Global Journal of Science Frontier Research
Mathematics and Decision Sciences
Volume 13 Issue 4 Version 1.0 Year 2013
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Some Further Developments in the Infinite Product Representation of Elementary Functions

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GJSFR-F Classification : *MSC 2010: 08B25 , 26A09*

SOME FURTHER DEVELOPMENTS IN THE INFINITE PRODUCT REPRESENTATION OF ELEMENTARY FUNCTIONS

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1. C. Caratheodory, Conformal representation, Cambridge university press, 1969.

Some Further Developments in the Infinite Product Representation of Elementary Functions

Viktor Reshniak

Abstract - An innovatory approach has been recently proposed for the derivation of infinite product representation of elementary functions. The approach is based on the comparison of different alternative forms of Green's functions for boundary-value problems stated for the two-dimensional Laplace equation. A number of new infinite product representations of elementary functions was actually derived within the scope of that approach. The present study continues the trend: it aims at an analysis of the approach and exploring ways for its extending to some other problem statements that might also be efficiently treated.

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I. Introduction

In a series of recent works (see, for example, [6] and [7]), an innovatory approach was proposed to the derivation of infinite product representation of elementary functions. The approach is based on the multiplicity of forms of Green's functions for some boundary-value problems. To introduce the key idea of the approach, we turn to one of the simplest problem settings of that kind. That is the Dirichlet problem

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \qquad (x,y) \in \Omega \tag{1}$$

$$u(x,0) = u(x,b) = 0, \qquad \lim_{x \to \pm\infty} u(x,y) < \infty \tag{2}$$

posed in the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < b\}$.
The classical representation

$$G(x,y;\xi,\eta) = \frac{1}{4\pi} \ln \frac{1 - 2e^{\omega(x-\xi)} \cos\omega(y+\eta) + e^{2\omega(x-\xi)}}{1 - 2e^{\omega(x-\xi)} \cos\omega(y-\eta) + e^{2\omega(x-\xi)}}, \qquad \omega = \frac{\pi}{b} \tag{3}$$

of the Green's function for the setting in (1)-(2) can be obtained by one of the two standard methods available in the field. Indeed, either the conformal mapping method [1, 2] algorithm, or the method of eigenfunction expansion [4, 5] procedure, with subsequent summation of trigonometric series that represents the Green's function, appear successful.

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In [6], it was recalled that if the method of images routine (that is also considered workable, but efficient in a limited number of problem settings) is used instead, then we arrive at an alternative to (3) expression

$$G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\prod_{n=-\infty}^{\infty}\sqrt{\frac{(x-\xi)^2+(y+\eta-2nb)^2}{(x-\xi)^2+(y-\eta+2nb)^2}} \tag{4}$$

Since the forms in (3) and (4) are equivalent, we obtain the multi-variable identity

$$\frac{1-2e^{\omega(x-\xi)}\cos\omega(y+\eta)+e^{2\omega(x-\xi)}}{1-2e^{\omega(x-\xi)}\cos\omega(y-\eta)+e^{2\omega(x-\xi)}}=\prod_{n=-\infty}^{\infty}\frac{(x-\xi)^2+(y+\eta-2nb)^2}{(x-\xi)^2+(y-\eta+2nb)^2} \tag{5}$$

which can be used as a starting point for the derivation of a number of infinite product representations of elementary functions.

Upon some trivial algebra, the identity in (5) can, for example, be transformed in the infinite product representation

$$\sin t=\frac{2t}{\pi}\prod_{k=1}^{\infty}\left[1+\frac{4t^2-\pi^2}{(1-4k^2)\pi^2}\right]$$

of the trigonometric sine function. The latter can be considered as an equivalent alternative form to the classical Euler infinite product expansion

$$\sin t=t\prod_{k=1}^{\infty}\left(1-\frac{t^2}{k^2\pi^2}\right)$$

of the sine function.

A score of new as well as alternative to existing infinite product representations of elementary functions was presented in [7] by the approach just described. Note, that the author of [7] has claimed that research in the area is still open. Accepting the challenge, we will show in the next section, that the approach also appears efficient in a few other cases.

II. Further Extension of the Approach

a) Semi-infinite Strip Region

We start with the mixed boundary-value problem

$$u(0,y)=\frac{\partial u(a,y)}{\partial x}=0 \tag{6}$$

$$u(x,0)=0,\qquad \lim_{y\to\infty}u(x,y)<\infty \tag{7}$$

posed for the two-dimensional Laplace equation in the semi-infinite strip region $\Omega=\{0<x<a,0<y<\infty\}$.

Following the technique proposed in [6] and [7], two equivalent representations of the Green's function for the problem in (6)-(7) can be found. The classical approach of eigenfunction expansion leads to a closed analytical form of its solution. On the other

Ref.

6. Yu. A. Melnikov, A new approach to the representation of some trigonometric and hyperbolic functions by infinite products, J. Math. Anal. Appl., 344, 1, 2008, 521-534.

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hand, another classical approach – the method of images - provides an infinite product expression of the required Green's function.

Method of eigenfunction expansion. Let the Poisson equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = -f(x,y), \qquad (x,y) \in \Omega \tag{8}$$

Ref

be subject to the boundary conditions in (6)-(7).
Once the solution of the stated problem is found as the integral

$$u(x,y) = \iint_{\Omega} G(x,y;\xi,\eta) f(\xi,\eta) d\Omega(\xi,\eta), \qquad (x,y) \in \Omega \tag{9}$$

the kernel function $G(x,y;\xi,\eta)$ of the above represents the Green's function to the boundary value problem in (6)-(7).

According to the theory of Fourier series, when the eigenfunctions are known, the solution of the given boundary-value problem can be obtained by the superposition [5]

$$u(x,y) = \sum_{k=1}^{\infty} Y_k(y) X_k(x) \tag{10}$$

where X_k is a complete system of eigenfunctions, orthogonal in [0,a]

$$X_k = \sin \lambda_k x,$$
$$\lambda_k = \frac{\pi(2k-1)}{2a}, \quad k = 1, 2, \ldots$$

Functions $Y_k(y)$ are to be determined from the Fourier series expansion of the right-hand side part in (8)

$$f(x,y) = \sum_{k=1}^{\infty} f_k(y) X_k(x), \qquad f_k(y) = \frac{2}{a}\int_0^a f(\xi,y) X_k(\xi) d\xi \tag{11}$$

resulting in

$$\frac{d^2 Y_k(y)}{dy^2} - \lambda_k^2 Y_k^2(y) = -\frac{2}{a}\int_0^a f(\xi,y)\sin\lambda_k\xi\, d\xi \tag{12}$$

$$Y_k(0) = 0 \tag{13}$$

$$\lim_{y\to\infty} |Y_k(y)| < \infty \tag{14}$$

Clearly, solution of the problem in (12)-(14) has the form

$$Y_k(y) = \int_0^{\infty} f_k(\eta) g_k(y;\eta) d\eta \tag{15}$$

5. R. Haberman, Applied Partial Differential Equations with Fourier Series and Boundary Value Problems, Pearson, 2012.

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where $g_k(y;\eta)$ is the Green's function for the homogeneous equation corresponding to (12) with boundary conditions in (13) and (14) ([4],[7])

$$g_k(y;\eta) = \frac{1}{2\lambda_k}\begin{cases} e^{\lambda_k(y-\eta)} - e^{-\lambda_k(y+\eta)}, & y \le \eta \\ e^{\lambda_k(\eta-y)} - e^{-\lambda_k(y+\eta)}, & y \ge \eta \end{cases}$$

Therefore the series in (10), along with (15) solves the problem posed in (6)-(8)

$$u(x,y) = \int_0^a \int_0^\infty \left[\frac{2}{a}\sum_{k=1}^{\infty} g_k(y;\eta)\ \sin\lambda_k\xi\ \sin\lambda_k x\right] f(\xi,\eta)d\xi d\eta$$

In light of (9) the kernel function of the above expression represents the Green's function to the problem in (6)-(7) posed for the Laplace equation in Ω

$$G(x,y;\xi,\eta) = \frac{2}{a}\sum_{k=1}^{\infty} g_k(y;\eta)\ \sin\lambda_k\xi\ \sin\lambda_k x \tag{16}$$

Upon the trigonometric identity

$$\sin\lambda_k\xi\ \sin\lambda_k x = \frac{1}{2}\left[\cos\lambda_k(\xi - x) - \cos\lambda_k(\xi + x)\right]$$

and (see, for example [3]) the summation formula

$$\sum_{k=1}^{\infty}\frac{p^{2k-1}\cos(2k-1)x}{2k-1} = \frac{1}{4}\ln\frac{1+2p\cos x+p^2}{1-2p\cos x+p^2}, \qquad 0 < x < 2\pi, \quad p^2 \le 1$$

the expression in (16) transforms into

$$\begin{aligned} G(x,y;\xi,\eta) = \frac{1}{4\pi}\Bigg(&\ln\frac{1+2e^{\omega(y-\eta)}\cos\ \omega(\xi-x)+e^{2\omega(y-\eta)}}{1-2e^{\omega(y-\eta)}\cos\ \omega(\xi-x)+e^{2\omega(y-\eta)}} \\ &-\ln\frac{1+2e^{\omega(y-\eta)}\cos\ \omega(\xi+x)+e^{2\omega(y-\eta)}}{1-2e^{\omega(y-\eta)}\cos\ \omega(\xi+x)+e^{2\omega(y-\eta)}} \\ &-\ln\frac{1+2e^{-\omega(y+\eta)}\cos\ \omega(\xi-x)+e^{-2\omega(y+\eta)}}{1-2e^{-\omega(y+\eta)}\cos\ \omega(\xi-x)+e^{-2\omega(y+\eta)}} \\ &+\ln\frac{1+2e^{-\omega(y+\eta)}\cos\ \omega(\xi+x)+e^{-2\omega(y+\eta)}}{1-2e^{-\omega(y+\eta)}\cos\ \omega(\xi+x)+e^{-2\omega(y+\eta)}}\Bigg) \end{aligned} \tag{17}$$

where

$$\omega = \frac{\pi}{2a}$$

After introducing the variables

$$\alpha = \frac{\pi}{2a}(y-\eta); \quad \beta = -\frac{\pi}{2a}(y+\eta); \quad \gamma = \frac{\pi}{2a}(x-\xi); \quad \theta = \frac{\pi}{2a}(\xi+x) \tag{18}$$

Ref.

