Pathway Integral Operator Associated with Aleph-Function and General Polynomials

By Dr. Rinku Jain & Dr. Kirti Arekar

K.J. Somaiya Institute of Management Studies & Research, India

Abstract - This paper is devoted to the study of a pathway fractional integral operator associated with the pathway model and pathway probability density for the $\aleph$-function and a generalized polynomial in the kernel. By specializing the coefficients and various parameters in the generalized polynomials and $\aleph$-function, our main theorem would readily yield several interesting results.

Keywords : pathway fractional integral operator, aleph function ($\aleph$-function), generalized polynomial.

GJSFR-F Classification : MSC 2010 : 11S23, 33C45

© 2013. Dr. Rinku Jain & Dr. Kirti Arekar. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/, permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.
Pathway Integral Operator Associated with Aleph-Function and General Polynomials

Dr. Rinku Jain & Dr. Kirti Arekar

Abstract - This paper is devoted to the study of a pathway fractional integral operator associated with the pathway model and pathway probability density for the \( \aleph \)-function and a generalized polynomial in the kernel. By specializing the coefficients and various parameters in the generalized polynomials and \( \aleph \)-function, our main theorem would readily yield several interesting results.

Keywords: pathway fractional integral operator, aleph function (\( \aleph \)-function), generalized polynomial.

I. Introduction

In the last three decades several authors have made significant contribution in the field of fractional calculus. Fractional calculus has been applied to almost every field of science, engineering, and Mathematics. The most popular one, we are based on here, is the Riemann-Liouville fractional integral operator [7]. The Pathway fractional integral operator, as an extension of Riemann-Liouville fractional integral operator, introduced by Nair [8] is defined in the following manner

\[
(P_{\alpha}^{\eta} f)(x) = x^{\eta} \int_{0}^{x} \left[ \frac{1 - a(1-\alpha)t}{1 - \alpha} \right] \frac{\eta}{(1-\alpha)} f(t) dt
\]

Where \( f(x) \in L(a,b), \eta \in \mathbb{C}, \text{Re}(\eta) > 0, a > 0 \) and ‘pathway parameter’ \( \alpha < 1 \).

The Pathway model is introduced by Mathai [1] and studied further by Mathai and Haubold[2], [3]. For real scalar \( \alpha \), the Pathway model for scalar random variables is represented by the following probability density function

\[
f(x) = c \left[ x^{\gamma - 1} \left[ 1 - a(1-\alpha) |x|^\delta \right] \right]^{\beta} \frac{\beta}{1-\alpha},
\]

\(-\infty < x < \infty, \delta > 0, \beta \geq 0, \left[ 1 - a(1-\alpha) |x|^\delta \right] > 0, \gamma > 0, \) where \( c \) is the normalizing constant and \( \alpha \) is called the pathway parameter. For real \( \alpha \), the normalizing constant is as follows:

Authors & E-mails: K.J. Somaiya Institute of Management Studies & Research Mumbai, India. E-mails: jainrinku5@gmail.com, deshmukh_k123@yahoo.com
1. For $\alpha<1$, it is a finite range density with $\left[1-a(1-\alpha)\right]x^{\beta} > 0$ and (1.2) remains in the extended generalized type-1 beta family. The Pathway density in (1.2), for $\alpha<1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.

For $\alpha>1$, we have

\[ f(x) = c \left| x \right|^{\gamma-1} \left[1 + a(\alpha - 1)\right]x^{\beta} - \frac{\beta}{\alpha - 1}, \]

for $-\infty < x < \infty$, $\delta > 0$, $\beta \geq 0$, $\left[1 - a(\alpha - 1)\right]x^{\beta} > 0$, $\gamma > 0$, which is extended generalized type-2 beta model for real $x$. It includes the type-2 beta density, the $F$ density, the Student-$t$ density, Cauchy density and many more.

Here, we consider only the case of Pathway parameter $\alpha<1$. For $\alpha \to 1$ both (1.2) and (1.6) take the exponential form, since

\[ \lim_{\alpha \to 1} c \left| x \right|^{\gamma-1} \left[1 - a(1-\alpha)\right]x^{\beta} = c \left| x \right|^{\gamma-1} \left[1 + a(\alpha - 1)\right]x^{\beta} - \frac{\beta}{\alpha - 1} \]

\[ = c \left| x \right|^{\gamma-1} e^{-\alpha \eta} \mid x \mid^\delta \]

This includes the generalized Gamma-, the Weibull-, the Chi-Square the Laplace and other related densities. For more details on the Pathway model, the reader is referred to the recent papers of [2], [3].

