \to \mathbb{R} \) and \( p \in J \). If \( f(p+) \) exists then \( f \) is cliquish at \( p \).

Proof: Let \( \varepsilon > 0 \) be given and let \( U \) be a neighborhood of \( p \) in \( J \). Then there exists a \( \delta_i > 0 \) such that \( (p - \delta_i, p + \delta_i) \cap J \subset U \).
Given \( f(p+) \) exists. So there exists \( \delta_2 > 0 \) such that
\[
|f(x) - f(p+)| < \frac{\varepsilon}{2} \quad \forall \ x \in (p, p + \delta_2) \subset J.
\]

Put \( \delta = \min\{\delta_1, \delta_2\} \) and \( W = (p, p + \delta) \).

Then for \( x, y \in W \), we have
\[
|f(x) - f(y)| = |f(x) - f(p+) + f(p+) - f(y)|
\leq |f(x) - f(p+)| + |f(y) - f(p+)|
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Thus for every \( \varepsilon > 0 \) and every neighborhood \( U \) of \( p \), there exists a non-empty open set \( W \subset U \) such that \( |f(x) - f(y)| < \varepsilon \quad \forall \ x, y \in W \)

\( \Rightarrow f \) is cliquish at \( p \).

**2.7 Remark:** From the above proposition it is clear that every \( ^+ \)Quasicontinuous function is cliquish. The converse is not true as is evident from the following example.

**2.8 Example:** Define \( f : [-1,1] \to \mathbb{R} \) as follows.
\[
f(x) = \begin{cases} 
1 & \text{if } 0 < x \leq 1 \\
x & \text{if } -1 \leq x \leq 0 \\
0 & \text{if } x = 0
\end{cases}
\]

Clearly \( f \) is cliquish at \( x = 0 \) but it is not \( ^+ \)Quasicontinuous.

**2.9 Theorem [2]:** If \( f : J \to \mathbb{R} \) is \( ^+ \)Quasicontinuous then the set of points of discontinuity of \( f \) is atmost countable.

c) **Maximal Ideals in \( \mathcal{C}^+(I) \)**

**3.1 Definition:** For each \( x \in I \), we define the following.
(a) \( M_x = \{f \in \mathcal{C}^+(I) \mid f(x) = 0\} \)  (b) \( M_x^+ = \{f \in \mathcal{C}^+(I) \mid f(x+) = 0\} \)

**3.2 Proposition:** For each \( x \in I \), the sets \( M_x \) and \( M_x^+ \) are maximal ideals in the commutative Banach algebra \( \mathcal{C}^+(I) \).

**Proof:** For \( x \in I \), define \( F_x \) and \( F_x^+ \) on \( \mathcal{C}^+(I) \) by \( F_x(f) = f(x) \) and \( F_x^+(f) = f(x+) \) for \( f \in \mathcal{C}^+(I) \).
Clearly $F_x$ and $F_x^+$ are multiplicative linear functionals in the dual space $\mathcal{B}_x$ with kernels $M_x$ and $M_x^+$ respectively. Hence $M_x$ and $M_x^+$ are ideals. Moreover $M_x$ and $M_x^+$ are maximal ideals in $\mathcal{C}^+(I)$.

3.3 Proposition: If $M$ is a maximal ideal in $\mathcal{C}^+(I)$ then either $M = M_x$ or $M = M_x^+$ for some $x \in I$.

Proof: For $x \in I$, define $F_x$ and $F_x^+$ on $\mathcal{C}^+(I)$ by $F_x(f) = f(x)$ and $F_x^+(f) = f(x^+)$ for $f \in \mathcal{C}^+(I)$.

Clearly $F_x$ and $F_x^+$ are multiplicative linear functionals in the dual space $\mathcal{B}_x$ with kernels $M_x$ and $M_x^+$ respectively. Hence $M_x$ and $M_x^+$ are ideals. Moreover $M_x$ and $M_x^+$ are maximal ideals in $\mathcal{C}^+(I)$.