4. Dean G. Duffy, Green's Functions with Applications, Chapman and Hall/CRC, 2001.

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the form in (17) reads

$$G(\alpha,\beta,\gamma,\theta)=\frac{1}{4\pi}\ln\left(\frac{\left(1+2e^{\alpha}\cos\,\gamma+e^{2\alpha}\right)}{\left(1-2e^{\alpha}\cos\,\gamma+e^{2\alpha}\right)}\frac{\left(1+2e^{\beta}\cos\,\theta+e^{2\beta}\right)}{\left(1-2e^{\beta}\cos\,\theta+e^{2\beta}\right)}\times\frac{\left(1-2e^{\alpha}\cos\,\theta+e^{2\alpha}\right)}{\left(1+2e^{\alpha}\cos\,\theta+e^{2\alpha}\right)}\frac{\left(1-2e^{\beta}\cos\,\gamma+e^{2\beta}\right)}{\left(1+2e^{\beta}\cos\,\gamma+e^{2\beta}\right)}\right) \tag{19}$$

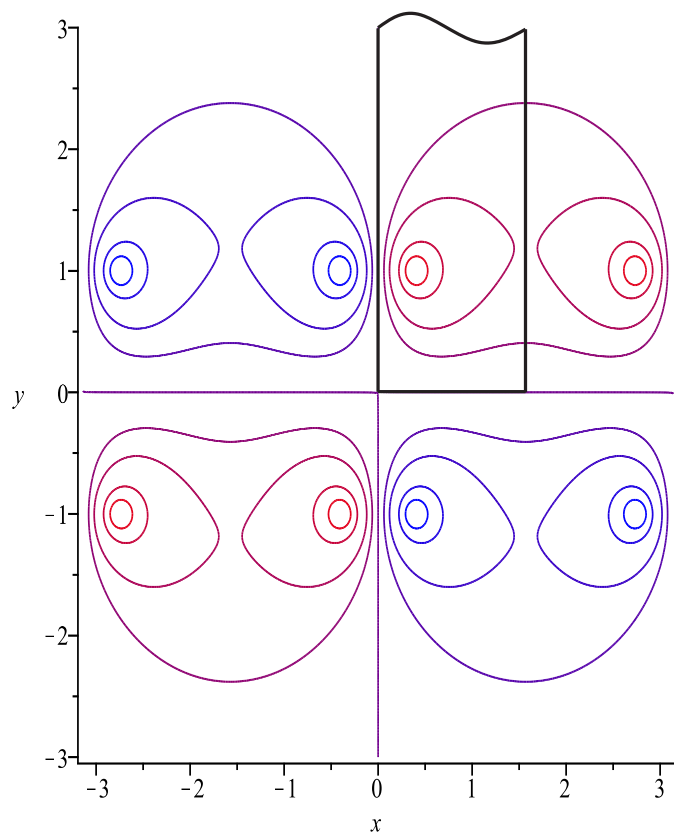

Figure 1 : Field generated in the semi-infinite strip $0\le x\le\pi/2,\ y\ge 0$ by a source located at $\xi=\pi/8,\ \eta=1$

By problem statement, the observation point (x,y) and the source point (ξ,η) are interior to the semi-infinite strip Ω. It means that parameters $\alpha,\beta,\gamma,\theta$ range as

$$-\infty<\alpha<\infty,\qquad -\infty<\beta<0,\qquad -\frac{\pi}{2}\le\gamma\le\frac{\pi}{2},\qquad 0\le\theta\le\pi \tag{20}$$

given that α and γ are not equal to zero at the same time.

But on the other hand the function in (19) is analytic everywhere except, of course, at the points of singularity, as it can be seen in Figure 1. Hence, the parameter ranges in (20) can be extended to the entire region of analyticity of (19).

Method of images. The Green's function for the two-dimensional Laplace equation has the form ([5, 7])

Ref

5. R. Haberman, Applied Partial Differential Equations with Fourier Series and Boundary Value Problems, Pearson, 2012.

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$$G(x,y;\xi,\eta) = -\frac{1}{2\pi}\ln r(x,y,\xi,\eta) + \mu(x,y;\xi,\eta)$$

where $r(x,y,\xi,\eta)$ is the distance between the field point and the source point, and $\mu(x,y;\xi,\eta)$ is a harmonic in Ω function.

According to the method of images, the regular component $\mu(x,y;\xi,\eta)$ is sought as a superposition of responses at a field point P from singularities Q_j^* placed outside the region Ω

$$\mu(x,y) = \sum_{j=1}^{m} \pm\frac{1}{2\pi}\ln|P-Q_j^*|$$

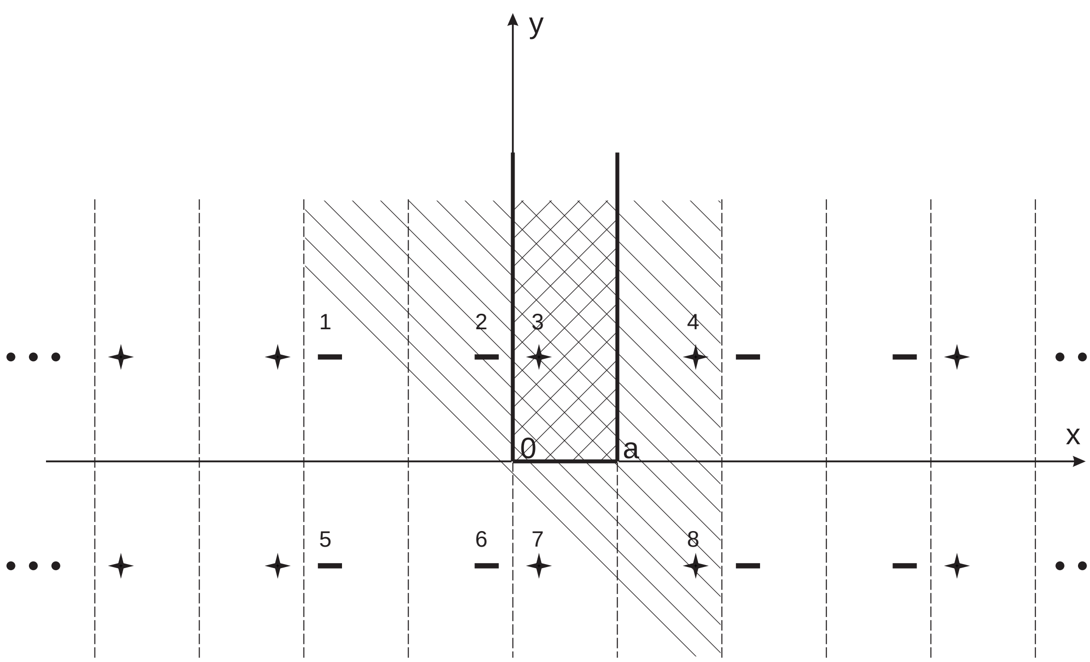

Figure 2 : Sequence of sources and sinks that arises in the method of images applied to the problem in (6)-(7)

The sign of each term in the above as well as the positions of the corresponding singularities are chosen to satisfy boundary conditions of the given boundary-value problem.

Figure 2 illustrates this approach to the problem in (6)-(7). Unit sources are labeled with the plussign and sinks with the minus sign. Geometry of Ω leads to the infinite sequence of singularities, where the shaded strip of width $4a$ is repeated periodically in the direction of the x-axis. The singularities have the following coordinates.

$(\xi_1^0 = -(2a-\xi);\ \eta_1^0 = \eta)$ $(\xi_5^0 = -(2a-\xi);\ \eta_5^0 = -\eta)$

$(\xi_2^0 = -\xi;\ \eta_2^0 = \eta)$ $(\xi_6^0 = -\xi;\ \eta_6^0 = -\eta)$

$(\xi_3^0 = \xi;\ \eta_3^0 = \eta)$ $(\xi_7^0 = \xi;\ \eta_7^0 = -\eta)$

$(\xi_4^0 = 2a-\xi;\ \eta_4^0 = \eta)$ $(\xi_8^0 = 2a-\xi;\ \eta_8^0 = -\eta)$

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Hence, the total effect on the field point P from the infinite sequence of suitably chosen sinks and sources Q_j^* is described by the following Green's function

$$G(x,y;\xi,\eta)=\sum_{k=-\infty}^{\infty}G^k(x,y;\xi,\eta)$$

$$=\frac{1}{2\pi}\sum_{k=0}^{\infty}\ln\sqrt{\frac{\left((x-\xi_1^k)^2+(y-\eta_1^k)^2\right)\left((x-\xi_2^k)^2+(y-\eta_2^k)^2\right)}{\left((x-\xi_3^k)^2+(y-\eta_3^k)^2\right)\left((x-\xi_4^k)^2+(y-\eta_4^k)^2\right)}}$$

$$\times\sqrt{\frac{\left((x-\xi_7^k)^2+(y-\eta_7^k)^2\right)\left((x-\xi_8^k)^2+(y-\eta_8^k)^2\right)}{\left((x-\xi_5^k)^2+(y-\eta_5^k)^2\right)\left((x-\xi_6^k)^2+(y-\eta_6^k)^2\right)}} \qquad (21)$$

with ξ_n^k representing the coordinates of singularities

$$\xi_n^k=\xi_n^0+4ak, \qquad k=\pm1,\pm2,\ldots$$

The form in (21) reads, in terms of the variables introduced in (18), as

$$G(\alpha,\beta,\gamma,\theta)=\frac{1}{4\pi}\ln\prod_{k=-\infty}^{\infty}\left(\frac{\left((\gamma+\pi(1-2k))^2+\alpha^2\right)\left((\theta-2\pi k)^2+\alpha^2\right)}{\left((\gamma+\pi(1-2k))^2+\beta^2\right)\left((\theta-2\pi k)^2+\beta^2\right)}\right.$$

$$\left.\times\frac{\left((\theta-\pi(1+2k))^2+\beta^2\right)\left((\gamma-2\pi k)^2+\beta^2\right)}{\left((\theta-\pi(1+2k))^2+\alpha^2\right)\left((\gamma-2\pi k)^2+\alpha^2\right)}\right) \qquad (22)$$

