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A Note on Basic Hypergeometric Function of N-Variable

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A NOTE ON BASIC HYPERGEOMETRIC FUNCTION OF N-VARIABLE

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A Note on Basic Hypergeometric Function of N-Variable

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Abstract- In this paper an attempt has been made to establish new transformation formula for the basic hypergeometric function of n variable.

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I. Introduction

Basic hypergeometric functions are among the most important functions with very diverse applications to Engineering,Physics and Mathematical Analysis.Nowadays, the importance of basic hypergeometric function and its unique role as the strategic resource in the Ramanujan's Mathematics became more important than the past time. The importance given to the Ramanujan's Mathematics has increased considerably in recent years and great deal of attention in basic hypergeometric function's literature is being given for the numerous topics that have been addressed by mathematicians working in the field of Basic hypergeometric functions, notably R.P Agarwal [1], G.E. Andrews and B.C. Berndt [2], G.E. Andrews [3, 4], R.Askey [5, 6],W.N. Bailey [7], S. Bhargava and Chandrashekar Adiga [8], R.Y.Denis [9], R.Y. Denis *et al.* [10], G.Gasper [11], V.K.jain [12, 13], M.S. Mahadeva Naika and B.N. Dharmendra [14], TH.M.Rassias and S.N. Singh [15], L.J.Slater [16], Pankaj Srivastava [17, 18], Pankaj Srivastava and Mohan Rudravarapu [19], A. Verma [20], G.N.Watson [21] and many others published large number of studies.In this paper, we are interested to develop certain new transformation formula for the basic hypergeometric function of n variable with the help of technique developed by Andrews [4], special cases also developed.

II. Notations and Definitions

A basic hypergeometric series of n-variables is defined as

$$\phi\left[\begin{matrix}(a_p):(b^1_{M_1});(b^2_{M_2});\ldots;(b^n_{Mn}) \\ \\ (c_t):(d^1_{N_1});(d^2_{N_2});\ldots;(d^n_{N_n})\end{matrix};x_1,x_2,\ldots;x_n\right]$$

Ref

[1] R.P.Agarwal, Certain basic hypergeometric identities associated with Mock theta functions, *Quarterly Journal Of Mathematics*, **20(1)** (1969), 121-128.

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$$= \sum_{m_1,m_2,\ldots,m_n\geq 0}^{\infty} \frac{[(a_p)]_{m_1+m_2+\ldots+m_n}[(b^1_{M_1})]_{m_1}\ldots[(b^n_{M_n})]_{m_n} x_1^{m_1}\ldots x_n^{m_n}}{[q]_{m_1}\ldots[q]_{m_n}[(c_t)]_{m_1+m_2+\ldots+m_n}[(d^1_{N_1})]_{m_1}\ldots[(d^n_{N_n})]_{m_n}}, \quad (2.1)$$

Where (a_p) stands for p-parameters $a_1, a_2, \ldots, a_p$. For the convergence of this series we require $\max(|q|, |x_1|, \ldots, |x_n|) < 1$. The q-shifted factorial is defined as

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\ldots(1-aq^{n-1}), & n = 1,2,3\ldots. \end{cases} \quad (2.2)$$

We also define

$$(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \text{ for } \quad |q| < 1. \quad (2.3)$$

The infinite product diverges when $a \neq 0$.
Also

$$(a_1, a_2, \ldots, a_r; q)_n = (a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n. \quad (2.4)$$

And

$$_1\phi_0\left[a; -; q, z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}, |z| < 1, |q| < 1. \quad (2.5)$$

The product formula

$$_1\phi_0\left(a; -; q, z\right) {}_1\phi_0\left(b; -; q, az\right) = {}_1\phi_0\left(ab; -; q, z\right). \quad (2.6)$$

$$_1\phi_1\left[a; c; q, c/a\right] = \frac{(c/a;q)_\infty}{(c;q)_\infty}. \quad (2.7)$$

III. Main Results

In this section we shall establish the following main result .

$$\phi\left[\begin{matrix} a, (a_p) : (b^1_{M_1}); (b^2_{M_2}); \ldots; (b^n_{Mn}) \\ c, (c_t) : (d^1_{N_1}); (d^2_{N_2}); \ldots; (d^n_{N_n}) \end{matrix} ; x_1, x_2, \ldots; x_n\right]$$

$$= \frac{[a]_\infty}{[c]_\infty} \sum_{r=0}^{\infty} \frac{[c/a]_r a^r}{[q]_r} \phi\left[\begin{matrix} (a_p) : (b^1_{M_1}); \ldots; (b^n_{Mn}) \\ (c_t) : (d^1_{N_1}); \ldots; (d^n_{N_n}) \end{matrix} ; x_1q^r, x_2q^r, \ldots; x_nq^r\right]. \quad (3.1)$$

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IV. Proof of (3.1)