3.3 Proposition: If $M$ is a maximal ideal in $\mathcal{C}^+(I)$ then either $M = M_x$ or $M = M_x^+$ for some $x \in I$.

Proof: Assume that $M \neq M_x$ and $M \neq M_x^+$ for any $x \in I$.

Then there exist $f_x$ and $g_x$ in $M$ such that $f_x \notin M_x$ and $g_x \notin M_x^+$.

Define $\varphi_x : I \to \mathbb{R}$ by $\varphi_x(t) = f_x^2(t) + g_x^2(t^+) \ \forall \ t \in I$.

Clearly $\varphi_x \in \mathcal{C}^+(I)$. Since $\varphi_x$ is $^+$ Quasicontinuous at $x$ and $\varphi_x(x) > 0$, there exists $\delta_x > 0$ such that $\varphi_x(t) > 0 \ \forall \ t \in [0, \delta_x)$ and for $x \neq 1$

We have $\varphi_1(t) = f_1^2(t) + g_1^2(t^+) \ \forall \ t \in I$.

Since $\varphi_1$ is continuous at $1$ and $\varphi_1(1) > 0$ there exists a $\delta > 0$ such that $\varphi_1(t) > 0 \ \forall \ t \in (1 - \delta, 1]$. Then $[0, 1] = \bigcup_{x \neq 1} [0, \delta_x) \cup (1 - \delta, 1]$. Since $I$ is compact, there exists $x_i \neq 1$ in $I$ such that $[0, 1] = [0, \delta_{x_i}) \cup (1 - \delta_i, 1]$.

Put $\varphi = \varphi_n^2 + \varphi_1^2$. Then $\varphi \in M$ and $\varphi(t) > 0 \ \forall \ t \in I \Rightarrow \frac{1}{\varphi} \in M$.

Then $e = \varphi \cdot \frac{1}{\varphi} \in M$. This is a contradiction.

Hence it follows that $M = M_x$ or $M = M_x^+$ for some $x \in I$. 
3.4 Remark: Let \( \mathcal{M}_x \) be the space of all maximal ideals in \( \mathcal{C}^* \). Then \( \mathcal{M}_x \) is a compact Hausdorff space with the weak* topology on \( \mathcal{C}^* \). Hence \( \mathcal{M}_x^2 = \mathcal{M}_x \times \mathcal{M}_x \) is a compact Hausdorff space with the product topology on \( \mathcal{C}^* \times \mathcal{C}^* \).

3.5 Proposition: Let \( \mathcal{A}^+ = \{(M_x, M^+_x) / x \in I\} \). Then there exists a one-to-one correspondence between \( I \) and \( \mathcal{A}^+ \).

Proof: Define \( \Psi^+: I \rightarrow \mathcal{A}^+ \) by \( \Psi^+(x) = (M_x, M^+_x) \).

Clearly \( \Psi^+ \) is surjective. If \( 0 \leq s < t \leq 1 \), the function

\[
\Psi^+_0(p) = \begin{cases} 
0 & \text{if } t \leq p \leq 1 \\
1 & \text{if } 0 \leq p < t \\
x - t & \text{if } \Psi^+(s) \neq \Psi^+(t)
\end{cases}
\]

satisfies \( \Psi^+_0 \in M_t \) and \( \Psi^+_0 \notin M_s \).

\( \Rightarrow \) \( M_s \neq M_t \)

\( \Rightarrow \) \( (M_x, M^+_x) \neq (M_t, M^+_t) \)

\( \Rightarrow \) \( \Psi^+(s) \neq \Psi^+(t) \)

Hence \( \Psi^+ \) is 1-1.

Hence \( \Psi^+ \) is a one-to-one correspondence between \( I \) and \( \mathcal{A}^+ \).

3.6 Remark: Each maximal ideal in \( \mathcal{C}^* \) is the kernel of some multiplicative linear functional on \( \mathcal{C}^* \), hence can be identified with a multiplicative linear functional on \( \mathcal{C}^* \). Let \( M_x \) and \( M^+_x \) be identified with the multiplicative linear functional \( F_x \) and \( F^+_x \) respectively. So we can write

\( \mathcal{A}^+ = \{(F_x, F^+_x) / x \in I\} \).