Infinite Products and Elementary Functions. As it follows directly from the equivalence of the representations of (19) and (22), the identity

$$\frac{\left(1+2e^{\alpha}\cos\gamma+e^{2\alpha}\right)}{\left(1-2e^{\alpha}\cos\gamma+e^{2\alpha}\right)}\frac{\left(1-2e^{\beta}\cos\gamma+e^{2\beta}\right)}{\left(1+2e^{\beta}\cos\gamma+e^{2\beta}\right)}$$

$$\times\frac{\left(1+2e^{\beta}\cos\theta+e^{2\beta}\right)}{\left(1-2e^{\beta}\cos\theta+e^{2\beta}\right)}\frac{\left(1-2e^{\alpha}\cos\theta+e^{2\alpha}\right)}{\left(1+2e^{\alpha}\cos\theta+e^{2\alpha}\right)}$$

$$=\prod_{k=-\infty}^{\infty}\left(\frac{\left((\gamma+\pi(1-2k))^2+\alpha^2\right)\left((\theta-2\pi k)^2+\alpha^2\right)}{\left((\gamma+\pi(1-2k))^2+\beta^2\right)\left((\theta-2\pi k)^2+\beta^2\right)}\right.$$

$$\left.\times\frac{\left((\theta-\pi(1+2k))^2+\beta^2\right)\left((\gamma-2\pi k)^2+\beta^2\right)}{\left((\theta-\pi(1+2k))^2+\alpha^2\right)\left((\gamma-2\pi k)^2+\alpha^2\right)}\right) \qquad (23)$$

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is valid at the region of analiticity of the function in (19).

The above identity involves four arbitrary parameters, and by a suitable choice of those some interesting representations of elementary functions can be obtained. To start with, we write its left-hand side in the form

$$\frac{(\cos\gamma + A_2)(\cos\gamma - A_1)(\cos\theta + A_1)(\cos\theta - A_2)}{(\cos\gamma - A_2)(\cos\gamma + A_1)(\cos\theta - A_1)(\cos\theta + A_2)}, \tag{24}$$

where

$$A_1 = \frac{1+e^{2\beta}}{2e^{\beta}}, \quad A_2 = \frac{1+e^{2\alpha}}{2e^{\alpha}}$$

This shows that it cannot be converted to an expression consisting of a single trigonometric function for any real value of the parameters $\alpha, \beta, \gamma, \theta$. It's also obvious from (24), that to avoid trivial results we should put some constraints on these parameters, specifically $\alpha \neq \pm\beta$, $\gamma \neq \pm\theta$.

It is worth noting that the trigonometric terms in the expression (24) depend only on the parameters γ and θ, and their successful choice allows that expression to be dependent only on the exponential terms.

As an illustration, let us consider the following parameter values: $\gamma = 0$, $\theta = \frac{\pi}{2}$, $\beta = 2u$, $\alpha = 2v$, which converts the expression in (23) into

$$\frac{\tanh^2 u}{\tanh^2 v} = \prod_{k=-\infty}^{\infty} \frac{\left(\pi^2(1+2k)^2 + 4v^2\right)\left(\pi^2 k^2 + u^2\right)}{\left(\pi^2(1+2k)^2 + 4u^2\right)\left(\pi^2 k^2 + v^2\right)} \tag{25}$$

Note that the above representation has already been derived in [6] and [7] upon considering a different problem.

If the parameter v in (25) is taken to infinity then we arrive at

$$\tanh^2 u = \prod_{k=-\infty}^{\infty} \frac{4\left(\pi^2 k^2 + u^2\right)}{\pi^2(1+2k)^2 + 4u^2} \tag{26}$$

When the parameter u is taken to infinity, the following expansion arises

$$\coth^2 v = \prod_{k=-\infty}^{\infty} \frac{\pi^2(1+2k)^2 + 4v^2}{4\left(\pi^2 k^2 + v^2\right)} \tag{27}$$

Relations in (26) and (27) have already been derived in [7] and their convergence has already been investigated.

Recalling the interrelation between hyperbolic and trigonometric functions

$$\tanh(x) = -i\ \tan(ix)$$

$$\coth(x) = i\ \cot(ix)$$

Ref.

7. Yuri A. Melnikov, Green's functions and Infinite Products: Bridging the Divide, Birkhauser-Springer, New York - London, 2011.

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the new infinite product representations

$$\tan^2 u = \prod_{k=-\infty}^{\infty} \frac{4\left(\pi^2 k^2 - u^2\right)}{4u^2 - \pi^2(1+2k)^2} \tag{28}$$

$$\cot^2 v = \prod_{k=-\infty}^{\infty} \frac{4v^2 - \pi^2(1+2k)^2}{4\left(\pi^2 k^2 - v^2\right)} \tag{29}$$

can be obtained.

To prove the convergence of the identity in (28), we rewrite it in the form

$$\tan^2 u = \frac{4u^2}{4u^2-\pi^2} \prod_{k=1}^{\infty} \frac{16\left(\pi^2 k^2 - u^2\right)^2}{\left(4u^2 - \pi^2(1+2k)^2\right)\left(4u^2 - \pi^2(1-2k)^2\right)}$$

$$= \frac{4u^2}{4u^2-\pi^2} \prod_{k=1}^{\infty} \left(1 + \pi^2 \frac{\pi^2(8k^2-1)+8u^2}{\left(4u^2 - \pi^2(1+2k)^2\right)\left(4u^2 - \pi^2(1-2k)^2\right)}\right) \tag{30}$$

Since the numerator in the second additive component in the product is a second degree polynomial in k, while the denominator is a polynomial of degree four the expansion in (30) converges uniformly for all $u \neq (1 \pm 2n)\pi/2$ at the rate of $1/k^2$.

A similar procedure shows the uniform convergence of the representation in (29)

$$\cot^2 v = \frac{\pi^2 - 4v^2}{4v^2} \prod_{k=1}^{\infty} \left(1 - \frac{\pi^2}{16}\left[\frac{\pi^2(8k^2+1)+8v^2}{(\pi^2 k^2 - v^2)^2}\right]\right)$$

for all $v \neq \pm \pi n$ at the rate of $1/k^2$.

The representations in (28) and (29) deliver

$$\tan u = \pm \prod_{k=-\infty}^{\infty} \sqrt{\frac{4\left(\pi^2 k^2 - u^2\right)}{4u^2 - \pi^2(1+2k)^2}} \tag{31}$$

where the minus sign is valid for $u \in (-\frac{\pi}{2}; 0)$ while the plus sign holds for $u \in [0; \frac{\pi}{2})$.

$$\cot v = \pm \prod_{k=-\infty}^{\infty} \sqrt{\frac{4v^2 - \pi^2(1+2k)^2}{4\left(\pi^2 k^2 - v^2\right)}} \tag{32}$$

where the plus sign is valid for $u \in (0; \frac{\pi}{2})$ while the minus corresponds to $u \in [\frac{\pi}{2}; \pi)$.

Figure 3 provides a sense of the convergence of the representations in (31).

Two other expansions can be obtained using the standard identities

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$$\sec u = \pm\sqrt{1+\tan^2 u} = \pm\sqrt{1+\prod_{k=-\infty}^{\infty}\frac{4\left(\pi^2k^2-u^2\right)}{4u^2-\pi^2(1+2k)^2}} \tag{33}$$

where the plus sign is valid for $u \in [-\frac{\pi}{2};\frac{\pi}{2})$, while the minus sign stays for $u \in [\frac{\pi}{2};\frac{3\pi}{2}]$, and

$$\csc v = \pm\sqrt{1+\cot^2 v} = \pm\sqrt{1+\prod_{k=-\infty}^{\infty}\frac{4v^2-\pi^2(1+2k)^2}{4\left(\pi^2k^2-v^2\right)}} \tag{34}$$

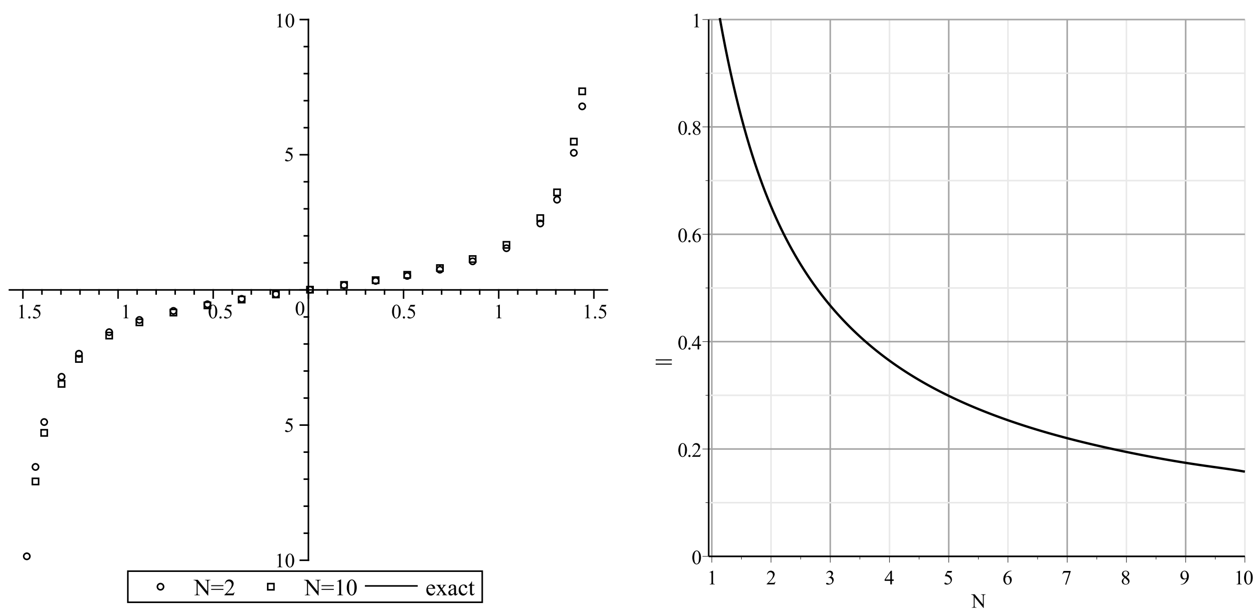
where the plus sign stays for $u \in [0;\pi)$ and the minus sign stays for $u \in [\pi;2\pi]$.