By using (2.1) the left hand side of (3.1) can be put in the following form

$$\sum_{m_1,m_2,\ldots,m_n\geq 0}^{\infty}\frac{[a]_{m_1+m_2+\ldots+m_n}[(a_p)]_{m_1+m_2+\ldots+m_n}}{[q]_{m_1}\ldots[q]_{m_n}[c]_{m_1+m_2+\ldots+m_n}[(c_t)]_{m_1+m_2+\ldots+m_n}}\times\frac{[(b^1_{M_1})]_{m_1}\ldots[(b^n_{M_n})]_{m_n}x_1^{m_1}\ldots x_n^{m_n}}{[(d^1_{N_1})]_{m_1}\ldots[(d^n_{N_n})]_{m_n}}$$

$$=\frac{[a]_\infty}{[c]_\infty}\sum_{m_1,m_2,\ldots,m_n\geq 0}^{\infty}\frac{[(a_p)]_{m_1+m_2+\ldots+m_n}[(b^1_{M_1})]_{m_1}\ldots[(b^n_{M_n})]_{m_n}}{[q]_{m_1}\ldots[q]_{m_n}[(c_t)]_{m_1+m_2+\ldots+m_n}}\times\frac{(cq^{m_1+\ldots+m_n})_\infty x_1^{m_1}\ldots x_n^{m_n}}{[(d^1_{N_1})]_{m_1}\ldots[(d^n_{N_n})]_{m_n}(aq^{m_1+m_2+\ldots+m_n})_\infty}$$

$$=\frac{[a]_\infty}{[c]_\infty}\sum_{m_1,\ldots,m_n\geq 0}^{\infty}\frac{[(a_p)]_{m_1+\ldots+m_n}[(b^1_{M_1})]_{m_1}\ldots[(b^n_{M_n})]_{m_n}x_1^{m_1}\ldots x_n^{m_n}}{[q]_{m_1}\ldots[q]_{m_n}[(c_t)]_{m_1+m_2+\ldots+m_n}[(d^1_{N_1})]_{m_1}\ldots[(d^n_{N_n})]_{m_n}}\times\sum_{r=0}^{\infty}\frac{(c/a)_r a^r q^{r(m_1+m_2+\ldots+m_n)}}{[q]_r}.$$

By changing the order of summation in the above equation, the right hand side of (3.1) can be obtained.

V. Particular Cases

Putting $p=t=M_1=\ldots=M_n=N_1=\ldots=N_n=0$ in (3.1), we get

$${}_1\phi_1\left[\begin{array}{ll} a:-;\ldots;-; & \\ & ;x_1,x_2,\ldots;x_n \\ c:-;\ldots;-; & \end{array}\right]$$

$$=\frac{[a]_\infty}{[c]_\infty}\sum_{r=0}^{\infty}\frac{[c/a]_r a^r}{[q]_r}\times{}_0\phi_0\left[-;-;x_1q^r\right]\ldots{}_0\phi_0\left[-;-;x_nq^r\right]$$

$$=\frac{[a]_\infty}{[c]_\infty}\sum_{r=0}^{\infty}\frac{[c/a]_r a^r}{[q]_r[x_1q^r]_\infty\ldots[x_nq^r]_\infty}$$

$$=\frac{[a]_\infty}{[c]_\infty}\sum_{r=0}^{\infty}\frac{[c/a]_r a^r[x_1]_r\ldots[x_n]_r}{[q]_r[x_1]_\infty\ldots[x_n]_\infty}$$

$$=\frac{[a]_\infty}{[c]_\infty[x_1]_\infty\ldots[x_n]_\infty}{}_{n+1}\phi_0\left[x_1,\ldots x_n,c/a;-;a\right],$$

which is valid if $|a|<1$. (5.1)

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Putting $p = t = N_1 = \ldots = N_n = 0, M_1 = \ldots = M_n = 1, b_1^1 = b_1, b_1^2 = b_2, \ldots, b_1^n = b_n$ in (3.1), we get:

$$\phi\left[\begin{array}{ll} a : b_1; \ldots ; b_n & \\ & ; x_1, x_2, \ldots ; x_n \\ c : -; \ldots ; - & \end{array}\right]$$

$$= \frac{[a]_\infty}{[c]_\infty} \sum_{r=0}^{\infty} \frac{[c/a]_r a^r}{[q]_r} \times {}_1\phi_0\left[b_1; -; x_1 q^r\right] \ldots {}_1\phi_0\left[b_n; -; x_n q^r\right]$$

$$= \frac{[a]_\infty}{[c]_\infty} \sum_{r=0}^{\infty} \frac{[c/a]_r a^r}{[q]_r} \frac{[b_1 x_1 q^r]_\infty}{[x_1 q^r]_\infty} \ldots \frac{[b_n x_n q^r]_\infty}{[x_n q^r]_\infty}$$

$$= \frac{[a]_\infty [b_1 x_1]_\infty \ldots [b_n x_n]_\infty}{[c]_\infty [x_1]_\infty \ldots [x_n]_\infty} \times \sum_{r=0}^{\infty} \frac{[c/a]_r a^r [x_1]_r \ldots [x_n]_r}{[q]_r [b_1 x_1]_r \ldots [b_n x_n]_r}$$

$$= \frac{[a]_\infty [b_1 x_1]_\infty \ldots [b_n x_n]_\infty}{[c]_\infty [x_1]_\infty \ldots [x_n]_\infty} \times {}_{n+1}\phi_n\left[x_1, \ldots, x_n, c/a; b_1 x_1, \ldots, b_n x_n; a\right], \quad (5.2)$$