3.7 Proposition: \( \mathcal{A}^+ \) is closed in \( \mathcal{B}^2 = \mathcal{B}_x \times \mathcal{B}_x \) and hence compact.

Proof: We prove that \( \mathcal{A}^+ \) is closed. Compactness is an immediate consequence of the Banach – Alaoglu theorem [5]. If \( F = (F_x, F^+_x) \in \mathcal{B}^2 \), we define

\[
\|F\| = \max \{|\|F_x\|, |\|F^+_x\|\}
\]

Then \( \mathcal{B}^2 \) is a Banach space under the above norm.

Let \( S = \{F / \|F\| \leq 1\} \subset \mathcal{B}^2 \). Put \( \mathcal{A} = \mathcal{A}^+ \cup \{O\} \).
The \( \mathcal{A}^+ \subset \mathcal{M}_+^2 \subset \mathcal{A} \subset S \subset B_+^2 \).

Define \( \mathcal{P}^+: \mathcal{A} \rightarrow \mathbb{R} \) by
\[
\mathcal{P}^+(F) = \begin{cases} 
1 & \text{if } F \in \mathcal{A} \text{ and } F \neq O \\
0 & \text{if } F = O
\end{cases}
\]

Since \( \mathcal{P}^+ \) is continuous, \( \mathcal{A}^+ \) and \( \mathcal{A} \) are closed in \( B_+^2 \).

d) Further Properties

4.1 Proposition: Fix \( f \in \mathcal{CE}^+(I) \). Define \( \psi_f : I \rightarrow \mathbb{R}^2 \) by
\[
\psi_f(x) = (f(x), f(x^+)), \quad \text{where } \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \text{ is considered with the norm }
\]

\[
\|(x_1, x_2)\| = \max \{|x_1|, |x_2|\}.
\]
Then \( \psi_f \) is continuous on \( I \) if and only if \( f \) is continuous on \( I \).

\[\text{Proof:}\] Assume that \( \psi_f \) is continuous on \( I \). Let \( p \in I \) and let \( \varepsilon > 0 \) be given.

Since \( \psi_f \) is continuous at \( p \), there exists a \( \delta > 0 \) such that
\[
\|\psi_f(x) - \psi_f(p)\| < \varepsilon \quad \forall \ x \in (p - \delta, p + \delta) \cap I.
\]
\[
\Rightarrow \|f(x) - f(p)\| < \varepsilon \quad \forall \ x \in (p - \delta, p + \delta) \cap I
\]
\[
\Rightarrow \|f(x^+) - f(p)\| < \varepsilon \quad \forall \ x \in (p - \delta, p + \delta) \cap I
\]
\[
\Rightarrow \max \{|f(x) - f(p)|, |f(x^+) - f(p)|\} < \varepsilon \quad \forall \ x \in (p - \delta, p + \delta) \cap I
\]
\[
\Rightarrow |f(x) - f(p)| < \varepsilon \quad \forall \ x \in (p - \delta, p + \delta)
\]
\[
\Rightarrow f \text{ is continuous at } p.
\]

Thus if \( \psi_f \) is continuous at \( p \) then \( f \) is continuous at \( p \).

Conversely suppose that \( f \) is continuous on \( I \).

Then \( \psi_f(x) = (f(x), f(x)) \quad \forall \ x \in I \).

Hence \( \psi_f \) continuous on \( I \).

4.2 Proposition: Let \( B = \{\psi_f / f \in \mathcal{CE}^+(I)\} \). Define \( F : \mathcal{CE}^+(I) \rightarrow B \) by
\[
F(f) = \psi_f.
\]
Then \( F \) is a one-to-one continuous multiplicative linear mapping from \( \mathcal{CE}^+(I) \) onto \( B \).
Proof: Clearly $F : \mathcal{C}^+(I) \to \mathcal{B}$ is surjective.