The convergence rate of the representation in (33) is illustrated in Figure 4.

Similarly to the work done for the problem stated in (6)-(7), we will consider another mixed boundary-value problem

$$u(0,y) = \frac{\partial u(a,y)}{\partial x} = 0 \tag{35}$$

$$\frac{\partial u(x,0)}{\partial x} = 0, \qquad \lim_{y\to\infty} u(x,y) < \infty \tag{36}$$



(a) Convergence with the 2nd and 10th partial products.

(b) L^2-norm of the error in $[-0.95\frac{\pi}{2}, 0.95\frac{\pi}{2}]$; N is a number of partial products

Figure 3 : Convergence of the representation in (31)

posed for the two-dimensional Laplace equation in the semi-infinite strip region $\Omega = \{0 < x < a, 0 < y < \infty\}$.

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Method of eigenfunction expansion. Upon implementing the routine applied to the problem in (6)-(7) and using the same sequence of eigenfunctions, one obtains the Green's function for the problem in (35)-(36) in the form

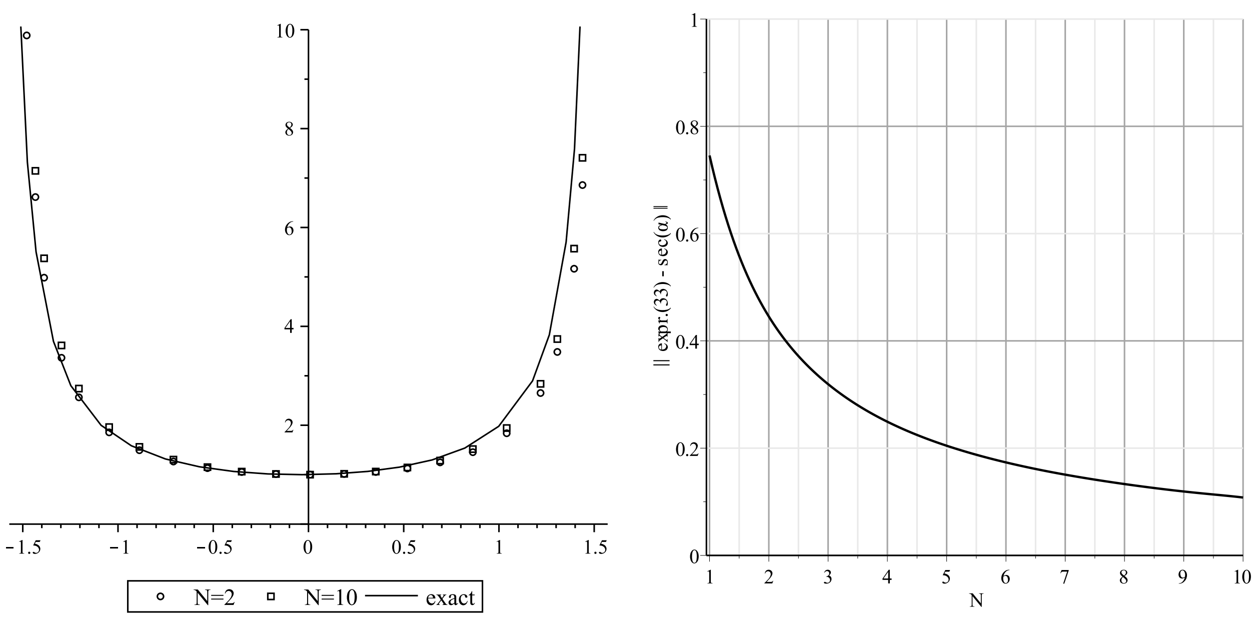
$$G(x,y;\xi,\eta)=\frac{2}{a}\sum_{k=1}^{\infty} g_k(y;\eta)\ \sin\lambda_k\xi\ \sin\lambda_k x$$

$$g_k(y;\eta)=\frac{1}{2\lambda_k}\begin{cases} e^{\lambda_k(y-\eta)}+e^{-\lambda_k(y+\eta)}, & y\le\eta \\ e^{\lambda_k(\eta-y)}+e^{-\lambda_k(y+\eta)}, & y\ge\eta \end{cases}$$

which reduces to

$$G(\alpha,\beta,\gamma,\theta)=\frac{1}{4\pi}\ln\left(\frac{\left(1+2e^{\alpha}\cos\gamma+e^{2\alpha}\right)}{\left(1-2e^{\alpha}\cos\gamma+e^{2\alpha}\right)}\frac{\left(1+2e^{\beta}\cos\gamma+e^{2\beta}\right)}{\left(1-2e^{\beta}\cos\gamma+e^{2\beta}\right)}\right.$$
$$\left.\times\frac{\left(1-2e^{\alpha}\cos\theta+e^{2\alpha}\right)}{\left(1+2e^{\alpha}\cos\theta+e^{2\alpha}\right)}\frac{\left(1-2e^{\beta}\cos\theta+e^{2\beta}\right)}{\left(1+2e^{\beta}\cos\theta+e^{2\beta}\right)}\right) \tag{37}$$

where we use the variables introduced earlier in (18).



(a) Convergence with the 2nd and 10th partial products.

(b) L^2-norm of the error in $[-0.95\frac{\pi}{2}, 0.95\frac{\pi}{2}]$; N is a number of partial products

Figure 4 : Convergence of the representation in (33)

Method of images. Distribution of singularities used in the method of images for the problem in (35)-(36) is shown in Figure 5. This gives rise to the infinite product version of the Green's function, written in terms of the variables introduced in (18), as

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$$G(x,y;\xi,\eta)=\frac{1}{4\pi}\ln\prod_{k=-\infty}^{\infty}\frac{\Big((\gamma+\pi(1-2k))^2+\alpha^2\Big)\Big((\theta-2\pi k)^2+\alpha^2\Big)}{\Big((\theta-\pi(1+2k))^2+\beta^2\Big)\Big((\gamma-2\pi k)^2+\beta^2\Big)}$$

$$\frac{\Big((\gamma+\pi(1-2k))^2+\beta^2\Big)\Big((\theta-2\pi k)^2+\beta^2\Big)}{\Big((\theta-\pi(1+2k))^2+\alpha^2\Big)\Big((\gamma-2\pi k)^2+\alpha^2\Big)}$$

Notes

which being compared with (37) constitutes another identity valid for $\alpha\neq\pm\beta,\ \gamma\neq\pm\theta$. Assuming $\gamma=0,\ \theta=\frac{\pi}{2},\ \beta=2u,\ \alpha=2v$, one arrives at

$$\coth^2 u\ \coth^2 v=\prod_{k=-\infty}^{\infty}\frac{\Big(\pi^2(1+2k)^2+4v^2\Big)\Big(\pi^2(1+2k)^2+4u^2\Big)}{\Big(\pi^2k^2+u^2\Big)\Big(\pi^2k^2+v^2\Big)}$$

Note that it has already been obtained earlier (see (25)) in this presentation.

b) Exterior of Circles

The key idea of the developments in the previous sections is to obtain an alternative to the classical expression for a Green's function as an appropriate arrangement of sinks and sources. For some regions such arrangements are periodic and described by infinite sequences of symmetrically placed images.

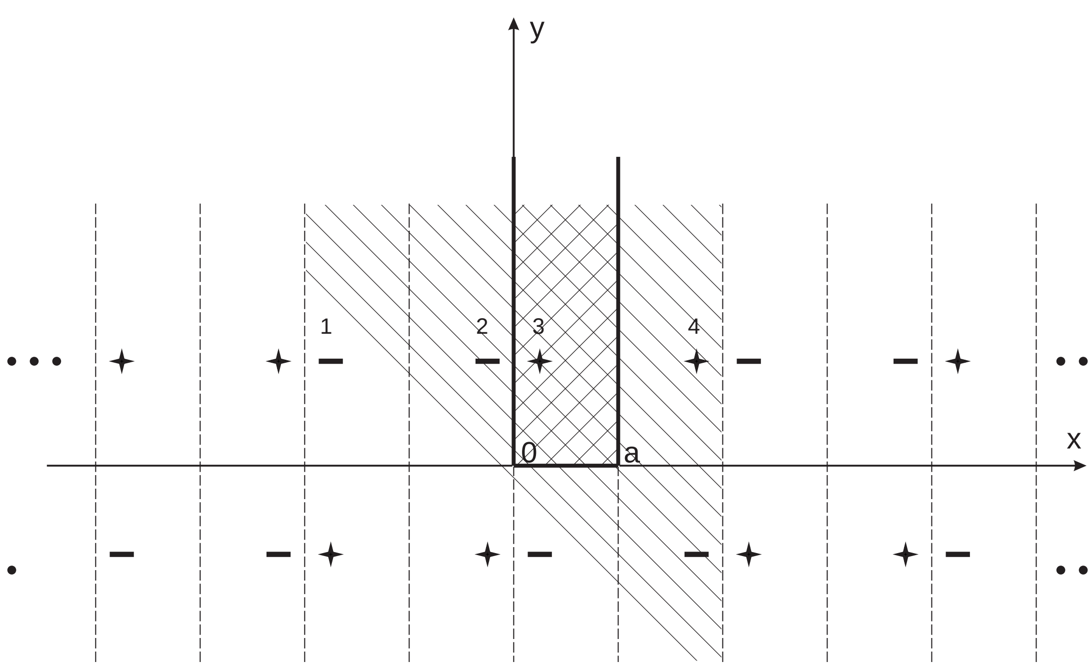

Figure 5 : Sequence of sources and sinks that arises in the method of images for the problem in (35)-(36)

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This approach was applied, for instance, in [8] to construct the potential due to a line charge in the infinite strip or in [2, 8] to construct the potential induced by a single line charge in the rectangular prism or in the circular ring. The same approach was used in [6, 7] to find a potential field induced by the Green's function to the boundary-value problem posed for the Laplace equation in infinite and semi-infinite strip region with different combinations of boundary conditions. Results obtained in the earlier works served as a basis for construction of infinite product representations of elementary functions. The present work is the logical continuation of [6, 7], and the expansions (31) - (32) and (33) - (34) complement the results derived there.