Putting $x_3 = x_4 = \ldots = x_n = 0$ in (5.2), we get

$$\phi\left[\begin{array}{ll} a : b_1; b_2; \ldots ; & \\ & ; x_1, x_2 \\ c : -; \ldots ; - & \end{array}\right] = \phi^{(1)}\left[a; b_1, b_2; c; x_1, x_2\right]$$

$$= \frac{[a]_\infty [b_1 x_1]_\infty [b_2 x_2]_\infty}{[c]_\infty [x_1]_\infty [x_2]_\infty} \times {}_3\phi_2\left[\begin{array}{ll} x_1, x_2, c/a; & \\ & ; a \\ b_1 x_1, b_2 x_2 & \end{array}\right], \quad (5.3)$$

This is the result due to Denis ([9],5.5).
Now, putting $b_2 = x_1/x_2, b_1 x_1 = c x_2$ in (5.3) and evaluating ${}_2\phi_1$ series on the right hand side with the help of Slater ([16];Appendix.IV.2), we get

$$\phi^{(1)}\left[a; c x_2/x_1, x_1/x_2; c; x_1, x_2\right] = \frac{[x_2 a]_\infty}{[x_2]_\infty}. \quad (5.4)$$

Now, substituting $p = t = 0,\ N_1 = \ldots = N_n = 1, M_1 = \ldots = M_n = 1, b_1^1 = b_1, b_1^2 = b_2, \ldots, b_1^n = b_n, d_1^1 = d_1, d_1^2 = d_2, \ldots, d_1^n = d_n$ in (3.1), we get:

$$\phi\left[\begin{array}{ll} a : b_1; \ldots ; b_n & \\ & ; x_1, x_2, \ldots ; x_n \\ c : d_1; \ldots ; d_n & \end{array}\right]$$

$$= \frac{[a]_\infty}{[c]_\infty} \sum_{r=0}^{\infty} \frac{[c/a]_r a^r}{[q]_r} \times {}_1\phi_1\left[b_1; d_1; x_1 q^r\right] \ldots {}_1\phi_1\left[b_n; d_n; x_n q^r\right], \quad (5.5)$$

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Taking $d_1 = b_1x_1q^r, d_2 = b_2x_2q^r, \ldots, d_n = b_nx_nq^r$ in (5.5), we get the following result after simplification

$$\phi\left[\begin{array}{l} a : b_1; \ldots ; b_n \\ \qquad\qquad\qquad\qquad ; x_1, x_2, \ldots ; x_n \\ c : b_1x_1q^r; \ldots ; b_nx_nq^r \end{array}\right]$$

$$= \frac{[a]_\infty [x_1]_\infty \ldots [x_n]_\infty}{[c]_\infty [b_1x_1]_\infty \ldots [b_nx_n]_\infty} \times {}_{n+1}\phi_n \left[b_1x_1, \ldots, b_nx_n, c/a; x_1, \ldots, x_n; a\right], \quad (5.6)$$

Now, putting $x_3 = x_4 = \ldots = x_n = 0$ in (5.6), we get

$$\phi\left[\begin{array}{l} a : b_1; b_2 \\ \qquad\qquad\qquad ; x_1, x_2 \\ c : b_1x_1q^r; b_2x_2q^r \end{array}\right]$$

$$= \frac{[a]_\infty [x_1]_\infty [x_2]_\infty}{[c]_\infty [b_1x_1]_\infty [b_2x_2]_\infty} \times {}_3\phi_2 \left[\begin{array}{l} b_1x_1, b_2x_2, c/a; \\ \qquad\qquad\qquad ; a \\ x_1, x_2 \end{array}\right], \quad (5.7)$$

Now, substituting $b_1x_1 = x_2, b_2x_2 = x_1/c$ in (5.7) and evaluating ${}_2\phi_1$ series in the right hand side with the help of Slater ([16]:Appendix IV.2), we get

$$\phi\left[\begin{array}{l} a : x_2/x_1; x_1/cx_2 \\ \qquad\qquad\qquad ; x_1, x_2 \\ c : x_2q^r; x_1q^r/c \end{array}\right] = \frac{(ax_1/c; q)_\infty}{(x_1/c; q)_\infty}. \quad (5.8)$$

A variety of similar interesting results can be scored.

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