For $f, g \in \mathcal{C}^+(I)$, $\psi_{f+g}(x) = ((f + g)(x), (f + g)(x+))$

$= (f(x), f(x+)) + (g(x), g(x+))$

$= \psi_f(x) + \psi_g(x) \quad \forall \ x \in I$

Hence $\psi_{f+g} = \psi_f + \psi_g \quad \forall \ f, g \in \mathcal{C}^+(I)$

$\Rightarrow F(f + g) = F(f) + F(g) \quad \forall \ f, g \in \mathcal{C}^+(I)$.

Let $c \in \mathbb{R}$.

Then it is easy to see that $F(cf) = \psi_{cf} = c\psi_f = cF(f) \quad \forall \ f \in \mathcal{C}^+(I)$.

Hence $F$ is linear.

Also we have $\psi_{fg}(x) = ((fg)(x), (fg)(x+))$

$= (f(x), f(x+)) (g(x), g(x+))$

$= \psi_f(x) \psi_g(x) \quad \forall \ x \in I$.

Hence $F(fg) = \psi_{fg} = \psi_f \psi_g = F(f)F(g)$.

$\Rightarrow F$ is multiplicative now we prove that $F$ is $1 - 1$. For this, suppose that $F(f) = F(g)$

$\Rightarrow \psi_f = \psi_g$

$\Rightarrow \psi_f(x) = \psi_g(x) \quad \forall \ x \in I$

$\Rightarrow (f(x), f(x+)) = (g(x), g(x+)) \quad \forall \ x \in I$

$\Rightarrow f(x) = g(x) \quad \forall \ x \in I$

$\Rightarrow f = g$.

Hence $F$ is $1 - 1$.

Suppose that $f_n \in \mathcal{C}^+(I)$, $n = 1, 2, 3, \ldots$, and $f \in \mathcal{C}^+(I)$.

Let $f_n \to f$ uniformly on $I$. Then for a given $\varepsilon > 0$ there exists an integer $N > 0$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $n \geq N$ and all $x \in I$.

Fix $x \in I$ and $n \geq N$. Since $f_n$ is $+\text{Quasicontinuous}$ there exists a $\delta > 0$ such that $|f_n(t) - f_n(x+)| < \frac{\varepsilon}{3}$ $\forall \ t \in (x, x + \delta)$.
Since \( f \) is also \(+\) Quasicontinuous at \( x \), there exists a \( \delta > 0 \) such that
\[
|f(t) - f(x+)| < \frac{\varepsilon}{3} \quad \forall \ t \in (x, x + \delta).
\]
Put \( \delta = \min\{\delta_1, \delta_2\} \). Then for \( t \in (x, x + \delta) \) and \( n \geq N \),
\[
|f_n(x+) - f(x+)| = |f_n(x+) - f_n(t) + f_n(t) - f(t) + f(t) - f(x+)|
\leq |f_n(x+) - f_n(t)| + |f_n(t) - f(t)| + |f(t) - f(x+)|
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]
Hence \( |f_n(x+) - f(x+)| < \varepsilon \) for all \( n \geq N \) and all \( x \in I \).

\[
n \geq N \implies \|F(f_n) - F(f)\| = \|\psi_{f_n} - \psi_f\|
= \sup\{\|\psi_{f_n}(x) - \psi_f(x)\| / x \in I\} < \varepsilon.
\]

\( \Rightarrow F(f_n) \to F(f) \) uniformly on \( I \).

Hence \( F \) is continuous on \( \mathcal{C}^+(I) \).

4.3 Proposition: The set \( \mathcal{B} = \{\psi_f / f \in \mathcal{C}^+(I)\} \) is a commutative Banach
algebra with identity \( \psi_e \) under the norm defined by \( \|\psi_f\| = \sup\{\|\psi_f(x)\| / x \in I\} \).

where \( \psi_e(x) = (1,1) \quad \forall \ x \in I \).

References