Working on further developments of the discussed approach, we have probably to turn to different geometries or different equations. One of such cases is described below.

Conformal mapping. Conformal mapping gives us a tool to derive Green's functions of the Dirichlet problem posed for the Laplace equation in a simply-connected region. If there is known a function $w(z,\zeta)$ which conformally maps the given region in the z-plane onto the interior of the unit disc in the w-plane with point ζ mapped on the center of the disc, the corresponding Green's function is presented ([1, 2, 7]) in terms of $w(z,\zeta)$ as

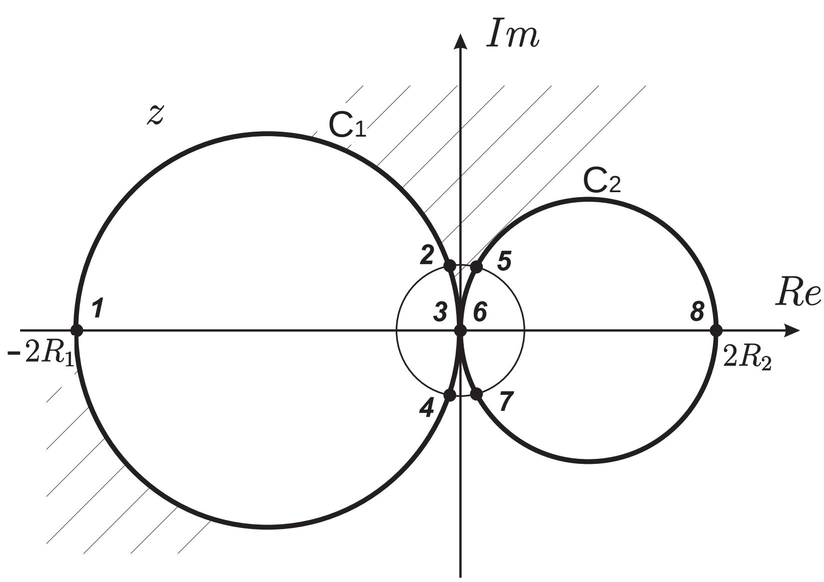
$$G(x,y;\xi,\eta) = -\frac{1}{2\pi}\ln|\omega(z,\zeta)| \qquad (38)$$

where

$$z = x + iy$$

$$\zeta = \xi + i\eta$$

As an example, consider the exterior of two circles having external contact (Figure 6). Function $f(z) = 1/z$ transforms this region into the infinite strip ([1, 2]) while the latter can be mapped onto the interior of a unit circle. Omitting cumbersome but trivial algebra, we just present the function that maps the exterior of two circles with external contact onto the interior of a unit circle as



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2. Z. Nehari, Conformal mapping, McGraw-Hill, New York, 1952.

Global Journal of Science Frontier Research (F) Volume XIII Issue IV Version I Year 2013

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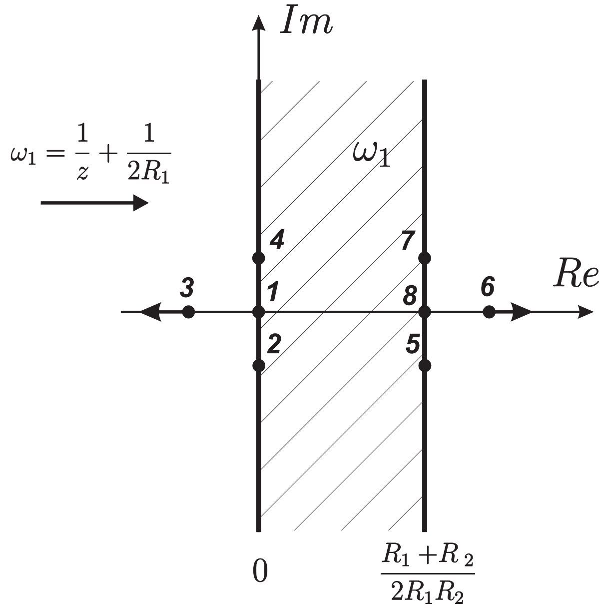

Figure 6 : Conformal mapping of the exterior of two circles with external contact onto the infinite strip

$$\omega(z,\zeta) = \frac{\exp\left(i\lambda\left(\frac{1}{z}+\frac{1}{2R_1}\right)\right) - \exp\left(i\lambda\left(\frac{1}{\zeta}+\frac{1}{2R_1}\right)\right)}{\exp\left(i\lambda\left(\frac{1}{z}+\frac{1}{2R_1}\right)\right) - \exp\overline{\left(i\lambda\left(\frac{1}{\zeta}+\frac{1}{2R_1}\right)\right)}}$$

where

$$\lambda = 2\pi\frac{R_1R_2}{R_1+R_2}$$

The modulus of the above reads as

$$|\omega(z,\zeta)| = \sqrt{\frac{e^{2\lambda\gamma}+e^{2\lambda\theta}-2e^{\lambda\gamma}e^{\lambda\theta}\cos\lambda(\alpha-\beta)}{e^{2\lambda\gamma}+e^{2\lambda\theta}-2e^{\lambda\gamma}e^{\lambda\theta}\cos\lambda\left(\alpha+\beta+\frac{1}{R_1}\right)}}$$

where

$$\alpha = \frac{x}{x^2+y^2}, \quad \beta = \frac{\xi}{\xi^2+\eta^2}, \quad \gamma = \frac{y}{x^2+y^2}, \quad \theta = \frac{\eta}{\xi^2+\eta^2} \tag{39}$$

Hence, in light of (38), the Green's function to the Dirichlet problem for the region depicted in Figure 6 appears as

$$G(x,y;\xi,\eta) = \frac{1}{4\pi}\ln\frac{e^{2\lambda\gamma}+e^{2\lambda\theta}-2e^{\lambda\gamma}e^{\lambda\theta}\cos\left(\lambda\left[\alpha+\beta+\frac{1}{R_1}\right]\right)}{e^{2\lambda\gamma}+e^{2\lambda\theta}-2e^{\lambda\gamma}e^{\lambda\theta}\cos\left(\lambda(\alpha-\beta)\right)} \tag{40}$$

Notes

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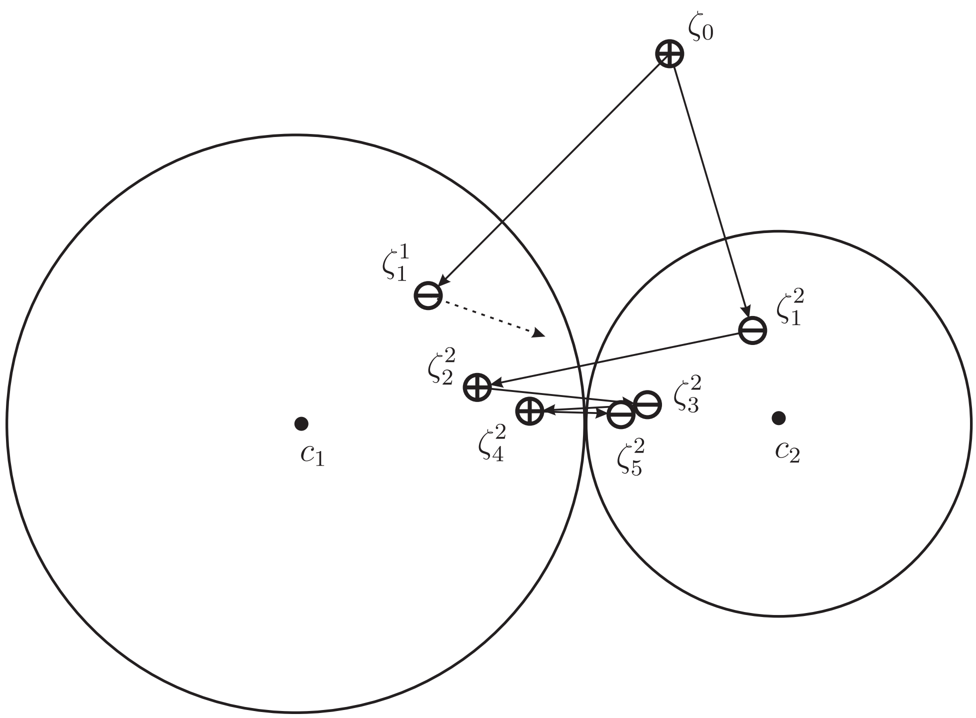

Figure 7: Sequence of sources and sinks that arises in the method of images

Method of images. In the case when boundaries of the region are formed by straight lines, images are placed symmetrically with respect to boundaries in a straightforward manner. But every straight line can be considered as a circumference with infinite radius. Thus the method of images can also be generalized to regions formed by circular arcs. In order to implement this idea, recall that an inversion with respect to a circle with radius R centered at a point c is given by the formula

$$z_1 = f(z_0) = \frac{R^2}{\overline{z_0 - c}} + c \tag{41}$$

where the points c, z_0, z_1 lie in the same plane.

Thus the Green's function to the Dirichlet problem posed for the Laplace equation in the exterior of a circle is given by the expression

$$G(x, y; \xi, \eta) = -\frac{1}{2\pi} \ln \frac{|z - z_0|}{|z - z_1|} + \mu(x, y; \xi, \eta) \tag{42}$$

The compensatory function $\mu(x, y; \xi, \eta)$ can easily be derived from the fact that the argument of the logarithm in (42) has a constant value on the circumference of a circle and is equal to $\frac{|z_0 - c|}{R}$.