II. Pathway Integral Operator of An $\aleph$-Function

The Aleph $\aleph$-function introduced by Sudland et al. [6] which is defined as a contour integral of Mellin Barnes Type:
\[ N[Z] = \sum_{p_1, q_1, \tau_1; r}^{m,n} \left[ z \left( \frac{x}{y} \right)^q \right] \left( a_j, A_j \right)_{1,n} \cdots \left( a_j, A_j \right)_{n+1, p_i} \left( b_j, B_j \right)_{1,m} \cdots \left( b_j, B_j \right)_{m+1, q_i} \]

\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{z^{m,n}}{p_1, q_1, \tau_1; r} (s) z^{-s} ds
\]

for all \( z \neq 0 \), \( \omega = \sqrt{-1} \) and

\[
\Omega \frac{m,n}{p_1, q_1, \tau_1; r} (s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\sum_{i=1}^{r} \tau_i \prod_{j=n+1}^{p_i} \Gamma(a_j + A_j s) \prod_{j=m+1}^{q_i} \Gamma(1 - b_j - B_j s)}
\]

(2.2)

The integration path \( \ell = \ell_{й\varepsilon}, \gamma \in R \) extends from \( \gamma - i\infty \) to \( \gamma + i\infty \), and is such that the poles, assumed to be simple, of \( \Gamma(1 - a_j - A_j s), j = 1, n \) do not coincide with the pole \( \Gamma(b_j + B_j s), j = 1, m \). The parameters \( p_i, q_i \) are non-negative integers satisfying \( 0 \leq n \leq p_i, 1 \leq m \leq q_i, \tau_i > 0 \) for \( i = 1, r \). The parameters \( A_j, B_j, A_{ji}, B_{ji} > 0 \) and \( a_j, b_j \in C \). The empty product in (1.3) is interpreted as unity. The existence conditions for the defining integral (1.1) are given below:

\[ \psi_i > 0, \quad |\arg(z)| < \frac{\pi}{2} \psi_i, \quad l = 1, r; \]

(3)

\[ \psi_i \geq 0, \quad |\arg(z)| < \frac{\pi}{2} \psi_i \quad \text{and} \quad \Re\{\zeta_i\} + 1 < 0, \]

(4)

Where

\[ \psi_i = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_i \left( \sum_{j=n+1}^{p_i} A_{ji} + \sum_{j=m+1}^{q_i} B_{ji} \right) \]

(5)

\[ \zeta_i = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_i \left( \sum_{j=n+1}^{q_i} b_{ji} - \sum_{j=m+1}^{p_i} a_{ji} \right) + \frac{1}{2} (p_i - q_i), \quad l = 1, r. \]

(6)

The general polynomials of \( R \) variables given by Srivastava [5] defined and represented as:
\[ S_{n_1,...,n_R}^{m_1,...,m_R} [x_1,...x_R] = \sum_{s_1=0}^{n_1/m_1} \ldots \sum_{s_R=0}^{n_R/m_R} \frac{(-n_1)^{m_1}s_1}{\angle s_1} \ldots \frac{(-n_R)^{m_R}s_R}{\angle s_R} \]  

(2.7)

\[ A[n_1,s_1;...;n_R,R_sR] x_1^{s_1} \ldots x_R^{s_R} \]

Where \( m_i \) is an arbitrary positive integer and coefficients \( A[n_1,s_1;...;n_R,R_sR] \) are arbitrary constants, real or complex.