Thus the formula (42) converts to

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{|z - z_1||z_0 - c|}{|z - z_0|R}$$

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In the case of two circles, each inversion of the form in (41) with respect to the first circle perturbs the potential on the circumference of the second circle and thus, should be compensated by another inversion with respect to the perturbed circle and so on. Consequently to satisfy the Dirichlet boundary conditions we should use the infinite sequence of inversions (Figure 7). This generates the potential field

$$F(z,\zeta_n^j)=\frac{1}{2\pi}\left(\ln\frac{|\zeta_1^1-z||\zeta_0^1-c_1|}{|\zeta_0^1-z|R_1}+\ln\frac{|\zeta_1^2-z||\zeta_0^2-c_2|}{|\zeta_0^2-z|R_2}\right.$$
$$-\ln\frac{|\zeta_2^1-z||\zeta_1^1-c_2|}{|\zeta_1^1-z|R_2}-\ln\frac{|\zeta_2^2-z||\zeta_1^2-c_1|}{|\zeta_1^2-z|R_1}$$
$$\left.+\ln\frac{|\zeta_3^1-z||\zeta_2^1-c_1|}{|\zeta_2^1-z|R_1}+\ln\frac{|\zeta_3^2-z||\zeta_2^2-c_2|}{|\zeta_2^2-z|R_2}-\ldots\right)$$

Clearly, all the singularities appear twice in the above expression and hence the Green's function has the following infinite product form

$$\begin{aligned}G(x,y;\xi,\eta)&=\frac{1}{2}F(z;\zeta_n^j)\\&=\frac{1}{4\pi}\ln\prod_{k=0}^{\infty}\frac{|\zeta_{2k+1}^1-z|^2|\zeta_{2k+1}^2-z|^2|\zeta_{2k}^1-c_1||\zeta_{2k}^2-c_2|}{|\zeta_{2k}^1-z||\zeta_{2k}^2-z||\zeta_{2k+2}^1-z||\zeta_{2k+2}^2-z||\zeta_{2k+1}^1-c_2||\zeta_{2k+1}^2-c_1|}\end{aligned}\tag{43}$$

where $z= x+iy$ is a field point, c_1, c_2, R_1, R_2 are centres and radii of the circles, $\zeta_0^1=\zeta_0^2$ $=z_0=\xi+i\eta$ is a source point and ζ_k^j are points at which singularities are placed

$$\zeta_k^1=\begin{cases}\dfrac{R_2^2}{\overline{\overline{\zeta_{k-1}^1-c_2}}}+c_2, & \text{if } k \text{ is even}\\[2ex]\dfrac{R_1^2}{\overline{\overline{\zeta_{k-1}^1-c_1}}}+c_1, & \text{if } k \text{ is odd}\end{cases}$$

$$\zeta_k^2=\begin{cases}\dfrac{R_1^2}{\overline{\overline{\zeta_{k-1}^2-c_1}}}+c_1, & \text{if } k \text{ is even}\\[2ex]\dfrac{R_2^2}{\overline{\overline{\zeta_{k-1}^2-c_2}}}+c_2, & \text{if } k \text{ is odd}\end{cases}\tag{44}$$

In our case, $c_1=-R_1, c_2=R_2$ and the expressions in (44) transform to

$$\zeta_k^1=\begin{cases}\dfrac{z_0}{\dfrac{k}{2}\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right)z_0+1}, & \text{if } k \text{ is even}\\[4ex]\dfrac{-\overline{z}_0}{\dfrac{1}{2}\left(\dfrac{k+1}{R_1}+\dfrac{k-1}{R_2}\right)\overline{z}_0+1}, & \text{if } k \text{ is odd}\end{cases}$$

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$$\zeta_k^2 = \begin{cases} \dfrac{-z_0}{\dfrac{k}{2}\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right)z_0 - 1}, & \text{if } k \text{ is even} \\[2ex] \dfrac{\overline{z}_0}{\dfrac{1}{2}\left(\dfrac{k-1}{R_1}+\dfrac{k+1}{R_2}\right)\overline{z}_0 - 1}, & \text{if } k \text{ is odd} \end{cases}$$

Notes

When k approaches infinity both values ζ_k^1 and ζ_k^2 tend to zero, implying that the limit of the general term in (43) equals unity. That is

$$\lim_{k\to\infty}\left(\frac{|\zeta_{2k+1}^1 - z|^2|\zeta_{2k+1}^2 - z|^2|\zeta_{2k}^1 + R_1||\zeta_{2k}^2 - R_2|}{|\zeta_{2k}^1 - z||\zeta_{2k}^2 - z||\zeta_{2k+2}^1 - z||\zeta_{2k+2}^2 - z||\zeta_{2k+1}^1 - R_2||\zeta_{2k+1}^2 + R_1|}\right) = 1$$

and thus the infinite product in (43) is absolutely and uniformly convergent.

Infinite Products and Elementary Functions. Obviously, in (40) and (43) one finds just alternative representations of the same Green's function, and we arrive at the multivariable identity

$$\frac{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta}\cos\left(\lambda\left[\alpha+\beta+\dfrac{1}{R_1}\right]\right)}{e^{2\lambda\gamma} + e^{2\lambda\theta} - 2e^{\lambda\gamma}e^{\lambda\theta}\cos\left(\lambda(\alpha-\beta)\right)}$$

$$= \prod_{k=0}^{\infty}\frac{|\zeta_{2k+1}^1 - z|^2|\zeta_{2k+1}^2 - z|^2|\zeta_{2k}^1 + R_1||\zeta_{2k}^2 - R_2|}{|\zeta_{2k}^1 - z||\zeta_{2k}^2 - z||\zeta_{2k+2}^1 - z||\zeta_{2k+2}^2 - z||\zeta_{2k+1}^1 - R_2||\zeta_{2k+1}^2 + R_1|} \quad (45)$$

Complex variables z and z_0 can be written in terms of real parameters α, β, γ, θ, which were introduced earlier in (39), as

$$z_0 = \xi + i\eta = \frac{\beta + i\theta}{\beta^2+\theta^2}$$

$$z = x + iy = \frac{\alpha + i\gamma}{\beta^2+\theta^2}$$

Thus, the expression in (45) has six arbitrary parameters: α, β, γ, θ, R_1 and R_2. If $\lambda = 1$, then $R_2 = R_1/(2\pi R_1 - 1)$ and the number of independent parameters reduces to five.

Trigonometric functions. For any fixed value of the parameters γ and θ the expression in (45)involves only trigonometric functions. Particularly, when $\gamma = 0$ and $\theta = 0$ the left-hand side of (45) represents the following elementary function

$$\frac{1-\cos\left(\alpha+\beta+\dfrac{1}{R_1}\right)}{1-\cos(\alpha-\beta)} \quad (46)$$

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If $R_1 = \dfrac{1}{n\pi}$ $\left(R_2 = \dfrac{1}{\pi(2-n)}, n = 0, 1, ...\right)$ the above expression converts to

$$\frac{1-(-1)^n \cos(\alpha+\beta)}{1-\cos(\alpha-\beta)} \tag{47}$$

Similarly, when $R_1 = \dfrac{2}{\pi(1+2n)}$ $\left(R_2 = \dfrac{2}{\pi(3-2n)}, n = 0, 1, ...\right)$, the function (46) converts to

$$\frac{1+(-1)^n \sin(\alpha+\beta)}{1-\cos(\alpha-\beta)}$$

So the functions (47) and (48) have the following infinite product representations

$$\frac{1-(-1)^n \cos(\alpha+\beta)}{1-\cos(\alpha-\beta)} = \prod_{k=0}^{\infty} \frac{\left[\alpha+\beta+\pi(2k+n)\right]^2 \left[\alpha+\beta-\pi(2(k+1)-n)\right]^2}{\left|(\alpha-\beta)^2-4\pi^2k^2\right|\left|(\alpha-\beta)^2-4\pi^2(k+1)^2\right|} \tag{49}$$

and

$$\frac{1+(-1)^n \sin(\alpha+\beta)}{1-\cos(\alpha-\beta)} = \prod_{k=0}^{\infty} \frac{\left[2(\alpha+\beta)+\pi(1+4k+2n)\right]^2 \left[2(\alpha+\beta)-\pi(3+4k-2n)\right]^2}{16\left|(\alpha-\beta)^2-4\pi^2k^2\right|\left|(\alpha-\beta)^2-4\pi^2(k+1)^2\right|} \tag{50}$$

Different interrelations between the parameters α and β are possible. For instance, when $\beta = \alpha \pm (2p+1)\pi$ $(p = 0, 1, 2, \dots)$, the expressions in (49) and (50) read

$$\frac{1+(-1)^n \cos(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{\left[\left(2\alpha+\pi(2p+n)\right)^2 - \pi^2(2k+1)^2\right]^2}{\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}, \tag{51}$$

$$\frac{1+(-1)^n \cos(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{\left[\left(2\alpha-\pi(2(p+1)-n)\right)^2 - \pi^2(2k+1)^2\right]^2}{\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|} \tag{52}$$

and

$$\frac{1-(-1)^n \sin(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{\left[\left(4\alpha+\pi(2(2p+n)+1)\right)^2 - 4\pi^2(2k+1)^2\right]^2}{16\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|},$$

$$\frac{1-(-1)^n \sin(2\alpha)}{2} = \prod_{k=0}^{\infty} \frac{\left[\left(4\alpha-\pi(2(2p-n)+3)\right)^2 - 4\pi^2(2k+1)^2\right]^2}{16\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}$$

which, in turn, provide us with infinite product representations of the sine and cosine functions

Notes

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Notes

$$\cos\alpha = (-1)^n\left(-1+2\prod_{k=0}^{\infty}\frac{\left[\left(\alpha+\pi\left(2p+n\right)\right)^2-\pi^2(2k+1)^2\right]^2}{\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}\right) \tag{53}$$

$$\cos\alpha = (-1)^n\left(-1+2\prod_{k=0}^{\infty}\frac{\left[\left(\alpha-\pi(2(p+1)-n)\right)^2-\pi^2(2k+1)^2\right]^2}{\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}\right) \tag{54}$$

and

$$\sin\alpha = (-1)^n\left(1-2\prod_{k=0}^{\infty}\frac{\left[\left(2\alpha+\pi\left(2(2p+n)+1\right)\right)^2-4\pi^2(2k+1)^2\right]^2}{16\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}\right) \tag{55}$$

$$\sin\alpha = (-1)^n\left(1-2\prod_{k=0}^{\infty}\frac{\left[\left(2\alpha-\pi\left(2(2p-n)+3\right)\right)^2-4\pi^2(2k+1)^2\right]^2}{16\pi^4\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}\right) \tag{56}$$

where $n, p = 0, 1, 2, ...$ For example, in the case of $n = 1$ and $p = 0$ the identities in (53)-(56) reduce to

$$\cos\alpha = 1+2\prod_{k=0}^{\infty}\frac{\left((\alpha+\pi)^2-\pi^2(2k+1)^2\right)^2}{\pi^4\left(4k^2-1\right)\left(4(k+1)^2-1\right)} \tag{57}$$

$$\cos\alpha = 1+2\prod_{k=0}^{\infty}\frac{\left((\alpha-\pi)^2-\pi^2(2k+1)^2\right)^2}{\pi^4\left(4k^2-1\right)\left(4(k+1)^2-1\right)} \tag{58}$$

and

$$\sin\alpha = -1-2\prod_{k=0}^{\infty}\frac{\left((2\alpha+3\pi)^2-4\pi^2(k+1)^2\right)^2}{16\pi^4\left(4k^2-1\right)\left(4(k+1)^2-1\right)} \tag{59}$$

$$\sin\alpha = -1-2\prod_{k=0}^{\infty}\frac{\left((2\alpha-\pi)^2-4\pi^2(k+1)^2\right)^2}{16\pi^4\left(4k^2-1\right)\left(4(k+1)^2-1\right)} \tag{60}$$

Figures 8 and 9 illustrate the sensitivity of the convergence of the representation in (53) to the choice of the parameters n and p. It is seen that the parameter values have the dramatical influence on the rate of convergence. The identities in (54)-(56) exhibit a similar behaviour.

By applying the power reduction formula to the expressions in (51) and (52) several new infinite product expansions for the sine and cosine functions can be obtained.

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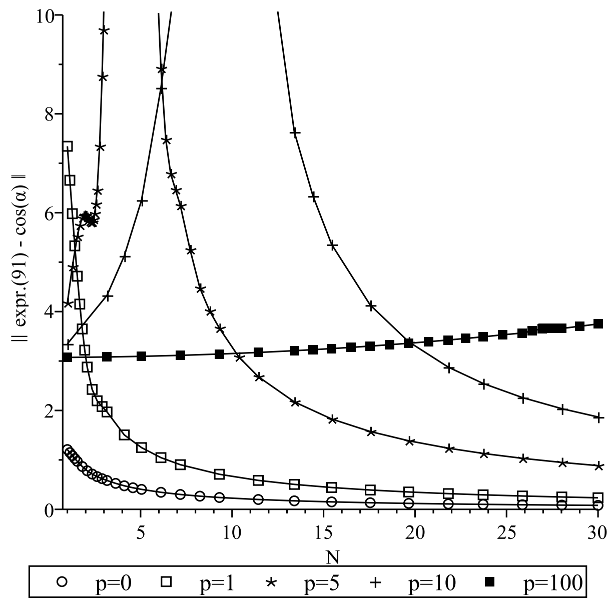
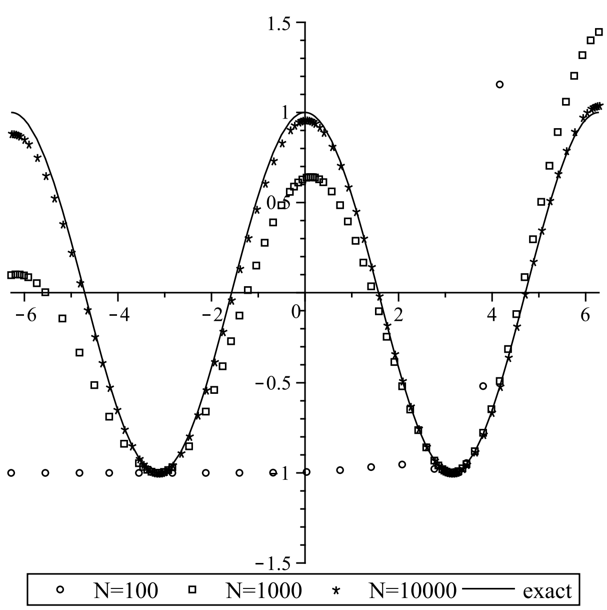
If n in (51) and (52) is an even number, i.e. $n = 2n$, then the expansions of the cosine function can be written as

$$\cos\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|4\Big(\alpha+\pi\big(p+n\big)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}} \tag{61}$$



(a) L^2-norm of the error in $[0, 2\pi]$; N is a number of partial products.



(b) Convergence of the infinite product expansion when $n = 0$, $p = 100$

Figure 8 : Convergence of the representation in (53) for different values of the parameter p; parameter $n = 0$ is fixed

Notes

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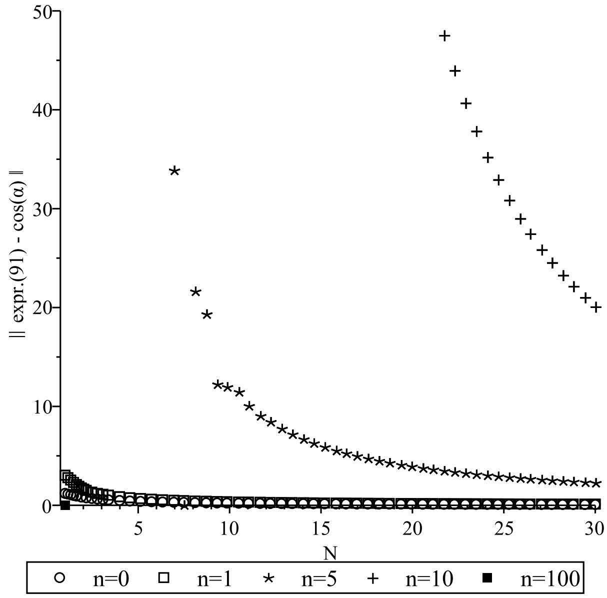
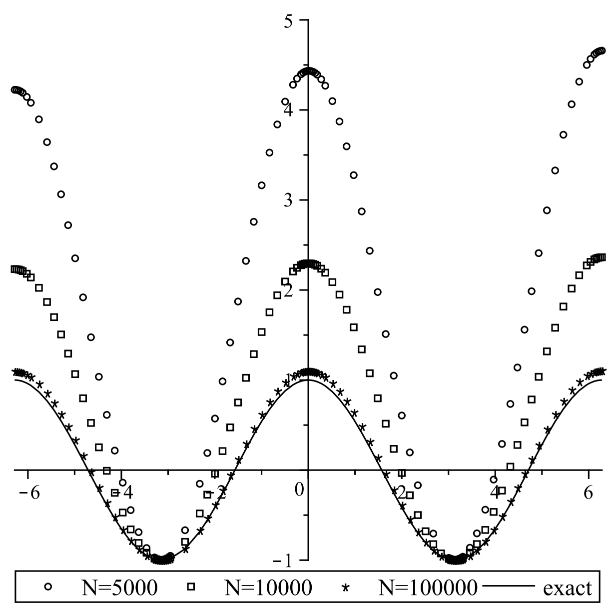


(a) L^2-norm of the error in $[0, 2\pi]$; N is a number of partial products



(b) Convergence of the infinite product expansion when $p = 0$, $n = 100$

Figure 9 : Convergence of the representation in (53) for different values of the parameter n; parameter $p = 0$ is fixed

$$\cos\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|4\Big(\alpha-\pi(p-n+1)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}} \tag{62}$$

where the plus sign corresponds to $\alpha\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right)$ and the minus sign corresponds to $\alpha\in\left[\frac{\pi}{2};\frac{3\pi}{2}\right]$.

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Similarly, when n in (51) and (52) is an odd number, i.e. $n= 2n+ 1$, then the expansion of the sine function can be written as

$$\sin\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha+\pi\big(2(p+n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}} \tag{63}$$

$$\sin\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha-\pi\big(2(p-n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(2p+1)^2-4k^2\right|\left|(2p+1)^2-4(k+1)^2\right|}} \tag{64}$$

where the plus sign corresponds to $\alpha\in[0;\pi)$ and the minus sign corresponds to $\alpha\in[\pi;2\pi]$.

The infinite product representations of the tangent function can also be obtained from the expansions in (61)-(64). Those are

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha+\pi\big(2(p+n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\left|4\Big(\alpha+\pi\big(p+n\big)\Big)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha+\pi\big(2(p+n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\left|4\Big(\alpha-\pi(p-n+1)\Big)^2-\pi^2(2k+1)^2\right|}$$

and

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha-\pi\big(2(p-n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\left|4\Big(\alpha+\pi\big(p+n\big)\Big)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\Big(2\alpha-\pi\big(2(p-n)+1\big)\Big)^2-\pi^2(2k+1)^2\right|}{\left|4\Big(\alpha-\pi(p-n+1)\Big)^2-\pi^2(2k+1)^2\right|}$$

where the plus sign corresponds to $\alpha\in\left[-\frac{\pi}{2};0\right)$ and the minus sign corresponds to $\alpha\in\left[0;\frac{\pi}{2}\right]$.