**Theorem 1.** With the set of sufficient conditions (2.3), (2.4), (2.5) and (2.6), let \( (\eta, u, u_1, \ldots, u_R, \beta \in \mathbb{C}, \text{Re}(1 + \frac{\eta}{1 - \alpha}) > 0, \alpha < 1, \text{Re}(\eta, u, u_1, \ldots, u_R, \beta) > 0 \) and \( m_i \) is an arbitrary positive integer and coefficients \( A[n_1,s_1;...;n_R,R_sR] \) are arbitrary constants, real or complex, then

\[ p_0(\eta, \alpha)^{(m,n)} x^{u-1} S_{n_1,...,n_R}^{m_1,...,m_R} [x_1, \ldots, x_R] \prod_{i=1}^{m,n} \left[ \frac{x^\beta}{[a(1-\alpha)]y + u_i^{k_i} + \ldots + u_R^{k_R}} \right] \]

\[ \frac{\eta + u + u_i^{k_i} + \ldots + u_R^{k_R} \Gamma\left(1 + \frac{\eta}{1 - \alpha}\right)}{[a(1-\alpha)]u + u_i^{k_i} + \ldots + u_R^{k_R}} \]

(2.8)

**Proof:** The Theorem 1 can be evaluated by using the definitions (1.1), (2.1) and (2.7) then by interchange the order of integrations and summations (which is permissible under the conditions stated above), evaluate the inner integral by making use of beta function formula, we arrive at the desired result.

**Theorem 2.** Let \( (\eta, u, u_1, \ldots, u_R, \beta \in \mathbb{C}, \text{Re}(1 + \frac{\eta}{1 - \alpha}) > 0, \alpha < 1, \text{Re}(\eta, u, u_1, \ldots, u_R, \beta) > 0 \) and \( m \) is an arbitrary positive integer and coefficients \( A[n_1,s_1;...;n_R,R_sR] \) are arbitrary constants, real or complex, then
\[ P_{0+}^{(\eta, \alpha)} \left[ x^{u-1} \sum_{n_1}^{m_1} \cdots \sum_{n_R}^{m_R} \prod_{i=1}^{u} x^{u_i} \right] \frac{H_{p, q, i}^{m, n} \left[ x^{\beta} \right]}{s_{n_1}^{m_1} \cdots s_{n_R}^{m_R}} \]

\[ = \sum_{s_1} \cdots \sum_{s_R} (-1)^{m_1 s_1} \cdots (-1)^{m_R s_R} A[n_1, s_1, \ldots, n_R s_R] \]

\[ x^{\eta + u + u_i k_i + \ldots + u R_k R} \Gamma \left( 1 + \frac{\eta}{1 - \alpha} \right) \]

\[ \frac{1}{[a(1 - \alpha)]^{u + u_i k_i + \ldots + u R_k R}} \]

Where \( H_{m, n}^{p, q, i} \left[ x \right] \) is the Fox’s H-Function [4].

**Proof:** The result in (2.2) can be derived from Theorem 1 by taking \( \tau_1 = \ldots = \tau_r = 1 \) and \( r = 1 \). We have the required result.

**Theorem 3.** Suppose that the conditions corresponding to Theorem 2 are satisfied. Then

\[ P_{0+}^{(\eta, \alpha)} \left[ x^{u-1} \sum_{n_1}^{m_1} \cdots \sum_{n_R}^{m_R} \prod_{i=1}^{u} x^{u_i} \right] \frac{H_{p, q, i}^{m, n} \left[ x^{\beta} \right]}{s_{n_1}^{m_1} \cdots s_{n_R}^{m_R}} \]

\[ = \sum_{s_1} \cdots \sum_{s_R} (-1)^{m_1 s_1} \cdots (-1)^{m_R s_R} A[n_1, s_1, \ldots, n_R s_R] \]

\[ x^{\eta + u + u_i s_i} \Gamma \left( 1 + \frac{\eta}{1 - \alpha} \right) \]

\[ \frac{1}{[a(1 - \alpha)]^{u + u_i k_i}} \]

\[ \sum_{s_1} \cdots \sum_{s_R} (-1)^{m_1 s_1} \cdots (-1)^{m_R s_R} A[n_1, s_1, \ldots, n_R s_R] \]

\[ H_{m, n}^{p, q, i} \left[ x \right] \]

where \( H_n(x) \) is the Hermite polynomials.

(2.10)
III. Special Cases

1. Letting $R = 1$ in the result (2.9), we get the result recently obtained by Chaurasia and Ghiya [9] for $\rho, \rho_1,$ and $\rho_2 \to 0$.

2. Letting $n_i \to 0, i' = 1, \ldots, R$, in the result (2.9), we get the result obtained by Nair in [8].

References Références Referencias


**Proof:** In Theorem 1, if we take $R = 1, m_1 = 2$ and $A_{n_1, s_1} = (-1)^{s_1}$, then we get the desired result.