So far we assumed $\beta=\alpha\pm(2p+1)\pi$. Different representations can be derived for the relation between α and β as

$$\beta=\alpha\pm(2p+1)\frac{\pi}{2},\qquad p=0,1,2,...$$

yielding in the following

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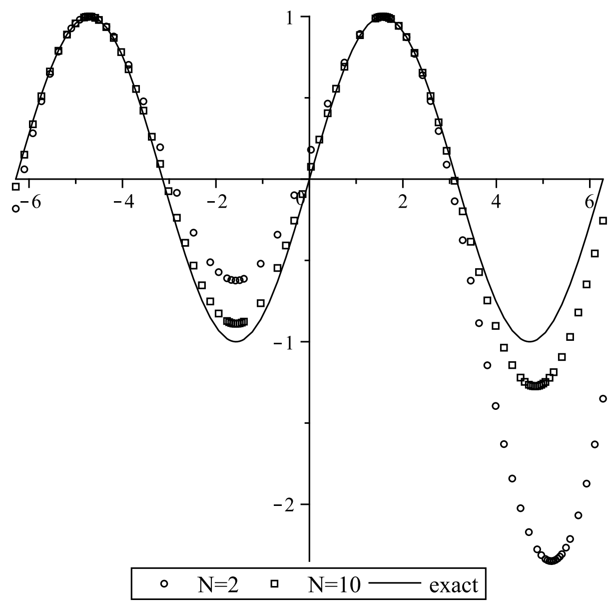
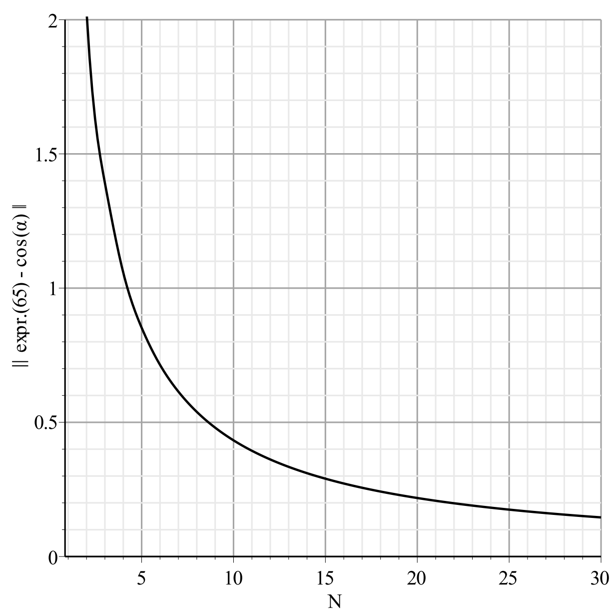


(a) Convergence with the 2nd and 10th partial products.



(b) L_2-norm of the error in $[0, 2\pi]$; N is a number of partial products

Figure 10 : Convergence of the representation in (65)

$$\sin\alpha = (-1)^{n+p}\left(-1+\prod_{k=0}^{\infty}\frac{\left[\left(2\alpha+\pi(2(p+n)-1)\right)^2-4\pi^2(2k+1)^2\right]^2}{\pi^4\left|(1+2p)^2-16k^2\right|\left|(1+2p)^2-16(k+1)^2\right|}\right) \tag{65}$$

$$\sin\alpha = (-1)^{n+p}\left(1-\prod_{k=0}^{\infty}\frac{\left[\left(2\alpha-\pi(2(p-n)+3)\right)^2-4\pi^2(2k+1)^2\right]^2}{\pi^4\left|(1+2p)^2-16k^2\right|\left|(1+2p)^2-16(k+1)^2\right|}\right) \tag{66}$$

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and

$$\cos\alpha = (-1)^{n+p}\left(-1+\prod_{k=0}^{\infty}\frac{16\left[\left(\alpha+\pi(p+n)\right)^2-\pi^2(2k+1)^2\right]^2}{\pi^4\left|(1+2p)^2-16k^2\right|\left|(1+2p)^2-16(k+1)^2\right|}\right) \tag{67}$$

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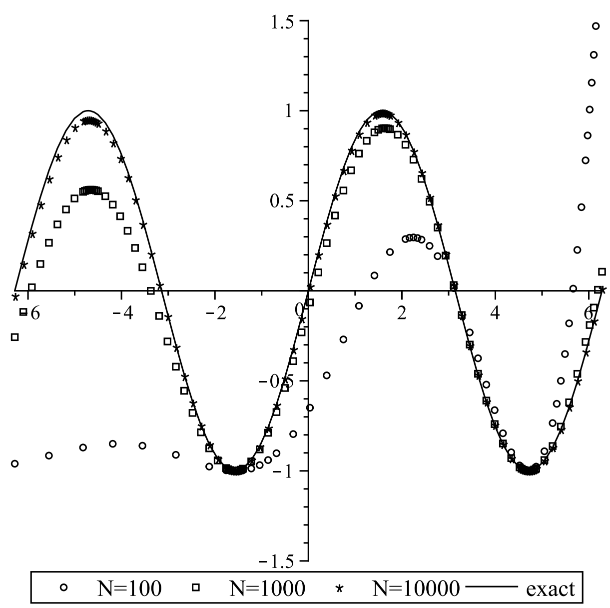
$$\cos\alpha = (-1)^{n+p}\left(1-\prod_{k=0}^{\infty}\frac{16\left[\left(\alpha-\pi(p-n+1)\right)^2-\pi^2(2k+1)^2\right]^2}{\pi^4\left|(1+2p)^2-16k^2\right|\left|(1+2p)^2-16(k+1)^2\right|}\right) \tag{68}$$

The convergence rate of the expansion in (65) with $n=1$, $p=0$ is illustrated in Figure 10. The sensitivity of the convergence in (65) to the choice of the parameters n and p is illustrated in Figure 11. The identities in (66)-(68) exhibit the similar behaviour.

By applying the power reduction formula to the identities in (67)-(68) one can derive alternative representations of the sine and cosine functions. When n and p are even numbers, i.e. $n=2n$ and $p=2p$, the expansions of the cosine and sine functions can be obtained in the form



(a) Convergence of the infinite product expansion when $p=0$, $n=100$; N is a number of partial products

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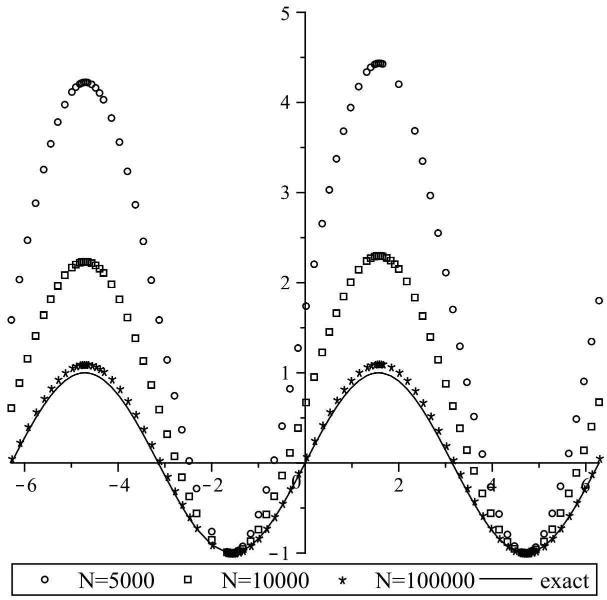

(b) Convergence of the infinite product expansion when $p=0$, $n=100$; N is a number of partial products

Figure 11 : Convergence of the representation in (65) for different values of the parameters n and $p=0$

$$\cos\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha+2\pi(p+n)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(1+4p)^2-16k^2\right|\left|(1+4p)^2-16(k+1)^2\right|}} \tag{69}$$

and

$$\sin\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha-\pi(2p-2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(1+4p)^2-16k^2\right|\left|(1+4p)^2-16(k+1)^2\right|}} \tag{70}$$

When n is odd and p is even, i.e. $n=2n+1$ and $p=2p$, one arrives at

$$\sin\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha+\pi(2p+2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(1+4p)^2-16k^2\right|\left|(1+4p)^2-16(k+1)^2\right|}} \tag{71}$$

and

$$\cos\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha-2\pi(p-n)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(1+4p)^2-16k^2\right|\left|(1+4p)^2-16(k+1)^2\right|}} \tag{72}$$

When n is even and p is odd, i.e. $n=2\,n$ and $p=2p+1$, we arrive at

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$$\sin\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha+\pi(2p+2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(3+4p)^2-16k^2\right|\left|(3+4p)^2-16(k+1)^2\right|}} \tag{73}$$

and

$$\cos\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha-2\pi\left((p-n)+1\right)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(3+4p)^2-16k^2\right|\left|(3+4p)^2-16(k+1)^2\right|}} \tag{74}$$

Finally, if both n and p are odd, i.e. $n= 2n+1$ and $p= 2p+1$, the expressions in (67)-(68) convert to

$$\cos\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha+2\pi(p+n+1)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(3+4p)^2-16k^2\right|\left|(3+4p)^2-16(k+1)^2\right|}} \tag{75}$$

and

$$\sin\alpha = \pm\sqrt{2}/2\prod_{k=0}^{\infty}\frac{4\left|\left(2\alpha-\pi(2p-2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\pi^2\sqrt{\left|(3+4p)^2-16k^2\right|\left|(3+4p)^2-16(k+1)^2\right|}} \tag{76}$$

In the above representations, the sign is selected similarly to those in (61)-(64)

Representations for the tangent function follow directly from (69)-(76). This provide us with the total number of sixteen expansions. Four of those are shown below.

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\left(2\alpha-\pi(2p-2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\left|\left(\alpha+2\pi(p+n)\right)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\left(2\alpha-\pi(2p-2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\left|\left(\alpha-2\pi(p-n)\right)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\left(2\alpha+\pi(2p+2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\left|\left(\alpha+2\pi(p+n)\right)^2-\pi^2(2k+1)^2\right|}$$

$$\tan\alpha = \pm\prod_{k=0}^{\infty}\frac{\left|\left(2\alpha+\pi(2p+2n+1)\right)^2-\pi^2(2k+1)^2\right|}{\left|\left(\alpha-2\pi(p-n)\right)^2-\pi^2(2k+1)^2\right|}$$

while the others can be derived in a similar manner.

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Acknowledgement

It is with gratitude that the author acknowledges the advises of Prof. Y.Melnikov in the process of preparation of this work.

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